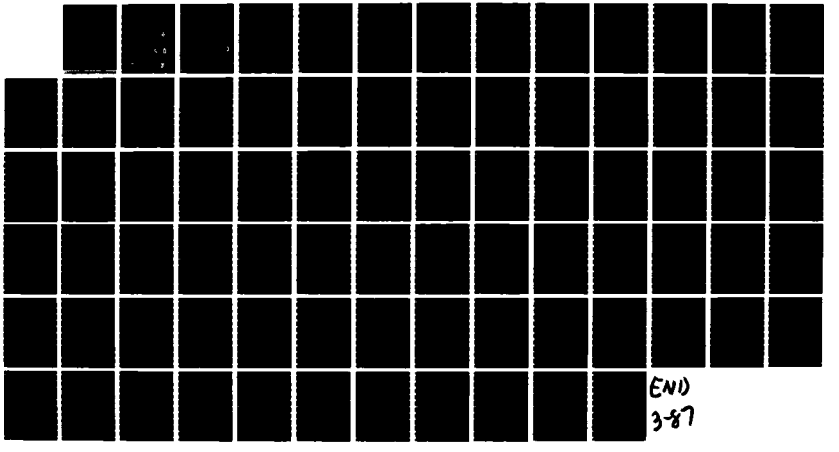
AD-A177 106 MODELING AND CONTROL OF FLEXIBLE STRUCTURES(U) SYSTEMS 1/1
ENGINEERING INC GREENBELT MD W H BENNETT ET AL.
16 DEC 86 SEI-TR-86-13 AFOSR-TR-87-0013
UNCLASSIFIED F49620-84-C-0115 F/G 20/11 NL



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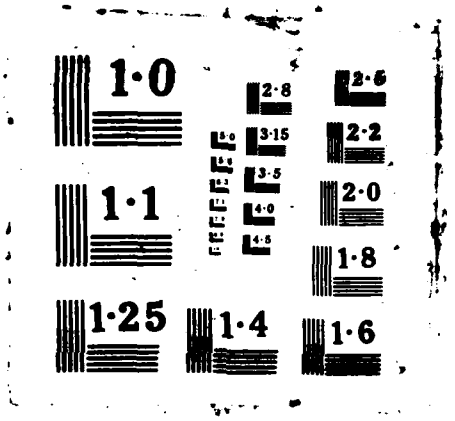
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AFOSR-TR- 87-0013

Modeling and Control of Flexible Structures

Annual Report
October 1984 - October 1985

SEI-TR-86-13

AFOSR Contract No. F49620-84-C-0115

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AFOSR-TR- 87-0013

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AFOSR Contract No. F49620-84-C-0115

Submitted to:

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Washington, DC 20332-6448
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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

AD-A177106

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release, distribution unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): SEI-TR-86-13	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR-87-0013
6a. NAME OF PERFORMING ORGANIZATION: Systems Engineering, Inc. / 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: Directorate of Aerospace Sciences, Air Force Office of Scientific Research/NA
6c. ADDRESS (City, State and ZIP Code): 7833 Walker Drive, Suite 308, Greenbelt, MD 20770	7b. ADDRESS (City, State and ZIP Code): Building 410, Bolling Air Force Base, Washington, D.C. 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: same as 7a. / 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: Contract No. F49620-84-C-0115
8c. ADDRESS (City, State and ZIP Code): same as 7b.	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	3005	A1	

11. TITLE (Include Security Classification): Modeling and Control of Flexible Structures (U)

12. PERSONAL AUTHOR(S): Drs. W. H. Bennett, G. L. Blankenship, H. G. Kwatny

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Annual Report	FROM 9/84 TO 10/85	86-12-16	

16. SUPPLEMENTARY NOTATION: The views, opinions, and/or findings contained in this report are those of the authors and should not be construed as an official AFOSR position, policy, or decision, unless so designated by other documentation.

17. COSATI CODES		
FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

In this report, we focus on the roles of models of flexible structures in the design and evaluation of control laws for the damping of vibrational motions in those structures. In the first section, we discuss a generic class of continuum models for flexible structures describing the abstract mathematical formulation of the models as a framework for the design of control laws. In the second section, we show how direct frequency domain designs for control laws may be achieved for this class of models based on a spectral factorization procedure which replaces the usual computation of Riccati equations. In the third section, we examine the problem of deriving transfer function representations of the structural models as required in the frequency domain design procedure. In section 4, we describe an analytical procedure for the derivation of continuum models for large scale structures with a regular infrastructure.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
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DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

Executive Summary

In this report we focus on the roles of models of flexible structures in the design and evaluation of control laws for the damping of vibrational motions in those structures.

In the first section we discuss a generic class of continuum models for flexible structures describing the abstract mathematical formulation of the models as a framework for the design of control laws.

In the second section we show how direct frequency domain designs for control laws may be achieved for this class of models based on a spectral factorization procedure which replaces the usual computation of Riccati equations. This procedure uses the infinite dimensional (continuum) model of the structure, and it leads to distributed (or localized) controls which impact the macroscopic behavior of the structure as a whole.

In the third section we examine the problem of deriving transfer function representations of the structural models as required in the frequency domain design procedure. We show that the analysis of certain types structures - beams with one or more degrees of freedom - can be "automated" (though this is far from trivial) using symbolic manipulation systems (SMP). Hybrid systems are also considered and models are developed based on the interconnection of components and subsystems with a careful delineation of the causal relationships among the components.

In section 4 we describe an analytical procedure for the derivation of continuum models for large scale structures with a regular infrastructure. We focus on the representation of dynamics of truss systems and the thermal transport properties of lattice structures. Control problems for typical systems of this type are also discussed.

1 Introduction

It is now generally accepted that large, low mass density lattice structures will be essential for several near term space applications. Moreover, it is apparent that active control of structural vibrations will be necessary to enhance their stiffness and damping properties. In this report we consider the construction of effective mathematical models for elastic dynamics of space structures with the objective of designing active control laws for these systems.

The success of active control for such structures will hinge to some extent on the ability of a control law to react to vibratory responses which may be initially localized before they propagate throughout a structure. This leads naturally to questions of how to implement active control so as to distribute the control effort spatially as it is needed. We argue that well known methods exist for the control of distributed parameter systems and can be effectively applied if continuum models for the candidate space structures can be computed. The nature of the required models is however quite different from the more standard finite element models which are popular for large structural analysis problems throughout the aerospace industry.

We begin in this section with a review of continuum models for active structural control. We highlight the nature of abstract state space models for these systems. In this study we have employed one method for the computation of a distributed control law for continuum dynamics based on a numerical procedure for spectral factorization. This method requires the computation of certain representation for an underlying state-space model for the structure to be controlled. In a later section we discuss the theoretical basis for computing effective distributed parameter models for large truss structures with regular lattice infastructure. The method which involves "homogenization" (an asymptotic analysis of multiple scales) leads to the well known Timeshenko model for beam dynamics. The analysis provides formulae for the effective beam parameters which are quite different than have been suggested by other averaging schemes [20,21,22].

Comprehensive models of flexible spacecraft dynamics will involve systems with fairly complex interconnections of lumped and distributed subsystems, and therefore, we intend to construct the overall models by first developing subsystem models and then combining them according to the required interconnection rules. These interconnections lead to basic questions of causality and well-posedness of certain standard models for beams. These questions are crucial to the computation of hybrid, state-space modeling of an integrated space platform.

Throughout this effort we have focused on the potential for automatic and computer-aided computation of the models by a combination of modern computer algebra [23] (symbolic manipulation) and numerical methods. In our efforts we have used the program SMP [24,25]. We will review the progress in using SMP for the computation of irrational transfer function models for hybrid problems in a later section.

1.1 Generic Models for Structural Dynamics

In this section, we discuss a generic model for elastic dynamics of structures from the point of view of continuum modeling. We will summarize the construction of a state space model and introduce a typical control problem for vibration suppression. We highlight the modal approximations which are popular for these problems and proceed to demonstrate an effective alternate technique for model construction and control computation based on the semi-group property [26] of a state space model. Effectively, modeling and control law computation can proceed in the frequency domain, based on transfer function methods, permitting the direct computation of a resolvent operator. We focus on the class of structural contro l problems for which the question of control of propagation of wave-like disturbances is important. In this framework, we can present the semi-group theory by concrete computations of practical interest to structural and control system engineers.

The standard linear continuum model for a flexible structure [27] is

given by a system of partial differential equations (PDE)

$$m(z)\frac{\partial^2 w(t,z)}{\partial t^2} + D_0\frac{\partial w(t,z)}{\partial t} + A_0 w(t,z) = F(t,z) \tag{1}$$

where $w(t,z)$ is an N-vector of displacements of a structure Ω with respect to some equilibrium for Ω in a bounded, open set in $\mathcal{R}^N$. The (vector) $z \in \Omega$ a coordinate in Ω. We assume the boundary $\partial\Omega$ is smooth. The mass density $m(z)$ is positive definite and bounded on $\partial\Omega$. The damping term $D_0\partial w/\partial t$ contains both (asymmetric) gyroscopic and (symmetric) structural damping effects. The internal restoring force $A_0 w$ is generated by a time-invariant, differential operator A_0 for the structure. For most common structural models, A_0 is an unbounded differential operator with domain $D(A_0)$ consisting of certain smooth functions satisfying appropriate boundary conditions on $\partial\Omega$. Thus, for these problems, $D(A_0)$ is typically dense in the Hilbert space $\mathcal{H}_0 = L^2(\Omega)$. Often (but not always), the spectrum of A_0, $\sigma(A_0)$, consists of discrete eigenvalues with associated eigenfunctions which constitute a basis for $L^2(\Omega)$.

The applied force distribution $F(t,z)$ can be thought of as consisting of three components

$$F(t,z) = F_d(t,z) + F_c(t,z) + F_a(t,z) \tag{2}$$

where F_d is N-vector of exogenous disturbances (possibly forces and torques), F_c is a continuous, distributed controlled force field (an available option in only some special applications), and F_a represents controlled forces due to localized actuation;

$$F_a(t,z) = \sum_{j=1}^{k} b_j(z)u_j(t) = B_0 u(t). \tag{3}$$

The actuator influence functions $b_j(z)$ are highly localized in Ω and can be approximated by delta functions. Measurements are available from a finite number p of sensors

$$y(t) = C_0 w + C_0'\frac{\partial w}{\partial t} \tag{4}$$

where $y(t)$ is a p-vector. The operators $B_0 : \mathcal{R}^m \rightarrow \mathcal{H}_0$, $C_0 : \mathcal{H}_0 \rightarrow \mathcal{R}^p$, and $C_0' : \mathcal{H}_0 \rightarrow \mathcal{R}^p$ are bounded.

The standard optimal regulator control problem for this model is to find the controls $u_j(t)$, $j = 1, \ldots, k$ (we ignore the possibility of F_c) given the observations $y(t)$ to maintain the system state, e.g.,

$$x(t,z) = \begin{pmatrix} w(t,z) \\ \frac{\partial w(t,z)}{\partial t} \end{pmatrix} \tag{5}$$

such that the performance index

$$J(u) = \int_0^\infty (\|x\|_Q^{\mathcal{H}_0} + \epsilon u^T u) dt \tag{6}$$

is minimized where $\|x\|_Q^{\mathcal{H}_0} = \langle x, Qx \rangle_{\mathcal{H}}$ defined in an appropriate Hilbert space for the state x. This is the generic control problem surveyed in Balas [28]. In this report, we will concentrate on the construction of state space models and computational aspects of equations of the form (1) and of optimal (discrete) controls $u_j(t)$ appearing in (3).

1.2 State Space Models

The choice of state space given by (5) is often attractive for models i n the generic form (1). (We will discuss later that attractive alternate state space models can arise in hybrid constructions.) A natural assumption for structural problems [27] is that A_0 is symmetric with compact resolvent and discrete (real) spectrum which is bounded from below. The state (5) can be considered as an element of a Hilbert space $\mathcal{H} = D(A_0)^{1/2} \times \mathcal{H}_0$ with the energy norm

$$\|x\|_E^2 = \langle x_1, A_0 x_1 \rangle_0 + \langle m x_2, x_2 \rangle_0 \tag{7}$$

where the first term represents potential energy and the second term is kinetic energy. Thus the (abstract) *state space model* can be written

$$\frac{\partial x(t,z)}{\partial t} = Ax(t,z) + Bu(t) \tag{8}$$

where $y(t) = Cx(t,z)$

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -D_0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \; C = [C_0, C_0'] . \tag{9}$$

For the elastic dynamics of space structures, there is always some (possibly small) damping D_0 appearing in (1) which causes A to b e dissipative. Thus the criteria of the Hille-Yoshida-Phillips theorem [26] are satisfied and A generates a C_0-semigroup with an operator which we write suggestively as e^{At}. Moreover, such models are "hyperbolic" [27] in the sense that the semigroup is a contraction, i.e., $\|e^{At}\| \leq 1$ and all but the zero frequency poles are only slightly damped; $\|e^{At}\| \leq e^{-\delta t}$ for some small $\delta > 0$. As we shall discuss in section 3, some popular models for structural elements such as beams do not fit in this framework. However, this framework includes models appropriate for considerations of wave-like dynamics which propagate causally in the spatial domain. For such models, the question of how to compute controls $u(t)$ and system response $x(t)$ focuses on the so-called *mild solution* of (8);

$$x(t,z) = e^{At}x(0,z) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma. \tag{10}$$

Various methods are available for approximation of the system (8) Burns and Cliff [29]. One popular method is based on a modal (eigen-) expansion of A which generates a sequence of finite dimensional subspaces $\mathcal{H}_k \subset \mathcal{H}, k = 1,2,\ldots$, where $\mathcal{H}_k = \text{span}\{\phi_j, j = 1,\ldots,k\}$ and the $\phi_j(z)$ are eigenfunctions (or mode shapes) for A. Based on this approximation, a sequence of finite dimensional models for (5) can be generated;

$$\dot{x}^{(k)}(t) = A^{(k)}x^{(k)}(t) + B^{(k)}u(t). \tag{11}$$

Using a truncated model (11) with k finite and the performance index J(u) (6) projected onto the space $\mathcal{H}_k$, one can solve the associated optimal control problem for the first k modes of A. However, as noted in Balas [28] in all but a few special cases, the control law when applied to the system (5) will excite higher order modes. The inherent robustness and stability properties as well as the degree of suboptimality of control laws based on such truncated modal approximations has received a great deal of attention in both the engineering and mathematics literature [27,28]. Various alternate approaches are available which deal directly with infinite dimensional control problem given by (6) and (10) – at least abstractly

Russell [30]. One method suggested by Davis [31,32] offers the advantage of a computational procedure for approximating the true optimal control in terms of the required control bandwidth. The method is based on an extension of a Wiener-Hopf solution [32] for the abstract control problem.

2 Wiener-Hopf Methods for Computation of Optimal, State-Feedback Control: The Davis-Stenger Algorithm

The connections between least squares optimization, spectral factorization, and algebraic Riccati equations have been considered important in control theory for many years. (See e.g., Anderson [66], Brockett [33], Willems [69], Helton [68], and the references therein). To see how the connection arises, consider the standard, finite-dimensional, infinite time regulator problem:

$$\min_{u \in U_{ad}} \int_0^\infty \|u(t)\|^2 + \|y(t)\|^2 dt \tag{1}$$

subject to the linear, time-invariant system model

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0 \tag{2}$$

and controlled output

$$y(t) = Cx(t),\ t \geq 0. \tag{3}$$

The transfer function relating the Laplace transform of the input vector $\hat{u}(s)$ to the output vector $\hat{y}(s)$ is

$$\begin{aligned} \hat{y}(s) &= G(s)\hat{u}(s), \\ G(s) &= C[sI - A]^{-1}B. \end{aligned} \tag{4}$$

The optimal control is known to be a linear state feedback

$$u(t) = -K_{opt}x(t) = -B^T P x(t) \tag{5}$$

where P is the unique, positive definite symmetric solution to algebraic (matrix) Riccati equation,

$$PA + A^T P - PB^T BP + C^T C = 0. \tag{6}$$

Standard algebraic manipulations based on (1)–(6) provide the spectral factorization relation

$$\begin{aligned} [I &+ K_{opt}(-sI - A)^{-1}B]^T[I + K_{opt}(sI - A)^{-1}B] \\ &= I + B^T(-s - A^T)^{-1}C^T C(sI - A)^{-1}B. \end{aligned} \tag{7}$$

[57, pp. 68]. Clearly (7) can be rewritten

$$H(s) = I + G^T(-s)G(s) = F^T(-s)F(s) \tag{8}$$

where $F(s) = I + K_{opt}[sI - A]^{-1}B$ is the causal spectral factor of $H(s)$. In fact many desirable properties of linear quadratic regulator design (e.g. inherent robustness properties) follow from (8) and the interpretation of $F(s)$ as the optimal return-difference operator for the problem (1)–(3).

Recall that for a closed loop control given by the transfer function relations

$$\begin{aligned} \hat{y}(s) &= Q(s)\hat{u}(s) \\ \hat{u}(s) &= \hat{r}(s) - \hat{e}(s) \end{aligned} \tag{9}$$

that the return-difference with respect to the returned signal $\hat{y}(s) = \hat{e}(s)$ is computed as the difference between $\hat{y}(s)$ and $\hat{u}(s)$ if the loop is broken there. Thus,

$$\hat{y}(s) - \hat{e}(s) = [1 + Q(s)]\,\hat{r}(s).$$

In (8) $F(s)$ is the optimal return-difference for loop breaking at the

control with state feedback. We remark that for continuum models for flexible structures that the underlying state space models are infinite dimensional and $F(s)$ represents an ideal limit on achievable control performance with state feedback.

In Nyquist stability theory the complex contour $1 + H(j\omega)$ for $\omega \in \Re$ is called the Nyquist contour. Practical aspects of Nyquist stability tests have been the cornerstone of control system design for at least 50 years. These stability tests have been extended to include an important class of distributed parameter systems where the resulting return difference is an irrational transfer function [58]. For our purposes in this study the Nyquist test will prove central and we will focus on the class of irrational transfer functions for which: (1) spectral factorization can be effectively computed at a finite number of samples and (2) the Nyquist theory of stability is well defined.

In addition to (7) a useful integral equation can be derived for the optimal state feedback under the additional assumption that the spectrum of the operator A in (2) is contained the open left half plane ($\mathbf{C}_-$). We will state the result as a theorem and outline a proof for the finite dimensional case. Next we consider the additional assumptions necessary for the result to hold for control of a distributed parameter problem in the form of (1)-(3).

Theorem 1 (Davis) *With the optimal linear regulator problem as defined in (1)-(3) the optimal feedback control (if it exists) $u(t) = K_{opt}x(t)$, can be computed as*

$$K_{opt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [F^*(i\omega)]^{-1} G^*(i\omega) C R(i\omega, A) d\omega. \tag{10}$$

under the assumption that A is a stable (matrix) operator.

In (10) we use the notation $G(s)$ as defined in (4), $F^*(i\omega) = F^T(-i\omega)$ with $F(s)$ the causal spectral factor given by (8) and $R(i\omega; A) = [i\omega I - A]^{-1}$ the (matrix) resolvent for A.

Proof: From standard results [33] the optimal control (if it exists) will stabilize the closed loop system so that $\sigma(A - BK_{opt}) \subseteq \mathbf{C}_-$ where $\mathbf{C}_-$ is the open left half of the complex plane. Construct a closed rectifiable contour Γ in the complex plane consisting of a relatively large portion of the $i\omega$-axis and a semicircular portion in the left half plane such that Γ encircles $\sigma(A - BK_{opt})$ in the positive sense. Let $A_c = A - BK_{opt}$. Then

$$\frac{1}{2\pi i}\oint_\Gamma [sI - A_c]^{-1} ds = I. \tag{11}$$

By assumption $\sigma(-A^T)$ is contained in $\mathbf{C}_+$ and

$$\frac{1}{2\pi i}\oint_\Gamma \left[sI + A^T\right]^{-1} ds = 0. \tag{12}$$

From (6) we write

$$A^T P + P A_c = -C^T C. \tag{13}$$

This leads to the relation

$$P(sI - A_c)^{-1} + (-sI - A^T)^{-1}P = (-sI - A^T)^{-1}C^TC(sI - A_c)^{-1}. \quad (14)$$

Now integrate (14) on Γ and use (11) and (12) to get

$$P = \frac{1}{2\pi i}\oint_\Gamma (-sI - A^T)^{-1}C^TC(sI - A_c)^{-1}ds. \quad (15)$$

Since the optimal gain is $K_{opt} = B^TP = [PB]^T$ we get

$$K_{opt} = \frac{1}{2\pi i}\oint_\Gamma B^T(sI - A_c)^{-1}C^TC(-sI - A^T)^{-1}ds. \quad (16)$$

Now from (7) and (8) we get that

$$C(sI - A_c)^{-1}B = C(sI - A)^{-1}B\left[I + K_{opt}(sI - A)^{-1}B\right]^{-1} \quad (17)$$

and therefore

$$C(sI - A_c)^{-1}B = G(s)F^{-1}(s).$$

Thus we can write

$$K_{opt} = \frac{1}{2\pi i}\oint_\Gamma F^{-T}(s)G^T(s)C(-sI - A)^{-1}ds. \quad (18)$$

Finally (10) is determined by substituting $s = i\omega$ under the observation that $G(i\omega) \to 0$ as $\omega \to \infty$.

For (10) to hold when $G(s)$ is irrational we must impose some constraints on the spectral properties of the associated infinite-dimensional operator A. Consideration for the computation of the Riccati operator P by the integral formula (15) suggests that the spectrum of A must consist of a countably infinite set of eigenvalues so that the path integrals can be computed as in (11) and (12). In addition, the spectral sets $\sigma(A_c)$ and $\sigma(A)$ must be separated by the imaginary axis (including the point at infinity). Also, the observation that $G(i\omega) \to 0$ as $\omega \to \infty$ for rational functions must be replaced by an equivalent assumption which further restricts the class of irrational transfer functions.

Effectively these additional assumptions will be guaranteed by the class of transfer functions for which spectral factorization can be performed. We will consider spectral factorization for distributed systems next. More importantly with the usual assumptions used in constructing continuum models for mechanical structures it appears that the resulting transfer functions will have the appropriate properties. However, in this study we have encountered much confusion in the literature. As discussed in the introduction a major goal of this study was to provide a consistent method for model construction leading to an appropriate class of transfer functions for models which are both well-posed and have appropriate spectral properties. In the next section we delineate the specific class of transfer functions for which spectral factorization can be computed efficiently by a numerical algorithm. We indicate the basis for the algorithm and discuss the implications of sampling and interpolation of the spectral factor. Finally we discuss the relationship between rational approximation and modal control.

2.1 Spectral Factorization

In this section we review the basis for an interactive algorithm for computation of a frequency sampled spectral factor. The algorithm (due to Davis and Dickinson [32]) provides an effective computational tool for obtaining the optimal gain K_{opt} via (10) without regard to computational difficulties associated with large dimensional Riccati equations. Convergence of the algorithm depends on certain technical assumptions which delineate the class of transfer functions. In the finite dimensional setting, a recursive algorithm for computation of the causal spectral factor $F(s)$ follows from a Newton-Raphson iteration for the matrix Riccati equation (6) given an initial stabilizing feedback $K_0 = B^*P_0$;

$$P_{n+1}(A - BB^*P_n) + (A - BB^*P_n)^*P_{n+1} = -C^*C - P_nBB^*P_n.$$

At the n^{th} iteration one can take an approximation to the causal spectral factor as

$$F_n(s) = I + B^*P_n(sI - A)^{-1}B.$$

Then following Davis and Dickinson [32] this leads to the form of the algorithm

$$F_{n+1}(i\omega) = P_+ \left\{ [F_n^*(i\omega)]^{-1} H(i\omega) [F_n(i\omega)]^{-1} \right\} F_n(i\omega), \tag{19}$$

where P_+ is the causal projection operator defined on the convolution algebra $I \oplus L_1$ or $I \oplus L_2$ by

$$P_+ \left\{ I + \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right\} = I + \int_0^{\infty} f(t) e^{-i\omega t} dt.$$

Computation of causal projection of a signal is a standard problem in signal processing which can be effectively solved through the use of a Hilbert transform [59], [60].

Definition .1 *The Hilbert transform of a signal $f(t)$ is given as*

$$\check{f}(t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau = f(t) * \frac{1}{\pi t}. \tag{20}$$

The basic utility of the Hilbert transform is evident by examination of its Fourier transform.

Fact 2 *Let $\check{F}(\omega)$ be the Fourier transform of $\check{f}(t)$ and $F(\omega)$ be the Fourier transform of $f(t)$.*

$$\check{F}(\omega) = \int_{-\infty}^{\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau \; e^{i\omega t} dt. \tag{21}$$

Since for $H(\omega) = \int_{-\infty}^{\infty} \frac{1}{\pi t} e^{i\omega t} dt$, we see that $H(\omega)$ is a complex function with the properties

$$\begin{aligned} |H(\omega)| &= 1 \\ \arg H(\omega) &= \begin{cases} -\pi/2 & \omega > 0 \\ \pi/2 & \omega < 0 \end{cases} . \end{aligned} \tag{22}$$

Therefore from (21) and (22) we get that

$$\check{F}(\omega) = \begin{cases} -iF(\omega) & \omega > 0 \\ iF(\omega) & \omega < 0 \end{cases} \tag{23}$$

Now to compute the causal projection of $f(t)$ (as in (19)) given frequency domain data $F(\omega)$ we first compute the Hilbert transform

$$\check{F}(\omega) = F(\omega) * \frac{1}{\pi\omega}.$$

By duality of the Fourier transform pair the property (23) holds in th e domain t;

$$\mathcal{F}^{-1}\left\{\check{F}(\omega)\right\} = -i\text{sgn}(t)f(t).$$

Finally causal projection can be computed as

$$P_+\left\{F(\omega)\right\} = \frac{1}{2}\left[F(\omega) + i\check{F}(\omega)\right]. \tag{24}$$

Before we consider computational issues further we review the extension of this algorithm to irrational transfer functions. The questions of existence and uniqueness of the spectral factorization of the transform $H(s) = I + G^T(s)G(s)$ naturally lead to conditions for which $S(\omega) = G^T(-i\omega)G(i\omega)$ is positive semidefinite for ω real. In the convergence proof of the iteration (19) Davis assumes that $G(s)$ is the transform of a rea l, vector-valued function which is both integrable and square integrable; i.e., $G(i\omega) \in \mathcal{F}(L_1 \cap L_2)$.

With these assumptions it is clear from the classical theory of Gohberg and Krein [54] that $H(s)$ has a unique spectral factorization as given in (8) with

$$F^{\pm}(i\omega) - I \in \mathcal{F}(L_1^+)$$

where $\mathcal{F}(L_1^+)$ is the class of Fourier transforms of functions in L_1 with positive support. As noted in [32], the assumptions on $G(s)$ in fact imply that

$$F^{\pm 1}(i\omega) - I \in \mathcal{F}(L_1^+ \cap L_2^+)$$

and $F(i\omega) = F(-i\omega)$.

Using this assumption Davis is able to show that the recursion (19) starting from an initial $F_0(i\omega) \in \mathcal{F}(L_1 \cap L_2)$ has all iterates $F_n(i\omega) \in \mathcal{F}(L_1 \cap L_2)$ and that:

$$\lim_{n\to\infty} F_n^*(\omega)F_n(\omega) = H(\omega) \quad \text{almost everywhere} \tag{25}$$

$$\lim_{n\to\infty} F_n(s) = F(s) \quad \text{for all } s \text{ with } \Re e\ s > 0. \tag{26}$$

These results should not be surprising for the class of transfer functions considered. The following theorems summarize well known properties of L_1 and L_2 functions.

Theorem 3 ([58]) *If $f \in L_1$ then*

1. *$\omega \mapsto \hat{f}(\omega)$ is uniformly continuous for $\omega \in \mathbf{R}$*
2. *$\hat{f}(\omega) \in L_\infty$*
3. *$|\hat{f}(\omega)| \to 0$ as $|\omega| \to \infty$*
4. *$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(i\omega)e^{i\omega t}d\omega$ almost everywhere in $\mathbf{R}$.*

Theorem 4 ((Parseval's theorem) [58]) *If $f \in L_2$ then*

1.
$$\int_{-\infty}^{\infty} |f|_2^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|d\omega;$$

2. *as $N \to \infty$*
$$\int_{-N}^{+N} f(t)e^{-i\omega t}dt \xrightarrow{L_2} \hat{f}(i\omega);$$

3. *as $N \to \infty$*
$$\frac{1}{2\pi}\int_{-N}^{+N} \hat{f}(i\omega)e^{i\omega t}d\omega \xrightarrow{L_2} f(t)$$

So we see that $f \in L_1$ means that $\hat{f}(i\omega)$ is bounded on ω and therefore the Fourier transform is well defined while $f \in L_2$ provides consistent approximation theory for band limited signals. We note that the third L_1 property means that such transfer functions are effectively band limited and strictly proper.

Application of spectral factor to the integral formula (10) will further restrict consideration to $G(s)$ both causal and stable so that $G(s)$ is analytic for $\Re e s > 0$. Thus the transfer functions we are considering belong to the Hardy space $G(s) \in \mathbf{H}^2 \cap \mathbf{H}^\infty$. Recall that by definition $\hat{f} \in \mathbf{H}^2$ if $\hat{f}$ is complex valued and analytic in $\mathbf{C}_+$ (the open right half of the complex plane) and

$$\sup_{\sigma>0} \int_{-\infty}^{\infty} |\hat{f}(\sigma + i\omega)| d\omega < \infty.$$

while $\hat{f} \in \mathbf{H}^\infty$ if $\hat{f}$ is complex valued and analytic in $\mathbf{C}_+$ and

$$\sup_{s \in \mathbf{C}_+} |\hat{f}(s)| < \infty.$$

The first property is inherited from $f \in L_2$ while the second comes from $f \in L_1$.

Finally, we remark that for certain mechanical structures that the transfer function models can be factorized in a "product expansion" [71]

$$\hat{f}(s) = \prod_{k=1}^{\infty} \frac{1 + (s^2/z_k^2)}{1 + (s^2/\omega_k^2)}. \tag{27}$$

Thus we conclude these remarks by indicating that the class of transfer function models for which spectral factorization can be computed by the present method are meromorphic and have representations as (27).

2.1.1 Remarks on Algorithm Construction

With regard to the integral formula for the optimal state feedback gain it is clear that we need $[F(i\omega)]^{-1}$. Thus as suggested by Davis it is convenient

to implement the iteration in the form

$$[F_{n+1}]^{-1} = [F_n]^{-1} \left(I + \mathcal{P}_+ \left\{ [F_n^*]^{-1} H [F_n]^{-1} - I \right\} \right)^{-1}. \tag{28}$$

Furthermore by initializing with F_0 a diagonal matrix with diagonal elements equal to the spectral factors of the diagonal elements of H the second term of (28) remains a perturbation of the identity (since $[F_n^*]^{-1} H [F_n]^{-1} - I \rightarrow 0$) which regularizes the computations. The diagonal initialization guarantees that the first residual $[F_0]^{-1} H [F_0]^{-1}$ has ones on the diagonal and all off diagonal elements less than one in magnitude. Using the properties of the Hilbert transform and the formula (24) one can readily compute the causal spectral factor for the individual scaler transfer functions directly (i.e. without iteration). In particular, the k^{th} diagonal element of $H(\omega), h_k(\omega)$ is a real valued function with $h_k(\omega) > 0$. Let $\check{h}_k(\omega)$ be the Hilbert transform of $h_k(\omega)$; viz.,

$$\check{h}_k(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(\sigma)}{\omega - \sigma} d\sigma \tag{29}$$

then the causal spectral factor $h_k(i\omega) = f_k(-i\omega) f_k(i\omega)$ is given by

$$f_k(\omega) = \sqrt{\mathcal{R}e\, h(\omega)}\; e^{-\frac{i}{2} \ln \check{h}(\omega)}. \tag{30}$$

Finally, we remark that numerical computation of the Hilbert transform can be achieved in several ways. Direct numerical integration of (29) is complicated by the fact that the integral is convergent in the Cauchy principal value sense. Effective quadrature algorithms for such problems have been coded and tested. A public domain version utilizing an adaptive quadrature procedure is available in the routine QAWC contained in a software package called QUADPACK [61].

Another approach is taken by Davis [32] based on an algorithm of Stenger [62]. This procedure essentially implements a discrete (sampled) version of the required computation using a digital Hilbert transform. For a finite number of sample points the Hilbert transform computation can be implemented by taking a discrete "fast fourier transform" (FFT) of the sampled data and shifting the imaginary part of the transformed data

according to (21). It is well known that control of error induced by the sampling process (Gibbs phenomenon) requires the careful choice of "data windows" or weighting functions for the computation. Although these considerations are well described in the literature on digital signal processing [60], [64], [63] it is not apparent that these considerations were contained in the work of Stenger.

In our experience, direct implementation of the algorithm described in Davis and Dickinson, based on the causal projection of F. Stenger [62] tends to be unreliable. Since detailed design of window functions for discrete Hilbert transforms was considered outside the scope of the present study we have found it expedient to use the adaptive quadrature software from QUADPACK [61].

2.1.2 Relationship to Modal Methods

In applications one typically sees the use of modal methods in the modeling and control of continuum models for structures with a procedure as follows [28],[72]:

1. Determine a finite-dimensional, modal (or finite element) approximation to the distributed parameter problem yielding

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with $x(t)$ a finite-dimensional, state vector representing displacements and velocities for a finite number of structural modes.

2. The finite-dimensional model may be further truncated to achieve a "reduced-order" approximation which will be used for control synthesis.

3. Standard linear quadratic regulation theory is applied to the reduced-order model. The resulting matrix Riccati equation can be solved (in principal) although at some significant computational cost for an arbitrary (large) number of modes.

4. Implementation of the active control law requires on-line estimation of the particular plant modes included for control design A subsequent analysis of the effect of control action on the residual (truncated) model may force a rather drastic reduction in the effective loop gain to avoid a potentially destabilizing phenomenon known as "spillover".

We remark that control laws of this type involve state feedback for a truncated model and therefore require real-time measurement (or estimation) of certain individual modes. Since both mode shapes and modal frequencies can change under the influence of feedback control such feedback can be difficult to realize. This is further complicated by the fact that typically, exact modal data is not available in the construction of step 1 using finite element procedures. The Wiener-Hopf methods we are investigating here are based on quite different modeling schemes and we will discuss modeling in more detail in later sections. However, since it is often desired to control "certain troublesome modes" in engineering design it is useful to consider the relationship of these methods with the objective of modal control.

Consider the state space continuum model for a structure considered in section 1.

$$\frac{\partial x(t,z)}{\partial t} = \mathbf{A}x(t,z) + \mathbf{B}u(t) \tag{31}$$

where $x(t,z) \in \mathcal{H}(\Omega)$, a real Hilbert space satisfying boundary conditions for $z \in \Omega$ is finite-dimensional control, $\mathbf{A}, \mathbf{B}$ are operators on $\mathcal{H}(\Omega)$ and we assume that for structures $\mathbf{A}$ is densely defined on $\mathcal{H}$ and maximally dissipative with a discrete spectrum consisting of its eigenvalues. The solution to (31) can therefore be given as

$$x(t,z) = e^{\mathbf{A}t}x(0,z) + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}u(s)ds \tag{32}$$

where $e^{\mathbf{A}t}$ is the semigroup operator. Roughly, since $\mathbf{A}$ is assumed to have discrete spectrum a modal analysis can be based on an eigen expansion of $\mathbf{A}$ in terms of eigenvalues λ_k and eigenfunctions $\varphi_k(z)$

$$\mathbf{A}\varphi_k(z) = \lambda_k\varphi_k(z).$$

Considering the Laplace transform of (32)

$$\hat{x}(s,z) = R(s;\mathbf{A})x(0,z) + H_{BC}(s,z)\hat{u}(s) \quad (33)$$

$$= \int_\Omega G_r(s,z,w)x(0,w)dw + H_{BC}(s,z)\hat{u}(s), \quad (34)$$

where $R(s;\mathbf{A})$ is the resolvent for $\mathbf{A}$ which can be given by the integral operator (as shown) with kernel called the Green's function.

In the framework of the distributed parameter (DP) control problem "modal control" as an objective means that we choose the output for control (2) by defining the projection $\pi_N : \mathcal{H} \to \mathcal{H}^N \subset \mathcal{H}$ where $\mathcal{H}^N$ is a finite dimensional "modal" subspace of H. Then taking

$$y(t) = C\pi_N x(t,z)$$

leads to

$$\hat{y}(s,z) = \int_\Omega \pi_N G_r(s,z,w)x(0,w)dw + \pi_N H_{BC}(s,z)\hat{u}(s),$$

where the elements of the matrix Green's function can be given in terms of a corresponding modal expansion;

$$CG_{r\,i,j}(s,z,w) = \sum_{k=1}^{N} \frac{\varphi_k(z)\varphi_k(w)}{p_k(s)}$$

with $p_k(s) = s^2 + 2\zeta_k\omega_k s + \omega_k^2$. Here ω_k, ζ_k are the frequency and damping of the eigenmodes and $\varphi_k(z)$ are the eigenfunctions (modeshapes).

To compute the optimal control by Wiener-Hopf methods we must perform spectral factorization on

$$H(s) = I + G^T(-s)G(s)$$

where $G(s) = CH_{BC}(s,z)$ so that $G(s)$ is a rational transfer function. Computation of the optimal state feedback via (10 requires the resolvent

$$CR(i\omega,\mathbf{A})\,x(0,w) = \int_\Omega \pi_N G_r(s,z,w)x(0,w)dw$$

which is also rational in the Laplace variable s. Under the assumption that all truncated modes are stable then the resulting control will stabilize the system. In this case the Davis method produces an effective distributed feedback control

$$u(t) = K_{opt}x(t,z) = \int_{\Omega} K(z)x(t,z)dz$$

for the truncated modal subspace which is effectively the same control one would get by the procedure described earlier assuming that exact modal data is available. However notice that the control is a full (distributed) state feedback so that (in principle) modal measurements are not required. Furthermore, computation of large dimensional Riccati equations is replaced by the procedure for spectral factorization. We remark that a variety of algorithms exist for spectral factorization of rational functions [70], [73].

The point to focus on is that the method of Davis completely separates the question of rational approximation (which may be based on modal expansion) of the model from the computational issues associated with determining the control. Instead, approximation is based on sampling and interpolation of the transfer functions is involved.

2.2 Importance of Damping in Models for Distributed Control

For the present study we have attempted to compute distributed controls for several examples. Several negative results lead us to reevaluate the theoretical basis for control and modeling problem. Our difficulties stem from precise computation of the irrational transfer functions for several distributed models which will be discussed later. Thus our control laws are computed and tested on bona fide distributed parameter models and not on finite dimensional approximations; therefore the idiosyncracies of various (academic) examples will lead to obvious difficulties. In this section we review our most recent conclusions about the assumptions in modeling for mechanical structures and resulting computation of optimal control laws.

The most pertinent comments in the literature appear to be summarized in the work of Gibson [65]. In particular, the importance of damping and the ability to compute distributed control laws for infinite dimensional models for mechanical structures. It is true that computation of control laws for such problems will always involve approximation in modeling, control problem formulation, and numerical computation. Gibson focuses on this question of approximation in modeling and control for distributed parameter problems via the use of finite dimensional models and control computation based on these models. Although on the surface the Davis method (based on spectral factorization of irrational transfer functions) is not subject to finite dimensional approximation (at least in the sense of modal expansions) it is clear that approximation via sampling and interpolation of the transfer functions involved is required.

Let us consider some observations of Gibson [65]. First, Consider the class of elastic mechanical structures modeled,

$$\ddot{x}(t,z) + C_0\dot{x}(t,z) + A_0x(t,z) = Bu(t) \tag{35}$$

where $x \in \mathcal{H}(\Omega)$ appropriately chosen to match required boundary conditions, $u(t)$ is a finite dimensional vector of controls, A_0 is a self adjoint operator $A_0 : D(A_0) \to \mathcal{H}$ densely defined on $\mathcal{H}$. Gibson further assumes that A_0 is *coercive*; i.e.

$$\langle A_0x, x\rangle_{\mathcal{H}} \geq \rho^2\|x\|^2_{\mathcal{H}}, \quad x \in D(A_0)$$

and A_0^{-1} is compact. C_0 is nonnegative, symmetric linear operator and there exists $\gamma \geq 0$ such that

$$\|C_0x\| \leq \gamma^2\|A_0x\|_{\mathcal{H}}, \quad x \in D(A_0). \tag{36}$$

Finally, B_0 is taken to be a bounded operator. Gibson shows that (36) is a necessary condition for the resulting semigroup operator for

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -C_0 \end{bmatrix}$$

on $\mathcal{H} \times \mathcal{H}$ to be *uniformly exponentially stable*; i.e., there exists $M > 0, \alpha > 0$ such that $\|e^{At}\| \leq Me^{-\alpha t}$ for $t \geq 0$. Such behavior corresponds to the case

when damping provides a uniform decay rate. Gibson addresses the question of convergence and stability of a sequence of finite dimensional (possibly modal) approximate control problems to the unique optimal control for the distributed model. His results indicate [65, thm 4.1] that for the quadratic linear regulator problem for the distributed model above (35) that if a solution exists the exact optimal feedback control provides a closed loop system whose infinitesimal generator $A_{cl} = A + BK_{opt}$ generates a strongly continuous semigroup which is uniformly exponentially stable.

For systems without damping; i.e. $C_0 \equiv 0$ in (35), Gibson shows that there can be no nonnegative, selfadjoint solution for the algebraic Riccati equation for the distributed model. This follows from the observation [65, theorem 5.13] that for $C_0 \equiv 0$, the semigroup generated by $A + \mathcal{E}$ for any $\mathcal{E}$ a compact linear operator, cannot be uniformly exponentially stable.

For systems with damping Gibson shows that the damping must be such that A (the generator for the open loop system) is uniformly exponentially stable in order that uniform exponential stability of the closed-loop system can be obtained by compact linear feedback. The significance of this fact is that the physical nature of damping must be considered in the computation of distributed parameter control and for problems where the available control affectors provide 'localized' (in the spatial domain) forces (or torques). Damping with a uniform decay rate (for essentially all modes) is required for such systems to be controlled with uniform exponential stability. Although it is generally agreed that physical structures will have this property is somewhat disconcerting that many standard models for simple structural elements like beams with internal damping *do not* have these properties. In fact with the exception of viscous damping there does not appear to be a single, universally excepted, well-defined mathematical model for beam dynamics which is obviously appropriate for the study of distributed parameter control of structures.

In a later section we will discuss damping models for structural elements in detail. At this point we remark that in this study it became apparent that the performance of active control of elastic structures was seen to be heavily dependent on the type of damping models used. If one is willing to

assume a sufficient amount of viscous damping then stabilizing control can be computed. Since viscous effects will not be present in space applications one is faced with a choice of several models for internal damping. Many of these models lead to problems for which either a uniform decay rate is not available or for which the spectrum of the operator contains more than mere eigenvalues. These models cannot be stabilized by compact linear feedback whether it is computed by Weiner-Hopf methods or by modal approximation and solution of a finite-dimensional quadratic regulator for the reduced model. However if one takes the second path the relevant stability questions are completely lost in the model reduction.

What is an appropriate choice for internal damping in space structures is definitely an open question. This issue has apparently not received a great deal of attention in the aerospace industry especially in that portion of the community involved in control of large flexible structures. These standard procedure here (as evidenced in for instance the ACOSS program) is to assume "low" modal damping can be added to the reduced-order model prior to control system design. In the present frequency domain computations one is forced to resolve these issues before proceeding with control computation.

3 State Space Models for Distributed Structural Elements

3.1 Standard Forms for Linear PDEs

In this section, we will consider the problem of deriving the transfer matrix description for typical distributed elements. It is our contention that the systems of interest to us–specifically beams with one space variable and perhaps several degrees of freedom–can be represented by one of two standard forms. Once identifying the structure of these standard models, it is straightforward, although far from trivial, to mechanize the construction of the required transfer matrices using symbolic computation. Moreover, in order to assemble hybrid system models by the interconnection of components or subsystems, it is essential to have a clear understanding of the causal requirements of the component mathematical models. The following paragraphs develop the required concepts in terms of commonly used structural elements. Since typical elements interact at physical boundaries, our foremost concern is with the formulation of appropriate boundary conditions for well-posed, initial-boundary value problems.

Before proceeding, we establish some basic terminology associated with systems of partial differential equations. Consider the system of first order partial differential equations defined for $t \geq 0$ and $0 \leq x \leq L$

$$E\frac{\partial w}{\partial t} = A^{\bullet}\frac{\partial w}{\partial x} + B^{\bullet}w, \quad w \in R^{n}. \tag{1}$$

If E is nonsingular, then (1) can be written

$$\frac{\partial w}{\partial t} = A\frac{\partial w}{\partial x} + Bw \tag{2}$$

where $A = E^{-1}A^{\bullet}$, $B = E^{-1}B^{\bullet}$. If A has only real eigenvalues and a complete set of eigenvectors, then the system is said to be *hyperbolic* (see, for example, Zauderer [40]). If there are multiple real eigenvalues and less than a complete set of eigenvectors, then the system is of (partial) *parabolic*

type. If all of the eigenvalues are complex, the system is of *elliptic* type. Systems with complex eigenvalues are not causal. Lyczkowski, et al. [41], and Sursock [42] provide an interesting discussion of this point in connection with a fluid flow problem. The underlying problem is that systems with complex eigenvalues are not well-posed as initial value problems, John [43], Lax [44]. We will not consider such problems any further.

If E is singular, (1) can give rise to mixed systems of all types. Some examples can be found in Firedly [45] and Lapidus and Pinder [46]. Our interest in this case will be limited to purely parabolic systems of the type

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} + A\frac{\partial w}{\partial x} + Bw \tag{3}$$

which commonly arise in engineering problems.

When the equations of motion for structural elements are derived from conservation laws–in particular, from variational principles–the resulting equations are typically of hyperbolic type (see, for example, Crandall, et al. [38]). However, further standard assumptions and approximations reduce the equations to parabolic systems in the form of (3). In the following paragraphs, several examples will be given. In summary, we are primarily interested in hyperbolic systems (1) and parabolic systems (3). In addition to equations (1) or (3), there are associated initial and boundary conditions. For equation (1), these conditions take the general form

$$\begin{array}{ll} \text{initial conditions} & w(x,0) = f(x) \\ \text{boundary conditions} & \Sigma w(0,t) + \Gamma w(L,t) = g(t) \end{array} \tag{4}$$

where dim(g)=dim(w). For equation (3), they take the general form

$$\begin{array}{ll} \text{initial conditions} & w(x,0) = f(x) \\ \text{boundary conditions} & \Sigma_1 w(0,t) + \Sigma_2 \frac{\partial w}{\partial x}(0,t) + \Gamma_1 w(L,t) + \Gamma_2 \frac{\partial w}{\partial x}(L,t) = g(t) \end{array} \tag{5}$$

where dim(g)=2dim(w).

It is well known that the coefficient matrices in (4), (5) must satisfy certain constraints if the problem formulation is to be well-posed. In the

hyperbolic case (equations (1) and (4)), these constraints essentially require that the boundary conditions be compatible with the wave directions. Further discussion can be found in Russell [30] and Agarwala [47].

3.2 The Timoshenko Beam Model

We will show how some conventional beam models can be reduced to the standard forms described in the preceding paragraphs. In particular, we will begin with the Timoshenko model and then consider two commonly used approximations which can be derived from it, the Euler-Bernoulli model and the "string" model.

Consider the beam illustrated in Figure 1. The equations of motion can be derived using Lagrange's equations and in the absence of dissipation take the form

$$\begin{aligned} \rho A \frac{\partial^2 \eta}{\partial t^2} &= \frac{\partial}{\partial x}\left[\kappa G A\left(\frac{\partial \eta}{\partial x} - \phi\right)\right] \\ \rho I \frac{\partial^2 \phi}{\partial t^2} &= \frac{\partial}{\partial x}\left[EI\frac{\partial \phi}{\partial x} + \kappa G A\left(\frac{\partial \eta}{\partial x} - \phi\right)\right] \end{aligned} \tag{6}$$

along with the natural boundary conditions for $\alpha = 0, L$

$$\begin{array}{ccc} \text{displacement} & \text{or} & \text{shear force} \\ \eta(\alpha,t) = \eta_\alpha(t) & & \kappa G A\left(\frac{\partial \eta(\alpha,t)}{\partial x} - \phi(\alpha,t)\right) = f_\alpha(t) \end{array} \tag{7}$$

and

$$\begin{array}{ccc} \text{rotation} & \text{or} & \text{moment} \\ \phi(\alpha,t) = \phi_\alpha(t) & & EI\left(\frac{\partial \phi(\alpha,t)}{\partial x}\right) = \tau_\alpha(t). \end{array} \tag{8}$$

The two equations (6) can be replaced by four first-order equations by introducing two new variables, ν(x,t) and γ(x,t):

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\partial \nu}{\partial x} \\ \frac{\partial \nu}{\partial t} &= \frac{\kappa G}{\rho}\left(\frac{\partial \eta}{\partial x} - \phi\right) \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{\partial \gamma}{\partial x} + \frac{A}{I}\gamma \\ \frac{\partial \gamma}{\partial t} &= \frac{E}{\rho}\frac{\partial \phi}{\partial x} \end{aligned}$$

These equations clearly represent a hyperbolic system and the natural boundary c onditions become for $\alpha = O, L$

$$\begin{array}{cc} \text{displacement} & \text{or} \quad \text{shear force} \\ \eta(\alpha,t) = \eta_\alpha(t) & \nu(\alpha,t) = \nu_\alpha(t), \dot{\nu}_\alpha(t) = \frac{f_\alpha(t)}{\rho A} \end{array} \tag{10}$$

and

$$\begin{array}{cc} \text{rotation} & \text{or} \quad \text{moment} \\ \phi(\alpha,t) = \phi_\alpha(t), & \gamma(\alpha,t) = \gamma_\alpha(t),\ \dot{\gamma}_\alpha(t) = \frac{\tau_\alpha(t)}{\rho I}. \end{array} \tag{11}$$

Note that the boundary conditions applied to the first order system (9) require the time integral of boundary forces or moments applied to the beam. It is easy to confirm that the transfer functions relating forces or moments to displacements or rotations as derived from either equations (6) or (9) are indeed identical and that the required integration is essential.

3.3 The Bernoulli-Euler Beam Model

The Bernoulli-Euler model is obtained from the Timoshenko model with two additional assumptions:

1. rotational inertia is neglected, $\rho I \to 0$
2. shear deformation is neglected, $\frac{\partial \eta}{\partial x} - \phi \to 0$

Assumption (1) reduces the second of equations (6) to

$$\kappa G A \left(\frac{\partial \eta}{\partial x} - \phi \right) = -\frac{\partial}{\partial x}\left(EI \frac{\partial \phi}{\partial x} \right) \tag{12}$$

Equation (12) and assumption (2) are now used to reduce the first equation of (6) to

$$\rho A \frac{\partial^2 \eta}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \eta}{\partial x^2}\right). \tag{13}$$

Note that equation (12) along with assumption (2) leads to the following expression for shear force

$$f = \kappa GA\left(\frac{\partial \eta}{\partial x} - \phi\right) = -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 \eta}{\partial x^2}\right). \tag{14}$$

Although (14) is commonly used in conjunction with the Bernoulli-Euler model (13), it should only be used with caution. Equation (13) is valid only in the limit $f \to 0$. We will return to this point below.

The boundary conditions (7) and (8) reduce to for $\alpha = 0, L$

$$\begin{array}{ccc} \text{displacement} & \text{or} & \text{shear force} \\ \eta(\alpha,t) = \eta_\alpha(t), & & -EI\frac{\partial^3 \eta(\alpha,t)}{\partial x^3} = f_\alpha(t), \end{array} \tag{15}$$

and

$$\begin{array}{ccc} \text{displacement} & \text{or} & \text{moment} \\ \frac{\partial \eta(\alpha,t)}{\partial x} = \phi(\alpha,t) = \phi_\alpha(t), & & EI\frac{\partial^2 \eta(\alpha,t)}{\partial x^2} = \tau_\alpha(t) \end{array} \tag{16}$$

Note that nonzero shear force is included as an admissible boundary condition; however, the remarks following equation (14) apply.

Equation (13) can be reduced to first-order form by introducing a new variable $\gamma(x,t)$

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= -\frac{I}{A}\frac{\partial^2 \gamma}{\partial x^2}, \\ \frac{\partial \gamma}{\partial t} &= \frac{E}{\rho}\frac{\partial^2 \eta}{\partial x^2}, \end{aligned} \tag{17}$$

and the boundary conditions associated with (17) are for $\alpha = 0, L$:

$$\begin{array}{ccc} \text{displacement} & \text{or} & \text{shear force} \\ \eta(\alpha,t) = \eta_\alpha(t), & & \frac{\partial \gamma(\alpha,t)}{\partial x} = \gamma'_\alpha(t), \dot{\gamma}'_\alpha = -\frac{A}{I} f_\alpha(t), \end{array} \tag{18}$$

and

$$\begin{array}{cc} \text{rotation} & \text{or} \\ \frac{\partial \eta(\alpha,t)}{\partial x} = \phi_\alpha(t), & \gamma(\alpha,t) = \gamma_\alpha(t),\ \dot{\gamma}_\alpha = \frac{\tau_\alpha(t)}{\rho I}. \end{array} \qquad \text{moment} \tag{19}$$

Observe that (17) is a parabolic system of the type (3) . Equations (17– 19) can be derived directly from (13) or from (9) upon invoking assumptions (1) and (2). We should also note that a corresponding expression for shear force obtains

$$f = -\left(\frac{I}{A}\right)\frac{\partial^2 \gamma}{\partial t \partial x}. \tag{20}$$

Because shear force is often used as a boundary input with the Bernoulli-Euler model, some further comment is warranted. It is a straightforward matter to compute the transfer function between any applied boundary input and the beam response, as is done, for example, by Kolousek [48]. In particular, consider the deflection response at $x = 0, H_0(s)$, to a force inputs at $x = L$, $F_L(s)$:

$$H_0(\cdot) = \left(\frac{L^3}{EI}\right)\frac{\cosh\lambda\cos\lambda - 1}{\lambda^3(\sinh\lambda + \sin\lambda)}F_L(s) \tag{21}$$

where

$$\lambda^4(s) = -\frac{\rho A L^4}{EI}s^2.$$

The transfer function in (21) has a branch point at the origin. This is true of any transfer function associated with a force input. On the other hand, all deflection responses to torque inputs have meromorphic transfer functions. Clearly, there is an issue concerning the Bernoulli-Euler model with force inputs which must be settled before it can be used with any confidence.

3.4 The "String" Model

In some situations, bending deformation may be negligible with respect to shear deformation, that is, $|\phi| \ll |\partial\eta/\partial x|$. In this case, the first equation

of (6) reduces to

$$\rho A \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial x}\left(\kappa G A \frac{\partial \eta}{\partial x}\right) \tag{22}$$

with boundary conditions for $\alpha = 0, L$

$$\begin{array}{cc} \text{displacement} & \text{or} \quad \text{shear force} \\ \eta(\alpha,t) = \eta_\alpha(t) & \kappa G A \frac{\partial \eta(\alpha,t)}{\partial x} = f_\alpha(t). \end{array} \tag{23}$$

This simple model is primarily useful for illustrative purposes. Again by introducing the new variable $\nu(x,t)$, equation (22) can be replaced by two first order equations

$$\begin{array}{rcl} \frac{\partial \eta}{\partial t} & = & \frac{\partial \nu}{\partial x} \\ \frac{\partial \nu}{\partial t} & = & \frac{\kappa G \partial \eta}{\rho \partial x} \end{array} \tag{24}$$

which is to be solved with boundary conditions for $\alpha = 0, L$

$$\begin{array}{ll} \text{displacement} \quad \text{or} & \text{shear force} \\ \eta(\alpha,t) = \eta_\alpha, & \nu(\alpha,t) = \nu_\alpha(t),\ \dot{\nu}_\alpha = \frac{f_\alpha(t)}{\rho A} \end{array} \tag{25}$$

Note that our first order model is slightly different from that of Burns and Cliff [29].

3.5 Distributed Elements with Wave Dynamics: Frequency Response Calculations for State Space Models

We will be concerned with the computation of certain irrational transfer functions and a resolvent operator. For our purposes, the res ·ent can be considered as an integral operator with kernel called the *Green's function.* In this section, we will consider explicitly the required computations for the abstract objects discussed previously. To do so we will focus on hyperbolic beam models for distributed elements. Such models can also be used for elastic dynamics of beams, cables, etc. In the next section, we will discuss

hybrid system models consisting of interconnections of these components with rigid bodies and other lumped parameter models.

Consider a class of hyperbolic partial differential equations in one space dimension $0 \leq z \leq L$, arising from models such as (??) which can be written (as in (2), (4))

$$\frac{\partial x(t,z)}{\partial t} = A\frac{\partial x(t,z)}{\partial z} + Bx(t,z) + Cu(t,z) \tag{26}$$

subject to boundary conditions

$$\Sigma x(t,0) + \Gamma x(t,L) = Df(t). \tag{27}$$

Here, x is an n-vector valued state $x \in \mathcal{H}^n(0,L)$, u is an m-vector valued distributed disturbance $u \in \mathcal{H}^m(0,L)$, A, B are real $n \times n$ matrices with A nonsingular and diagonalizable [30], and Σ, Γ are $n \times n$ matrices. Controlability questions for systems of this type are considered in Russell [30]. We remark that (28) is a concrete example of the transform of the abstract formula (??). Thus it is clear that the resolvent for the operator $\mathbf{A} : \mathcal{H}^n(0,L) \to \mathcal{H}^n(0,L)$ defined by (26) and (27) is the integral operator $\int_0^L G_r(s,z,w)\cdot dw$. After taking Laplace transforms in the temporal variable t, we obtain an equation

$$\hat{X}(s,z) = \int_0^L G_r(s,z,w)\hat{M}(s,w)dw + H_{BC}(s,z)\hat{F}(s) \tag{28}$$

where

$$\hat{M}(s,w) = x(0,w) - C\hat{U}(s,w),$$

and $\hat{X}, \hat{U}, \hat{F}$ are the Laplace transforms of x, u, f, respectively. The function $G_r(s,z,w)$ is the Green's function [49],[50] for (26), (27) and $H_{BC}(s,z)$ is a transfer function from boundary control to state. Since in most cases of practical interest the control of flexible structures will be effected by actuators whose influence functions are highly localized, we have formulated our model with boundary control only.

A straightforward calculation leads to the following form for H_{BC}

$$H_{BC}(s,z) = \Phi(s,z)\left[\Sigma + \Gamma\Phi(s,L)\right]^{-1} D \tag{29}$$

where

$$\Phi(s,z) = e^{[A^{-1}(sI-B)z]}. \tag{30}$$

The Green's function for (26), (27) is the solution to

$$\frac{\partial G_r(s,z,w)}{\partial z} = A^{-1}\left[sI - B\right] G_r(s,z,w) + I_n\delta(z-w) \tag{31}$$

subject to the boundary conditions

$$\Sigma G_r(s,0,w) + \Gamma G_r(s,L,w) = 0 \tag{32}$$

where $\delta(\cdot)$ is the Dirac delta function [49], [50]. From (31) we see that the solution is discontinuous at the point $z = w$. After some computation, we can write

$$G_r(s,z,w) = \begin{cases} G_{r\,LEFT}(s,z,w), & \text{for } 0 \le z \le w \\ G_{r\,RIGHT}(s,z,w), & \text{for } w \le z \le L \end{cases} \tag{33}$$

with

$$G_{r\,LEFT}(s,z,w) = H_{BC}(s,z)\Gamma\Phi(s,-w) \tag{34}$$

$$G_{r\,RIGHT}(s,z,w) = -H_{BC}(s,z)\Gamma\Phi^{-1}(s,L-w) \tag{35}$$

3.6 Modeling of Hybrid Systems

In most applications, models for the dynamics of flexible structures will involve interaction between various elastic and rigid elements. In the particular case of flexible structures associated with large space structures, the potential topological configurations can be quite complex. Various elements such as beams, truss structures, cables, membranes, etc., may have dominant distributed parameter effects. Typically a central body or bodies represent large concentrations of mass with respect to the overall low mass density of the flexible structure. These are most effectively represented by lumped parameter models of their rigid body dynamics. Additionally, various attitude control actuators can add concentrated inertia elements which can be effectively modeled as lumped systems. Thus, carefully chosen linear, hybrid models can provide an effective tool for analysis of dynamics

of vibrations and their effect on small angle motions for complex space platforms. In this section, we consider the structures and computations of certain resulting transfer functions and the resolvent operator for the composite system along the lines of Section 1.

The concept of a mechanical impedance (terminology borrowed from electrical network theory) has been used in structural dynamic modeling for many years [48]. The dynamic stiffness method (application to space structure modeling is reviewed in Piche [51]) uses this notion to compute effective transfer function models for interconnected structures [52]. Our approach here will follow along similar lines except that we will focus on computing the resolvent operator for a hybrid structure by direct manipulation of its kernel; viz, a Green's function. xxx A hybrid state space model is constructed in Burns and Cliff [29] (where considerations are given for approximation ad computation in the hybrid state space). We will consider a hybrid state space as consisting of a direct sum of spaces $\mathcal{X} = \mathcal{X}_\ell \oplus \mathcal{X}_d$ where $\mathcal{X}_d = \mathcal{H}^{N_d}$ is the distributed part constructed on an appropriate Hilbert space of N_d-vector valued functions with the "energy" inner product of (??) for a distributed parameter system (DPS) written in abstract form (as in (??))

$$\frac{\partial x_d(t,z)}{\partial t} = \mathbf{A}x_d(t,z) + \mathbf{B}u(t), \tag{36}$$

with $x_d^0(z) = x_d(0,z) \in D(\mathbf{A}) \subseteq \mathcal{H}_d^N$. We assume that $\mathbf{A}$ generates a C_0-semigroup which is contractive (so that (??) is well defined). We ignore contribution to (??) of the possible distributed control force. By taking Laplace transforms in (36), we can write

$$\hat{X}_d(s,z) = [sI - \mathbf{A}]^{-1}\mathbf{B}\hat{U}(s) \tag{37}$$

which is an abstract formulation for the transform of (??).

For structural control, we restrict attention to the hyperbolic problem of section 1.3:

$$\frac{\partial x_d(t,z)}{\partial t} = A_d\frac{\partial x_d(t,z)}{\partial z} + B_d x_d(t,z) + C_d v(t,z) \tag{38}$$

$0 \leq z \leq L$, subject to natural boundary conditions.

$$\Sigma x_d(t,0) + \Gamma x_d(t,L) = D f_d(t). \tag{39}$$

For this problem, equation (37) can be written

$$\hat{X}_d(s,z) = \int_0^L G_r(s,z,w)\hat{M}(s,w)dw + H_{BC}(s,q)\hat{F}_d(s) \qquad (40)$$

where

$$\hat{M}(s,w) = x_d^0(w) - C\hat{V}(s,w), \qquad (41)$$

A_d, C_d, B_d are matrices defined as in (2), $\hat{V}(s,w)$ is the Laplace transform o f a distributed exogenous disturbance, and $\hat{F}_d(s)$ is an m_d-vector of inputs to the DPS. Clearly, (38)-(39) can represent a disjoint collection of distributed elements such as beams, cables, etc. (Conceptually, a version of (40) can also be written for higher dimensional spatial domains, but we feel for the current presentation that the required complexity of notation can mask the simplicity of the underlying concepts (see Butkovskiy for details [50].)

Similarly, all lumped parameter (LPS) component models are combined into a LPS state space model as

$$\dot{x}_\ell(t) = A_\ell x_\ell(t) + B_\ell f_\ell(t),\ x_\ell^o = x_\ell(0) \qquad (42)$$

with $x_\ell \in R^{N_\ell} \equiv \mathcal{X}_\ell$ a finite dimensional real space. By taking Laplace transforms in (42), we write (analogous to (40))

$$\hat{X}_\ell(s) = R_\ell(s)x_\ell^0 + H_\ell(s)\hat{F}_\ell(s), \qquad (43)$$

where $R_\ell(s) = [sI_{N_\ell} - A_\ell]^{-1}$ is the resolvent for the (matrix) operator A_ℓ and $H_\ell(s) = R_\ell(s)B$.

The hybrid state space $\mathcal{X} = \mathcal{X}_\ell \oplus \mathcal{X}_d$ consists of elements

$$x(t,z) = \begin{bmatrix} x_\ell(t) \\ x_d(t,z) \end{bmatrix} \qquad (44)$$

which are $N = N_d + N_\ell$-valued functions of $z \in [0,L]$, $t > 0$. Finally, the interconnection of component systems is resolved through a topological constraint relation consisting of $m = m_d + m_\ell$ linear equations;

$$f(t) + K_1 x_d(t,0) + K_2 x_d(t,L) + K_3 x_\ell(t) = Eu(t) \qquad (45)$$

where $u(t)$ is a k-vector of control inputs to the hybrid system, K_1, K_2 are $m \times N_d$, K_3 is $m \times N_\ell, E$ is $m \times k$ real matrices. The hybrid modeling problem is to find an equation of the form (40) by solving (40), (43)–(45) simultaneously for the hybrid state $x(t,z)$. We provide the resulting model in the following form:

$$\hat{X}(s,z) = \int_0^L \tilde{G}_r(s,z,w)\hat{M}(s,w)dw + \tilde{R}(s,z)x_\ell^0 + \tilde{H}(s,z)\hat{U}(s), \qquad (46)$$

where $\hat{M}(s,w)$ is given in (41). The resolvent operator for the hybrid system is

$$R(s;\mathbf{A}) = \left[\int_0^L \tilde{G}_r(s,z,w)\cdot dw,\ \tilde{R}(s,z)\right] \qquad (47)$$

where $R(s;\mathbf{A}) : \mathcal{X} \to D(\mathbf{A}) \subseteq \mathcal{X}$, $\tilde{G}_r$ is $N \times N_d$ and $\tilde{R}$ is $N \times N_\ell$ are matrix valued functions given by

$$\tilde{R}(s,z) = \begin{bmatrix} I_{N_\ell} - H_\ell(s)\tilde{Q}_1(s) \\ -H_{BC}(s,z)\tilde{Q}_2(s) \end{bmatrix} K_3 R_\ell(s), \qquad (48)$$

$$\tilde{G}_r(s,z,w) = \begin{bmatrix} -H_\ell(s)\tilde{Q}_1(s) \\ G_r(s,z,w) - H_{BC}(s,z)\tilde{Q}_2(s) \end{bmatrix} P(s,w) \qquad (49)$$

where

$$\tilde{Q}(s) = [I_M + Q(s)]^{-1} = \begin{bmatrix} \tilde{Q}_1(s) \\ \tilde{Q}_2(s) \end{bmatrix}, \qquad (50)$$

$$Q(s) = [K_3 H_\ell(s),\ K_1 H_{BC}(s,0) + K_2 H_{BC}(s,L)], \qquad (51)$$

$$P(s,w) = K_1 G_r(s,0,w) + K_2 G_r(s,L,w). \qquad (52)$$

Finally, the $N \times k$ transfer function matrix from boundary control to hybrid state is

$$\tilde{H}(s,z) = \begin{bmatrix} H_\ell(s) & 0 \\ 0 & H_{BC}(s,z) \end{bmatrix} \tilde{Q}(s)E. \qquad (53)$$

The derivation of (46)–(53) is straightforward and proceeds as follows. Substitute (40), (42) into (45) and solve for the interconnecting force $\hat{F}(s)$. This identifies the terms $Q(s), P(s,w)$ above. Now substitute the appropriate components of $\hat{F}(s)$ into (40), (42) and use the hybrid state model (44).

In the next section we examine a modeling technique which permits the identification of continuum models for extended structures with regular infrastructure.

4 Homogenization of regular structures

In this section we consider the problem of modeling and control of a class of lattice structures, e.g., trusses. Their large size and repetitive infrastructure require special techniques for structural analysis to cope with the large number of degrees of freedom. Approximations of such systems by continua provide a simple means for comparing structural characteristics of lattices with different configurations, and they are effective in representing macroscopic vibrational modes and structural response due to temperature and load inputs. Our approach to the construction of such models is based on a technique for asymptotic analysis called *homogenization*. It has been widely used in mathematical physics for the treatment of composite systems like porous media for which one wishes to have an effective approximating system with parameters which are constant across the structure.[1]

Before developing the general features of the method and applying it to the treatment of lattice structures, we shall make a few remarks on other work on continuum models which has appeared in the recent structural mechanics literature.

Noor, et. al. [20] use an energy method to derive a continuum approximation for trusses with triangular cross sections in which the modal displacements of the truss are related to a linearly varying displacement field for an equivalent bar. Plates with a lattice infrastructure are also treated. In Dean and Tauber [9] and Renton [19] exact analytical expressions for the solutions of trusses under load were derived using finite difference calculus. By expressing the difference operators in terms of Taylor's series Renton [37] was able to derive continuum approximations to the finite difference equations resulting in expressions for equivalent plate stiffnesses, for example. In a recent paper Renton [37] used this approach to give equivalent beam properties for trusses, which complements the earlier work of Noor, Anderson and Greene [22], and Nayfeh and Hefzy [21]. (See also (Anderson [22]).)

[1] See, for example, the papers of Larsen [14], Keller [13], and the reports of Babuska [2] for applications and discussions of design techniques.

In most cases a continuum model is associated with the original (lattice) structure by averaging the parameters of the lattice over some natural volume (e.g., of a "cell" of the structure) and identifying the averaged parameter value (mass density, stress tensor, etc.) with the corresponding distributed parameter in the continuum model. A specific form for the continuum model is postulated at the outset of the analysis; e.g., a truss with lattice structure will be approximated by a beam, with the beam dynamical representation assumed in advance. While this approach has an appealing directness and simplicity, it has some problems.

First, it is very easy to construct an example in which the "approximate model" obtained by averaging the parameters over a cell is not a correct approximation to the system behavior. This is done in subsection 4.1.[2] Second, one cannot use this procedure to obtain "corrections" to the approximation based on higher order terms in an expansion, which may sometimes be done in an asymptotic analysis. These terms can be used to describe the microscopic behavior (e.g., local stresses) in the structure. Third, the averaging method (averaging the parameters over space) does not apply in a straightforward way to systems with a random structure, since the appropriate averaging procedure may not be obvious.[3] Fourth, the method cannot be naturally imbedded in an optimization procedure; and controls and state estimates based on the averaged model may not be accurate reflections of controls and state estimates derived in the course of a unified optimization - averaging procedure. In particular, the method does not provide a systematic way of estimating the degree of suboptimality of controls and state estimates computed from the idealized model.

In this work we use a totally different technique called *homogenization* from the mathematical theory of asymptotic analysis to approximate the dynamics of structures with a repeating cellular structure. Homogenization produces the distributed model as a consequence of an asymptotic analysis carried out on a rescaled version of the physical system model.

[2] See the numerical experiments in (Bourgat [7]).

[3] Homogenization methods do apply to systems with a randomly heterogeneous structure, see (Papanicolaou and Varadhan [16]) and (Kunnemann [39]). We shall treat such systems in a subsequent report.

Unlike the averaging method, homogenization can be used in combination with optimization procedures; and it can yield systematic estimates for the degree of suboptimality of controls and estimators derived from idealized models. While our results are stated in terms of simple structures, they demonstrate the feasibility of the method; and they suggest its potential in the analysis of structures of realistic complexity.

In subsection 4.1 we give an example derived from (Bensoussan, Lions, and Papanicolaou [5]) illustrating some of the subtleties of homogenization, particularly in the context of control problems. In subsection 4.2 we derive a homogenized representation for the dynamics of a lattice structure undergoing transverse deflections. We show that the behavior of the lattice is well approximated by the Timenshenko beam equation; and we show that this equation arises naturally as the limit of the lattice dynamics when the density of the lattice structure goes to infinity in a well defined way. The problem of vibration control of a lattice is posed and discussed in subsection 4.3. In subsection 4.4 we derive a diffusion approximation for the thermal conductivity of a one-dimensional lattice structure. This property is useful in analyzing new materials for large space structures.

Acknowledgements: We are grateful to Professor George Papanicolaou for bringing Kunnemann's paper to our attention and to Drs. A. Amos and R. Lindberg for their comments on an earlier version of this work.

4.1 A one-dimensional example

From (Bensoussan, Lions, and Papanicolaou [5]) we have the following example:

$$-\frac{d}{dx}[a^\epsilon(x)\frac{du^\epsilon(x)}{dx}] = f(x), x \in (x_0, x_1) \tag{1}$$
$$u^\epsilon(x_0) = 0 = u^\epsilon(x_1)$$

where $a^\epsilon(x) \stackrel{\text{def}}{=} a(x/\epsilon)$, and $a(y)$ is periodic in y with period Y_0, $a(y) \geq \alpha > 0$. It is simple to show that

$$||u^\epsilon||^2_{H^1} \stackrel{\text{def}}{=} \int_{x_0}^{x_1} |u^\epsilon(x)|^2 + |\frac{du^\epsilon(x)}{dx}|^2 dx \leq c \tag{2}$$

and so, $u^\epsilon \rightarrow u$ weakly in the Hilbert space H^1.[4] Moreover,

$$a^\epsilon \rightarrow M(a) \stackrel{\text{def}}{=} \frac{1}{Y_0} \int_0^{Y_0} a(y) dy \tag{3}$$

and it is natural to suppose that $u^\epsilon \rightarrow u$ with the limit defined by

$$-\frac{d}{dx}[M(a)\frac{d}{dx}u(x)] = f(x), x \in (x_0, x_1) \tag{4}$$

$$u(x_0) = u(x_1)$$

This is untrue in general (Bensoussan, Lions, and Papanicolaou [5], pp. 8-10). The correct limit is given by

$$-\frac{d}{dx}[\bar{a}\frac{d}{dx}u(x)] = f(x), x \in (x_0, x_1) \tag{5}$$

$$u(x_0) = u(x_1)$$

with

$$\bar{a} \stackrel{\text{def}}{=} [M(\frac{1}{a})]^{-1} \tag{6}$$

In general, $M(a) > \bar{a}$; and so, the error in identifying the limit, (4) versus (5), is fundamental.

The system (4) corresponds to averaging the parameter $a^\epsilon(x)$ over a natural *cell;* a procedure similar to that used in the past to define continuum models for lattice structures. As (5) shows, the actual averaging process can be more subtle than one might expect, even for simple problems.

[4] Here $H^1 \stackrel{\text{def}}{=} \{u \in L^2(x_0, x_1) : ||u||_{H^1} < \infty\}$

4.1.1 Homogenization of the example

To see how (5) arises, we can use the method of *multiple scales* which applies to a variety of perturbation problems. Suppose

$$u^\epsilon(x) = u^\epsilon(x, \frac{x}{\epsilon}) = u_0(x, \frac{x}{\epsilon}) + \epsilon u_1(x, \frac{x}{\epsilon}) + \dots \tag{7}$$

that is, we suppose that u^ϵ depends on the "slow" scale x and the "fast" scale $y \stackrel{\text{def}}{=} x/\epsilon$; and we adopt an *ansatz* which reflects this dependence. Using the identity

$$\frac{d}{dx}[u(x, \frac{x}{\epsilon})] = \frac{\partial u}{\partial x} + \frac{1}{\epsilon}\frac{\partial u}{\partial y}, \text{ with } y = \frac{x}{\epsilon}. \tag{8}$$

then (1) may be rewritten as

$$-(\frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y})\{a(y)(\frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y})[u_0 + \epsilon u_1 + ...]\} = f \tag{9}$$

Simplifying and equating coefficients of like powers of ϵ, we find first that

$$-\frac{1}{\epsilon^2}\frac{\partial}{\partial y}[a(y)\frac{\partial}{\partial y}u_0] = 0. \tag{10}$$

The assumptions on *a(y)* imply

$$u_0(x, y) = u_0(x) \tag{11}$$

i.e., no y-dependence. The coefficients of ϵ^{-1} satisfy

$$\{\frac{\partial}{\partial y}[a(y)\frac{\partial}{\partial x}u_0] + \frac{\partial}{\partial x}[a(y)\frac{\partial}{\partial y}u_0] - \frac{\partial}{\partial y}[a(y)\frac{\partial}{\partial y}u_1]\} = 0 \tag{12}$$

or

$$\frac{\partial}{\partial y}[a(y)\frac{\partial}{\partial y}u_1] = -\frac{\partial a}{\partial y}\frac{\partial u_0}{\partial x} \tag{13}$$

If we look for u_1 in the form

$$u_1(x, y) = -\chi(y)\frac{\partial u_0}{\partial x} + \hat{u}_1(x), \tag{14}$$

then the *corrector* $\chi(y)$ must satisfy

$$-\frac{d}{dy}[a(y)\frac{d}{dy}\chi(y)] = -\frac{da}{dy} \tag{15}$$

and be periodic. That is,

$$a(y)\frac{d\chi}{dy} = a(y) + c \tag{16}$$

which has a periodic solution (unique up to an additive constant in y) if and only if

$$\frac{1}{Y_0}\int_0^{Y_0}[1 + \frac{c}{a(y)}]dy = 0 \tag{17}$$

which implies

$$c = -[M(\frac{1}{a})]^{-1} \stackrel{\text{def}}{=} \bar{a} \tag{18}$$

We obtain an equation for $u_0(x)$ from the solvability condition for $u_2(x,y)$. Equating the coefficients of ϵ^0 in the expansion, we have

$$-\frac{\partial}{\partial y}[a(y)\frac{\partial}{\partial y}u_2] - \frac{\partial}{\partial y}[a(y)\frac{\partial}{\partial x}u_1] \tag{19}$$

$$-\frac{\partial}{\partial x}[a(y)\frac{\partial}{\partial y}u_1] - \frac{\partial}{\partial x}[a(y)\frac{\partial}{\partial x}u_0] = f(x)$$

This has a solution $u_2(x,y)$, periodic in y if and only if

$$\{\frac{1}{Y_0}\int_0^{Y_0}[a(y) + \frac{\partial}{\partial y}[a(y)\chi(y)] - a(y)\frac{\partial}{\partial y}\chi(y)]dy\} \cdot \frac{d^2u_0(x)}{dx^2} \tag{20}$$

$$+f(x) = 0$$

where we have used (14). The integral of the second term is zero, since it is the integral of the derivative of a periodic function over one period. Using (16) and (18), (20) reduces to

$$-\bar{a}\frac{d^2u_0}{dx^2} + f(x) = 0 \tag{21}$$

(plus the boundary conditions) which is (5) (6).

4.1.2 Control and homogenization of the one dimensional system

One of the simplest (stochastic) control problems associated with the preceding system is defined by the Hamilton - Jacobi - Bellman equation

$$-a(\frac{x}{\epsilon})\frac{d^2u^\epsilon}{dx^2} - \frac{1}{\epsilon}b(\frac{x}{\epsilon})\frac{du^\epsilon}{dx} = \inf_{v\in\mathcal{R}}[\frac{1}{2}v^2 + g(x,y)v\frac{du^\epsilon}{dx} - cu^\epsilon] \tag{22}$$

$$x \in \mathbf{O}, u^\epsilon(x) = 0 \text{ on } \Gamma \stackrel{\text{def}}{=} \partial\mathbf{O}$$

where $\mathbf{O}$ is an open interval in $\mathcal{R}$, and each function $a(y), b(y)$, and $g(x,y)$ is periodic in y with period Y_0. We assume that $a(y) \geq \alpha > 0$ and that $c > 0$, and that the controls v take values in $\mathcal{R}$.

This Bellman equation corresponds to the stochastic control problem

$$u^\epsilon(x) = \inf_{v(\cdot)} J^\epsilon[v(\cdot)]$$

$$J^\epsilon[v(\cdot)] = E_x\{\int_0^{\tau_a} l(x^\epsilon, \frac{x^\epsilon}{\epsilon}, v)[e^{-\int_0^t c(x^\epsilon, \frac{x^\epsilon}{\epsilon}, v)ds}]dt\} \tag{23}$$

$$dx^\epsilon(t) = \sigma(x^\epsilon, \frac{x^\epsilon}{\epsilon})dw(t) + \frac{1}{\epsilon}b(x^\epsilon, \frac{x^\epsilon}{\epsilon})dt + G(x^\epsilon, \frac{x^\epsilon}{\epsilon}, v)dt$$

$$x^\epsilon(0) = x \in \mathbf{O}, t \geq 0.$$

with $\sigma^2(x,y) \stackrel{\text{def}}{=} a(y)$, $b(x,y) = b(y)$, $G(x,y,v) = g(x,y)v$, $l(x,y,v) = \frac{1}{2}v^2$, and $c(x,y,v) = c$, a constant in (22). Each function in (23) is assumed to be periodic in y with period one. We are interested in the behavior of the optimal cost and control law for (22) in the limit as $\epsilon \to 0$. The stochastic control problem (23) was treated in (Bensoussan, Boccardo, and Murat [4]); the analysis here uses different arguments which emphasize the computational aspects of the system.

Evaluating the infimum in (22), we have the nonlinear system

$$a(\frac{x}{\epsilon})u^\epsilon_{xx} + \frac{1}{\epsilon}b(\frac{x}{\epsilon})u^\epsilon_x - cu^\epsilon_x - \frac{1}{2}g^2(x, \frac{x}{\epsilon})(u^\epsilon_x)^2 = 0 \tag{24}$$

$$x \in \mathbf{O}, u^\epsilon(x)|_\Gamma = 0.$$

The analysis of the control problem involves homogenization of this system.

Let

$$A_1 = a(y)\partial_{yy} + b(y)\partial_y \tag{25}$$

with its formal adjoint defined by

$$A_1^* = \partial_y[a(y)\partial_y \cdot] - \partial_y[(b(y) - a_y(y))\cdot]. \tag{26}$$

The problem

$$A_1^* m = 0, y \to m(y) \text{ periodic} \tag{27}$$

$$m > 0, \int_Y m(y)dy = 1$$

has a unique solution $m(\cdot)$ on $Y = S^0$, the unit circle, with

$$0 < \underline{m} \leq m(y) \leq \bar{m} < \infty. \tag{28}$$

So $m(\cdot)$ is a density on Y. We *assume* that $b(\cdot)$ is *centered*

$$\int_Y m(y)b(y)dy = 0. \tag{29}$$

As a consequence the system

$$A_1\chi(y) = b(y) \tag{30}$$

$$y \to \chi(y) \text{ periodic }, \int_Y \chi(y)dy = 0$$

has a well defined solution. $\chi(\cdot)$ is the *corrector* associated with the problem.

As before we set $y = x/\epsilon$ and look for u^ϵ in the form

$$u^\epsilon(x) = u^\epsilon(x,y) = u_0(x,y) + \epsilon u_1(x,y) + \ldots, \tag{31}$$

and we use

$$\partial_x\phi(x,y) = \phi x(x,y) + \frac{1}{\epsilon}\phi_y(x,y), y = x/\epsilon \tag{32}$$

$$\partial_{xx}\phi(x,y) = \phi_{xx}(x,y) + 2\epsilon\phi_{xy}(x,y) + \phi_{yy}(x,y).$$

Substituting in (24), we have

$$a(y)[u_{0xx} + 2\epsilon u_{0xy} + \frac{1}{\epsilon^2}u_{0yy}] + a(y)[\epsilon u_{1xx} + 2u_{1xy} + \frac{1}{\epsilon}u_{1yy}]$$

$$+a(y)[\epsilon^2 u_{2xx} + 2\epsilon u_{2xy} + u_{2yy}]$$

$$+\frac{1}{\epsilon}b(y)[u_{0x} + \frac{1}{\epsilon}u_{0y}] + \frac{1}{\epsilon}b(y)[\epsilon u_{1x} + u_{1y}] \tag{33}$$

$$+\frac{1}{\epsilon}b(y)[\epsilon^2 u_{2x} + \epsilon u_{2y}] - c[u_0 + \epsilon u_1 + \epsilon^2 u_2]$$

$$-\frac{1}{2}g^2(x,y)[(u_{0x} + \epsilon u_{1x} + \epsilon^2 u_{2x}) + \frac{1}{\epsilon}(u_{0y} + \epsilon u_{1y} + \epsilon^2 u_{2y})]^2 = O(\epsilon^2)$$

The last term is

$$-\frac{1}{2}g^2(x,y)[(\frac{1}{\epsilon^2}u_{0y}^2 + 2\epsilon u_{0x}u_{0y} + 2u_{0x}u_{1y} + u_{0x}^2) \tag{34}$$

$$+\epsilon(2u_{0x}u_{1x} + 2u_{1x}u_{1y} + 2u_{1y}u_{2y} + 2u_{0x}u_{2y})] + O(\epsilon^2)$$

Equating coefficients of like powers of ϵ, we obtain

$$(\epsilon^{-2})\{a(y)u_{0yy} + b(y)u_{0y} - \frac{1}{2}g^2(x,y)u_{0y}^2 = 0\} \tag{35}$$

$$(\epsilon^{-1})\{a(y)u_{1yy} + b(y)u_{1y} + 2a(y)u_{0xx} \tag{36}$$

$$+b(y)u_{0x} - g^2(x,y)u_{0x}u_{0y} = 0\}$$

$$(\epsilon^0)\{a(y)u_{2yy} + b(y)u_{2y} + 2a(y)u_{1xy} + b(y)u_{1x} \tag{37}$$

$$+a(y)u_{0xx} - cu_0 - \frac{1}{2}g^2(x,y)u_{0x}^2 - g^2(x,y)u_{0x}u_{1x} = 0\}.$$

Choosing $u_0(x,y) = u_0(x)$, which must be justified, satisfies (35). We can then solve (36) by choosing

$$u_1(x,y) = -\chi(y)u_{0x}(x) + \hat{u}_1(x). \tag{38}$$

Equation (37) has a solution for $u_2(x,y)$ if

$$\int_Y m(y)\{-2a(y)\chi_y u_{0xx} - b(y)\chi_y u_{0x} + a(y)u_{0xx} \tag{39}$$

$$-cu_0 - \frac{1}{2}g^2(x,y)(1-2\chi_y)u_{0x}^2\}dy = 0.$$

This gives an equation for $u_0(x)$

$$qu_{0zz} - cu_0 - \frac{1}{2}\Gamma u_{0x}^2 = 0$$

where

$$q \stackrel{\text{def}}{=} \int_Y m(y)\{a(y)[1-2\chi_y(y)] - \chi(y)b(y)\}dy$$

$$\Gamma \stackrel{\text{def}}{=} \int_Y m(y)g^2(x,y)[1-2\chi_y(y)]dy.$$

Remark. From the definition of A_1 and the corrector $\chi(y)$ we have

$$\int_Y m(y)b(y)\chi(y)dy = \int_Y [a(y)\chi_{yy} + b(y)\chi_y]\chi(y)m(y)dy \tag{42}$$

$$= \int_Y \chi(y)\partial_{yy}[a(y)\chi(y)m(y)]dy - \int_Y \chi(y)\partial_y[b(y)\chi(y)m(y)]dy$$

Also, using (30),

$$\int_Y m(y)b(y)\chi(y)dy = \int_Y \chi(y)a(y)m(y)\chi_{yy}(y)dy \tag{43}$$

$$-\int_Y \chi(y)b(y)m(y)\chi_y dy + 2\int_Y \chi(y)\chi_y\partial_y[a(y)m(y)]dy$$

Adding these two expressions, we have

$$2\int_Y m(y)b(y)\chi(y)dy$$

$$= 2\int_Y \chi(y)a(y)m(y)\chi_{yy}dy + 2\int_Y \chi(y)\chi_y[am]_y dy \tag{44}$$

$$= -2\int_Y \partial_y[\chi am]\chi_y dy + 2\int_Y \chi(y)\chi_y[am]_y dy - 2\int_Y \chi_y a(y)m(y)\chi_y dy$$

Thus, q may be rewritten as

$$q = \int_Y m(y)\{a(y)[1-2\chi_y+\chi_y^2]\}dy \tag{45}$$

$$= \int_Y m(y)a(y)[1-\chi_y]^2 dy$$

and clearly $q \geq 0$.

The term q in (41) summarizes the effects of the averaging process on the uncontrolled system. The homogenization process interacts with the control system through the term Γ, whose form would be difficult to "guess" from simple averaging procedures.

4.2 Continuum Model for a Simple Structural Mechanical System

4.2.1 Problem definition

Consider the truss shown in Figure 1 (undergoing an exaggerated deformation). We shall assume that the truss has a regular (e.g., triangular) cross-section and no "interlacing" supports. We assume that the displacements of the system are "small" in the sense that no components in the system buckle. We are interested in describing the dynamical behavior of the system when the number of cells (a unit between two (triangular) cross sections) is large; that is, in the limit as

$$\epsilon \stackrel{\text{def}}{=} \ell/L \to 0. \tag{46}$$

We shall make several assumptions to simplify the analysis. First, we shall assume that the triangular sections are essentially rigid, and that all mobility of the system derives from the flexibility of the members connecting the triangular components. Second, we shall ignore damping and frictional effects in the system. Third, we shall confine attention to small transverse displacements $\eta(t,x)$ and small in plane rotations $\phi(t,x)$ as indicated in Figure 1, ignoring longitudinal and out of plane motions and torsional twisting. Fourth, we shall assume that the mass of the triangular cross members dominates the mass of the interconnecting links.

Systems of this type have been considered in several papers including [20], [21], [22], and [37]. In those papers a continuum beam model was

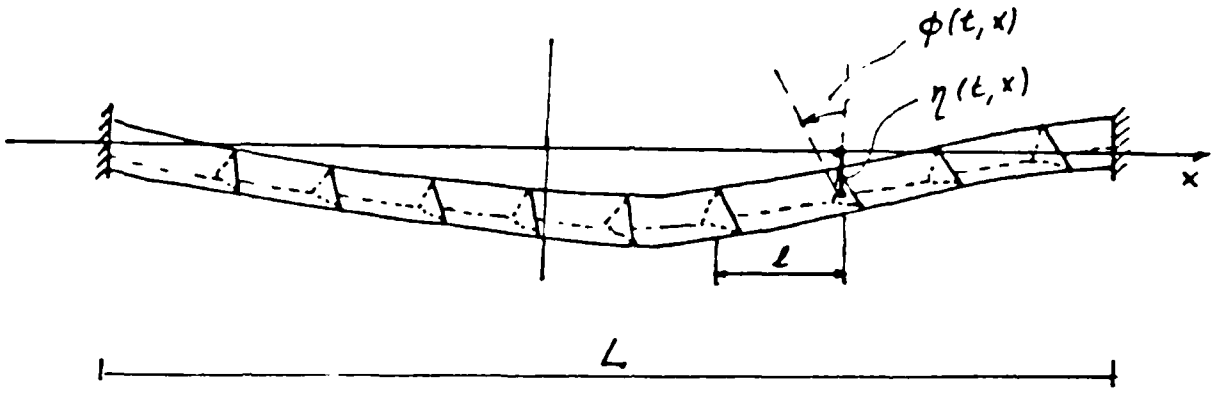

Figure 1: Deformed truss with regular cross-section.

hypothesized and effective values for the continuum system parameters were computed by averaging the associated parameters of the discrete system. Our approach to the problem is based on homogenization – asymptotic analysis and is quite different.

The assumptions simplify the problem substantially, by suppressing the geometric structure of the truss. We can retain this structure by writing dynamical equations for the nodal displacements of the truss members. For triangular cross sections nine parameters describe the displacements of each sectional element. The analysis which follows may be carried over to this case, but the algebraic complexity prevents a clear presentation of the main ideas. As suggested in (Noor et al. [20]) one should use a symbolic manipulation program like MACSYMA, SMP, Reduce, etc., to carry out the complete details of the calculations. We shall take up this problem on another occasion; for now we shall treat the highly simplified problem which, as we shall see, leads to the Timoshenko beam.

We shall begin by reformulating the system in terms of a discrete element model as suggested in (Crandal et al. [38]); see Figure 2. In this model we follow the displacement $\eta_i(t)$ and rotation $\phi_i(t)$ of the i^{th} mass M. The bending springs (k_b^i) tend to keep the system straight by keeping

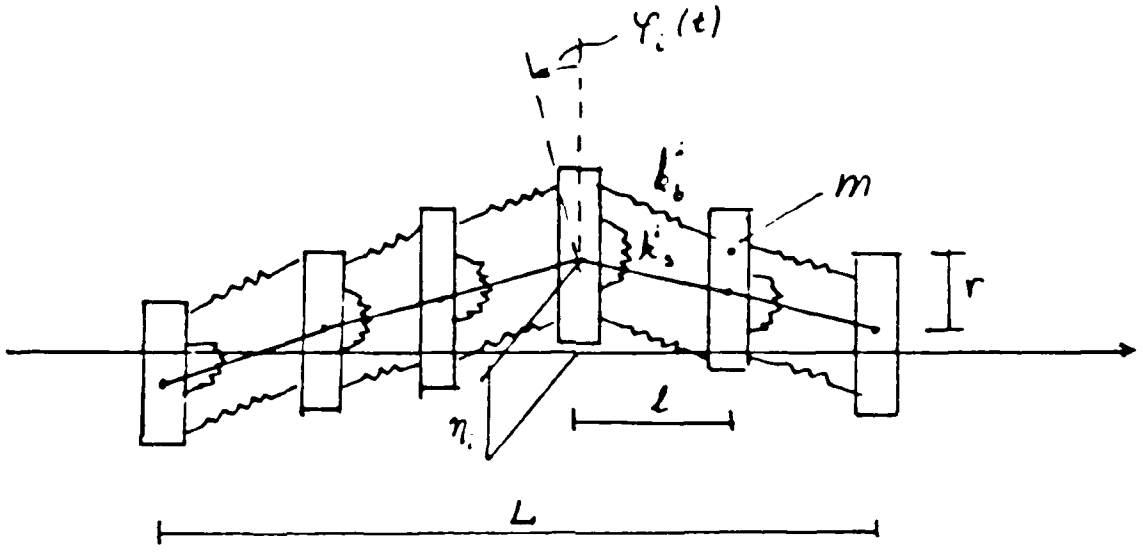

Figure 2: A lumped parameter model of the simplified truss system.

the masses parallel and the shearing springs (k_s^i) tend to keep the masses perpendicular to the connecting links. We assume small displacements and rotations so the approximations

$$\sin\phi_i(t) \approx \phi_i(t) \tag{47}$$

$$\tan^{-1}[\eta_i(t)/\ell] \approx \eta_i(t)/\ell$$

are valid.

In this case the (approximate) equations of motion of the i^{th} mass are[5]

$$\frac{d^2\phi_i}{dt^2} \approx \frac{1}{r}k_s^i\{[\frac{\eta_{i+1}(t)-\eta_i(t)}{\ell}] - \phi_i(t)\} \tag{48}$$

$$+S_\ell^-\{k_b^i[\frac{\phi_{i+1}(t)-\phi_i(t)}{\ell}]\}$$

$$\frac{d^2\eta_i}{dt^2} = S_\ell^-\{k_s^i[\frac{\eta_i(t)-\eta_{i+1}(t)}{\ell}] - \phi_i(t)\} \tag{49}$$

[5]The spring constants depend on i since they represent the restorative forces of flexed bars, bent by different amounts.

where we have normalized $M = 1$ and defined

$$S_\ell^- \eta_i \stackrel{\text{def}}{=} \frac{1}{\ell}[\eta_{i-1} - \eta_i] \tag{50}$$

and similarly for $S_\ell^- \phi_i$.

To proceed, we shall introduce the nondimensional variable $\epsilon = \ell/L$ and rewrite the system (48)(49) as

$$r\frac{d^2\phi_i^\epsilon}{dt^2} = \frac{1}{r}K_s^i\{\nabla^{\epsilon+}\eta_i^\epsilon(t) - \phi_i^\epsilon\} + \nabla^{\epsilon+}\{K_b^i\nabla^{\epsilon-}\phi_i^\epsilon(t)\} \tag{51}$$

$$\frac{d^2\eta_i^\epsilon}{dt^2} = -\nabla^{\epsilon-}\{K_s^i[\nabla^{\epsilon+}\eta_i^\epsilon(t) - \phi_i^\epsilon(t)]\}$$

where

$$K_s^i = k_s^i\ell, K_b^i = k_b^i\ell \tag{52}$$

$$\nabla^{\epsilon+}\eta_i = \frac{1}{\epsilon}[\eta_{i+1} - \eta_i], \nabla^{\epsilon-}\eta_i = \frac{1}{\epsilon}[\eta_i - \eta_{i-1}].$$

Normalizing $\ell = 1$, we associate a position $x \in [-1/2, 1/2]$ with each mass; and we introduce the notation

$$\eta(t, x_i) = \eta_i(t), \phi(t, x_i) = \phi_i(t). \tag{53}$$

Having normalized $\ell = 1$, we have $\epsilon = \ell$ and $x_{i+1} = x_i + \ell = x_i + \epsilon$. Let $\tilde{Z} = \{x_i\}$ be the set of all points in the system. In this notation

$$(\nabla^{\epsilon+}\eta)(t, x) = \frac{1}{\epsilon}[\eta(t, x + \epsilon) - \eta(t, x)] \tag{54}$$

$$(\nabla^{\epsilon-}\eta)(t, x) = \frac{1}{\epsilon}[\eta(t, x) - \eta(t, x - \epsilon)], x \in \tilde{Z}$$

and the system is

$$\frac{d^2\phi^\epsilon(t, x_i)}{dt^2} = K_s(x_i)\{\nabla^{\epsilon+}\eta^\epsilon(t, x_i) - \phi^\epsilon(t, x_i)\}$$

$$+ r\nabla^{\epsilon+}\{K_b(x_i)\nabla^{\epsilon-}\phi^\epsilon(t, x_i)\} \tag{55}$$

$$\frac{d^2\eta^\epsilon(t, x_i)}{dt^2} = -\nabla^{\epsilon-}\{K_s(x_i)[\nabla^{\epsilon+}\eta^\epsilon(t, x_i) - \phi^\epsilon(t, x_i)]\}, x \in \tilde{Z}$$

The scaling of (55) may be interpreted in the following way: Formally, at least, the right sides of both terms in (55) are $O(\epsilon^{-2})$. This implies that the time variations are taking place in the "fast time scale" $\tau = t/\epsilon$. Also, the spatial variations are taking place in the "microscopic scale" x which varies in ϵ-increments (e.g., $x_{i+1} = x_i + \epsilon$). Introducing the macroscopic scale $z = \epsilon x$, and the slow time scale $\sigma = \epsilon\tau$, we may rescale (55) and observe its dynamical evolution on the large space-time scale on which macroscopic events (e.g., "distributed phenomena") take place.

Rewritten in this spatial scale, the system becomes

$$\frac{d^2\phi^\epsilon(t,z_i{}^\epsilon)}{dt^2} = \frac{1}{\epsilon}K_s(z_i{}^\epsilon)\{\delta^{\epsilon+}\eta(t,z_i{}^\epsilon) - \phi^\epsilon(t,z_i\epsilon)\}$$

$$+ \frac{1}{\epsilon^2}\delta^{\epsilon+}\{rK_b(z_i{}^\epsilon)\delta^{\epsilon-}\phi^\epsilon(t,z_i{}^\epsilon)\} \tag{56}$$

$$\frac{d^2\eta^\epsilon(t,z_i{}^\epsilon)}{dt^2} = \frac{1}{\epsilon^2}\delta^{\epsilon-}\{K_s(z_i{}^\epsilon)[\delta^{\epsilon+}\eta(t,z_i{}^\epsilon) - \epsilon\phi^\epsilon(t,z_i{}^\epsilon)]\} \tag{57}$$

where

$$\delta^{\epsilon\pm} = \epsilon\nabla^{\epsilon\pm} = O(1) \text{ in } \epsilon. \tag{58}$$

The essential mathematical problem is to analyze the solutions $\phi^\epsilon, \eta^\epsilon$ of (56)(57) in the limit as $\epsilon \to 0$.

4.2.2 Mathematical analysis

To proceed, we shall generalize the problem (56)(57) slightly by allowing K_s and K_b to depend on z as well as z/ϵ. This permits the restoring forces in the model system to depend on the large scale shape of the structure as well as on local deformations. We use the method of multiple scales; that is, we introduce $y = x/\epsilon$ and look for solutions of (56)(57) in the form

$$\eta^\epsilon(t) = \eta^\epsilon(t,z,y), \phi^\epsilon(t) = \phi^\epsilon(t,z,y), \tag{59}$$

and we have

$$K_s = K_s(z,y), K_b = K_b(z,y), y = \frac{z}{\epsilon}$$

On smooth functions $\psi(z, z^\epsilon)$ the operators $\delta^{\epsilon\pm}$ satisfy

$$(\delta^{\epsilon+}\psi)(z,y) = \psi(z+\epsilon, y+1) - \psi(z,y)$$

$$= \psi(z,y+1) - \psi(z,y) + \psi(z+\epsilon, y+1) - \psi(z,y+1) \tag{61}$$

$$= (S^+\psi)(z,y) + \epsilon\frac{\partial\psi}{\partial z}(z,y+1) + \frac{1}{2}\epsilon^2\frac{\partial^2\psi}{\partial z^2}(z,y+1) + O(\epsilon^3)$$

$$(\delta^{\epsilon-}\psi)(z,y) = \psi(z,y) - \psi(z-\epsilon, y-1)$$

$$= \psi(z,y) - \psi(z,y-1) + \psi(z,y-1) - \psi(z-\epsilon, y-1)$$

$$= (S^-\psi)(z,y) - \epsilon\frac{\partial\psi}{\partial z}(z,y+1) + \frac{1}{2}\epsilon^2\frac{\partial^2\psi}{\partial z^2}(z,y-1) + O(\epsilon^3) \tag{62}$$

We assume that ϕ^ϵ and η^ϵ may be represented by

$$\phi^\epsilon(t,z,y) = \phi_0(t,z) + \epsilon\phi_1(t,z,y) + \ldots \tag{63}$$

$$\eta^\epsilon(t,z,y) = \eta_0(t,z) + \epsilon\eta_1(t,z,y) + \ldots$$

and substituting (63) in (56)(57) and using (60) (61)(62), we arrive at a sequence of equations for $(\phi_0,\eta_0),(\phi_1,\eta_1),\ldots$ by equating the coefficients of like powers of ϵ.

Starting with $\epsilon^{-2},\epsilon^{-1},\epsilon^0,\ldots$, we have

$$\frac{1}{\epsilon^2}S^+[rK(z,y)S^-\phi_0(t,z)] = 0 \tag{64}$$

which is trivially true from (62) (63). The same term involving $\eta_0(t,x)$ from (57) is trivially satisfied by the assumption (63). Continuing

$$\frac{1}{\epsilon}[S^+\{rK_b(z,y)S^-\phi_1(t,z,y)\} \tag{65}$$

$$+K_s(z,y)\{S^+\eta_0(t,z) - \phi_0(t,z)\}] = 0$$

which may be solved by using the corrector $\chi_\phi(z,y)$ and taking

$$\phi_1(t,z,y) = \chi_\phi(z,y)\phi_0(t,z) \tag{66}$$

with

$$S^+\{rK_b(z,y)S^-\chi_\phi(z,y)\} = K_s(z,y) \tag{67}$$

If we regard z as a parameter in (67), then there exists a solution χ_ϕ, unique up to an additive constant, if $K_b(z,\cdot), K_s(z,\cdot)$ are periodic in y, if there exist constants A and B so that

$$0 < A \leq K_b(z,y) \leq B < \infty \tag{68}$$

and if the average of $K_s(z,\cdot)$ is zero

$$\frac{1}{\ell}\int_{-L/2}^{L/2} K_s(s,y)dy = 0. \tag{69}$$

Let us *assume* that (68) (69) hold, and

$$0 < A \leq K_s(z,y) \leq B < \infty. \tag{70}$$

Considering (57), the $O(\epsilon^{-1})$ term in the asymptotic expansion is

$$\frac{1}{\epsilon}[S^-\{K_s(z,y)(S^+\eta_1(t,z,y) - \phi_0(t,z))\}] = 0. \tag{71}$$

Again we introduce the corrector $\chi_\eta(z,y)$, and take η_1 in the form

$$\eta_1(t,z,y) = \chi_\eta(z,y)\phi_0(t,z) \tag{72}$$

which gives the equation for the corrector

$$S^-\{K_s(z,y)[S^+\chi_\eta(z,y) - 1]\} = 0 \tag{73}$$

or

$$S^-\{K_s(z,y)S^+\chi_\eta(z,y)\} = K_s(z,y) - K_s(z,y-1) \tag{74}$$

By hypothesis the right side in (74) is periodic in y and has zero average (69). Hence, (74) has a periodic solution, unique to an additive constant.

Continuing, the $O(\epsilon^0)$ term in (56) is

$$S^+\{rK_b(z,y)S^-\phi_2(t,z,y)\} + K_s(z,y)[S^+\eta_1(t,z,y) - \phi_1(t,z,y)]$$

$$+ K_s(z,y)\frac{\partial}{\partial z}\eta_0(t,z) + S^+\{rK_b(z,y)\frac{\partial}{\partial z}\phi_1(t,z,y)\} \tag{75}$$

$$+S^+\{rK_b(z,y)\frac{\partial^2}{\partial z^2}\phi_0(t,z,y)\} + \frac{\partial}{\partial z}\{rK_b(z,y+1)\}\frac{\partial}{\partial z}\phi_0(t,z)$$

$$+\frac{\partial^2}{\partial z^2}\{rK_b(z,y+1)\}\phi_0(t,z) - \frac{\partial^2\phi_0}{\partial t^2} = 0.$$

This should be regarded as an equation for ϕ_2 as a function of y with *(t,z)* as parameters. In this sense the solvability condition is as before, the average of the sum of all terms on the left in (75), except the first, should be zero. We must choose ϕ_0 so that this in fact occurs; and that defines the *limiting system.*

Using the correctors (66) (72), we must have

$$\text{Average}_{(y)}\{\frac{\partial^2\phi_0}{\partial t^2} - \frac{\partial^2\phi_0}{\partial z^2}[S^+(rK_b(z,y)) + S^+(rK_b(z,y)\chi_\phi(z,y))]$$

$$- \frac{\partial\phi_0}{\partial z}[\frac{\partial}{\partial z}(rK_b(z,y+1))] - \frac{\partial\eta_0}{\partial z}K_s(z,y) \tag{76}$$

$$-\phi_0[\frac{\partial^2}{\partial z^2}(rK_b(z,y+1)) + S^+(rK_b(z,y)\frac{\partial}{\partial z}\chi_\phi(z,y))$$

$$+K_s(z,y)(S^+\chi_\eta(z,y) - \chi_\phi(z,y))]\} = 0$$

Defining the functions $EI(z), G(z)$ by the associated averages in (76), the averaged equation is

$$\frac{\partial^2\phi_0}{\partial t^2} = \frac{\partial}{\partial z}(EI(z)\frac{\partial\phi_0}{\partial z}) + G(z)\frac{\partial\eta_0}{\partial z} - H(z)\phi_0 \tag{77}$$

which is the angular component of the Timoshenko beam system (Crandall et al. [38] p. 348).

Arguing in a similar fashion, we can derive the equation for the macroscopic approximation displacement of the lattice system in terms of the "equivalent" displacement $\eta_0(t,z)$ in the Timoshenko beam system

$$\frac{\partial^2\eta_0}{\partial t^2} = \frac{\partial}{\partial z}[N(z)(\frac{\partial\eta_0}{\partial z} - \phi_0(t,z))] \tag{78}$$

4.2.3 Summary

We have shown that a simplified model of the dynamics of the truss with rigid cross sectional area may be well approximated by the Timoshenko beam model in the limit as the number of cells (proportional to L/ℓ) becomes large. The continuum beam model emerges naturally in the analysis, as a consequence of the periodicity and the scaling.

To compute the approximate continuum model, one must solve (67) and (74) (numerically) for the correctors and then compute the parameters in (77) (78) by numerically averaging the quantities in (76) (and its analog for (57)) which involve the correctors and the data of the problem.

4.3 Homogenization and Stabilizing Control of Lattice Structures

In this subsection we show that the process of deriving effective "continuum" approximations to complex systems may be developed in the context of optimal control designs for those systems. This procedure is more effective than the procedure of first deriving homogeneous – continuum approximations for the structure, designing a control algorithm for the idealized model, and then adapting the algorithm to the physical model. In fact, separation of optimization and asymptotic analysis can lead to incorrect algorithms or ineffective approximations, particularly in control problems where nonlinear analysis (e.g., of the Bellman dynamic programming equation) is required.

We shall apply the combined homogenization - optimization procedure described in subsection 4.1 (based on (Bensoussan, Boccardo, and Murat [4])) to the problem of controlling the dynamics of lattice structures like the truss structure analyzed in the previous subsection. We shall only formulate a prototype problem of this type and discuss its essential features.

Consider the model for the lattice structure analyzed in subsection 4.3

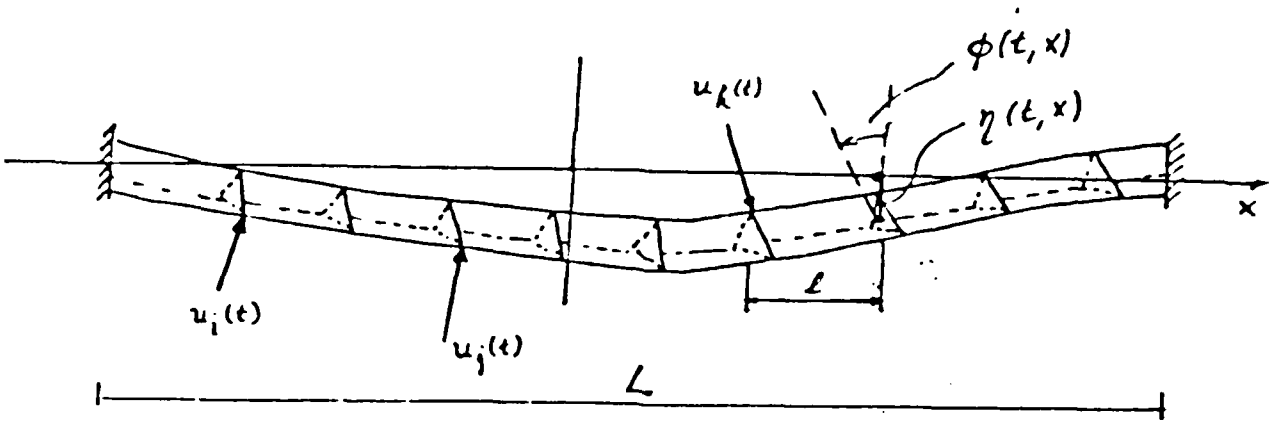

Figure 3: Truss with transverse actuator forces.

with control actuators added. The truss shown in Figure 1 is again constrained to move in the plane and torsional motion is excluded to simplify the model and confine attention to the basic ideas. Now, however, we include a finite number of actuators acting to cause transverse motions. The truss with actuator forces indicated by arrows is shown in Figure 3. The corresponding discrete element model is shown in Figure 4.

Suppose that the physical actuators act along the local normal to the truss midline as shown in the figures, and that the forces are small so that linear approximations to transcendental functions (e.g., $\sin\phi_i \approx \phi_i$, etc.) are valid. Then the controlled equations of motion of the discrete element system are (recall equation (51))

$$r\frac{d^2\phi_i^\epsilon}{dt^2} = \frac{1}{r}K_a^i\{\nabla^{\epsilon+}\eta_i^\epsilon(t) - \phi_i^\epsilon\} + \nabla^{\epsilon+}\{K_b^i\nabla^{\epsilon-}\phi_i^\epsilon(t)\} \tag{79}$$

$$\frac{d^2\eta_i^\epsilon}{dt^2} = -\nabla^{\epsilon-}\{K_a^i[\nabla^{\epsilon+}\eta_i^\epsilon(t) - \phi_i^\epsilon(t)]\} + \sum_{j=1}^{m}\delta(i,i_j)u_j(t)$$

where the notation in (52) has been used,

$$\delta(i,j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{80}$$

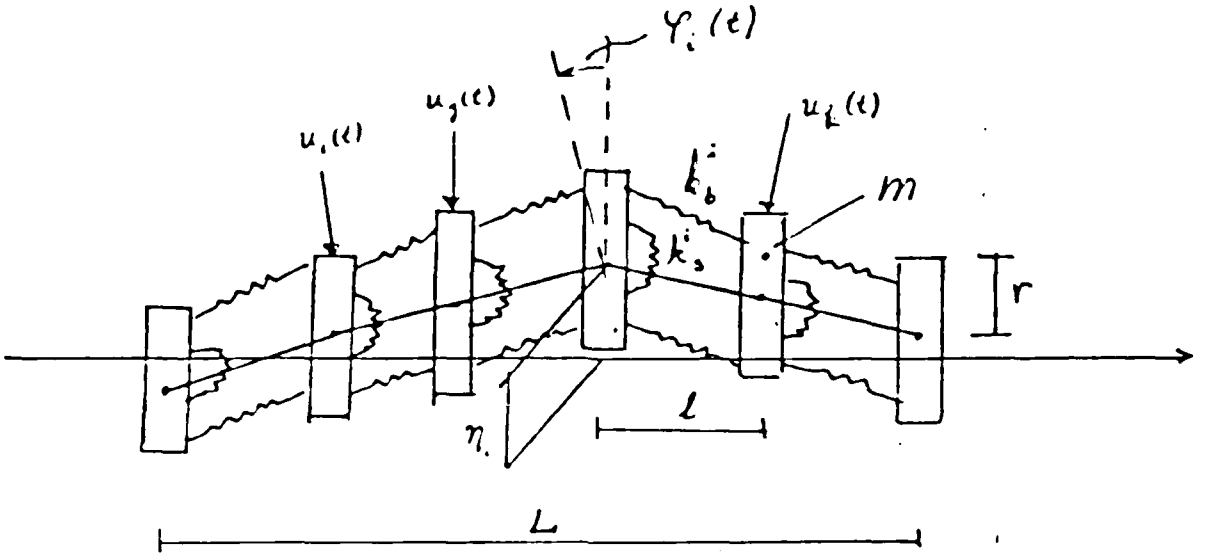

Figure 4: Discrete element model of the controlled truss.

and $i_j, j = 1, \ldots, m$ are the locations of the actuators. Hence, if $\delta(i, i_j) = 0$ for all $j = 1, \ldots, m$ there is no actuator located at the i^{th} point which corresponds to the physical point $x \in [0, L]$. The number m of actuators is given at the outset and does not, of course, vary with the scaling. The control problem is to select the actuator forces as functions of the displacements and velocities of components of the structure to damp out motions of the structure. Measurements would typically be available from a finite number of sensors located along the structure. We shall not elaborate on this component of the model, and shall instead assume that the entire state can be measured. To achieve the stabilization, we shall associate a cost functional with the system (79). Let

$$u(t) = [u_1(t), \ldots, u_m(t)]^T \tag{81}$$

be the vector of control forces, and

$$J^\gamma[u(\cdot)] = \int_0^\infty \sum_{i=1}^{N} \{a_i[\phi_i^\epsilon(t)]^2 + b_i[\eta_i^\epsilon(t)]^2 + \alpha_i[\dot{\phi}_i^\epsilon(t)]^2 + \beta_i[\dot{\eta}_i^\epsilon(t)]^2 \tag{82}$$

$$+\sum_{j=1}^{m}\delta(i,i_j)u^2(t)\}e^{-\gamma t}dt$$

where (a_i, b_i) and (α_i, β_i) are non-negative weights. Formally, the control problem is to select $\delta(i,i_j)u_j(t), i = 1,\ldots,N, j = 1,\ldots,m$ to achieve

$$\inf_{u(\cdot)} J^\gamma[u(\cdot)] \tag{83}$$

subject to (79) (80) and the appropriate boundary conditions. The case $\gamma \to 0$ corresponds to stabilization by feedback.

The analysis of this control problem is based on the scaling used in subsection 4.3, equations (51) - (58). Let $\tau = t/\epsilon$ be the fast time scale, then

$$\begin{aligned} J^\gamma[u(\cdot)] = \int_0^\infty \epsilon \sum_{i=1}^{N} \{ & a_i[\hat{\phi}_i^\epsilon(\tau)]^2 + b_i[\hat{\eta}_i^\epsilon(\tau)]^2 \\ & + \alpha_i\epsilon^2[\hat{\dot{\phi}}_i^\epsilon(\tau)]^2 + \beta_i\epsilon^2[\hat{\dot{\eta}}_i^\epsilon(\tau)]^2 \\ & + \sum_{j=1}^{m}\delta(i,i_j)u^2(\tau)\}e^{-\epsilon\gamma\tau}d\tau \end{aligned} \tag{84}$$

with $\hat{\phi}_i^\epsilon(\tau) = \phi_i^\epsilon(\epsilon\tau)$, etc.

Let $(\phi, \dot{\phi}, \eta, \dot{\eta})$ be the state vector of the system (79) with $\phi = [\phi_1,\ldots,\phi_N]^T$ and similarly for the other terms. Let $V = V^{\epsilon,\gamma}(\phi,\dot{\phi},\eta,\dot{\eta})$ be the optimal value function for the problem (79) (84). Then the Bellman equation associated with (79) (84) is

$$\begin{aligned} & \epsilon\sum_{i=1}^{N}[\dot{\phi}V_{\phi_i} + \dot{\eta}_i V_{\eta_i}] \\ & +\epsilon\sum_{i=1}^{N}\{\frac{1}{r^2}K_s^i[\nabla^{\epsilon+}\eta_i - \phi_i] + \frac{1}{r}\nabla^{\epsilon+}[K_b^i\nabla^{\epsilon-}\phi_i]\}V_{\dot{\phi}_i} \\ & + \epsilon\sum_{i=1}^{N}\{-\nabla^{\epsilon-}[K_s^i(\nabla^{\epsilon-}\eta_i - \phi_i)]\}V_{\dot{\eta}_i} \\ & \min_{u_j\in U_{ad}}\{\epsilon\sum_{i=1}^{N}\sum_{j=1}^{m}[\delta(i,i_j)\cdot_{ij}V_{\dot{\eta}_i} + \delta(i,i_j)u_j^2]\} \end{aligned} \tag{85}$$

$$+\epsilon \sum_{i=1}^{N} [a_i \phi_i^2 + b_i \eta_i^2 + \epsilon^2 (\alpha_i \dot{\phi}^2 + \beta_i \dot{\eta}_i^2)] - \epsilon \gamma V = 0.$$

Remarks:

1. Note that the minimization in (85) is well defined if the admissible range of the control forces is convex since the performance measure has been assumed to be quadratic in the control variables $\delta(i, i_j)u_j$.

2. Since we have not included the effects of noise in the model, the state equations are deterministic and the Bellman equation (85) is a first order system. To "regularize" the analysis, at least along the lines followed in conventional homogenization analysis, it is useful to include the effects of noise in the model and exploit the resulting coercivity properties in the asymptotic analysis.

3. If we introduce the macroscopic spatial scale $z = \epsilon x$, the mesh $\{x_i\}$, and the variables

$$\phi(t, z_i) = \phi_i^\epsilon(t), \dot{\phi}(t, z_i) = \dot{\phi}_i^\epsilon(t), \text{ etc.} \tag{86}$$

then the sums may be regarded as Riemann approximations to integrals over the macroscopic spatial scale z. The asymptotic analysis of (85) with this interpretation defines the mathematical problem constituting simultaneous homogenization - optimization for this case.

4.4 Effective conductivity of a periodic lattice

In this subsection we consider a version of a heat conduction problem treated by Kunnemann. Simple expressions for thermal properties of composite materials, have been derived in the past using homogenization techniques. The derivation of effective conductivities for discrete structures is useful for assessing the thermal response of such structures in variable environmental conditions.

4.4.1 Problem definition

Let $Z = \{0, \pm 1, \pm 2, \ldots\}$ and $Z^d = Z \times \ldots \times Z$ (d times) be a d-dimensional lattice. Let $\epsilon > 0$ be a number small relative to 1. We want to describe the effective conduction of thermal energy on the ϵ-spaced lattice ϵZ^d. Let $e_i = (0, 0, \ldots 0, 1, 0, \ldots 0)^T$ with 1 in the i^{th} position, $i = 1,2, \ldots, d$. If x is a point in ϵZ^d, then $x \pm \epsilon e_i, 1 \leq i \leq d$, are the nearest neighbors of x. Let $a_{\pm}(x), x \in Z^d, 1 \leq i \leq d$, be the two functions defined on the lattice, and *assume*

$$a_i(x) \stackrel{\text{def}}{=} a_+(x) = a_{i-}(x + e_i), x \in Z^d, 1 \leq i \leq d \tag{87}$$

$$0 < A \leq a_i(x) \leq B < \infty, \forall x \in Z^d, 1 \leq i \leq d \tag{88}$$

$$a_i(x) \text{ is periodic with period } \ell \geq 1 \tag{89}$$

$$\text{in each direction, } 1 \leq i \leq d.^6$$

Next let

$$a^{\epsilon}_{i\pm}(x) = a_{i\pm}(\frac{x}{\epsilon}), x \in \epsilon Z^d, 1 \leq i \leq d. \tag{90}$$

Equation (88) means that the conduction process is reversible and that the conductivity $a_i(x)$ is a "bond conductivity," i.e., independent of the direction in which the bond $(x, x + e_i)$ is used by the process. Equation (90) means that the configuration of bond conductivities $a^{\epsilon}_{i\pm}(\cdot)$ on ϵZ^d is simply $a_{i\pm}(\cdot)$ on ϵZ^d "viewed from a distance." Assumption (89) imposes a regularity condition on the physics of the conduction process. An assumption like this is essential for existence of a limit as $\epsilon \to 0$. In one dimension the situation is illustrated in Figure 5 and Figure 6. A system similar to this with random bond conductivities was treated by Kunnemann [39] by imposing some ergodicity properties on the bond conductivities.

One can associate with this system a random (jump) process

$$\{X^{\epsilon}(t, x), t \geq 0, x \in \epsilon Z^d\}$$

[6] The period may be different in different directions.

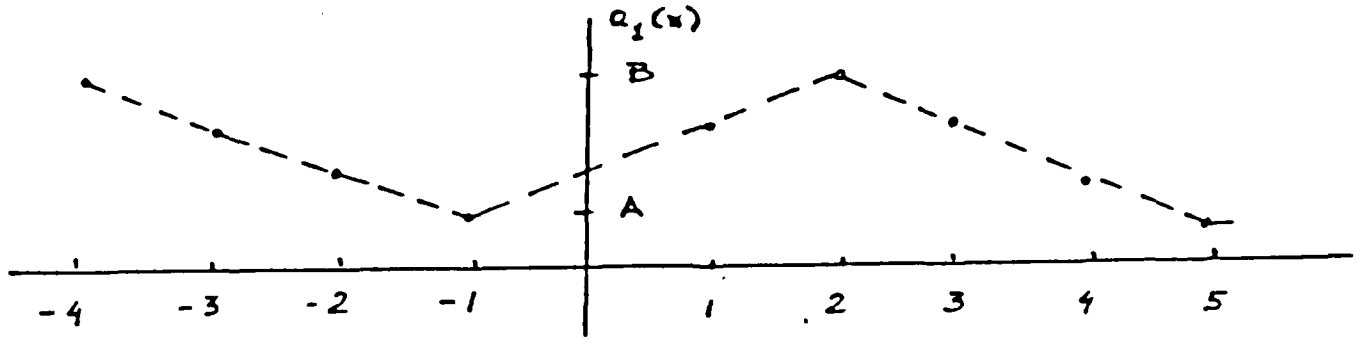

Figure 5a. Conductivity on unscaled lattice with period $l = 6$.

Figure 5: Conductivity on unscaled lattice with period $\ell = 6$.

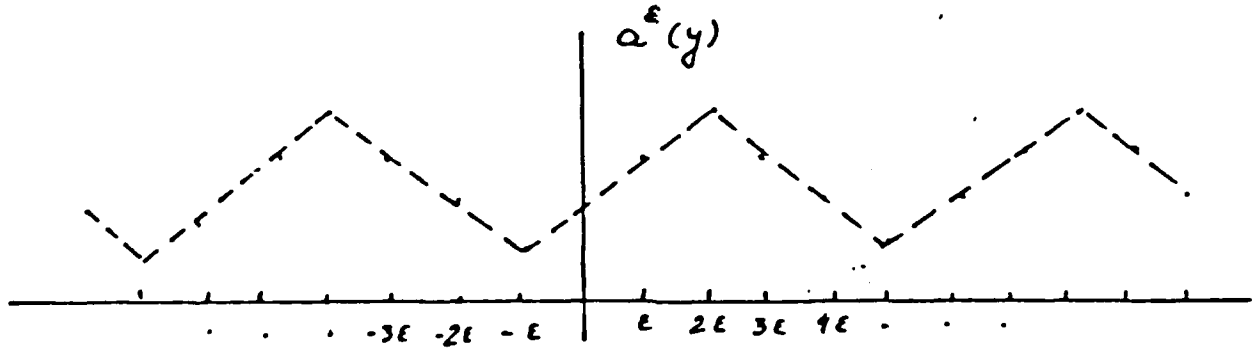

Figure 5b. Conductivity on ϵ-scaled lattice,
$y = \epsilon x$, $x \in Z$, period $\epsilon l = 6\epsilon$.

Figure 6: Conductivity on ϵ-scaled lattice, $y = \epsilon x$, $x \in Z$, period $\epsilon\ell = 6\epsilon$.

on the ϵ-spaced lattice.[7] In effect, as $\epsilon \to 0$, $\{X^\epsilon\}$ converges to a Brownian motion on the lattice; and the main result of the analysis is an expression for the diffusion matrix $Q \stackrel{\text{def}}{=} [q_{ij}; i,j = 1,2,\ldots,d]$ of this process. This matrix describes the macroscopic diffusion of thermal energy in the system. It is the *effective conductivity*.

We shall carry out the asymptotic analysis of this system in the limit as $\epsilon \to 0$ using homogenization. Let

$$(\nabla_i^{\epsilon-} u)(x) \stackrel{\text{def}}{=} \frac{1}{\epsilon}[u(x - \epsilon e_i) - u(x)] \tag{91}$$

$$(\nabla_i^{\epsilon+} u)(x) \stackrel{\text{def}}{=} \frac{1}{\epsilon}[u(x + \epsilon e_i) - u(x)]$$

$$x \in \epsilon Z^d, 1 \le i \le d,$$

for any u square summable on ϵZ^d or square integrable on $\mathcal{R}^d$ with e_i the i^{th} natural basis vector in $\mathcal{R}^d$. Then

$$\frac{\partial u^\epsilon(t,x)}{\partial t} = -\sum_{i=1}^{d} \nabla_i^{\epsilon-}[a_i(\frac{x}{\epsilon})\nabla_i^{\epsilon+} u^\epsilon(t,x)] \tag{93}$$

$$\stackrel{\text{def}}{=} L^\epsilon u^\epsilon(t,x)$$

is the diffusion equation on the ϵ-spaced lattice with density $u^\epsilon(t,x)$ and conductivity $a_i(x/\epsilon)$. We are interested in an effective parameter representation of the thermal conduction process as $\epsilon \to 0$.

Remark: Although probabilistic methods are not required in the analysis, the associated probabilistic framework has a great deal of intuitive appeal. The operator L^ϵ may be identified as the infinitesimal generator of a pure jump process $X^\epsilon(s)$ in the "slow" time scale $s \stackrel{\text{def}}{=} \epsilon^2 t$; (Breiman [8]). Moreover, L^ϵ is selfadjoint on ϵZ^d with the inner product

$$(f,g) \stackrel{\text{def}}{=} \sum_{x \in Z^d} f(x)g(x). \tag{94}$$

[7] Definition of this process is not necessary for the analysis, but it bolsters the intuition.

Hence, the backward and forward equations for the process $X^\epsilon(s)$ are, respectively,

$$\frac{\partial p^\epsilon(y,t|x)}{\partial t} = [L^\epsilon p^\epsilon(y,t|\cdot)](x) \tag{95}$$

$$\frac{\partial p^\epsilon(y,t|x)}{\partial t} = [L^\epsilon p^\epsilon(\cdot,t|x)](y)$$

So the process is "symmetric" in the sense of Markov processes (Breiman [8]).

The asymptotic analysis of (93), when interpreted in this context, means that as the bond lattice is contracted by ϵ and time is sped up by ϵ^{-2}, the jump process $\{X^\epsilon(s)\}$ approaches a diffusion process with diffusion matrix Q. In other words, on the microscopic scale thermal energy is transmitted through the lattice by a jump process; but when viewed on a macroscopic scale the energy appears to diffuse throughout the lattice. The microscopic physics are described in (Kirkpatrick [11]) and (Kittel [12]). The approximation developed belo w for a periodic lattice is similar to the one developed by Kunnemann for a random lattice. This similarity demonstrates the robustness of the method, and the limited dependence of the macroscopic properties of the medium on the details of the microscopic variations of the structure.

Because the basic problem (93) is "parabolic," we can introduce the probabilistic mechanism and make use of it in the analysis. In the "hyperbolic," structural mechanical problems we treated before this device was not readily available.

4.4.2 Asymptotic analysis-homogenization

The essential mathematical step is to show strong convergence of the semigroup of L^ϵ, say

$$T^\epsilon(t) \stackrel{\text{def}}{=} e^{L^\epsilon t} \longrightarrow T(t) \stackrel{\text{def}}{=} e^{Lt} \text{ as } \epsilon \to 0 \tag{96}$$

and to identify the limiting operator

$$q_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{97}$$

This is accomplished by proving convergence of the resolvents

$$,[-L^\epsilon + \alpha]^{-1} \longrightarrow [-L + \alpha]^{-1} \text{ as } \epsilon \to 0 \tag{98}$$

That is, if f is a given function and

$$u^\epsilon(\cdot) \stackrel{\text{def}}{=} [-L^\epsilon + \alpha]^{-1} f \tag{99}$$

$$u(\cdot) \stackrel{\text{def}}{=} [-L + \alpha]^{-1} f$$

then $u^\epsilon \to u$ in an appropriate sense.

The method of multiple scales will be used to compute the limit. Because the conductivities $a_i(x)$ in (93) do not depend on time, we may work directly with L^ϵ rather than the parabolic PDE (93) (cf. (Bensoussan, Lions, and Papanicolaou [5]) Remark 1.6, p. 242). The method of multiple scales is convenient because it is a systematic way of arriving at the "right answers" - something which is not always simple in this analysis.

Bearing in mind (99), we consider

$$(L^\epsilon u^\epsilon)(x) = f(x) \tag{100}$$

with $u^\epsilon(x)$ in the form

$$u^\epsilon(x) = u_0(x, \frac{x}{\epsilon}) + \epsilon u_1(x, \frac{x}{\epsilon}) + \epsilon^2 u_2(x, \frac{x}{\epsilon}) + \ldots \tag{101}$$

with the functions $u_j(x, y)$ periodic in $y \in \epsilon Z^d$ for every $j = 0, 1, \ldots$. (As it turns out the boundary conditions are somewhat irrelevant to the construction of "right answers.") To present the computations in a simple form, it is convenient to introduce $y = x/\epsilon$, to treat x and y as independent variables, and to replace y by x/ϵ at the end.

Recall the operators $\nabla_i^{\epsilon\pm}$ from (93). Applied to a smooth function $u = u(x, x/\epsilon)$, we have

$$(\nabla_i^{\epsilon -} u)(x, y) = \frac{1}{\epsilon}[u(x - \epsilon e_i, y - e_i) - u(x, y)] \tag{102}$$

$$= \frac{1}{\epsilon}[u(x, y - e_i) - u(x, y)] + \frac{1}{\epsilon}[u(x - \epsilon e_i, y - e_i) - u(x, y - e_i)]$$

$$= \frac{1}{\epsilon}(\nabla_i^- u)(x, y) - \frac{\partial u}{\partial x_i}(x, y - e_i) + \epsilon \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(x, y - e_i) + O(\epsilon^2)$$

where on functions $\phi = \phi(y)$

$$(\nabla_i^- \phi)(y) = \phi(y - e_i) - \phi(y) \tag{103}$$

Defining

$$(\nabla_i^+ \phi)(y) = \phi(y + e_i) - \phi(y) \tag{104}$$

we also have

$$(\nabla_i^{\epsilon +} u)(x, y) = \frac{1}{\epsilon}(\nabla_i^+ u)(x, y) + \frac{\partial u}{\partial x_i}(x, y + e_i) \tag{105}$$

$$+\epsilon \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(x, y + e_i) + O(\epsilon^2).$$

Now we substitute (101) into (100) and use the rules (103) (104). Equating coefficients of like powers of ϵ, this leads to a sequence of equations for $u_0, u_1, \ldots$. Specifically, (using the summation convention)

$$(L^\epsilon u^\epsilon)(x, y) = -\nabla_i^{\epsilon -}[a_i(y)\nabla_i^{\epsilon +} u^\epsilon]$$

$$= \frac{1}{\epsilon^2}\nabla_i^-[a_i(y)\nabla_i^+ u_0(x, y)] - \nabla_i^{\epsilon -}[a_i(y)\frac{\partial u_0}{\partial x_i}(x, y + e_i)]$$

$$- \frac{1}{2}\epsilon \nabla_i^{\epsilon -}[a_i(y)\frac{\partial^2 u_0}{\partial x_i^2}(x, y + e_i)] + O(\epsilon) \tag{106}$$

$$-\frac{1}{\epsilon}\nabla_i^-[a_i(y)\nabla_i^+ u_1(x, y)]$$

$$-\epsilon \nabla_i^{\epsilon -}[a_i(y)\frac{\partial u_1}{\partial x_i}(x, y + e_i)] + O(\epsilon)$$

$$-\nabla_i^-[a_i(y)\nabla_i^+ u_2(x, y)] + O(\epsilon) = f(x)$$

That is, labeling each term by its order in ϵ

$$(\epsilon^{-2}) \quad \nabla_i^-[a_i(y)\nabla_i^+ u_0] = 0 \tag{107}$$

$$(\epsilon^{-1}) \quad \epsilon\nabla_i^{\epsilon-}[a_i(y)\frac{\partial u_0}{\partial x_i}(x, y+e_i)] + \nabla_i^-[a_i(y)\nabla_i^+ u_1(x,y)] = 0 \tag{108}$$

and (recall $\epsilon\nabla_i^{\epsilon\pm}$ is $O(1)$ in ϵ)

$$(\epsilon^0) \quad \frac{1}{2}\epsilon\nabla_i^{\epsilon-}[a_i(y)\frac{\partial^2 u_0}{\partial x_i^2}(x, y+e_i)] - \epsilon\nabla_i^{\epsilon-}[a_i(y)\frac{\partial u_1}{\partial x_i}(x, y+e_i)] \tag{109}$$

$$-\nabla_i^-[a_i(y)\nabla_i^+ u_2(x,y)] = f(x)$$

From (107) we have

$$a_i(y-e_i)[u_0(x,y) - u_0(x, y-e_i)] \tag{110}$$

$$-a_i(y)[u_0(x, y+e_i) - u_0(x,y)] = 0$$

If we take $u_0(x,y) = u_0(x)$, this is trivially true; and (108) simplifies to

$$\epsilon\nabla_i^{\epsilon-}[a_i(y)\frac{\partial u_0}{\partial x_i}(x)] + \nabla_i^-[a_i(y)\nabla_i^+ u_1(x,y)] = 0. \tag{111}$$

At this point we introduce "correctors." That is, we assume

$$u_1(x,y) = \sum_{k=1}^{d} \chi_k(y)\frac{\partial u_0}{\partial x_k} + \hat{u}_1(x) \tag{112}$$

with $\chi_k(\cdot)$ the correctors. Using this in (111), we have (again using the summation convention)

$$\nabla_i^-[a_i(y)\nabla_i^+\chi_k(y)]\frac{\partial u_0}{\partial x_k} + [a_k(y-e_k) - a_k(y)]\frac{\partial u_0}{\partial x_k} = 0 \tag{113}$$

If we take $\chi_k(y)$ as the solution of

$$\nabla_i^-[a_i(y)\nabla_i^+\chi_k(y)] + [a_k(y-e_k) - a_k(y)] = 0 \tag{114}$$

(we have to verify the well-posedness of (114)), then (113) is satisfied. (The term $\hat{u}_1(x)$ is determined (formally) from the $O(\epsilon)$ term in the system (101) (106).)

Regarding the well-posedness of (114), note that

$$\nabla_i^-[a_i(y)\nabla_i^+\phi(y)] = \psi(y) \tag{115}$$

has a *periodic* solution on ϵZ which is unique up to an additive constant iff the average of the function $\psi(y)$ over a period $(\epsilon\ell)$ is zero; i.e.,

$$\bar{\psi} \stackrel{\text{def}}{=} \frac{1}{L}\sum_{k=1}^{L}\psi(y+ke_N) = 0, n = 1,2,\ldots,d. \tag{116}$$

This condition clearly holds in (114), and so, $\chi_k(y)$ is well defined (up to an additive constant).

We shall determine the equation for $u_0(x)$ by using (112) (114) in (109). Using the Kronecker delta function δ_{ik}, we have

$$\begin{aligned}
&\frac{1}{2}\epsilon\nabla_i^{\epsilon-}[a_i(y)\delta_{ik}]\frac{\partial^2 u_0}{\partial x_i\partial x_k} - \epsilon\nabla_i^{\epsilon-}[a_i(y)\chi_k(y+e_i)]\frac{\partial^2 u_0}{\partial x_i\partial x_k} = f(x)\\
&= \{\frac{1}{2}\nabla_i^-[a_i(y)\delta_{ik}] - \nabla_i^-[a_i(y)\nabla_i^+\chi_k]\}\frac{\partial^2 u_0}{\partial x_i\partial x_k} \qquad (117)\\
&-\nabla_i^-[a_i(y)\chi_k(y)]\frac{\partial^2 u_0}{\partial x_i\partial x_k} - \nabla_i^-[a_i(y)\nabla_i^+ u_2] = f(x)
\end{aligned}$$

The term in braces is zero from (114). To obtain the solvability condition (116) for u_2, we introduce the average

$$\stackrel{\text{def}}{=} \text{ symmetric part } \{\overline{-\nabla_i^-[a_i(y)\chi_k(y)]}\} \tag{118}$$

Then solvability of the equation for u_2 gives the equation

$$q_{ik}\frac{\partial^2 u_0}{\partial x_i\partial x_k} = f(x). \tag{119}$$

And this is the diffusion equation which defines the limiting behavior of the system (100) in the macroscopic x-scale in the limit as $\epsilon \to 0$.

We can justify the asymptotic analysis by using energy estimates or probabilistic methods as in (Bensoussan, Lions, and Papanicolaou [5]). (See also Kunnemann [39]).) We shall omit this analysis here.

4.4.3 Summary

Returning to the original problem (93) for the evolution of thermal energy on a microscopic scale, we have shown that the thermal density $u^\epsilon(t,x) \to u_0(t,x)$ as $\epsilon \to 0$ (in an appropriate norm) where

$$\frac{\partial u_0}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} q_{ij}\frac{\partial u_0}{\partial x_i \partial x_j} \tag{120}$$

with

$$q_{ij} = -\frac{1}{L}\sum_{k=1}^{L}\{\nabla_i^+[a_i(y)\chi_k(y)] + \nabla_k^-[a_i(y)\chi_i(y)]\} \tag{121}$$

with the correctors $\chi_k, k = 1,2,\ldots,d$, given by

$$\sum_{i=1}^{d}\nabla_i^-[a_i(y)\nabla_i^+\chi_i(y)] = -[a_k(y-e_k) - a_k(y)] \tag{122}$$

$$k = 1,2,\ldots,d$$

To compute the limiting "homogenized" model (120), one must solve the system (122) (numerically) and then evaluate the average (121).

The fact that the original problem (93) is "parabolic" (i.e., it describes a jump random process), enables us to exploit the associated probabilistic structure to anticipate and structure the analysis. In this way we can anticipate that the limit problem will involve a diffusion process. In fact, the arguments used are entirely analytical and the limiting diffusion (120) is constructed in a systematic way. It is not postulated.[8]

[8]Probabilistic arguments can be used (Bensoussan, Lions, and Papanicolaou [5], Chapter 3); and they have some advantages.

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