

AD-A176 951

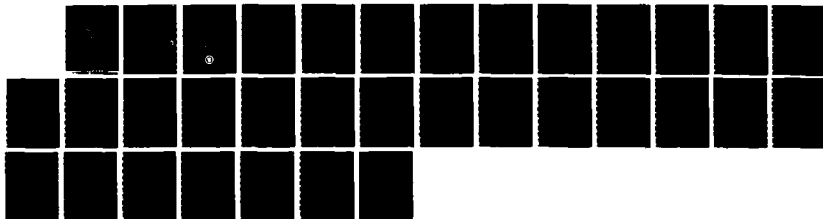
SOLUTION OF DENY CONVOLUTION EQUATION RESTRICTED TO
HALF LINE VIA A RANDO (U) PITTSBURGH UNIV PA CENTER
FOR MULTIVARIATE ANALYSIS A A ALZAID ET AL SEP 86

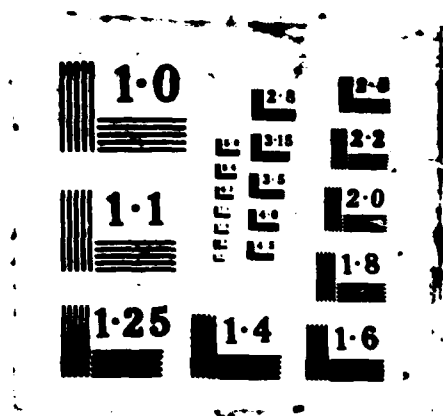
1/1

UNCLASSIFIED

TR-86-24 AFOSR-TR-87-0030 F49620-85-C-0000 F/G 12/1

NL





2

unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. AGENCY USE ONLY 87-0030	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Solution of Deny convolution equation restricted to half line via a random approach		5. TYPE OF REPORT & PERIOD COVERED technical - September 1986
		6. PERFORMING ORG. REPORT NUMBER 86-24
7. AUTHOR(s) Abdulhamid A. Alzaid, Ka-Sing Lau, C. Radhakrishna Rao and D.N. Shanbhag		8. CONTRACT OR GRANT NUMBER(s) F49620-85-C-0008
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Multivariate Analysis 515 Thackeray Hall University of Pittsburgh, Pittsburgh, PA 15260		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F, 2304/45
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research Dept. of the Air Force Bolling Air Force Base, DC 20332		12. REPORT DATE September
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 37112-11		13. NUMBER OF PAGES 29
		15. SECURITY CLASS. (of this report) unclassified
		16. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Deny's equation, ladder variables, modified Rao-Rubin condition, order statistics, queueing systems, random walk, Wiener-Hopf factorization.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A general solution of the Deny convolution equation restricted to half line is obtained using the concepts of random walk theory. The equation in question arises in several places in applied probability such as in queueing and storage theories and characterization problems of probability distributions. Some of the important applications are briefly discussed.

DD FORM 1 JAN 73 1473

unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD-A176 951

DTIC FILE COPY

DTIC
EXTRACTED
FEB 20 1987
E

AFOSR-TR- 87-0080

**SOLUTION OF DENY CONVOLUTION EQUATION
RESTRICTED TO HALF LINE VIA A RANDOM APPROACH**

BY

**Abdulhamid A. Alzaid,
King Saud University, Saudia Arabia**

Ka-Sing Lau —

C. Radhakrishna Rao*

University of Pittsburgh, U.S.A.

D.N. Shanbhag*

University of Sheffield, U.K.

Approved for public release;
Distribution is unlimited.

**AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF CONFIDENTIALITY TO DTIC**

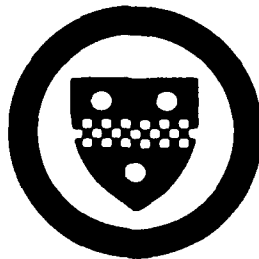
**This technical report has been reviewed and is
approved for public release IAW AFR 190-12.
Distribution is unlimited.**

MATTHEW J. KEEPER

Chief, Technical Information Division

Center for Multivariate Analysis

University of Pittsburgh



87 2 20_234

SOLUTION OF DENY CONVOLUTION EQUATION
RESTRICTED TO HALF LINE VIA A RANDOM APPROACH

BY

Abdulhamid A. Alzaid,
King Saud University, Saudia Arabia

Ka-Sing Lau —
C. Radhakrishna Rao*

University of Pittsburgh, U.S.A.

D.N. Shanbhag*
University of Sheffield, U.K.

September 1986

Technical Report 86-24

Center for Multivariate Analysis
515 Thackeray Hall
University of Pittsburgh
Pittsburgh, PA 15260

Accession For	
NTIS GRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	
By _____	
Distribution/ _____	
Availability Codes	
Dist	Special
A-1	

* The work of these authors was supported by Contract F49620-85-C-0008 of the Air Force Office of Scientific Research. The United States Government is authorized to reproduce and distribute reprints for governmental purposes, notwithstanding any copyright notation hereon.



1

SOLUTION OF DENY CONVOLUTION EQUATION
RESTRICTED TO HALF LINE VIA A RANDOM WALK APPROACH

by

Abdulhamid A. Alzaid, King Saud University, Saudi Arabia,

Ka-Sing Lau and C. Radhakrishna Rao,*
University of Pittsburgh, U.S.A. and

D. N. Shanbhag, University of Scheffield, U.K.

ABSTRACT

A general solution of the Deny convolution equation restricted to half line is obtained using the concepts of random walk theory. The equation in question arises in several places in applied probability such as in queueing and storage theories and characterization problems of probability distributions. Some of the important applications are briefly discussed.

Key Words and Phrases: Deny's equation, Ladder variables, Modified Rao-Rubin condition, Order statistics, Queueing systems, Random walk, Wiener-Hopf factorization.

AMS classification index: 60E05, 62E10

* Research sponsored by the Air Force Office of Scientific Research (AFSC) under Contract F49620-85-C-0008. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

1. INTRODUCTION

Let S be such that it equals either $R=(-\infty, \infty)$ or $R_+([0, \infty))$, σ be a σ -finite measure on S such that $\sigma(\{0\}^c) > 0$ and $H: S \rightarrow R_+$ be a Borel measurable function that is locally integrable (w.r.t. Lebesgue measure), satisfying the integral equation

$$H(x) = \int_S H(x+y)\sigma(dy) \text{ for a.a. } [L]x \in S, \quad (1.1)$$

where a.a.[L] refers to 'almost all w.r.t. Lebesgue measure'. The integral equation (1.1) when $S = R$ or $S = R_+$ has been extensively studied by many authors. In particular Deny (1961) has identified the general solution to the equation when $S = R$ and Lau and Rao (1982) when $S = R_+$. [Indeed, Deny (1961) arrives at the solution to the equation with $S = R$ assuming H to be continuous; however, as a straight forward corollary of Deny's theorem; the general solution when H is not necessarily continuous follows as pointed out by Rao and Shanbhag (1986).] A special case of Deny's result when σ is a probability measure and H is bounded was established earlier by Choquet and Deny (1960), while the Lau-Rao result subsumes various partial results given earlier by Marsaglia and Tubilla (1975), Shanbhag (1977), Shimizu (1978), Ramachandran (1979) and several others.

There also exist by now a number of alternative approaches for arriving at the solution to (1.1) either in the case of $S = R$ or $S = R_+$. [See, for example, Ramachandran (1982), Davies and Shanbhag (1984), Prakasa Rao and Ramachandran (1984), Lau and Rao (1984), Rao and Shanbhag (1986), and Alzaid, Rao and Shanbhag (1985)]. Both Deny's theorem and its variant given by Lau and Rao have applications in characterization problems of probability distributions and branches of applied probability such as reliability and renewal theories. For the details concerning

applications, the reader is referred to Feller (1966, Vol. 2, p.351), Shimizu (1978), Shanbhag (1977), Lau and Rao (1982), M.B. Rao and Shanbhag (1982), Rao (1983), Alzaid (1983) and Rao and Shanbhag (1986) and relevant references therein.

Consider now the integral equation

$$H(x) = \int_R H(x+y)\sigma(dy) \text{ for a.a. } [L]x \in R_+ \quad (1.2)$$

where σ is a σ -finite measure on R such that $\sigma(\{0\}^c) > 0$ and H is a non-negative locally integrable Borel measurable function on R . [It can be shown by choosing H and σ properly that the Lau-Rao (1982) equation is a special case of (1.2).] We show that H satisfying (1.2) has the representation

$$H(x) = \int_{(-\infty, 0)} H(x+y)\rho\delta(y) + \xi(x)e^{\eta x} \text{ for a.a. } [L]x \in R_+ \quad (1.3)$$

for some real η , a certain measure ρ concentrated on $(-\infty, 0)$ and a periodic function ξ having every nonzero support point of σ to be its period. Further the measure ρ arrived at in (1.3) is such that if $\sigma((-\infty, 0)) > 0$, then either both ρ and σ are non-arithmetic or they are arithmetic with the same span.

In view of this result, it is shown in Corollary 1 of Section 2, that Deny's (1961) result or its generalized versions given by Lau and Rao (1984) and Prakasa Rao and Ramachandran (1984) follow from the Lau-Rao (1982) theorem. Since the proof of (1.3) given in this paper is rather simple, the observation just made concerning the alternative proof of Deny's (1961) result is of importance, especially in the light of the elementary proof based on exchangeability of the Lau-Rao theorem given recently by Alzaid, Rao and Shanbhag (1985b); the possibility of

such an approach to Deny's theorem was pointed out by Alzaid, Rao and Shanbhag (1985). [The reader may find it instructive to compare the present proof based on the Lau-Rao theorem of Deny's theorem with the proof of the Lau-Rao theorem based on Deny's theorem as given by Rao and Shanbhag (1986).]

2. THE MAIN THEOREM

Let us consider the equation (1.2), and define, relative to σ , the measures σ_{1n} and σ_{2n} on R such that for every Borel set B of R and every integer $n \geq 1$

$$\begin{aligned}\sigma_{1n}(B) &= \sigma^n(\{(x_1, \dots, x_n) \in R^n: s_m \in R_+, m=1, \dots, n-1, s_n \in B\}) \\ \text{and } \sigma_{2n}(B) &= \sigma^n(\{(x_1, \dots, x_n) \in R^n: s_m \in (-\infty, 0), m=1, \dots, n-1, s_n \in B\}) \\ &= \sigma^n(\{(x_1, \dots, x_n) \in R^n: s_m > s_n, m=1, \dots, n-1, s_n \in B\}),\end{aligned}$$

where $s_m = x_1 + \dots + x_m$, $m \geq 1$ and σ^n is the product measure $\prod_{i=1}^n \sigma_i$ with $\sigma_i = \sigma$ in the notation of Burrill (1972). Following the analogy with concepts in random walk in probability theory, we may refer to the measures ρ and τ defined below respectively as the descending ladder height measure and the (weak) ascending ladder height measure relative to σ :

$$\rho(\cdot) = \sum_{n=1}^{\infty} \sigma_{1n}((-\infty, 0) \cap \cdot) \quad (2.1)$$

and

$$\tau(\cdot) = \sum_{n=1}^{\infty} \sigma_{2n}(R_+ \cap \cdot). \quad (2.2)$$

It is easy to check that if $\sigma((-\infty, 0)) > 0$, then the closed subgroup of R generated by $\text{supp}[\rho]$ is the same as that generated by $\text{supp}[\sigma]$. Also, if $\sigma(R_+) > 0$, then the closed subgroup of R generated by $\text{supp}[\tau]$ is the same as the one generated by $\text{supp}[\sigma]$. These observations in turn, imply

that if $\sigma((-\infty, 0)) > 0$, then either both ρ and σ are non-arithmetic or both are arithmetic with the same span, and if $\sigma(R_+) > 0$, then either both τ and σ are non-arithmetic or both are arithmetic with the same span. The main result of the paper is the following theorem.

THEOREM. Let the function H satisfying (1.2) be not a function that is equal to 0 a.e. $[L]$ on (α, ∞) , where $\alpha = \inf(\text{supp}[\sigma])$. Then ρ and τ as defined in (2.1) and (2.2) are Lebesgue-Stieltjes measures. [This is equivalent to the statement that they are both Radon measures and also to the statement that they are both regular measures.] Moreover,

$$H(x) = \int_{(-\infty, 0)} H(x+y)\rho(dy) + \xi(x)e^{\eta x} \text{ for a.a. } [L] x \in R_+ \quad (2.3)$$

where ξ is a non-negative periodic function with $\xi(\cdot) = \xi(\cdot+s)$ for every $s \in \text{supp}[\sigma]$, and η is a real number such that

$$\int_{R_+} e^{\eta x} \tau(dx) = 1. \quad (2.4)$$

[If η satisfying (2.4) does not exist, then we take ξ in (2.3) to be identically zero with η as an arbitrary real number.]

Proof. The case $\sigma((0, \infty)) = 0$ is trivial, since (1.2) implies, in view of the assumptions, that σ is a Lebesgue-Stieltjes measure in this case. We shall therefore assume that $\sigma((0, \infty)) > 0$. The problem remains invariant if H and $\sigma(dy)$ are replaced respectively by $H(x)e^{-\delta x}$ and $e^{\delta y}\sigma(dy)$ for any $\delta \in R$. In that case, there is no loss of generality in assuming that $\sigma((0, \infty)) > 1$ and hence in assuming that

$$0 < H^*(x) = \int_x^\infty H(y)dy < \infty \text{ for all } x \in \mathbb{R}$$

following essentially the arguments of Alzaid et al (1985). Since (1.2) is satisfied by H^* , the theorem follows if we prove it by replacing H by H^* . We can then assume without loss of generality that H itself is a positive decreasing continuous function. Now (1.2) gives us inductively

$$H(x) = \int_{(-\infty, 0)} H(x+y) \left(\sum_{n=1}^k \sigma_{1n} \right) (dy) + \int_{\mathbb{R}_+} H(x+y) \sigma_{1k} (dy) \\ k = 1, 2, \dots, \quad x \in \mathbb{R}_+. \quad (2.5)$$

This, in turn, implies that for each $x \in \mathbb{R}_+$, the sequence of the latter integrals on the rhs of (2.5) is a decreasing (nonnegative real) sequence and hence has a limit. Consequently, we can write

$$H(x) = \int_{(-\infty, 0)} H(x+y) \rho(dy) + \hat{H}(x), \quad x \in \mathbb{R}_+, \quad (2.6)$$

where

$$\hat{H}(x) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}_+} H(x+y) \sigma_{1k} (dy). \quad (2.7)$$

Since H is assumed to be positive continuous and the integral appearing on the r.h.s. of (2.7) has to be finite, it follows immediately that ρ is a Lebesgue-Stieltjes measure. Indeed (2.6) implies, in view of the decreasing nature of H , that $\rho(\mathbb{R}) \leq 1$. The fact that τ is a Lebesgue-Stieltjes measure is obvious in the case of $\sigma((-\infty, 0)) = 0$, in view of the assumption on H , since

$$\tau = \sigma \text{ and } \int_{\mathbb{R}_+} H(y) \sigma(dy) < \infty.$$

Furthermore, if $\sigma((-\infty, 0)) > 0$, then given any $s_0 \in (-\infty, 0) \cap \text{supp}[\sigma]$ and $y \in R_+$, we have

$$\begin{aligned} & \tau((-\infty, y + \frac{ns_0}{2})) \sigma((\frac{3s_0}{2}, \frac{s_0}{2})) \\ & \leq \tau((-\infty, y + \frac{(n+1)s_0}{2})) + \rho((\frac{3s_0}{2}, 0)), \quad n=0, 1, 2, \dots, \end{aligned} \quad (2.8)$$

which implies that

$$\tau((-\infty, y + \frac{ns_0}{2})) < \infty \text{ if } \tau((-\infty, y + \frac{(n+1)s_0}{2})) < \infty.$$

Since $\tau((-\infty, 0)) = 0$, we have then by reverse induction that $\tau((-\infty, y)) < \infty$ for each y . This implies that τ is a Lebesgue-Stieltjes measure. (For obtaining (2.8), the identity given in brackets immediately after the definition of σ_{2n} may be used.) This establishes the first part of the theorem.

Define now, for any Borel set $A(\subseteq R)$, $\sigma_{1n}^{(A)}$ to be the measure σ_{1n} with $s_1 \in R_+$ in its definition replaced by $s_1 \in R_+ \cap A$. Clearly then for every $x \in R_+$ and Borel set A the sequence

$$\left\{ \int_{R_+} H(x+y) \sigma_{1n}^{(A)}(dy) : n=1, 2, \dots \right\}$$

is a decreasing sequence of non-negative real numbers and hence has a limit. Consequently, we have for each $c \in R_+$

$$\hat{H}(x) = \lim_{n \rightarrow \infty} \int_{R_+} H(x+y) \sigma_{1n}^{[0, c]}(dy) + \lim_{n \rightarrow \infty} \int_{R_+} H(x+y) \sigma_{1n}^{(c, \infty)}(dy), \quad x \in R_+. \quad (2.9)$$

Since for every $x, c \in R_+$ and $n=1, 2, \dots$

$$\int_{R_+} H(x+y) \sigma_{1n}^{(c, \infty)}(dy) \leq \int_{(c, \infty)} H(x+y) \sigma(dy)$$

it follows that the latter limit on the r.h.s. of (2.9) tends to zero as $c \rightarrow \infty$. We can therefore write

$$\hat{H}(x) = \lim_{c \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{R_+} H(x+y) \sigma_{1n}^{[0,c]}(dy), \quad x \in R_+. \quad (2.10)$$

Observe that the interpretation of σ_{2n} given under brackets immediately after its definition yields for each $c \in R_+$ and $n=1,2,\dots$

$$\int_{R_+} H(x+y) \sigma_{1n}^{[0,c]}(dy) = \sum_{m=1}^n \int_{[0,c]} \int_{R_+} H(x+y+z) \sigma_{1,n-m}(dz) \sigma_{2m}(dy), \quad x \in R_+ \quad (2.11)$$

where σ_{10} is the probability concentrated on $\{0\}$, because

$$R^n = \bigcup_{m=1}^n \{(x_1, \dots, x_n) : s_r > s_m, r=1,2,\dots,m-1; s_r \geq s_m, r=m,\dots,n\}.$$

Since τ is a Lebesgue-Stieltjes measure and

$$0 \leq \int_{R_+} H(x+y+z) \sigma_{1n}(dz) \leq H(x), \quad x, y \in R_+, \quad n=1,2,\dots,$$

we get from (2.10) and (2.11), using the Lebesgue dominated convergence theorem in particular, that

$$\begin{aligned} \hat{H}(x) &= \lim_{c \rightarrow \infty} \int_{[0,c]} \hat{H}(x+y) \tau(dy) \\ &= \int_{R_+} \hat{H}(x+y) \tau(dy), \quad x \in R_+. \end{aligned} \quad (2.12)$$

Then (2.7) and (2.12) imply the second part of the theorem, in view of the Lau-Rao (1982) theorem, for which an elementary proof exists as shown by Alzaid et al (1985).

Remark 1. The above theorem remains valid with trivial changes in its proof even when the local integrability of H on R is replaced by that on (α, ∞) , where α is as defined in the statement of the Theorem.

Remark 2. The proof of the above theorem simplifies considerably if (1.2) is replaced by (1.1) with $S = R$. In that case, essentially by symmetry, the fact that ρ is a Lebesgue-Stieltjes measure implies that τ is a Lebesgue-Stieltjes measure. Also, the validity of the identity in (2.6) for all $x \in R$ implies that

$$\int_{(-\infty, 0)} H(x+y) \rho(dy) < \infty \text{ for all } x \in R$$

and hence, by symmetry again, that

$$\int_{R_+} H(x+y) \tau(dy) < \infty \text{ for all } x \in R.$$

In view of these observations, it follows that, in the present case, the Theorem follows even when the portion of the proof following the observation that ρ is a Lebesgue-Stieltjes measure until the identity (2.10) is deleted, provided $[0, c]$ and $\sigma_{1n}^{[0, c]}$ in (2.11) are replaced by R_+ and σ_{1n} and portions not relevant are deleted.

COROLLARY 1. [Deny (1961) and Lau and Rao (1984)]: If H satisfying (1.1) with $S = R$ is not a function that is equal to zero for a.e. $[L]$, then it has the representation

$$H(x) = \xi_1(x) \exp(\eta_1 x) + \xi_2(x) \exp(\eta_2 x) \text{ for a.a. } [L] x \in R$$

where ξ_1 and ξ_2 are periodic functions with $\xi_i(\cdot) = \xi_i(\cdot + s)$, $i=1,2$, for each point $s \in \text{supp } [\sigma]$ and η_i such that

$$\int_R \exp(\eta_i x) \sigma(dx) = 1, \quad i=1,2.$$

[For the uniqueness of the representation, one may assume for example that $\xi_2 \equiv 0$ if $\eta_1 = \eta_2$; Rao and Shanbhag (1986) have implicitly assumed this to be so in their proof of the Lau-Rao (1984) result based on Deny's theorem.]

Proof. As in the proof of the main Theorem, it is clear that there is no loss of generality in assuming H to be continuous and decreasing. In such a case, we get

$$H(x) = \int_{(-\infty, 0)} H(x+y)\rho(dy) + \xi(x)\exp(\eta x) \text{ for all } x \in \mathbb{R} \quad (2.13)$$

with ξ as a non-negative continuous periodic function on $\mathbb{R}$ with every non-zero point of $\text{supp}[\sigma]$ as its period and η as a real number. Clearly η in (2.13) satisfies

$$\int_{\mathbb{R}} \exp(\eta x) dx = 1$$

if $\xi \not\equiv 0$. Define now c^* to be equal to zero if $\xi \equiv 0$ and to be the supremum of the set C defined as the set of all non-negative c 's for which $H_c(x) \stackrel{\Delta}{=} H(x) - c\xi(x)\exp(\eta x) \geq 0$ for all $x \in \mathbb{R}$ and

$$H_c(x) = \int_{(-\infty, 0)} H_c(x+y)\rho(dy) + d \xi(x)\exp(\eta x), \quad x \in \mathbb{R}$$

with $d \geq 0$. Clearly C is compact by the dominated convergence theorem (on noting in particular that $0 \leq c, d \leq H(x_0)/\xi(x_0) < \infty$ when $\xi(x_0) \neq 0$) and hence $c^* \in C$; also

$$H_c(x) = \int_{(-\infty, 0)} H_{c^*}(x+y)\rho(dy), \quad x \in \mathbb{R}. \quad (2.14)$$

In view of the Lau-Rao (1982) theorem (which is now an obvious corollary of the main Theorem) and the definition of H_{c^*} , the asserted representa-

tion for H follows.

Remark 3. If σ is any Lebesgue-Stieltjes measure on R (not necessarily associated with an integral equation), then, using the arguments employed in the theorem to establish τ to be a Lebesgue-Stieltjes measure, we have that both ρ and τ as defined in (2.1) and (2.2) respectively are Lebesgue-Stieltjes measures when at least one of them is given to be so. In that case it is easily seen that

$$\sigma(B) + \tau * \rho(B) = \begin{cases} \rho(B) & \text{if } B \text{ is a Borel set of } (-\infty, 0) \\ \tau(B) & \text{if } B \text{ is a Borel set of } R_+ \end{cases}$$

which implies that

$$\sigma + \tau * \rho = \rho + \tau. \quad (2.15)$$

The equation (2.15) implies that for every relatively compact Borel subset B of R

$$\sigma(B) = \rho(B) + \tau(B) - \tau * \rho(B) \quad (2.16)$$

which may be viewed as Wiener-Hopf factorization of σ . In the proof of the Theorem, after observing that ρ and τ are Lebesgue-Stieltjes measures, one could have obviously appealed to either (2.15) or (2.16) to arrive at the result, given the condition

$$\int_{R_+} H(x+y) \tau(dy) < \infty \text{ for all } x \in R_+,$$

since this yields

$$\begin{aligned}
\hat{H}(x) &= H(x) - \int_{(-\infty, 0)} H(x+y) \rho(dy) \\
&= \int_R H(x+y) \sigma(dy) - \int_R H(x+y) \rho(dy) \\
&= \int_R [H(x+y) - \int_R H(x+y+z) \rho(dz)] \tau(dy) \\
&= \int_{R_+} \hat{H}(x+y) \tau(dy), \quad x \in R_+.
\end{aligned}$$

In the case of (1.1) with $S = R$, we can take

$$\int_{R_+} H(x+y) \tau(dy) < \infty$$

as observed in Remark 2, and hence the present approach remains valid.

This provides us with a further proof of the result of Corollary 1.

Remark 4. In view of Remark 3, we may raise the question as to whether there exists, under the hypothesis of the Theorem, a situation in which for some Borel $B \subset R_+$ with positive Lebesgue measure

$$\int_{R_+} H(x+y) \tau(dy) = \infty \text{ for all } x \in B$$

so that the argument based on the Wiener-Hopf factorization as it stands does not remain valid. The answer to this question is in the affirmative as is shown in the following example.

EXAMPLE 1. Let σ be the probability measure concentrated on $\{-1, 0, 1, 2, \dots\}$ such that its moment generating function is given by

$$M(t) = 1 + \alpha e^{-t} (1 - e^{-t})^\beta, \quad t \leq 0,$$

where α and β are fixed positive numbers such that $1 < \beta < 2$ and $0 < \alpha\beta \leq 1$.

[This moment generating function but for a location change was considered

earlier by Seneta (1968) in connection with a certain problem in branching processes.] Let H be such that

$$H(x) = \begin{cases} [x+1] & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $[x+1]$ is the integer part of $x + 1$. Hence it follows that H satisfies the hypothesis of the theorem and

$$H(x) = \int_{\mathbb{R}} H(x+y)\sigma(dy), \quad x \in \mathbb{R}_+.$$

In this case ρ is the probability measure concentrated on $\{-1\}$ and, in view of the Wiener-Hopf factorization of σ , τ is the probability measure with the moment generating function

$$M^*(t) = 1 - \alpha(1 - e^t)^{\beta-1}, \quad t \leq 0.$$

The expression for $M^*(t)$ implies that τ in question has an infinite mean and hence we have here

$$\int_{\mathbb{R}_+} H(x+y)\tau(dy) = \infty \text{ for each } x \in \mathbb{R}_+.$$

Thus we have a simple counter-example supporting the claim made.

From this example we can obviously produce examples with H satisfying an additional condition of being positive continuous and decreasing.

[It is also worth pointing out here that this example illustrates the validity of the statement of Feller (1966) concerning ladder height means in a random walk that is induced by variables with zero mean on page 380 immediately above his Theorem 2: in the present case, we have the mean of σ to be zero, the mean of the descending ladder height to be finite, and the mean of the ascending ladder height to be infinite.]

Remark 5. In (2.3), we can choose ξ to be a constant if σ is non-arithmetic and as a periodic function with period λ if σ is arithmetic with span λ .

Remark 6. For the σ appearing in the Theorem, the Wiener-Hopf factorization given by (2.15) implies that for every real θ and x, y such that $0 \leq x < y < \infty$

$$\int_{[x,y]} \exp(\theta z) \tau(dz) \int_{[-x,0]} \exp(\theta z) \rho(dz) \leq \int_{[0,y]} \exp(\theta z) \tau(dz)$$

yielding that

$$\tau^*(\theta) \triangleq \int_{R_+} \exp(\theta z) \tau(dz) < \infty$$

whenever

$$\rho^*(\theta) \triangleq \int_{(-\infty,0)} \exp(\theta z) \rho(dz) > 1.$$

By symmetry, we have also $\rho^*(\theta) < \infty$ whenever $\tau^*(\theta) > 1$. The Wiener-Hopf factorization of σ also gives

$$\sigma^*(\theta) + \tau^*(\theta) \rho^*(\theta) = \tau^*(\theta) + \rho^*(\theta), \quad \theta \in R \quad (2.17)$$

where

$$\sigma^*(\theta) = \int_R \exp(\theta x) \sigma(dx).$$

Then (2.17) implies the following. If we assume that $\tau^*(\theta) > 1$ and $\rho^*(\theta) > 1$ (and hence that $1 < \tau^*(\theta) < \infty$ and $1 < \rho^*(\theta) < \infty$), then we get that $1 - \sigma^*(\theta) = (1 - \tau^*(\theta))(1 - \rho^*(\theta)) > 0$ and hence $\sigma^*(\theta) < 1$. However, the definitions of ρ^* and τ^* imply that $\rho^*(\theta)$ and $\tau^*(\theta)$ denote respectively $\rho((-\infty, 0))$ and $\tau(R_+)$ corresponding to the case with $e^{\theta x} \sigma(dx)$ in place of $\sigma(dx)$. If $\sigma^*(\theta) < 1$, we have obviously the measure relative to $e^{\theta x} \sigma(dx)$

(i.e., the measure for which each Borel set B receives the value $\int_B e^{\theta x} \sigma(dx)$) to be bounded by 1 and hence the ladder height measure interpretations of τ and ρ yields that $\rho^*(\theta) < 1$ and $\tau^*(\theta) < 1$. Consequently it follows that for each θ at least one of $\tau^*(\theta)$ and $\rho^*(\theta)$ should be less than or equal to 1. (Otherwise, we arrive at a contradiction.) This, in turn, implies on using in particular the dominated or monotone convergence theorem according as the case that for a θ_0 such that $\tau^*(\theta_0)$ (under $\sigma((0, \infty)) > 0$) or $\rho^*(\theta_0)$ equals 1, we have either $\tau^*(\theta_0) = 1$ and $\rho^*(\theta_0) \leq 1$ or $\rho^*(\theta_0) = 1$ and $\tau^*(\theta_0) \leq 1$ and hence $\sigma^*(\theta_0) = 1$; from this we conclude that (2.4) in the case of $\sigma((0, \infty)) > 0$ implies $\sigma^*(n) = 1$.

Remark 7. From what is given in Remark 6, we conclude that the measure σ in the Theorem satisfies either $\sigma((-\infty, 0)) \leq 1$ or $\sigma(R_+) \leq 1$.

COROLLARY 2. Let H be as in the Theorem and for each $x \in R_+$ and $\rho^{(x)}$ denote the ρ measure on R with an alteration that the s_r 's involved in its definition are replaced by $s_r + x$. Then, for each $x \in R_+$, $\rho^{(x)}$ is a Lebesgue-Stieltjes measure and, given that

$$H(x) \text{ and } \int_{(-\infty, 0)} H(x+y) \rho(dy)$$

are right continuous on R_+ and also that $H(x)$ is locally bounded on R_+ , we have for each $x \in R_+$

$$H(x) = \int_{(-\infty, 0)} H(y) \rho^{(x)}(dy) + \xi(x) \int_{[-x, 0)} e^{n(x+y)} \left(\sum_{n=0}^{\infty} \rho^*(n) \right) (dy) \quad (2.18)$$

for some non-negative real periodic function ξ such that it is a constant if σ is non-arithmetic and a function with period λ if σ is arith-

metric with span λ , and n is as defined in the Theorem. (If n satisfying (2.4) does not exist, we take n to be any arbitrary real number with $\xi \equiv 0$; also, we define ρ^{*0} to be the probability measure that is degenerate at zero.) Moreover, if f is any Borel measurable function on $(-\infty, 0)$ such that $\int_{(-\infty, 0)} |f(y)| \rho^{(x)}(dy) < \infty$ for each $x \in R_+$ and ξ is any Borel measurable function of the form of ξ defined above, then the function $\hat{H}$ given by

$$\begin{aligned} \hat{H}(x) &= \int_{(-\infty, 0)} f(y) \rho^{(x)}(dy) + \xi(x) \int_{[-x, 0)} e^{n(x+y)} \left(\sum_{n=0}^{\infty} \rho^{*n} \right)(dy), \quad x \in R_+ \\ &= f(x), \quad x \in (-\infty, 0) \end{aligned}$$

satisfies (1.2) (even with 'a.a.[L]' replaced by 'all'.).

Proof. We have that ρ is a Lebesgue-Stieltjes measure concentrated on $(-\infty, 0)$. Clearly, for each $x \in R_+$ and Borel set B

$$\rho^{(x)}(B+x) = \sum_{n=1}^{\infty} \rho_n^{(x)}((-\infty, -x) \cap B) \quad (2.19)$$

where $\rho_1^{(x)} = \rho$ and for each $n \geq 2$, $\rho_n^{(x)}$ is the convolution of measures $\rho_{n-1}^{(x)}([-x, 0) \cap \cdot)$ and ρ (and hence is a well defined Lebesgue-Stieltjes measure concentrated on $(-\infty, -x)$). For any bounded interval $[\alpha, \beta]$, we have

$$\rho^{(x)}([\alpha, \beta] + x) \leq \sum_{n=1}^{\infty} \rho^{*n}([\alpha, \beta]) < \infty$$

(with ρ^{*n} as the n -fold convolution of ρ with itself) and hence it follows that $\rho^{(x)}$ is a Lebesgue-Stieltjes measure. Now, if $H(x)$ and $\int_{(-\infty, 0)} H(x+y) \rho(dy)$ are right-continuous on R_+ , then (2.1) is valid with

'a.a.[L]' replaced by 'all' and ξ as defined in (2.18). From this we get for each positive integer k and $x \in R_+$.

$$H(x) = \int_{(-\infty, -x)} H(x+y) \left(\sum_{n=1}^k \rho_n^{(x)} \right) (dy) + \int_{[-x, 0)} H(x+y) \rho^{*k}(dy) \\ + \xi(x) \int_{[-x, 0)} e^{n(x+y)} \left(\sum_{n=0}^{k-1} \rho^{*n} \right) (dy) \quad (2.20)$$

on noting that the restrictions of ρ^{*n} and $\rho_n^{(x)}$ to $[-x, 0)$ are identical for each $n \geq 1$. On taking, under the assumption of local boundedness of $H(x)$ on R_+ , the limit as $k \rightarrow \infty$ in (2.20), we arrive at the identity (2.18). It is easy to verify that $\hat{H}$ of the Theorem satisfies (1.2) (even with 'a.a.[L]' replaced by 'all') on using the Wiener-Hopf factorization and the observation in Remark 6 that (2.4) implies $\sigma^*(n) = 1$ (and $\rho^*(n) \leq 1$).

Remark 7. If we have $H(x)$ to be monotonic on R (or on $[\alpha, \beta]$ if α defined in the Theorem is finite), then the above Corollary 2 remains valid with the assumption of the right-continuity of $H(x)$ and $\int_{(-\infty, 0)} H(x+y) \rho(dy)$ on R_+ replaced by the assumption that $H(x)$ is right-continuous on R (or on $[\alpha, \infty)$ if $\alpha > -\infty$). Moreover, if $\alpha > -\infty$, the corollary remains valid with the right continuity of $H(x)$ and $\int_{(-\infty, 0)} H(x+y) \rho(dy)$ on R_+ together with the local boundedness of $H(x)$ on R_+ replaced by the assumption that $H(x)$ is locally bounded and right-continuous on $[\alpha, \infty)$. It is also worth noting here that if the local boundedness of H is deleted from the statement of the corollary, then the corollary remains valid subject to the alteration that 'for every $x \in R_+$ ' immediately above (2.18) is replaced by "for a.a.[L] $x \in R_+$ '

since by Fatou's lemma and the local integrability of H , we have in (2.20), as $k \rightarrow \infty$

$$\int_{[-x,0)} H(x+y) \rho^{*k}(dy) \rightarrow 0 \text{ for a.a. } [L]x \in R_+.$$

COROLLARY 3. Let σ and H be as in the theorem with $\alpha > -\infty$ and let $\rho^{(x)}$ for each $x \in R_+$ be as defined in Corollary 2. Assume that the restriction of H to $[\alpha, \infty)$ is locally bounded and right-continuous. Define $\Theta = \{\theta: \sigma^*(\theta) = 1\}$ with σ^* as defined in Remark 6. Then Θ is non-empty (without having more than two members), and if $\sigma((-\infty, 0)) > 0$ and θ_0 is the only member in Θ or is the smaller of the two members in Θ , we have

$$\mu_{\theta_0, \sigma} \stackrel{\Delta}{=} \int_{[\alpha, \infty)} x e^{\theta_0 x} \sigma(dx) \leq 0.$$

Moreover, in this case, we have the assertions of Corollary 2 to be valid with

- (i) $\xi \equiv 0$ if Θ is a singleton and $\mu_{\theta_0, \sigma} < 0$,
- (ii) $\int_{[-x,0)} e^{n(x+y)} \left(\sum_{n=0}^{\infty} \rho^{*n} \right) (dy)$ replaced by $x e^{\theta_0 x} - \int_{(-\infty,0)} y e^{\theta_0 y} \rho^{(x)}(dy)$ if Θ is a singleton and $\mu_{\theta_0, \sigma} = 0$,
- (iii) $\int_{[-x,0)} e^{n(x+y)} \left(\sum_{n=0}^{\infty} \rho^{*n} \right) (dy)$ replaced by $e^{\theta_1 x} - \int_{(-\infty,0)} e^{\theta_1 y} \rho^{(x)}(dy)$ with $\theta_1 \in \Theta$ and $\theta_1 > \theta_0$ if Θ is a doubleton.

Proof. If $\mu((-\infty, 0)) > 0$, we have obviously in view of the fact that $\alpha > -\infty$ some real number δ such that $\rho^*(\delta) = 1$ and if $\mu((-\infty, 0)) = 0$, we have clearly $\xi \neq 0$ and, hence from the Theorem, n exists such that $\tau^*(n) = 1$, where ρ^* and τ^* are as defined in Remark 6. From what is mentioned in

Remark 6, we have $\sigma^*(\delta) = 1$ if $\rho^*(\delta) = 1$ and $\sigma^*(\eta) = 1$ if $\tau^*(\eta) = 1$. Consequently it follows that Θ is non-empty. Clearly the assumptions of the Theorem imply $\sigma(\{0\}) < 1$ and hence in view of Remark 6 we have $\delta \leq \eta$ whenever $\rho^*(\delta) = 1$ and $\tau^*(\eta) = 1$. From the Wiener-Hopf factorization concerning probability measures it is clear that from any λ , $\sigma^*(\lambda) = 1$ if and only if either $\rho^*(\lambda) = 1$ or $\tau^*(\lambda) = 1$. It is therefore clear that Θ can at most have two points. (This latter fact also follows directly from properties of moment generating functions of probability measures.) From Feller's (1966) Theorem 2 on page 380 and the remark following Theorem 2 on page 396 it follows that if $\rho^*(\theta_0) = 1$, then $\mu_{\theta_0, \sigma} \leq 0$. If θ_0 is as defined in the statement of the corollary and $\sigma((-\infty, 0)) > 0$, then, from the fact that $\rho^*(\delta) = 1$ and $\tau^*(\eta) = 1$ imply that $\delta \leq \eta$, we have $\rho^*(\theta_0) = 1$ and hence $\mu_{\theta_0, \sigma} \leq 0$. We shall now establish the assertions (i), (ii) and (iii). Clearly, the assertion (i) is obvious from Corollary 2 since the remark in Feller (1966) following Theorem 2 on page 396 implies $\tau^*(\theta_0) < 1$ if $\mu_{\theta_0, \sigma} < 0$ and hence there is no η such that $\tau^*(\eta) = 1$. On the other hand, if we have the situation either as in the assertion (ii) or the assertion (iii), we have an η such that $\tau^*(\eta) = 1$. In the case of $\mu_{\theta_0, \sigma} = 0$, Feller's Theorem 2 on page 380 implies the existence of $\eta = \theta_0$ and in the case of Θ containing two points clearly we have $\eta = \theta_1 > \theta_0$ with $\theta_1 \in \Theta$ in view of what we have observed earlier. Let ξ be as in Corollary 2. We shall consider here this to be a function defined on $\mathbb{R}$. Clearly the restriction of ξ to $[\alpha, \infty)$ is right-continuous, locally integrable and locally bounded. Define now in the case of assertion (ii)

$$H(x) = \begin{cases} \xi(x)(x+\alpha)e^{\theta_0 x} & \text{if } x \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

and in the case of assertion (iii)

$$H(x) = \begin{cases} \xi(x)e^{\theta_1 x} & \text{if } x \geq \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Observe that in both the cases H satisfies (1.2) and we get

$$H(x) = \int_{(-\infty, 0)} H(x+y)\rho(dy) + c\xi(x), \quad x \in R_+$$

for some positive constant c (which need not be the same in the two cases). In both the cases, we have H satisfying the conditions required to arrive at (2.18) of Corollary 2 and hence the equation in question to be valid with $c\xi$ replacing ξ . Since $c^{-1}\xi$ is of the form of ξ , the assertions (ii) and (iii) easily follow.

COROLLARY 4. Let $\{(v_n, w_n): n=0,1,2,\dots\}$ be a sequence of vectors of non-negative real components such that at least one $v_n \neq 0$ and $w_0 > 0$ and $\lambda > 0$. Further, let k and τ be positive integers such that the largest common divisor of k and those n for which $w_n > 0$ be τ . Then the sequence $\{(v_n, w_n)\}$ as defined satisfies the recurrence equations

$$v_n + k = \sum_{m=0}^{\infty} w_m v_{m+n}, \quad n=0,1,\dots \quad (2.21)$$

and for some integer k_1 that is an integer multiple of τ such that $k_1 > k$,

$$\lambda v_{n+k} = v_{n+k_1}, \quad n=0,1,\dots,k \quad (2.22)$$

only if

$$v_n = q_n \lambda^{n/(k_1-k)}, \quad n=0,1,\dots$$

for some non-negative periodic sequence q_n with period τ .

Proof. It is sufficient if the result is established for $\tau = 1$.

Defining H and σ such that

$$H(x) = \begin{cases} v_{[x]+k}, & x \geq k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sigma(\{n\}) = w_{n+k}, \quad n \geq -k, \quad \sigma(\{-k, -k+1, \dots\}^c) = 0,$$

where $[x]$ denotes the integer part of x , we see that the conditions for the validity of (2.3) with 'a.a.[L]' replaced by 'all' are met. The equation in question implies, in view of (2.22) and the fact that $\sigma(\{-k\}) > 0$, that $H(x+k_1-k) = \lambda H(x)$ for all $x \geq -k$. Since we can find an integer r such that $r(k_1-k) \geq k$, we have subsequently

$$H(x) = \int_{[-k, \infty)} H(x+r(k_1-k)+y) \lambda^{-r} \sigma(dy), \quad x \in R_+ \quad (2.23)$$

with $r(k_1-k) - k \geq 0$. There is no loss of generality in assuming that $\sigma(\{0\}) > 0$ and hence that the measure σ^* defined such that

$$\sigma^*(B) = \lambda^{-r} \sigma(B-r(k_1-k)) \text{ for every Borel set } B$$

is arithmetic with unit span. Consequently, the Lau-Rao Theorem (which is now a corollary of our theorem) implies, in view of (2.23) and the periodicity of $H(x) \lambda^{-x/(k_1-k)}$, $x \geq -k$, the required result.

Remark 8. It is also possible to prove the result of the corollary by using the Perron-Frobenius Theorem given in Seneta (1973, pp.1-2). For the details, the reader is referred to Alzaïd (1983).

Remark 9. If H is as in the Theorem such that it is locally bounded and right-continuous and α defined in the Theorem is finite and $\sigma(\{\alpha\}) > 0$, then, for some $\beta > |\alpha|$ with β as an integer multiple of the span of σ in case σ is arithmetic, and $\lambda > 0$,

$$\lambda H(x+|\alpha|) = H(x+\beta) \text{ for all } x \in [0, |\alpha|]$$

only if $H(x) = \xi(x)\exp(\eta x)$, $x \in [\alpha, \infty)$ with ξ as a periodic function with every non-zero support point of σ to be its period and η such that

$\int_{[\alpha, \infty)} \exp(\eta x) \sigma(dx) = 1$ and $\eta = -(\beta - \alpha) \log \lambda$. This result is a version of the above Corollary 4 in the case of arithmetic σ , and it follows via a modified version of the proof of the corollary on using the result of Corollary 2 when $H(\cdot)$ is replaced by $H(x_0 + \cdot)$ with $x_0 \in \mathbb{R}_+$ in the case of nonarithmetic σ . Whether or not the result of the present remark is valid without the condition $\sigma(\{\alpha\}) > 0$ is an interesting question to which we have not found any answer as yet.

Remark 10. Now, if we have k to be a positive integer and $\{(v_n, w_n): n=0,1,2,\dots\}$ to be a sequence of vectors with non-negative real components such that $w_0 > 0$, $v_n \neq 0$ for some n , and the largest common divisor of k and those n for which $w_n > 0$ equals 1, then it follows that the result of Corollary 3 identifies the solution to the system of equations

$$v_{n+k} = \sum_{m=0}^{\infty} w_m v_{m+n}, \quad n=0,1,2,\dots \quad (2.24)$$

given the sequence $\{w_n: n=0,1,2,\dots\}$ and the values of $v_0, v_1, \dots, v_{k-1}$.

Indeed, if we define $D = \{b: b > 0, b^k = \sum_{n=0}^{\infty} b^n w_n\}$ and $\{f_{i,r}: i \geq k, r=0,1,\dots,k-1\}$ to be the sequence of absorption measures corresponding to the non-negative matrix T (in the sense of Seneta (1973)) with state space $\{0,1,2,\dots\}$ and the (i,j) th element as

$$T_{ij} = \begin{cases} \delta_{ij} & \text{if } i,j = 0,1,\dots,k-1 \\ w_{j-i+k} & \text{if } i = k,k+1,\dots \text{ and } j \geq i-k \\ 0 & \text{otherwise,} \end{cases}$$

where δ_{ij} is the Kronecker delta, then the corollary yields that the system of equations is satisfied if and only if D is nonempty, $f_{n,0}, \dots, f_{n,k-1}$ are finite for each $n \geq k$, and one of the following condition holds:

(i) D has only one point, $\sum_{n=0}^{\infty} (n-k)b^n w_n < 0$ for $b \in D$, and

$$v_n + f_{n,0}v_0 + \dots + f_{n,k-1}v_{k-1}, \quad n=k,k+1,\dots$$

(ii) D has only one point, $\sum_{n=0}^{\infty} (n-k)b^n w_n = 0$ for $b \in D$ and for

some $c \geq 0$

$$v_n = f_{n,0}(v_0 - c \cdot 0 \cdot b^0) + \dots + f_{n,k-1}(v_{k-1} - c(k-1)b^{k-1}) + cnb^n,$$

$$n=k,k+1,\dots,$$

with $b \in D$.

(iii) D contains two points and for some $c \geq 0$

$$v_n = f_{n,0}(v_0 - cb^0) + \dots + f_{n,k-1}(v_{k-1} - cb^{k-1}) + cb^n, \quad n=k,k+1,\dots,$$

with b as the larger of the two members of D .

A direct and substantially simpler proof of the result here without involving the Wiener-Hopf factorization can obviously be given (see, for example, Alzaïd (1983)).

3. SOME COMMENTS ON PREVIOUS RESULTS

We shall now discuss in the light of our findings some existing results and suggest possible extensions.

3.1 ARNOLD (1980):

Let $Y_{1,n} \leq \dots \leq Y_{nn}$ denote the n ordered observations in a random sample of size n from a non-degenerate distribution F concentrated on the set of non-negative integers. Arnold (1980) raised the question as to whether the independence of the random variable $Y_{2,n} - Y_{1,n}$ and the event $\{Y_{1,n} = m\}$ for a fixed integer $m > 1$ implies the distribution F to be geometric (or shifted geometric). In this case, the property is equivalent to a recurrence relation of the type (2.24) subject to a modification that w_n 's are given to be certain functions of v_n 's. In view of the modification involved, we can not obviously apply our result in Remark 10 directly to identify the solution v_n to the system of equations in the present case. However, the result of Corollary 4 shows that, under some mild conditions assuring certain points to be atoms of F , the independence of $Y_{2,n} - Y_{1,n}$ and $\{Y_{1,n} = m\}$ together with the condition

$$P\{Y_{2,n} - Y_{1,n} > j | Y_{1,n} = m\} = P\{Y_{2,n} - Y_{1,n} > j | Y_{1,n} = m + m'\}, \quad j=0,1,\dots,m-1$$

for some fixed integer $m' > 0$ characterizes a geometric distribution.

This extends a result of Sreehari (1983) showing that the independence of $Y_{2,n} - Y_{1,n}$ and $\{Y_{1,n} = m\}$ and the independence of $Y_{2,n} - Y_{1,n}$ and $\{Y_{1,n} = m + m'\}$ for some fixed integer $m' > 0$ characterizes, under some mild conditions, a geometric distribution.

3.2 KRISHNAJI (1974):

In view of the result in Remark 10, it is seen that Theorem 4 of Krishnaji (1974) is not correct. This also follows from the counter-example given by Patil and Taillie (1979). The error in Krishnaji's argument appears in the last sentence of the proof in which it is claimed that since $X = \Lambda \exp\{\Lambda(\theta-1)\}$ is degenerate, Λ has to be degenerate. We may however point out here that Krishnaji's Theorem with the portion " $G(t)$ is non-negative for all real t " in it replaced by " $G(t)$ is infinitely divisible" is valid.

3.3 SHANBHAG AND TAILLIE (1979):

The following result is an extended version of the Shanbhag-Taillie (1979) result and it is a trivial corollary of our result of Corollary 4:

"Let $\{(a_x, b_x): x=0,1,\dots\}$ be a sequence vectors with non-negative real components such that $a_x > 0$ for all x and $b_0 > 0$. Let (X,Y) be a random vector with non-negative integer-valued components such that for each x with $P\{X=x\} > 0$, we have

$$P\{Y=y|X=x\} = \frac{a_y b_{x-y}}{c_x}, \quad y=0,1,\dots,x,$$

where $\{c_x\}$ is the convolution of $\{a_x\}$ and $\{b_x\}$. Assume that $P\{X-Y=k_0\} > 0$ and $P\{X-Y=k_0+k_1\} > 0$. Then the following conditions are equivalent.

- (i) Y and $X-Y$ are independent.
- (ii) $P\{Y=y\} = P\{Y=y|X-Y=k_0\}$, $y=0,1,\dots$;
 $P\{Y=y|X-Y=k_0\} = P\{Y=y|X-Y=k_0+k_1\}$, $y=0,1,\dots,k_0$.
- (iii) For some $\theta > 0$ and some periodic sequence $\{q_x: x=0,1,\dots\}$

with the largest common divisor of the x for which $b_x > 0$
as one of its periods

$$P\{X=x\} = q_x c_x \theta^x, \quad x=0,1,2,\dots "$$

Shanbhag-Taillie (1979) result may be considered to be a variant of Shanbhag's (1977) extension of the Rao-Rubin (1964) Theorem on damage models and is itself an extension of Patil-Taillie (1979) result. Patil and Taillie (1979) have considered a specialized version of Shanbhag-Taillie model with $a_x = \frac{\pi^x}{x!}$, $x=0,1,\dots$ and $b_x = \frac{(1-\pi)^x}{x!}$, $x=0,1,\dots$ when $\pi \in (0,1)$ and fixed, and hence with

$$P\{Y=y|X=x\} = \binom{x}{y} \pi^y (1-\pi)^{x-y}, \quad y=0,1,\dots,x; \quad x \geq 0. \quad (3.1)$$

These latter authors have also shown that if (3.1) is valid together with

$$P\{Y=y\} = P\{Y=y|X-Y=k\}, \quad y=0,1,\dots; \quad P\{X-Y=k\} > 0 \quad (3.2)$$

for some fixed $k > 0$, then it is not necessary that X be Poisson thus disproving a conjecture of Srivastava and Singh (1975). Our result of the last remark not only shows that Srivastava-Singh conjecture is false but also identifies under Shanbhag-Taillie model the class of distributions relative to which the condition (3.2) is valid. In particular, it follows from our result that under the model in question if (3.2) is valid with $k = 1$, then either $g_x/c_x = \theta \lambda_1^x + (1-\theta) \lambda_2^x$ for certain $\theta \in [0,1]$, $\lambda_1, \lambda_2 > 0$ or $g_x/c_x = \theta \lambda_1^x + (1-\theta) x \lambda_2^x$ for certain $\theta \in [0,1]$, $\lambda_1, \lambda_2 > 0$, where $g_x = P\{X=x\}$.

3.4 KENDALL (1951, 1953), LINDLEY (1952) AND OTHERS:

The restricted Deny equation (1.2) appears in several places in

queueing and storage theories (see, for example, Kendall (1951, 1953), Lindley (1952) and Wishart (1956)). In particular, Lindley (1952) has shown that the stationary waiting time distribution function corresponding to a GI/G/1 queueing system satisfies (1.2) with $H(x) = 0$ for $x < 0$. Indeed our Corollary 2 gives the expression for H in this case and shows that the distribution in question exists if and only if the relative traffic intensity of the system is less than 1 and that when it exists the distribution is compound geometric; it may, however, be noted here that the results cited are not new and these have appeared in Lindley (1952), Feller (1966) and elsewhere. The result of our last remark could be applied to obtain certain conclusions of Kendall (1953) and Wishart (1956) concerning GI/M/s and GI/ E_k /1 systems respectively. The result implies that, in either of the two cases, the stationary queue length distribution exists if and only if the corresponding relative traffic intensity is less than 1. Also it yields the known results that in a GI/M/1 queueing system the stationary queue length distribution is geometric and in a GI/M/s system the stationary waiting time distribution is exponential but for a discontinuity at zero.

REFERENCES

1. Alzaid, A.A. (1983) Some contributions to characterization theory. Ph.D. thesis, University of Sheffield.
2. Alzaid, A.A., Rao, C. Radhakrishna and Shanbhag, D.N. (1986). An application of Perron-Frobenius theorem to a damage model problem. Sankhyā A, 48, 43-50.
3. Alzaid, A.A., Rao, C. Radhakrishna and Shanbhag, D.N. (1985). Solution of the integrated Cauchy equation using exchangeability. Tech. Rept. 85-10, Center for Multivariate Analysis, Pittsburgh.
4. Arnold, B.G. (1980). Two characterizations of the geometric distribution. J. Appl. Prob. 17, 570-573.
5. Burrill, C.W. (1972). Measure Integration and Probability, McGraw Hill, New York.
6. Choquet, G. and Deny, J. (1960). Sur l'équation de convolution $\mu = \mu * \sigma$. C.R. Acad. Sci. Paris 250, 799-801.
7. Davies, P.L. and Shanbhag, D.N. (1984). A generalization of a theorem of Deny with applications in characterization theory. (Submitted for publication).
8. Deny, J. (1961). Sur l'équation de convolution $\mu * \sigma$. Semin. Theor. Potent M. Brelot, Fac. Sci. Paris, 1959-60 e ann.
9. Feller, W. (1966). An introduction to probability theory and its applications, Volume 2. Wiley, New York.
10. Kendall, D.G. (1951). Some problems in the theory of queues. J. Roy. Statist. Soc. B, 13, 151-185.
11. Kendall, D.G. (1958). Stochastic processes occurring in the theory of queues and their analysis by the method of imbedded Markoff chain. Ann. Math. Statist. 24, 338-354.
12. Krishnaji, N. (1974). Characterization of some discrete distributions based on a damage model. Sankhyā A, 36, 204-213.
13. Lau, Ka-Sing and Rao, C. Radhakrishna (1982). Integrated Cauchy functional equation and characterizations of the exponential law. Sankhyā A, 44, 72-90
14. Lau, Ka-Sing and Rao, C. Radhakrishna (1984a). On the integral equation $f(x) = \int_R f(x+y)d\mu(y)$, $x \geq 0$. Tech. Rept. 84-09, University of Pittsburgh.
15. Lau, Ka-Sing and Rao, C. Radhakrishna (1984b). Integrated Cauchy functional equation on the whole line. Sankhyā, 46, 311-319.
16. Lindley, D.V. (1952). Theory of queues with a single server. Proc. Cambridge Philos. Soc., 48, 277-279.
17. Marsaglia, G. and Tubilla, A. (1975). A note on the lack of memory property of the exponential distribution. Ann. Prob. 3, 352-354.
18. Patil, G. and Taillie, C. (1979). On a version of the Rao-Rubin theorem. Sankhyā, Series A, 41, 129-132.

19. Ramachandran, B. (1979). On the strong memoryless property of the exponential and geometric laws. Sankhyā A, 41, 244-251.
20. Ramachandran, B. (1982). On the equation $f(x) = \int_{[0,\infty)} f(x+y) d\mu(y)$. Sankhyā, Series A, 44, 364-371.
21. Ramachandran, B. and Prakasa Rao, B.L.S. (1983). On the equation $f(x) = \int_{-\infty}^{\infty} f(x+y) d\mu(y)$. Sankhyā, 46, 326-338.
22. Rao, C. Radhakrishna (1983). An extension of Deny's theorem and its application to characterization of probability distributions. In A Festschrift for Eric Lehmann (Eds. P.J. Bickel et al.), Wadsworth Monterey, CA. 348-365.
23. Rao, C. Radhakrishna and Rubin, H. (1964). On a characterization of the Poisson distribution. Sankhyā A, 26, 295-298.
24. Rao, C. Radhakrishna and Shanbhag, D.N. (1986). Recent results on characterization of probability distributions: a unified approach through extensions of Deny's theorem. To appear in Appl. Prob.
25. Rao, M.B. and Shanbhag, D.N. (1982). Damage models. In Encyclopedia of Statistical Sciences, Vol. 2, Wiley, New York. 262-265.
26. Seneta, E. (1968). The stationary distribution of a branching process allowing immigration: A remark on the critical case. J. Roy. Statist. Soc., Ser. B, 30, 176-179.
27. Seneta, E. (1973). Non-negative Matrices, George Allen and Unwin Ltd., London.
28. Shanbhag, D.N. (1977). An extension of Rao-Rubin characterization of the Poisson distribution. J. Appl. Prob. 14, 640-646.
29. Shanbhag, D.N. and Taillie, C. (1979). An extension of the Patil-Taillie characterization of the Poisson distribution (unpublished).
30. Shimizu, R. (1978). Solution to a functional equation and its application to some characterization problems. Sankhyā, Series A, 40, 319-332.
31. Sreehari, M. (1983). A characterization of the geometric distribution. J. Appl. Prob. 20, 209-212.
32. Srivastava, R.C. and Singh, J. (1975). On characterizations of the binomial and Poisson distributions based on a damage model. In Statistical Distributions in Scientific Work, 3, Ed. G.P. Patil et al. Reidel, Dordrecht, 271-278.
33. Wishart, D.M. G. (1956). A queueing system with chisquare service time distribution. Ann. Math. Statist. 27, 768-779.

END

3-87

DTIC