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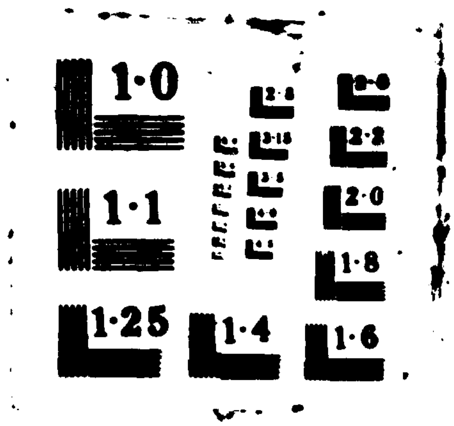
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This problem is motivated by studying dissipative dynamic systems under uncertainty, which in the simplest case will be a model of the automatic cruise control of an aircraft under rain down wind conditions.

We consider a new type of controlled diffusion models in which there are no restrictions on the drift (which is under our control), moreover the drift can take on infinite values. This results in the so called "singular" or "free boundary" control.

The optimal policy in these models is different from the ones seen in the classical cases. It consists of keeping the process within certain boundaries with minimal efforts. It is shown that one can find optimal boundaries by considering a special stopping game of two players with opposite interests.

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FREE BOUNDARY CONTROL OF BROWNIAN MOTION
AND A RELATED OPTIMAL STOPPING PROBLEM

BY

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FREE BOUNDARY CONTROL OF BROWNIAN MOTION AND A RELATED OPTIMAL STOPPING PROBLEM

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1. We consider a controlled stochastic linear system

$$Z_s = x + X_{s-t} + R_{s-t} - L_{s-t}. \tag{1}$$

Here X is a (μ,σ^2)-Brownian motion and R and L are the control functionals, which are increasing and adapted to the σ-field generated by the process X. For the policy S = (L,R) the expected cost takes the form

$$K_S(x,t) = E\{\int_t^T h(Z_s,s)e^{-\gamma(s-t)}ds + \ell\int_t^T e^{-\gamma(s-t)}\, dL_{s-t} + r\int_t^T e^{-\gamma(s-t)}\, dR_{s-t}\}. \tag{2}$$

Here h, ℓ and r stand for holding cost and unit cost of displacement to the left and to the right respectively, and $\gamma > 0$ is the discount factor. Our objective is to characterize the optimal cost (the value function)

$$V^*(x,t) = \max_S K_S(x,t) \tag{3}$$

and to describe the optimal policy $S^* = (L^*,R^*)$ for which $V^* = K_{S^*}$. The function V satisfies the Hamilton-Jacobi-Bellman equation (cf. [2])

$$\begin{aligned} 0 &= \min\{\frac{\partial V(x,t)}{\partial t} + \Gamma V(x,t) - \gamma V(x,t) + h(x,t), \\ &\qquad DV(x,t) + r, -DV(x,t) + \ell\}, \\ 0 &= V(x,T), \end{aligned} \tag{4}$$

where $D = \frac{\partial}{\partial x}$ and

$$\Gamma = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}. \tag{5}$$

Our main technical assumptions are similar to the ones in [2]. We assume that

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$$T < \infty, \tag{6}$$

h(x,t) is a nonnegative function such that there exists constants m and $0 \le c \le C$, such that for every x, x', t, t'

$$\begin{aligned} & c|x|^m - C \le h(x,t) \le C(1+|x|^m), \\ & |h(x,t) - h(x',t)| \le C(1+|x|^{m-1}+|x'|^{m-1})\ |x - x'|, \\ & |h(x,t) - h(x,t')| \le C(1+|x|^m)\ |t - t'|, \\ & 0 \le \frac{\partial^2 f}{\partial x^2}(x,t) \le C(1+|x|^q),\ q = (m-2)^+. \end{aligned} \tag{7}$$

<u>Theorem 1</u>. Under the assumption (6), (7), there exists a unique solution V^* to the equation (4). This solution is the value function (3) of the control problem (1), (2).

There exists an optimal policy $S^* = (L^*,R^*)$ for which $V^* = K_{S^*}$. If

$$x_1^*(t) = \min\{x: DV(x,t) = \ell\}$$

$$x_2^*(t) = \max\{x: DV(x,t) = -r\}$$

then for Z_s^* given by (1)

$$x_2^*(x) \le Z_s^* \le x_1^*(s), \tag{8}$$

and

$$\begin{aligned} & \int_t^T 1_{Z_s^* < x_1^*(s)} dL^*_{s-t} = 0 \\ & \int_t^T 1_{Z_s^* > x_2^*(s)} dR_{s-t} = 0. \end{aligned} \tag{9}$$

The above theorem shows that the optimal control consists of reflecting of the control process Z^* from time-dependent (a` priori unknown) boundary.

Let $\mathcal{D} = \{(x,t): x_2^*(t) \le x \le x_1^*(t)\}$ and let $W = DV(x,t)$. By formally differentiating (4) we get

$$\frac{\partial W}{\partial t}(x,t) + \Gamma W(x,t) - \gamma W(x,t) + H(x,t) = 0, \tag{10}$$

$$\text{if } (x,t) \in \mathcal{D},$$

$$W(x,t) \le r, \text{ for all } x \in \mathbb{R},\ 0 \le t < T, \tag{11}$$

$$W(x,t) \ge -\ell, \text{ for all } x \in \mathbb{R},\ 0 \le t < T, \tag{12}$$

$$W(x,T) = 0. \tag{13}$$

where all equalities and inequalities are understood in the sense of generalized function.

Assume that $H(0,t) = 0$, i.e. $0 = \operatorname{argmin} h(x,t)$. Consider the following minmax problem (game of two persons)

$$W(x,t) = \sup_{\sigma} \inf_{\tau} E\{\int_t^{\tau\wedge\sigma\wedge T} e^{-\gamma(s-t)} H(x+X_{s-t})\,ds + \ell\, e^{-\gamma(\tau-t)} 1_{\tau<T} 1_{\tau<\sigma} - r\, e^{-\gamma(\sigma-t)} 1_{\sigma<T} 1_{\sigma<\tau}\}, \tag{14}$$

where sup is taken over all stopping times $\sigma \ge t$ such that $x + X_{\sigma-t} < 0$ and inf is taken over all stopping times $\tau \ge t$ such that $x + X_{\tau-t} > 0$.

<u>Theorem 2</u>. The optimal stopping game described above has value that is the right hand side of (14) does not change if $\sup_\sigma \inf_\tau$ is replaced by $\inf_\tau \sup_\sigma$. The value of the game W satisfies (10) - (13) and it relates to the value function V by

$$W = DV.$$

2. Suppose h does not depend on t and we consider an infinite horizon optimization problem

$$V(x) = \sup_{R,L} E\{\int_0^\infty e^{-\gamma s} h(Z_s)\,ds + \int_0^\infty r\, e^{-\gamma s}\,dR_s + \int_0^\infty \ell\, e^{-\gamma s}\,dL_s\} \tag{15}$$

where Z_s is given by (1) with $t = 0$.

The Hamilton-Jacobi-Bellman equation for the value function V given by (15) reduces to an ordinary differential equation with gradient constrains

$$0 = \min\{\Gamma V(x) - \gamma V(x) + h(x),\ V'(x) + r,\ \ell - V'(x)\} \tag{16}$$

In case of infinite horizon control, we can loosen the assumption on h, namely we assume that h is a nonnegative convex C^1 function and

$$|h'(x)| \to \infty \text{ as } |x| \to \infty. \tag{17}$$

Theorem 3. Assume that (17) holds and r, $\ell > 0$. Then there exists a unique solution $V^*(x)$ to (16). There exists a unique optimal policy R^*, L^*. If $a = \inf\{x: \frac{dV^*}{dx}(x) > -r\}$ and $b = \sup\{x: \frac{dV^*}{dx}(x) < \ell\}$ then for all $t > 0$

$$a \le Z_t^* \le b$$

where $Z_t^* = x + X_t + R_t^* - L_t^*$. Moreover

$$\int_0^\infty 1_{Z_t^* \neq a}\, dR_t^* = \int_0^\infty 1_{Z_t^* \neq b}\, dL_t^* = 0.$$

The above theorem shows that the optimal control in the infinite horizon problem consists of keeping the controlled process Z^* inside the interval [a,b] reflecting it at the boundaries.

We want to establish the correspondence between optimal control problems and game of optimal stopping of two persons. For simplicity, we assume that h attains its minimum at point 0.

Consider an optimal stopping game of two persons.

$$W(x) = \sup_\sigma \inf_\tau E\{\int_0^{\tau\wedge\sigma} e^{-\gamma t} h(x+X_t)dt + \ell e^{-\gamma\tau} 1_{\tau<\sigma} - r e^{-\gamma\tau} 1_{\sigma<\tau}\} \tag{18}$$

where sup is taken over all stopping times σ such that $x + X_\sigma > 0$ and inf is taken over all stopping times τ such that $x + X_\tau < 0$.

Theorem 4. The quantity in the right hand side of (18) does not change if $\sup_\sigma \inf_\tau$ is changed to $\inf_\tau \sup_\sigma$. The value of the game W given by (18) is equal to the derivative of V^* given by (16). The optimal policies σ^* and τ^* in (18) are given by

$$\sigma^* = \inf\{t:\ x+X_t \le b\}$$

$$\tau^* = \inf\{t:\ x+X_t \ge a\}$$

where a and b are the same as in the theorem 3.

Similar results were obtained in [7] for the problem with average (per unit of time) criterion.

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