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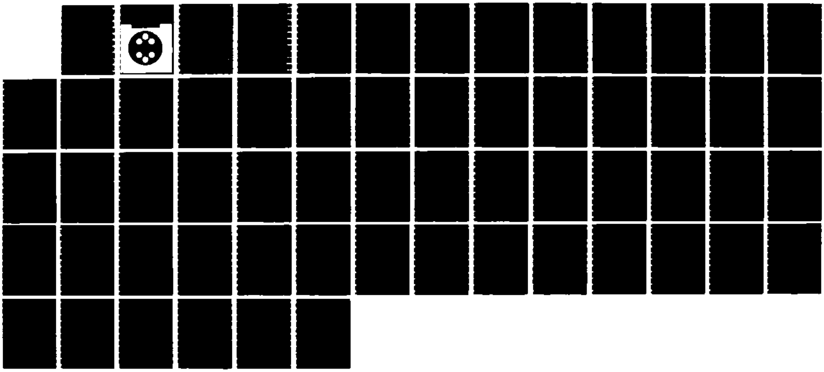

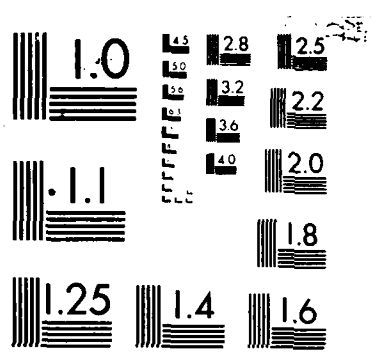

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APPROXIMATION OF INFINITE DELAY AND VOLTERRA TYPE EQUATIONS

by

K. Ito and F. Kappel

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Approximation of Infinite Delay and Volterra Type Equations

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"Approximation of infinite delay and Volterra type equations"

by K. Ito and F. Kappel

Summary

Linear autonomous functional differential equations of neutral type include Volterra integral and integrodifferential equations as special cases. The paper considers numerical approximation of solutions to these equations by first converting the initial value problem to an abstract Cauchy problem in a product space ($\mathbb{R}^n \times$ weighted L^2-space) and then using abstract approximation results for C_o-semigroups combined with Galerkin type ideas. In order to obtain concrete schemes subspaces of Legendre and Laguerre polynomials are used. The convergence properties of the algorithms are demonstrated by several examples.

Running head: Volterra type equations

Subject classification: 34G10, 34K99, 45D05, 45J05,45L10

1. Introduction

The purpose of this paper is to develop an approximation scheme based on L^2-approximation by orthogonal polynomials for the following problem:

$$
\begin{aligned}
y(t) &= \phi^0 + \sum_{j=0}^{p} A_j \int_0^t x(\tau-h_j)d\tau + \int_0^t \int_{-\infty}^0 A(\sigma)x(\tau+\sigma)d\sigma d\tau \\
&\quad + \int_0^t f(\tau)d\tau, \quad t \geq 0, \qquad (1.1) \\
x(t) &= y(t) + \sum_{j=1}^{p} B_j x(t-h_j) + \int_{-\infty}^0 B(\tau)x(t+\tau)d\tau \text{ a.e. on } t \geq 0, \\
x(t) &= \phi^1(t) \text{ a.e. on } t < 0,
\end{aligned}
$$

where $0 = h_0 < h_1 < \ldots < h_p = h$, $\phi^0 \in \mathbb{R}^n$ and ϕ^1 resp. f is an $\mathbb{R}^n$-valued function on $(-\infty,0]$ resp. $[0,\infty)$. Furthermore, A_j, B_j are $n\times n$-matrices and $A(\cdot)$, $B(\cdot)$ are $n\times n$-matrix valued functions on $(-\infty,0]$. It will be convenient to define

$$
\begin{aligned}
D(x_t) &= x(t) - \sum_{j=1}^{p} B_j x(t-h_j) - \int_{-\infty}^0 B(\tau)x(t+\tau)d\tau, \\
L(x_t) &= \sum_{j=0}^{p} A_j x(t-h_j) + \int_{-\infty}^0 A(\tau)x(t+\tau)d\tau,
\end{aligned}
$$

where as usual for a function $x: \mathbb{R} \to \mathbb{R}^n$ the functions $x_t: (-\infty,0] \to \mathbb{R}^n$, $t \geq 0$, are defined by $x_t(\tau) = x(t+\tau)$, $\tau \leq 0$.

The state of problem (1.1) at time t naturally is the pair $(y(t),x_t)$. Correspondingly we choose as a state space $Z = \mathbb{R}^n \times L^2_g(-\infty,0;\mathbb{R}^n)$ with norm

$$
|\phi|_Z^2 = |\phi^0|^2 + \int_{-\infty}^0 |\phi^1(\tau)|^2 g(\tau)d\tau, \quad \phi = (\phi^0,\phi^1) \in Z,
$$

where the weighting function g is of the form

$$
g(\tau) = e^{\beta\tau}, \quad \tau < 0, \qquad (1.2)
$$

with $\beta > 0$, $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$.

Before we discuss the solution semigroup of problem (1.1) in the next section we indicate some special cases covered by (1.1).

It is clear that $y(t;\phi)$ is always continuous on $t \geq 0$, whereas $x(t;\phi)$ need not to be continuous on $t \geq 0$. This motivates to introduce the pair $(y(t),x_t)$ as the state of (1.1) at time t and not the pair $(x(t),x_t)$ (see [3]). If for a solution of (1.1) $x(t;\phi)$ is continuous on $\mathbb{R}$ then $y(t) = D(x_t)$ for all $t \geq 0$, $\phi^0 = D(\phi^1)$, $y(t)$ is locally absolutely continuous a.e. on $t \geq 0$ and $x(t)$ is a solution of

$$\frac{d}{dt} D(x_t) = L(x_t) + f(t) \quad \text{a.e. on } t \geq 0, \qquad (1.3)$$
$$x(t) = \phi^1(t) \quad \text{for all } t < 0.$$

Equation (1.3) is a functional-differential equation of neutral type. Of course, if $B_j = 0$, $j = 1,\dots,p$, and $B \equiv 0$ then (1.3) is an equation of retarded type. Solutions of (1.1) for general $\phi \in Z$ could be considered as generalized solutions of (1.3). In this case $y(t) = D(x_t)$ only a.e. on $t \geq 0$. Further important types of equations covered by (1.1) are Volterra integro-differential and integral equations,

$$\dot{x}(t) = \int_0^t A_1(t-\tau)x(\tau)d\tau, \quad t \geq 0 \qquad (1.4)$$
$$x(0) = \phi^0,$$

$(A_j = 0,\ B_j = 0,\ B \equiv 0,\ A(\sigma) = A_1(-\sigma),\ \phi_1 = 0)$ and

$$x(t) = \tilde{f}(t) + \int_0^t B_1(t-\tau)x(\tau)d\tau, \quad t \geq 0, \qquad (1.5)$$

$(A_j = 0,\ B_j = 0,\ A \equiv 0,\ B(\sigma) = B_1(-\sigma),\ \phi_1 = 0,\ \tilde{f}(t) = \phi^0 + \int_0^t f(\tau)d\tau)$, where $\tilde{f}$ is locally absolutely continuous on $t \geq 0$.

2. The solution semigroup

We impose the following condition:

$$A,\ B \in L^2_{1/g}(-\infty,0;\mathbb{R}^{n\times n}). \tag{2.1}$$

Since the weighting function g defined in (1.2) trivially satisfies hypotheses (H1) and (H2) of [9] we get immediately from [9 ; Thm 2.1].

Proposition 2.1. Assume $f \equiv 0$. Then the family $T(t)$, $t \geq 0$, of operators defined by

$$T(t)\phi = (y(t;\phi), x_t(\phi)),\quad t \geq 0,\ \phi \in Z,$$

where $y(t;\phi)$, $x(t;\phi)$ is the solution of (1.1) corresponding to ϕ, is a C_0-semigroup on Z.

Let A be the infinitesimal generator of $T(\cdot)$. Then

Proposition 2.2. A is given by

$$\text{dom } A = \{(\phi^0,\phi^1) \in Z \mid \phi^1 \text{ is locally absolutely continuous on } (-\infty,0],\ \dot\phi^1 \in L^2_g(-\infty,0;\mathbb{R}^n) \text{ and } D(\phi^1) = \phi^0\},$$

$$A(\phi^0,\phi^1) = (L(\phi^1),\dot\phi^1),\quad (\phi^0,\phi^1) \in \text{dom } A.$$

Proof: Let $\phi \in D(A)$ and choose $\lambda \in \mathbb{R}$ sufficiently large. Then $\phi = (\lambda I - A)^{-1}\psi = \int_0^\infty e^{-\lambda t}T(t)\psi dt$ for a $\psi \in Z$ which is equivalent to

$$\phi^0 = \int_0^\infty e^{-\lambda t} y(t;\psi)dt, \tag{2.2}$$

$$\phi^1(\tau) = \int_0^\infty e^{-\lambda t} x(t+\tau;\psi)dt = e^{\lambda\tau}\int_\tau^\infty e^{-\lambda t} x(t;\psi)dt,\ \tau \leq 0. \tag{2.3}$$

Equation (2.3) shows that ϕ^1 is locally absolutely continuous on $(-\infty,0]$ and

$$\dot{\phi}^1 = \lambda\phi^1 - \psi^1 \in L^2_g(-\infty,0;\mathbb{R}^n). \tag{2.4}$$

From $\lambda\phi^1 - [A\phi]^1 = \psi^1$ and (2.4) we see

$$[A\phi]^1 = \dot{\phi}^1.$$

Taking Laplace-transforms in the second equation of (1.1) and observing (2.2), (2.3) we get

$$\phi^0 = D(\phi^1).$$

Differentiating the first equation in (1.1) and then taking Laplace-transforms we obtain analogously

$$\lambda\phi^0 - \psi^0 = L(\phi^1).$$

This and $\lambda\phi^0 - [A\phi]^0 = \psi^0$ shows

$$[A\phi]^0 = L(\phi^1).$$

Thus we have shown that the operator given in the proposition is an extension of the infinitesimal generator of $T(\cdot)$. Call this extension for the moment C and choose $\phi \in \text{dom } C$. We put, for λ sufficiently large, $\psi = (\lambda I - C)\phi$ and $\phi_\lambda = (\lambda I - A)^{-1}\psi \in \text{dom } A$, i.e. $\psi = (\lambda I - A)\phi_\lambda$. Then

$$\psi^0 = \lambda D(\phi^1) - L(\phi^1) = \lambda D(\phi^1_\lambda) - L(\phi^1_\lambda),$$

$$\psi^1 = \lambda\phi^1 - \dot{\phi}^1 = \lambda\phi^1_\lambda - \dot{\phi}^1_\lambda.$$

These two equations imply

$$\phi^1(\tau) - \phi^1_\lambda(\tau) = e^{\lambda\tau}(\phi^1(0) - \phi^1_\lambda(0)),$$

and

$$[\lambda D(e^{\lambda\cdot}I) - L(e^{\lambda\cdot}I)](\phi^1(0) - \phi^1_\lambda(0)) = 0.$$

The estimate

$$|I - D(e^{\lambda\cdot}I) + \frac{1}{\lambda} L(e^{\lambda\cdot}I)|$$

$$\leq \sum_{j=1}^{p} |B_j| e^{-\lambda h_j} + \left(\frac{1}{2\lambda+\beta}\right)^{1/2} |B|_{L^2_{1/g}} + \frac{1}{\lambda} \sum_{j=0}^{p} |A_j| e^{-\lambda h_j}$$

$$+ \frac{1}{\lambda} \left(\frac{1}{2\lambda+\beta}\right)^{1/2} |A|_{L^2_{1/g}}$$

shows that $[\lambda D(e^{\lambda\cdot}I) - L(e^{\lambda\cdot}I)]^{-1}$ exists for λ sufficiently large, which implies $\phi^1(0) = \phi^1_\lambda(0)$ and therefore also $\phi = \phi_\lambda \in \text{dom } A$ ∎

It will be necessary to consider problem (1.1) in state spaces with weighting functions different from g as defined in (1.2). Let $0 < \gamma < \beta$,

$$\tilde{g}(\tau) = e^{\gamma\tau}, \quad \tau \leq 0, \tag{2.5}$$

and put $\tilde{Z} = \mathbb{R}^n \times L^2_{\tilde{g}}(-\infty,0;\mathbb{R}^n)$ with the usual norm. Then obviously $A, B \in L^2_{1/\tilde{g}}(-\infty;0;\mathbb{R}^{n\times n})$ and therefore Propositions 2.1 and 2.2 are also true for problem (1.1) considered in the state space $\tilde{Z}$. Let $\tilde{T}(\cdot)$ and $\tilde{A}$ denote the solution semigroup of (1.1) in $\tilde{Z}$ and its infinitesimal generator, respectively.

<u>Lemma 2.3.</u> a) $\tilde{Z}$ is dense in Z and the embedding $\tilde{Z} \subset Z$ is continuous.

b) For any $v \in L^2_g(-\infty,0;\mathbb{R}^n)$ and any $\alpha > 0$ we have

$$|\tau|^\alpha v \in L^2_g(-\infty,0;\mathbb{R}^n).$$

c) For any $v \in L^2_{\tilde{g}}(-\infty,0;\mathbb{R}^n)$ such that also $\dot{v} \in L^2_{\tilde{g}}(-\infty,0;\mathbb{R}^n)$ and any $\alpha > 0$ we have

$$\lim_{\tau\to-\infty} e^{\frac{\beta}{2}\tau}|\tau|^\alpha v(\tau) = 0.$$

d) The sets $\tilde{D}_k := \operatorname{dom}\tilde{A}^k \subset \operatorname{dom} A^k$, $k = 1,2,\dots,$ are dense in $\tilde{Z}$ and invariant with respect to $T(\cdot)$.

Proof: a) Density of $\tilde{Z}$ follows from density of $L^2_{\tilde{g}}(-\infty,0;\mathbb{R}^n)$ in $L^2_g(-\infty,0;\mathbb{R}^n)$. The latter property is obvious, because all continuous functions with compact support are contained in $\tilde{Z}$. Since $\gamma < \beta$, we obviously have $|\phi|_Z \le |\phi|_{\tilde{Z}}$ for all $\phi \in \tilde{Z}$.

b) This follows from

$$\begin{aligned}||\tau|^\alpha v|^2_{L^2_g} &= \int_{-\infty}^{0} e^{(\beta-\gamma)\tau}|\tau|^{2\alpha}e^{\gamma\tau}|v(\tau)|^2 d\tau \\ &\le \left(\frac{2\alpha}{(\beta-\gamma)e}\right)^{2\alpha}|v|^2_{L^2_{\tilde{g}}}, \quad v \in L^2_{\tilde{g}}(-\infty,0;\mathbb{R}^n).\end{aligned}$$

c) Using $v(\tau) = v(0) + \int_0^\tau \dot{v}(\tau)d\tau$ we obtain

$$\begin{aligned}e^{\frac{\beta}{2}\tau}|\tau|^\alpha|v(\tau)| &\le e^{\frac{\beta}{2}\tau}|\tau|^\alpha|v(0)| \\ &\quad + e^{\frac{\beta}{2}\tau}|\tau|^\alpha\left(\int_\tau^0 e^{-\gamma\sigma}d\sigma\right)^{1/2}\left(\int_\tau^0 e^{\gamma\sigma}|\dot{v}(\sigma)|^2 d\sigma\right)^{1/2} \\ &\le e^{\frac{\beta}{2}\tau}|\tau|^\alpha|v(0)| + \sqrt{\frac{1}{\gamma}}\, e^{\frac{1}{2}(\beta-\gamma)\tau}|\tau|^\alpha|\dot{v}|_{L^2_g}, \quad \tau \le 0,\end{aligned}$$

which implies the result.

d) Clearly, $\tilde{D}_k$ is dense in $\tilde{Z}$ and invariant with respect to $\tilde{T}(\cdot)$ (cf. for instance [14]).

From a) it follows that $\tilde{D}_k$ is dense in $\tilde{Z}$. Since $T(t)|\tilde{Z} = \tilde{T}(t)$, it is clear that $\tilde{D}_k$ is also invariant with respect to $T(\cdot)$. The inclusion $\tilde{D}_k \subset D_k$ is obvious from a) and Proposition 2.2 ∎

For the nonhomogeneous equation we have

Proposition 2.4. Let $f \in L^1_{loc}(0,\infty;\mathbb{R}^n)$ and let $x(t)$, $y(t)$ be the solution of (1.1). Then

$$(y(t),x_t) = T(t)\phi + \int_0^t T(t-\tau)(f(\tau),0)d\tau, \quad t \geq 0. \tag{2.6}$$

For equations with finite delays (2.6) was proved in [3] (see also [15], Section 2.3). The proof for the infinite delay case is quite analogous and is left to the reader.

3. Legendre and Laguerre polynomials

In this section we state repectively prove convergence results concerning L^2-approximation by Legendre and Laguerre polynomials.

3.1. Legendre polynomials

For the convenience of the reader we first collect some well-known facts on Legendre polynomials (see for instance [1]). The n-th Legendre polynomial $P_n(t)$, $n = 0,1,\dots$, is of degree n and satisfies

$$\frac{d}{dt}[(1-t^2)P_n'] + n(n+1)P_n = 0, \quad P_n(1) = 1.$$

For all $n = 0,1,\dots$

$$\begin{aligned} &P_n(1) = 1, \quad P_n(-1) = (-1)^n, \\ &|P_n(t)| \le 1 \quad \text{on} \quad [-1,1], \\ &\dot{P}_n(1) = \frac{n(n+1)}{2}, \quad \dot{P}_n(-1) = (-1)^n \frac{n(n+1)}{2}. \end{aligned} \tag{3.1}$$

The sequence $P_n(t)$, $n = 0,1,\dots$, forms a complete orthogonal set in $L^2(-1,1;\mathbb{R})$,

$$\int_{-1}^{1} P_n(t)P_m(t)dt = \begin{cases} \frac{2}{2n+1} & \text{for} \quad n = m, \\ 0 & \text{for} \quad n \ne m. \end{cases} \tag{3.2}$$

The derivative of $P_n(t)$ is a polynomial of degree $n-1$ and thus a combination of $P_0(t),\dots,P_{n-1}(t)$,

$$\begin{aligned} &\dot{P}_{2k}(t) = \sum_{j=0}^{k-1} (4j+3)P_{2j+1}(t), \\ &\dot{P}_{2k+1}(t) = \sum_{j=0}^{k} (4j+1)P_{2j}(t), \end{aligned} \tag{3.3}$$

$k = 0,1,2,\dots$.

Using the orthogonality relations (3.2) we get for $f \in L^2(-1,1;\mathbb{R})$ the expansion

$$f = \sum_{j=0}^{\infty} f_j P_j, \quad f_j = \frac{2j+1}{2} \int_{-1}^{1} f(t) P_j(t)\,dt.$$

Let f^N be the orthogonal projection of f onto span$(P_0,\ldots,P_n)$ in $L^2(-1,1;\mathbb{R})$, i.e.

$$f^N = \sum_{j=0}^{N} f_j P_j.$$

Then we have the following convergence results. When there is no danger of confusion D denotes differentiation.

Theorem 3.1. a) For any $k = 0,1,\ldots$ there exists a constant $c = c(k)$ such that

$$|f - f^N|_{L^2} \le \frac{c}{N^k} |f|_{W^{k,2}}$$

for all $f \in W^{k,2}(-1,1;\mathbb{R})$.

b) For any $k = 1,2,\ldots$ there exists a constant $c = c(k)$ such that

$$|D(f - f^N)|_{L^2} \le \frac{c}{N^{k-3/2}} |f|_{W^{k,2}}$$

for all $f \in W^{k,2}(-1,1;\mathbb{R})$.

c) For any $k = 1,2,\ldots$ there exists a constant $c = c(k)$ such that

$$|f(\pm 1) - f^N(\pm 1)| \le \frac{c}{N^{k-1/2}} |f|_{W^{k,2}}$$

for all $f \in W^{k,2}(-1,1;\mathbb{R})$.

Proof. a) and b) are special versions of results obtained in [4]. c) is given in [7] ∎

3.2. Laguerre polynomials

The Laguerre polynomial L_n, $n = 0,1,2,\dots$, is of degree n and satisfies

$$tL_n'' + (1-t)L_n' + nL_n = 0, \quad L_n(0) = 1. \tag{3.4}$$

The sequence of Laguerre polynomials $L_0, L_1, \dots$ is complete and orthogonal in $L_w^2(0,\infty;\mathbb{R})$ with weight $w(t) = e^{-t}$, $t \geq 0$,

$$\int_0^\infty e^{-t} L_m(t) L_n(t)\,dt = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases} \tag{3.5}$$

For the derivatives we have in analogy to (3.3) the formula

$$\frac{d}{dt} L_n = -\sum_{j=0}^{n-1} L_j, \quad n = 0,1,\dots, \tag{3.6}$$

(as usual $\sum_{j=0}^{-1} = 0$).

In order to derive convergence results analogous to those of Theorem 3.1 we need some preparation. On the linear subspaces

$$\begin{aligned} \mathcal{B}_k = \{v \in C^{2k-1}(0,\infty;\mathbb{R} \mid\ & v^{(2k-1)} \text{ locally absolutely continuous} \\ & \text{on } [0,\infty),\ t^m v^{(j)} \in L_w^2(0,\infty;\mathbb{R}),\ m = 0,1,\dots,\ j = 0,\dots,2k, \\ & \text{and } \lim_{t\to\infty} e^{-t/2} t^m v^{(j)}(t) = 0,\ m = 0,1,\dots,\ j = 0,\dots,2k-1\}, \end{aligned}$$

$k = 1,2,\dots$, we define the operator B by

$$(Bv)(t) = t\ddot{v}(t) + (1-t)\dot{v}(t), \quad v \in \mathcal{B}_k.$$

Lemma 3.2. a) $L_n \in \mathcal{B}_k$ and $L_n = -\frac{1}{n} BL_n$ for $n = 0,1,2,\dots$, $k = 1,2,\dots$. Moreover

$$B^j v \in \mathcal{B}_1,\ j = 0,\dots,k-1, \quad \text{for all} \quad v \in \mathcal{B}_k. \tag{3.7}$$

b) Let $v \in C^1(0,\infty;\mathbb{R})$. Then $\dot{v} \in \mathcal{B}_k$ implies $v \in \mathcal{B}_k$, $k = 1,2,\dots$.
c) B is symmetric in $L^2_w(0,\infty;\mathbb{R})$.

Proof. The first part of a) is trivial because of (3.4). Let $v \in \mathcal{B}_k$. Then $B^j v$ is a linear combination of terms of the form $t^\mu v^{(\nu)}$, $\mu = 0,1,2,\dots$, $\nu = 0,1,\dots,2j$, and therefore $t^m(B^j v)'$ and $t^m(B^j v)''$ are linear combinations of terms of the form $t^\mu v^{(\nu)}$, $\mu = 0,1,\dots$, $\nu = 0,1,\dots,2j+2 \le 2k$. Then the result is obvious.
b) We only have to prove $t^m v \in L^2_w(0,\infty;\mathbb{R})$, $m = 0,1,\dots$, and $e^{-t/2}t^m v(t) \to 0$ as $t \to \infty$, $m = 0,1,\dots$. Since trivially $t^m v(0) \in L^2_w(0,\infty;\mathbb{R})$ and $e^{-t/2}t^m v(0) \to 0$ as $t \to \infty$ we only have to investigate $t^m \int_0^t \dot{v}(\tau)d\tau$. The result then follows from the estimates

$$\int_0^\infty e^{-t}t^{2m} \left|\int_0^t \dot{v}(\tau)d\tau\right|^2 dt \le \int_0^\infty e^{-t}t^{2m+1} \int_0^t |\dot{v}(\tau)|^2 d\tau dt$$

$$= \int_0^\infty e^{-\tau}|\dot{v}(\tau)|^2 p(\tau)d\tau < \infty,$$

where $p(\tau)$ is a polynomial of degree $2m+1$, and

$$\left|e^{-t/2}t^m \int_0^t \dot{v}(\tau)d\tau\right| \le e^{-t/2}t^{m+1/2}|\dot{v}|_{L^2_w}.$$

c) Density of $\mathcal{B}_k$ is clear by a). Let $u,v \in \mathcal{B}_k$. Then

$$e^{-t}(1-t)u(t)v(t)$$

$$= e^{-t/2}u(t)e^{-t/2}v(t) - e^{-t/2}tu(t)e^{-t/2}v(t) \to 0 \quad \text{as} \quad t \to \infty$$

and also

$$te^{-t}\dot{u}(t)v(t) = e^{-t/2}t\dot{u}(t)e^{-t/2}v(t) \to 0,$$

$$te^{-t}u(t)\dot{v}(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore by integration by parts

$$\langle Bu,v\rangle_{L^2_w} = \int_0^\infty e^{-t}[t\ddot{u}(t) + (1-t)\dot{u}(t)]v(t)dt$$

$$= - u(0)v(0) - \int_0^\infty e^{-t}[(1-t)v(t) + t\dot{v}(t)]\dot{u}(t)dt$$

$$- \int_0^\infty e^{-t}[(1-t)\dot{v}(t) - (2-t)v(t)]u(t)dt$$

$$= \int_0^\infty e^{-t}((1-t)\dot{v}(t) + t\ddot{v}(t))u(t)dt = \langle u,Bv\rangle_{L^2_w} \blacksquare$$

We now are in a position to prove convergence results for Laguerre polynomials similar to those of Theorem 3.1 for Legendre polynomials following the approach taken in [4]. For $u = \sum_{k=0}^{\infty} u_k L_k \in L^2_w(0,\infty;\mathbb{R})$, $u_k = \int_0^\infty e^{-t}u(t)L_k(t)dt$, $k = 0,1,2\ldots$, let u^N be the image of u under the orthogonal projection $P^N: L^2_w(0,\infty;\mathbb{R}) \to \mathrm{span}(L_0,\ldots L_N)$, i.e.

$$u^N = P^N u = \sum_{k=0}^{N} u_k L_k .$$

Theorem 3.3. Let $u \in \mathcal{B}_k$. Then

$$|u - u^N|_{L^2_w} \leq \frac{1}{(N+1)^k} |B^k u|_{L^2_w} , \quad N = 0,1,\ldots .$$

Proof. Using Lemma 3.2 we get

$$u_j = \langle u,L_j\rangle_{L^2_w} = - \frac{1}{j} \langle u,BL_j\rangle_{L^2_w} = - \frac{1}{j} \langle Bu,L_j\rangle_{L^2_w}$$

$$= \ldots = (- \frac{1}{j})^k \langle B^k u,L_j\rangle_{L^2_w} .$$

This implies

$$|u - u^N|^2_{L^2_w} = \sum_{j=N+1}^{\infty} |u_j|^2 = \sum_{j=N+1}^{\infty} \frac{1}{j^{2k}} |\langle B^k u, L_j \rangle_{L^2_w}|^2$$
$$\le \frac{1}{(N+1)^{2k}} |B^k u|^2_{L^2_w}, \quad N = 0,1,\ldots \blacksquare$$

It will be convenient to use the notations $D = \frac{d}{dt}$ and $K^N = P^N D - DP^N$.

Lemma 3.4. a) For any $u \in \mathrm{span}(L_0,\ldots,L_N)$

$$|Du|_{L^2_w} \le N|u|_{L^2_w}, \quad N = 0,1,2,\ldots .$$

b) For any u such that $\dot{u} \in \mathcal{B}_k$

$$|K^N u|_{L^2_w} \le \frac{\sqrt{2}}{N^{k-1/2}} |B^k \dot{u}|_{L^2_w}, \quad N = 1,2,\ldots .$$

Proof. a) We have $u = \sum_{j=0}^{N} u_j L_j$ and using (3.6)

$$Du = \sum_{j=1}^{N} u_j \dot{L}_j = -\sum_{j=1}^{N} u_j \sum_{i=0}^{j-1} L_i = -\sum_{i=0}^{N-1} \Big(\sum_{j=i+1}^{N} u_j\Big) L_i .$$

Therefore by Cauchy's inequality

$$|Du|^2_{L^2_w} = \sum_{i=0}^{N-1} \Big(\sum_{j=i+1}^{N} u_j\Big)^2 \le \sum_{i=0}^{N-1} \Big(\sum_{j=i+1}^{N} 1\Big)\Big(\sum_{j=i+1}^{N} |u_j|^2\Big)$$
$$= \sum_{i=0}^{N-1} (N-i) \sum_{j=i+1}^{N} |u_j|^2 = \sum_{j=1}^{N} |u_j|^2 \sum_{i=0}^{j-1} (N-i)$$
$$\le \Big(\sum_{j=0}^{N} j\Big) \sum_{j=1}^{N} |u_j|^2 \le \frac{N(N+1)}{2} |u|^2_{L^2_w} \le N^2 |u|^2_{L^2_w}.$$

b) Let $\dot{u} = \sum_{j=0}^{\infty} z_j L_j$. Then

$$P^N Du = \sum_{j=0}^{N} z_j L_j . \tag{3.8}$$

We next compute the Fourier-coefficients of $\int_0^\tau \dot{u}(\sigma)d\sigma$. Since $\dot{u} \in B_k$ we have for some constant $c > 0$

$$|\dot{u}(\tau)| \leq ce^{\tau/2}, \ \tau \geq 0.$$

This implies

$$\int_0^\infty e^{-\tau}|L_j(\tau)| \int_0^\tau |\dot{u}(\sigma)|d\sigma \leq 2c \int_0^\infty e^{-\tau}|L_j(\tau)|(e^{\tau/2}-1)d\tau$$

$$\leq 2c \int_0^\infty e^{-\tau/2}|L_j(\tau)d\tau < \infty.$$

Therefore we can use Fubini's theorem in the following computations. For the moment we put $w_j = \sum_{i=0}^{j} z_i L_i$ and get

$$\int_0^\infty e^{-\tau}L_j(\tau) \int_0^\tau \dot{u}(\sigma)d\sigma d\tau = \int_0^\infty \dot{u}(\sigma) \int_\sigma^\infty e^{-\tau}L_j(\tau)d\tau d\sigma$$

$$= \int_0^\infty e^{-\sigma}\dot{u}(\sigma)(\sum_{i=0}^{j} L_j^{(i)}(\sigma))d\sigma = <\dot{u}, \sum_{i=0}^{j} L_j^{(i)}>_{L_w^2}$$

$$= <\dot{w}_j, \sum_{i=0}^{j} L_j^{(i)}>_{L_w^2} = \int_0^\infty e^{-\sigma}\dot{w}_j(\sigma)(\sum_{i=0}^{j} L_j^{(i)})d\sigma$$

$$= \int_0^\infty e^{-\tau}L_j(\tau) \int_0^\tau \dot{w}_j(\sigma)d\sigma d\tau.$$

From (3.6) we obtain

$$\int_0^\tau L_i(\sigma)d\sigma = L_i(\tau) - L_{i+1}(\tau)$$

and therefore

$$\int_0^\tau \dot{w}_j(\sigma)d\sigma = \sum_{i=0}^{j} z_i \int_0^\tau L_i(\sigma)d\sigma = \sum_{i=0}^{j} z_i(L_i(\tau) - L_{i+1}(\tau)).$$

Using this above we get

$$\int_0^\infty e^{-\tau} L_j(\tau) \int_0^\tau \dot{u}(\sigma)d\sigma d\tau = \sum_{i=0}^{j} z_i \langle L_j, L_i - L_{i+1}\rangle_{L_w^2}$$

$$= \begin{cases} z_j - z_{j-1} & \text{for } j = 1,2,\dots, \\ z_0 & \text{for } j = 0. \end{cases}$$

From $u(\tau) = u(0) + \int_0^\tau \dot{u}(\sigma)d\sigma$ we obtain

$$u = (u(0) + z_0)L_0 + \sum_{j=1}^{\infty} (z_j - z_{j-1})L_j$$

and

$$u^N = P^N u = (u(0) + z_0)L_0 + \sum_{j=1}^{N} (z_j - z_{j-1})L_j .$$

Using (3.6) this implies

$$DP^N u = \sum_{j=1}^{N} (z_{j-1} - z_j) \sum_{i=0}^{j-1} L_i = \sum_{i=0}^{N-1} \Big(\sum_{j=i+1}^{N} (z_{j-1} - z_j)\Big) L_i$$
$$= \sum_{i=0}^{N-1} (z_i - z_N) L_i .$$

This and (3.8) give

$$K^N u = \sum_{j=0}^{N} z_j L_j - \sum_{j=0}^{N-1} (z_j - z_N) L_j$$
$$= z_N L_N + \sum_{j=0}^{N-1} z_N L_j = z_N \sum_{j=0}^{N} L_j .$$

Using Theorem 3.3 for $N-1$ and $\dot{u}$ we get

$$|K^N u|^2_{L_w^2} = (N+1)|z_N|^2 \le (N+1) \sum_{j=N}^{\infty} |z_j|^2 = (N+1)|\dot{u} - P^{N-1}\dot{u}|^2_{L_w^2}$$
$$\le \frac{N+1}{N^{2k}} |B^k \dot{u}|^2_{L_w^2} \le \frac{2}{N^{2k-1}} |B^k \dot{u}|^2_{L_w^2} \quad \blacksquare$$

<u>Theorem 3.5.</u> a) Let u be such that $u^{(\ell)} \in \mathcal{B}_k$. Then the following estimate is true:

$$|D^{\ell}(u-u^N)|_{L^2_w} \leq \frac{c}{N^{k-\ell+1/2}}, \quad N = 1,2,\ldots,$$

for $\ell = 1,2,\ldots$, where $c = c(|B^k\dot{u}|_{L^2_w},\ldots,|B^k u^{(\ell)}|_{L^2_w})$.

b) Let u be such that $\dot{u} \in \mathcal{B}_k$. Then for any $T > 0$

$$|u - u^N|_{L^\infty(0,T;\mathbb{R})} \leq \frac{\tilde{c}}{N^{k-1/2}},$$

where $\tilde{c} = \tilde{c}(T,|B^k\dot{u}|_{L^2_w})$.

c) Let u be such that $\dot{u} \in \mathcal{B}_k$. Then

$$|u(0) - (P^N u)(0)| \leq \frac{1}{N^k}|B^k\dot{u}|_{L^2_w}, \quad N = 1,2,\ldots.$$

<u>Proof.</u> a) An easy induction shows

$$D^{\ell} - D^{\ell}P^N = D^{\ell} - P^N D^{\ell} + \sum_{j=0}^{\ell-1} D^j K^N D^{\ell-1-j}, \quad \ell = 1,2,\ldots.$$

Using this formula, Theorem 3.3 for $u^{(\ell)}$ and Lemma 3.4 (note, that $K^N u \in \mathrm{span}(L_0,\ldots,L_N)$) we get

$$\begin{aligned} |D^{\ell}(u-u^N)|_{L^2_w} &\leq \frac{1}{(N+1)^k}|B^k u^{(\ell)}|_{L^2_w} \\ &+ \sum_{j=0}^{\ell-1} N^j \frac{\sqrt{2}}{N^{k-1/2}}|B^k u^{(\ell-j)}|_{L^2_w} \\ &\leq \frac{\sqrt{2}}{N^{k-\ell+1/2}} \sum_{j=1}^{\ell} |B^k u^{(j)}|_{L^2_w}. \end{aligned}$$

b) Since $|\dot{u}|_{L^2(0,T;\mathbb{R})} \leq e^{T/2}|\dot{u}|_{L^2_w}$, the result is an immediate consequence of part a) for $\ell = 1$ and Sobolev's embedding theorem.

c) Let $u = \sum_{j=0}^{\infty} u_j L_j$ and $\dot{u} = \sum_{j=0}^{\infty} z_j L_j$. Then for $j = 1,2,\ldots$

$$u_j = \langle u, L_j \rangle_{L_w^2} = -\frac{1}{j} \langle u, BL_j \rangle_{L_w^2}$$

$$= -\frac{1}{j} \int_0^{\infty} \frac{d}{dt} (te^{-t} \frac{d}{dt} L_j) u(t) dt = \frac{1}{j} \int_0^{\infty} te^{-t} (\frac{d}{dt} L_j) \dot{u}(t) dt.$$

Using

$$t \frac{d}{dt} L_j = j(L_j - L_{j-1})$$

we get

$$u_j = \int_0^{\infty} e^{-t} (L_j - L_{j-1}) \dot{u} dt = z_j - z_{j-1}.$$

Then

$$u(0) - (P^N u)(0) = u(0) - \sum_{j=0}^{N} u_j = u(0) - u_0 - \sum_{j=1}^{N} (z_j - z_{j-1})$$

$$= u(0) - u_0 + z_0 - z_N$$

$$= u(0) - \int_0^{\infty} e^{-t} u(t) dt + \int_0^{\infty} e^{-t} \dot{u}(t) dt - z_N$$

$$= -z_N.$$

From $\dot{u} \in \mathcal{B}_k$ we get

$$z_j = -\frac{1}{j^k} \langle B^k \dot{u}, L_j \rangle_{L_w^2}, \quad j = 1,2,\ldots .$$

Therefore

$$|u(0) - (P^N u)(0)|^2 = |z_N|^2 \leq \sum_{j=N}^{\infty} |z_j|^2$$

$$= \sum_{j=N}^{\infty} \frac{1}{j^{2k}} |\langle B^k \dot{u}, L_j \rangle_{L_w^2}|^2 \leq \frac{1}{N^{2k}} |B^k \dot{u}|_{L_w^2}^2, \quad N = 1,2,\ldots \blacksquare$$

4. Formulation of the approximation scheme

Following the general outline given in [12] or [11] we formulate the approximation scheme for problem (1.1) using Legendre and Laguerre polynomials. Throughout this section we replace the weighting function g as defined in (1.2) by

$$g(\tau) = \begin{cases} 1 & \text{for } -h \leq \tau \leq 0, \\ e^{\beta(\tau+h)} & \text{for } \tau < -h, \end{cases} \tag{4.1}$$

where as before $\beta > 0$. This weighting function gives the same space with an equivalent norm as the one defined in (1.2). Let $r_i = h_i - h_{i-1}$, $i = 1,\dots,p$, and define the functions ζ_i, $i = 1,\dots,p+1$, by

$$\zeta_i(\tau) = \frac{1}{r_i}(h_{i-1}+h_i+2\tau), \quad -h_i \leq \tau < -h_{i-1}, \tag{4.2}$$

for $i = 1,\dots,p$ and by

$$\zeta_{p+1}(\tau) = -\beta(\tau+h), \quad \tau < -h. \tag{4.3}$$

Trivial computations show that the mappings Θ_i defined by $\Theta_i\chi = \chi \circ \zeta_i$, $i = 1,\dots,p+1$, are metric isomorphisms $L^2(-1,1;\mathbb{R}^n) \to L^2(-h_i,-h_{i-1};\mathbb{R}^n)$, $i = 1,\dots,p$, and $L^2_w(0,\infty;\mathbb{R}^n) \to L^2_g(-\infty,-h;\mathbb{R}^n)$, respectively.

We put

$$e_{ij}(\tau) = \begin{cases} P_j(\zeta_i(\tau))I & \text{for } -h_i \leq \tau < -h_{i-1}, \\ 0 & \text{elsewhere}, \end{cases} \tag{4.4}$$

$i = 1,\dots,p$, $j = 0,\dots,N$, and

$$e_{p+1,j}(\tau) = \begin{cases} L_j(\zeta_{p+1}(\tau))I & \text{for } \tau < -h, \\ 0 & \text{elsewhere} \end{cases} \tag{4.5}$$

$j = 0,\ldots,N$. Furthermore, let

$$\hat{e}_{00} = (I,0), \quad \hat{e}_{ij} = (0,e_{ij}),\ i = 1,\ldots,p+1,\ j = 0,\ldots,N,$$

$$E^N = (e_{10},\ldots,e_{p+1,N}), \quad \hat{E}^N = (\hat{e}_{00},\hat{e}_{10},\ldots,\hat{e}_{p+1,N})$$

and

$$Y_i^N = \text{span}(e_{i0},\ldots,e_{iN}),\ i = 1,\ldots,p+1,$$

$$Y^N = Y_1^N \times \ldots \times Y_{p+1}^N,$$

$$Z^N = \mathbb{R}^n \times Y^N = \text{span}(\hat{e}_{00},\hat{e}_{10},\ldots,\hat{e}_{p+1,N}).$$

Let $p^N: Z \to Z^N$ be the orthogonal projection. The coordinate vector $\alpha^N(p^N\phi)$ of $\phi = (\phi^0,\phi^1) \in Z$ is given by

$$\alpha^N(p^N\phi) = (Q^N)^{-1}\text{col}(\phi^0,\langle e_{01},\phi^1\rangle_{L_g^2},\ldots,\langle e_{p+1,N},\phi^1\rangle_{L_g^2}),$$

where

$$Q^N = \langle \hat{E}^N,\hat{E}^N\rangle_Z = \text{diag}(I,r_1q^N,\ldots,r_pq^N,\tfrac{1}{\beta}\, I_{n(N+1)}), \tag{4.6}$$

where $I_{n(N+1)}$ denotes the $n(N+1)\times n(N+1)$-identity matrix and $q^N := \frac{1}{r_i}(\langle e_{ij},e_{ik}\rangle_{L_g^2})_{j,k=0,\ldots,N}$ is given by

$$q^N = \text{diag}(1,\tfrac{1}{3},\ldots,\tfrac{1}{2N+1}) \otimes I.$$

For later use we note that for $\phi,\ \psi \in Z^N$ with coordinate vectors $\alpha^N(\phi)$, $\alpha^N(\psi)$, respectively, we have

$$\langle \phi, \psi \rangle_Z = \alpha^N(\phi)^T Q^N \alpha^N(\psi). \tag{4.7}$$

Following the general scheme as outlined in [12] or [11] we introduce the approximating delta-impulses δ_i^N by

$$\delta_i^N = E^N \gamma_i^N, \quad \gamma_i^N = (Q^N)^{-1} E^N(-h_i-0)^T,$$

$i = 0,\dots,p$. Easy computations using (4.4), (4.5) and (4.6) show that

$$\gamma_i^N = \frac{1}{r_{i+1}} \operatorname{col}(0,\dots,0,1,3,\dots,2N+1,0,\dots,0) \otimes I, \; i = 0,\dots,p-1,$$

where the nonzero entries occur at positions i to $i+N$, and

$$\gamma_p^N = \beta \operatorname{col}(0,\dots,0,1,\dots,1) \otimes I.$$

For later use we compute the norm of δ_i^N considered as an operator $\mathbb{R}^n \to L_g^2$. Using (4.7) and the explicit representation for γ_i^N given above we get for $x \in \mathbb{R}^n$

$$|\delta_i^N x|^2_{L_g^2} = \langle \delta_i^N x, \delta_i^N x \rangle_{L_g^2} = x^T (\gamma_i^N)^T Q^N \gamma_i^N x$$

$$= \frac{1}{r_{i+1}} \sum_{k=0}^{N} (2k+1)|x|^2 = \frac{(N+1)^2}{r_{i+1}} |x|^2$$

for $i = 0,\dots,p-1$, i.e.

$$|\delta_i^N| = \left(\frac{1}{r_{i+1}}\right)^{1/2} (N+1), \quad i = 0,\dots,p-1. \tag{4.8}$$

Analogous computations give

$$|\delta_p^N| = \left(\frac{N+1}{\beta}\right)^{1/2}. \tag{4.9}$$

As in [12] (see proof of Lemma 5.1) we obtain

$$<\delta_i^N x,\phi^1>_{L^2_g} = x^T\phi^1(-h_i-0), \quad i = 0,\dots,p, \tag{4.10}$$

for $x \in \mathbb{R}^n$ and $\phi^1 \in Y^N$. In analogy to the definition given in [12] for the retarded finite delay case we define the approximating operators A^N by

$$\begin{aligned}(A^N\phi)^0 &= A_0(\phi^0 + \sum_{i=1}^{p} B_i\phi^1(-h_i) + \int_{-\infty}^{0} B(\tau)\phi^1(\tau)d\tau) \\ &\quad + \sum_{i=1}^{p} A_i\phi^1(-h_i) + \int_{-\infty}^{0} A(\tau)\phi^1(\tau)d\tau \qquad (4.11) \\ &= A_0\phi^0 + \sum_{i=1}^{p}(A_i+A_0B_i)\phi^1(-h_i) + \int_{-\infty}^{0}[A(\tau)+A_0B(\tau)]\phi^1(\tau)d\tau,\end{aligned}$$

$$\begin{aligned}(A^N\phi)^1 &= \frac{d^+}{d\tau}\phi^1 + \delta_0^N(\phi^0-\phi^1(0) + \sum_{i=1}^{p} B_i\phi^1(-h_i)+\int_{-\infty}^{0} B(\tau)\phi^1(\tau)d\tau) \\ &\quad + \sum_{i=1}^{p}\delta_i^N(\phi^1(-h_i) - \phi^1(-h_i-0)),\end{aligned} \tag{4.12}$$

for $\phi = (\phi^0,\phi^1) \in Z^N$.

The approximation $z^N(t)$ to $(y(t),x_t)$ is given by

$$z^N(t) = e^{A^N t}P^N\phi + \int_0^t e^{A^N(t-s)}(f(s),0)ds, \quad t \geq 0, \tag{4.13}$$

i.e.

$$\dot{z}^N(t) = A^N z^N(t) + (f(t),0), \quad t \geq 0,$$

$$z^N(0) = P^N\phi.$$

For the implementation of the scheme we have to compute matrix representations $[A^N]$ for the operators A^N. As in [12] we get

$$[A^N] = (Q^N)^{-1}H^N, \tag{4.1-}$$

where

$$H^N = \langle \hat{E}^N, A^N\hat{E}^N \rangle_Z .$$

Using the definition of the basis elements and (4.11), (4.12) we get

$$A^N\hat{e}_{00} = (A_0, \delta_0^N),$$

$$A^N\hat{e}_{ij} = ((-1)^{j+1}(A_i + A_0B_i) + A_{ij} + A_0B_{ij}, \frac{d^+}{d\tau} e_{ij} + \delta_0^N((-1)^{j+1}B_i + B_{ij}) + (-1)^{j+1}\delta_i^N - \delta_{i-1}^N),$$

$i = 1,\dots,p$, $j = 0,\dots,N$,

$$A^N\hat{e}_{p+1,j} = (A_{p+1,j} + A_0B_{p+1,j}, \frac{d^+}{d\tau} e_{p+1,j} + \delta_0^N B_{p+1,j} - \delta_p^N),$$

$j = 0,\dots,N$, where

$$A_{ij} = \begin{cases} \int_{-h_i}^{-h_{i-1}} A(\tau)e_{ij}(\tau)d\tau & \text{for } i = 1,\dots,p, \\ \int_{-\infty}^{-h} A(\tau)e_{p+1,j}(\tau)d\tau & \text{for } i = p+1, \end{cases}$$

and

$$B_{ij} = \begin{cases} \int_{-h_i}^{-h_{i-1}} B(\tau)e_{ij}(\tau)d\tau & \text{for } i = 1,\dots,p, \\ \int_{-\infty}^{-h} B(\tau)e_{p+1,j}(\tau)d\tau & \text{for } i = p+1. \end{cases}$$

Since $\frac{d^+}{d\tau} e_{ik}$ is a polynomial of degree $k-1$ on $(-h_i,-h_{i-1})$ for $i = 1,\ldots,p$ and on $(-\infty,-h)$ for $i = p+1$, respectively, we get immediately from the orthogonality relations (3.2) and (3.5)

$$\langle e_{ij}, \frac{d^+}{d\tau} e_{ik}\rangle_{L_g^2} = 0 \quad \text{for} \quad j \geq k. \tag{4.15}$$

In case $j < k$ we get

$$\begin{aligned}\langle e_{ij}, \frac{d^+}{d\tau} e_{ik}\rangle_{L_g^2} &= \int_{-h_i}^{-h_i-1} e_{ij}(\tau)\dot{e}_{ik}(\tau)d\tau = (1-(-1)^{j+1})I \\ &- \langle e_{ik}, \frac{d^+}{d\tau} e_{ij}\rangle_{L_g^2} = (1-(-1)^{j+k})I,\end{aligned}$$

$i = 1,\ldots,p$, and similarly

$$\langle e_{p+1,j}, \frac{d^+}{d\tau} e_{p+1,k}\rangle_{L^2} = I. \tag{4.16}$$

Using (4.10) and the definition of the basis elements we obtain

$$\langle \hat{e}_{00}, A^N\hat{e}_{00}\rangle_Z = A_0,$$

$$\langle \hat{e}_{00}, A^N\hat{e}_{ij}\rangle_Z = (-1)^j\,(A_i+A_0B_i) + A_{ij} + A_0B_{ij},$$

$i = 1,\ldots,p$, $j = 0,\ldots,N$,

$$\langle \hat{e}_{00}, A^N e_{p+1,j}\rangle_Z = A_{p+1,j} + A_0B_{p+1,j}, \quad j = 0,\ldots,N,$$

$$\langle \hat{e}_{1j}, A^N\hat{e}_{00}\rangle_Z = I, \quad j = 0,\ldots,N,$$

$$\langle \hat{e}_{1j}, A^N\hat{e}_{1k}\rangle_Z = \langle e_{1j}, \frac{d^+}{d\tau} e_{1k}\rangle_{L_g^2} - I + (-1)^k B_1 + B_{1k},$$

$j,k = 0,\ldots,N$,

$$\langle \hat{e}_{1j}, A^N\hat{e}_{ik}\rangle_Z = (-1)^k B_i + B_{ik},$$

$j,k = 0,\ldots,N, \quad i = 2,\ldots,p,$

$$\langle \hat{e}_{1j}, A^N \hat{e}_{p+1,k} \rangle_Z = B_{p+1,k}, \quad j,k = 0,\ldots,N,$$

$$\langle \hat{e}_{ij}, A^N \hat{e}_{ik} \rangle_Z = \langle e_{ij}, \frac{d+}{d\tau} e_{ik} \rangle_{L_g^2} - I,$$

$j,k = 0,\ldots,N, \quad i = 2,\ldots,p+1,$

$$\langle \hat{e}_{i+1,j}, A^N \hat{e}_{ik} \rangle_Z = (-1)^k I,$$

$j,k = 0,\ldots,N, \quad i = 1,\ldots,p$. All other inner products are zero. From this and (4.15), (4.16) it follows that H^N is given by

$$H^N = \begin{pmatrix} A_0 & \alpha_1^N & \cdots & \cdots & \cdots & \alpha_{p+1}^N \\ \varepsilon^N & h^N \otimes I + \beta_1^N & \beta_2^N & \cdots & \cdots & \beta_{p+1}^N \\ 0 & k^N \otimes I & h^N \otimes I & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & h^N \otimes I & 0 \\ 0 & 0 & \cdots & 0 & k^N \otimes I & h_{p+1}^N \otimes I \end{pmatrix}$$

where

$$\alpha_i^N = (A_i + A_0 B_i + A_{i0} + A_0 B_{i0}, -(A_i + A_0 B_i) + A_{i1} + A_0 B_{i1}, \ldots, (-1)^N (A_i + A_0 B_i) + A_{iN} + A_0 B_{iN}),$$

$i = 1,\ldots,p,$

$$\alpha_{p+1}^N = (A_{p+1,0} + A_0 B_{p+1,0}, \ldots, A_{p+1,N} + A_0 B_{p+1,N}),$$

$$k^N = \begin{pmatrix} 1 & -1 & \dots & (-1)^N \\ | & | & & | \\ 1 & -1 & \dots & (-1)^N \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}, \qquad \varepsilon^N = \begin{pmatrix} 1 \\ | \\ 1 \end{pmatrix} \otimes I,$$

$$\beta_i^N = k^N \otimes B_i + \begin{pmatrix} B_{i0} & \dots & B_{iN} \\ | & & | \\ B_{i0} & \dots & B_{iN} \end{pmatrix}, \qquad i = 1,\dots,p,$$

$$\beta_{p+1}^N = \begin{pmatrix} B_{p+1,0} & \dots & B_{p+1,N} \\ | & & | \\ B_{p+1,0} & \dots & B_{p+1,N} \end{pmatrix},$$

$$h^N = \begin{pmatrix} -1 & 1 & -1 & \dots & (-1)^{N-1} \\ | & \ddots & \ddots & \ddots & \vdots \\ & & & & -1 \\ & & & & 1 \\ -1 & & \text{---} & & -1 \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}$$

and

$$h_{p+1}^N = \begin{pmatrix} -1 & 0 & \text{---} & 0 \\ | & \ddots & \ddots & | \\ & & & 0 \\ -1 & \text{---} & & -1 \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)} .$$

The coordinate vector $w^N(t)$ of the approximations $z^N(t) \in Z^N$ to $(y(t),x_t)$ is the solution of

$$\dot{w}^N(t) = [A^N]w^N(t) + f(t)\mathrm{col}(I,0,\dots,0),$$
$$w^N(0) = p^N\phi, \tag{4.17}$$

where $\mathrm{col}(I,0,\dots,0) \in \mathbb{R}^{n((N+1)p+1)\times n}$.

Of course, the subspaces Y_i^N need not to have the same dimension, $i = 1,\dots,p+1$. We could also take $Y_i^N = \mathrm{span}(e_{i0},\dots,e_{iN_i})$, $i = 1,\dots,p+1$. The resulting modifications are obvious.

Of course, it is also possible to choose $g(\tau) = e^{\beta\tau}$, $\beta > 0$, and to defined $\zeta(\tau) = -\beta\tau$, $\tau < 0$. Then $\Theta\chi = \chi \circ \zeta$ defines an isomorphism $L_w^2(0,\infty;\mathbb{R}^n) \to L_g^2(-\infty,0;\mathbb{R}^n)$. The basis elements of Z^N are defined by

$$e_j(\tau) = L_j(\zeta(\tau))I, \quad \tau < 0, \quad j = 0,\dots,N,$$

and

$$\hat{e}_{00} = (I,0), \quad \hat{e}_j = (0,e_j), \quad j = 0,\dots,N.$$

Of course, $Y^N = \mathrm{span}(e_0,\dots,e_N)$ and $Z^N = \mathbb{R}^n \times Y^N$. In this case we have

$$Q^N = \mathrm{diag}(I,q^N),$$

where $q^N = \frac{1}{\beta}\,\mathrm{diag}(I,\dots,I) \in \mathbb{R}^{n(N+1)\times n(N+1)}$. We only have to introduce $\delta_0^N = (e_0,\dots,e_N)\gamma_0^N$ with $\gamma_0^N = \beta\,\mathrm{col}(I,\dots,I)$. Analogously to (4.9) we get

$$|\delta_0^N| = \left(\frac{N+1}{\beta}\right)^{1/2}.$$

The operators A^N are defined by

$$(A^N\phi)^0 = A_0\phi^0 + \sum_{i=1}^{p}(A_i+A_0B_i)\phi^1(-h_i) + \int_{-\infty}^{0}[A(\tau)+A_0B(\tau)]\phi^1(\tau)d\tau$$

$$(A^N\phi)^1 = \dot{\phi}^1 + \delta_0(\phi^0-\phi^1(0)) + \sum_{i=1}^{p}B_i\phi^1(-h_i) + \int_{-\infty}^{0}B(\tau)\phi^1(\tau)d\tau) \tag{4.18}$$

for $\phi = (\phi^0,\phi^1) \in Z^N$. Again $[A^N]$ is given by (4.14). By analogous computations as in the previous case we obtain

$$H^N = \begin{pmatrix} A_0 & \alpha^N \\ \varepsilon^N & h^N + \beta^N \end{pmatrix},$$

where

$$\alpha^N = (\alpha_0, \ldots, \alpha_N), \tag{4.19}$$

$$\alpha_j = \sum_{k=1}^{p} (A_k + A_0 B_k) e_j(-h_k) + A_{1j} + A_0 B_{1j},$$

$$j = 0, \ldots, N, \quad A_{1j} = \int_{-\infty}^{0} A(\tau) e_j(\tau) d\tau, \quad B_{1j} = \int_{-\infty}^{0} B(\tau) e_j(\tau) d\tau,$$

$$\varepsilon^N = \mathrm{col}(I, \ldots, I) \in \mathbb{R}^{n(N+1) \times n}$$

$$h^N = \begin{pmatrix} -1 & & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ & & & & 0 \\ -1 & \cdots & & & -1 \end{pmatrix} \in \mathbb{R}^{n(N+1) \times n(N+1)},$$

$$\beta^N = \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_N \\ \vdots & \vdots & & \vdots \\ \beta_0 & \beta_1 & \cdots & \beta_N \end{pmatrix} \in \mathbb{R}^{n(N+1) \times n(N+1)}, \tag{4.20}$$

$$\beta_j = \sum_{k=1}^{p} B_k e_j(-h_k) + B_{1j}, \quad j = 0, \ldots, N.$$

Again the coordinate vector for the approximations is governed by (4.17), where now $\mathrm{col}(I,0,\ldots,0) \in \mathbb{R}^{n(N+1) \times n}$.

As will be indicated at the end of Section 5 the scheme determined by (4.18) will need more smoothness on the initial data compared to the scheme defined by (4.11), (4.12) in order to achieve the same rate of convergence.

3. Proof of convergence for the approximation scheme

The following assumption on the matrices $B_1,\dots,B_p$ will be used:

(A) There exist numbers $\lambda_j > 0$, $j = 1,\dots,p$, such that the $np\times np$-matrix

$$(B_1\dots B_p)^T(B_1\dots B_p) - \frac{1}{2(1+\sum_{j=1}^{p}\lambda_j)}\operatorname{diag}(\lambda_1 I,\dots,\lambda_p I)$$

is negative definit.

Hypothesis (A) is certainly satisfied if $|B_j|$, $j = 1,\dots,p$, is sufficiently small. Moreover, by a transformation

$$y(t) = e^{-\alpha t}x(t), \quad \alpha > 0,$$

we can always transform system (1.1) into a system such that (A) is satisfied. Let A, B satisfy (2.1) and let $\phi^1 \in L^2_g(-\infty,0;\mathbb{R}^n)$ where $g(\tau) = e^{\beta\tau}$ with some $\beta \in \mathbb{R}$. System (1.1) is equivalent to

$$Dx_t = \phi^0 + \int_0^t Lx_s ds + \int_0^t f(s)ds, \quad t \geq 0 \quad \text{a.e.},$$

$$x(t) = \phi^1(t), \quad t < 0 \quad \text{a.e.}$$

Then $y(t)$ satisfies

$$D_\alpha y_t = e^{-\alpha t}\phi^0 + \int_0^t e^{-\alpha(t-s)}L_\alpha y_s dx + e^{-\alpha t}\int_0^t f(s)ds, \quad t \geq 0 \quad \text{a.e.},$$

$$y(t) = e^{-\alpha t}\phi^1(t), \quad t < 0 \quad \text{a.e.},$$

where

$$D_\alpha y_t = y(t) - \sum_{j=1}^{p} B_j e^{-\alpha h_j} y(t-h_j) - \int_{-\infty}^{0} B(\tau)e^{\alpha\tau}y(t+\tau)d\tau,$$

$$L_\alpha y_t = \sum_{j=0}^{p} A_j e^{-\alpha h_j} y(t-h_j) + \int_{-\infty}^{0} A(\tau)e^{\alpha\tau} y(t+\tau)d\tau .$$

Using the equation for y we get

$$\alpha \int_0^t D_\alpha y_s ds = \int_0^t \alpha e^{-\alpha s} ds \phi^0 + \int_0^t \alpha \int_0^s e^{-\alpha(s-\tau)} L_\alpha y_\tau d\tau ds$$

$$= \phi^0 - e^{-\alpha t}\phi^0 - \int_0^t e^{-\alpha(t-s)} L_\alpha y_s ds + \int_0^t L_\alpha y_s ds$$

$$- e^{-\alpha t} \int_0^t f(\tau)d\tau + \int_0^t e^{\alpha s} f(s)ds.$$

Therefore y satisfies the equation

$$D_\alpha y_t = \phi^0 + \int_0^t (L_\alpha y_s - \alpha D_\alpha y_s)ds + \int_0^t e^{-\alpha s} f(s)ds, \quad t \geq 0 \quad \text{a.e.},$$

$$y(t) = e^{-\alpha t}\phi^1(t), \quad t < 0 \quad \text{a.e.}$$

Easy computations show that $e^{\alpha\tau}A - \alpha e^{\alpha\tau}B$ and $e^{\alpha\tau}B$ are in $L^1_{\bar g}(-\infty,0;\mathbb{R}^n)$ and $e^{-\alpha t}\phi^1 \in L^2_{\bar g}(-\infty,0;\mathbb{R}^n)$ where $\bar g(\tau) = e^{(r+\alpha)\tau}$, $\tau \leq 0$.

Let $T^N(t) = e^{A^N t}$, $t \geq 0$. The main result of this paper is contained in

<u>Theorem [illegible].1.</u> Suppose that (A) is satisfied. Then

$$\lim_{N\to\infty} T^N(t)P^N\phi = T(t)\phi, \quad t \geq 0, \quad \phi \in Z,$$

the limit being uniform on bounded intervals.

The result immediately follows from the Trotter-Kato-Theorem as for instance contained in [14]. The assumptions of this theorem will be verified in the following subsections.

5.1. Stability of the scheme

In order to prove $|T^N(t)| \le Me^{\omega t}$, $t \ge 0$, $N = 1,2,\dots$, we show that

$$<A^N\phi,\phi>_{\hat{g}} \le \omega|\phi|^2_{\hat{g}}, \quad \phi \in Z^N,\ N = 1,2,\dots, \tag{5.1}$$

where ω is independent of N and $<\cdot,\cdot>_{\hat{g}}$ resp. $|\cdot|_{\hat{g}}$ is the inner product resp. corresponding norm on Z for a weighting function $\hat{g}$ such that $|\cdot|_{\hat{g}}$ and $|\cdot|_Z$ are equivalent, $|\phi|^2_{\hat{g}} = |\phi^0|^2 + \int_{-\infty}^{0} \hat{g}(\tau)|\phi^1(\tau)|^2 d\tau$. We choose

$$\hat{g}(\tau) = g(\tau) + \sum_{j=k}^{p} \lambda_j \quad \text{for} \quad \tau \in (-h_k,-h_{k-1}),$$

$k = 1,\dots,p+1$ $(h_{p+1} = \infty)$. It is convenient to put $\alpha_k = 1 + \sum_{j=k}^{p} \lambda_j$, $k = 1,\dots,p$. Since $(\phi^0, 1+\frac{\hat{g}-g}{g})\phi^1) \in Z^N$ for $\phi = (\phi^0,\phi^1) \in Z^N$ we immediately get

$$<\delta_j^N x,\phi^1>_{L^2_{\hat{g}}} = <\delta_j^N x,(1+\frac{\hat{g}-g}{g})\phi^1>_{L^2_g} = \alpha_j x^T\phi^1(-h_j-0),$$

$j = 0,\dots,p$. Using this formula we get for $\phi \in Z^N$

$$\begin{aligned}
<A^N\phi,\phi>_{\hat{g}} = &\ [A_0\phi^0 + \sum_{j=1}^{p}(A_j+A_0B_j)\phi^1(-h_j) \\
&+ \int_{-\infty}^{0}(A(\tau)+A_0B(\tau))\phi^1(\tau)d\tau]^T\phi^0 \\
&+ \int_{-\infty}^{0} \hat{g}(\tau)(\frac{d^+}{d\tau}\phi^1)^T\phi^1(\tau)d\tau \\
&+ \alpha_1(\phi^0-\phi^1(0) + \sum_{j=1}^{p} B_j\phi^1(-h_j) + \int_{-\infty}^{0} B(\tau)\phi^1(\tau)d\tau)^T\phi^1(0) \\
&+ \sum_{j=1}^{p}\alpha_{j+1}[\phi^1(-h_j)-\phi^1(-h_j-0)]^T\phi^1(-h_j-0) \\
= &\ I_1 + \dots + I_4.
\end{aligned}$$

Here and in the following $\alpha_{p+1} = 1$. We shall need the estimates

$$
\begin{aligned}
&\left|\int_{-\infty}^{0} (A(\tau)+A_0B(\tau))\phi^1(\tau)d\tau\right| \\
&\leq \int_{-\infty}^{0}|A(\tau)+A_0B(\tau)|\,\frac{1}{g(\tau)^{1/2}}\,g(\tau)^{1/2}|\phi^1(\tau)|d\tau \qquad (5.2)\\
&\leq |A + A_0B|_{L^2_{1/g}}\,|\phi|^1_{L^2_g} \leq |A + A_0B|_{L^2_{1/g}}\,|\phi^1|_{L^2_{\hat{g}}}
\end{aligned}
$$

and

$$
\left|\int_{-\infty}^{0} B(\tau)\phi^1(\tau)d\tau\right| \leq |B|_{L^2_{1/g}}\,|\phi^1|_{L^2_{\hat{g}}}. \qquad (5.3)
$$

Furthermore

$$
\begin{aligned}
&\int_{-\infty}^{0}\hat{g}(\tau)\left(\frac{d^+}{d\tau}\,\phi^1\right)^T(\tau)\phi^1(\tau)d\tau = \frac{1}{2}\cdot\int_{-\infty}^{0}\hat{g}(\tau)\,\frac{d^+}{d\tau}\,|\phi^1(\tau)|^2d\tau \\
&= \frac{1}{2}\int_{-\infty}^{-h} e^{\beta(h+\tau)}\,\frac{d}{d\tau}\,|\phi^1(\tau)|^2d\tau + \frac{1}{2}\sum_{j=1}^{p}\alpha_j\int_{-h_j}^{-h_{j-1}}\frac{d}{d\tau}|\phi^1(\tau)|^2d\tau \\
&= \frac{1}{2}|\phi^1(-h-0)|^2 - \frac{\beta}{2}\int_{-\infty}^{-h} e^{\beta(h+\tau)}|\phi^1(\tau)|^2d\tau \\
&+ \frac{1}{2}\sum_{j=1}^{p}\alpha_j[|\phi^1(-h_{j-1}-0)|^2 - |\phi^1(-h_j)|^2].
\end{aligned}
$$

From this we get

$$
\begin{aligned}
I_2 + I_4 = &-\frac{1}{2}\sum_{j=1}^{p}\alpha_{j+1}|\phi^1(-h_j) - \phi^1(-h_j-0)|^2 \\
&+ \frac{1}{2}\sum_{j=1}^{p}(\alpha_{j+1}-\alpha_j)|\phi^1(-h_j)|^2 \\
&+ \frac{1}{2}\sum_{j=1}^{p}\alpha_j|\phi^1(-h_{j-1}-0)|^2 - \frac{1}{2}\sum_{j=1}^{p}\alpha_{j+1}|\phi^1(-h_j-0)|^2 \\
&+ \frac{1}{2}|\phi^1(-h-0)|^2 - \frac{\beta}{2}\int_{-\infty}^{-h} e^{\beta(h+\tau)}|\phi^1(\tau)|^2d\tau
\end{aligned}
$$

$$= \frac{1}{2}\alpha_1|\phi^1(0)|^2 - \frac{1}{2}\sum_{j=1}^{p}\alpha_{j+1}|\phi^1(-h_j)-\phi^1(-h_j-0)|^2$$

$$-\frac{1}{2}\sum_{j=1}^{p}\lambda_j|\phi^1(-h_j)|^2 - \frac{\beta}{2}\int_{-\infty}^{-h} e^{\beta(h+\tau)}|\phi^1(\tau)|^2 d\tau.$$

For I_3 we get (also using (5.3))

$$I_3 = \frac{1}{2}\alpha_1[|\phi^0 + \sum_{j=1}^{p} B_j\phi^1(-h_j) + \int_{-\infty}^{0} B(\tau)\phi^1(\tau)d\tau|^2$$

$$-|\phi^0 - D\phi^1|^2 - |\phi^1(0)|^2]$$

$$\leq 2\alpha_1(1+|B|^2_{L^2_{1/g}})|\phi|^2_{\hat{g}} + \alpha_1|\sum_{j=1}^{p} B_j\phi^1(-h_j)|^2$$

$$-\frac{\alpha_1}{2}|\phi^0 - D\phi^1|^2 - \frac{\alpha_1}{2}|\phi^1(0)|^2.$$

Using (5.2) we obtain for I_1:

$$I_1 \leq |A_0||\phi^0|^2 + |A + A_0 B|_{L^2_{1/g}}|\phi^0||\phi^1|_{L^2_{\hat{g}}}$$

$$+\frac{1}{4\varepsilon}|\phi^0|^2\sum_{j=1}^{p}|A_j + A_0 B_j|^2 + \varepsilon\sum_{j=1}^{p}|\phi^1(-h_j)|^2.$$

By hypothesis (A) we can choose ε so small such that

$$\alpha_1|\sum_{j=1}^{p} B_j\phi^1(-h_j)|^2 - \frac{1}{2}\sum_{j=1}^{p}\lambda_j|\phi^1(-h_j)|^2 + \varepsilon\sum_{j=1}^{p}|\phi^1(-h_j)|^2 \leq C.$$

Alltogether we have

$$\langle A^N\phi,\phi\rangle_{\hat{g}}$$

$$\leq (|A_0| + |A+A_0B|_{L^2_{1/g}} + \frac{1}{4\varepsilon}\sum_{j=1}^{p}|A_j+A_0B_j|^2 + 2\alpha_1(1+|B|^2_{L^2_{1/g}}))|\phi|^2_{\hat{g}} \tag{5.4}$$

for all $\phi \in Z^N$.

The proof of (5.1) in case of (4.18) is quite analogous. Again one uses the weighting function $\hat{g}$ (instead of $g(\tau) = e^{\beta\tau}$, $\tau \leq 0$, in this case).

5.2. Consistency of the scheme

In this section we prove

$$p^N\phi \to \phi \quad \text{as} \quad N \to \infty \quad \text{for all} \quad \phi \in Z \tag{5.5}$$

and

$$A^N p^N\phi \to A\phi \quad \text{as} \quad N \to \infty \quad \text{for all} \quad \phi \in \mathcal{D} \tag{5.6}$$

where $\mathcal{D} \subset \operatorname{dom} A$ is such that $(\lambda I - A)\mathcal{D}$ is dense in Z for λ sufficiently large.

By Lemma 2.3, d) the sets $\tilde{D}_k$, $k = 1,2,\ldots$, satisfy $\tilde{D}_k \subset \operatorname{dom} \tilde{A}$ and are dense in Z. Since $(\lambda I - A)\tilde{D}_k = (\lambda I - \tilde{A})\tilde{D}_k$ and $\tilde{D}_k = \operatorname{dom} \tilde{A}^k$ we immediately get

$$(\lambda I - A)\tilde{D}_k = \tilde{D}_{k-1}.$$

Therefore the sets $\tilde{D}_k$, $k = 1,2,\ldots$, are appropriate candidates for $\mathcal{D}$.

Let $p^N\phi = (\phi^0, \phi^N)$ for $\phi = (\phi^0, \phi^1)$. Then using Proposition 2.2, (4.11), (4.12) and $\phi \in \operatorname{dom} A$ we get for $\phi \in \tilde{D}_k$

$$\begin{aligned}(A^N p^N\phi - A\phi)^0 = &\sum_{i=1}^{p} (A_i + A_0 B_i)(\phi^N(-h_i) - \phi^1(-h_i)) \\ &+ \int_{-\infty}^{0} (A(\tau) + A_0 B(\tau))(\phi^N(\tau) - \phi^1(\tau))\,d\tau,\end{aligned}$$

$$(A^N p^N \phi - A\phi)^1 = \frac{d^+}{d\tau}\phi^N - \dot{\phi}^1 + \delta_0^N(\phi^1(0) - \phi^N(0))$$
$$- \sum_{i=1}^{p} B_i(\phi^1(-h_i) - \phi^N(-h_i)) - \int_{-\infty}^{0} B(\tau)(\phi^1(\tau) - \phi^N(\tau))d\tau$$
$$+ \sum_{i=1}^{p} \delta_i^N(\phi^N(-h_i) - \phi^1(-h_i)).$$

Therefore by (4.8) and (4.9)

$$|A^N p^N \phi - A\phi|_g \le (|A + A_0 B|_{L^2_{1/g}} + \frac{N+1}{r_1^{1/2}}|B|_{L^2_{1/g}})|\phi^1 - \phi^N|_{L^2_g}$$
$$+ \frac{N+1}{r_1^{1/2}}|\phi^1(0) - \phi^N(0)|$$
$$+ \sum_{i=1}^{p-1}(|A_i + A_0 B_i| + \frac{N+1}{r_1^{1/2}}(|B_i| + 1))|\phi^1(-h_i) - \phi^N(-h_i)|$$
$$+ (|A_p + A_0 A_p| + \frac{N+1}{r_1^{1/2}}|B_p| + (\frac{N+1}{\beta})^{1/2})|\phi^1(-h) - \phi^N(-h)|$$
$$+ (N+1)\sum_{i=1}^{p-1}\frac{1}{r_{i+1}^{1/2}}|\phi^1(-h_i) - \phi^N(-h_i - 0)|$$
$$+ (\frac{N+1}{\beta})^{1/2}|\phi^1(-h) - \phi^N(-h-0)| + |\frac{d^+}{d\tau}(\phi^N - \dot{\phi}^1)|_{L^2_g}.$$

For a function $\psi \in L^2_g(-\infty,0;\mathbb{R}^n)$ we introduce $\psi_i = \psi|[-h_i,-h_{i-1}]$, $i = 1,\ldots,p$, and $\psi_{p+1} = \psi|(-\infty,-h)$. Let π_i^N, $i = 1,\ldots,p$, resp. π_{p+1}^N be the orthogonal projections $L^2(-h_i,-h_{i-1};\mathbb{R}^n) \to Y_i^N$ resp. $L^2_g(-\infty,-h;\mathbb{R}^n) \to Y_{p+1}^N$. Furthermore, we denote by σ^N resp. $\tilde{\sigma}^N$ the orthogonal projections $L^2(-1,1;\mathbb{R}^n) \to \mathrm{span}(P_0 I,\ldots,P_N I)$ resp. $L^2_w(0,\infty;\mathbb{R}^n) \to \mathrm{span}(L_0 I,\ldots,L_N I)$ (recall $w(t) = e^{-t}$).

Since $\pi_i^N \phi_i^1 = \phi_i^N$, $i = 1,\ldots,p+1$, we have to prove that as $N \to \infty$

$$N|\phi_i^1 - \pi_i^N \phi_i^1|_{L^2(-h_i,-h_{i-1};\mathbb{R}^n)} \to 0,\ i = 1,\ldots,p. \qquad (5.7)$$

$$N|\phi_{p+1}^1 - \pi_{p+1}^N \phi_{p+1}^1|_{L^2_g(-\infty,-h;\mathbb{R}^n)} \to 0, \qquad (5.8)$$

$$N|\phi_i^1(-h_i) - (\pi_{i+1}^N\phi_i^1)(-h_i-0)| \to 0, \quad i = 0,\ldots,p-1,$$
$$N|\phi_i^1(-h_i) - (\pi_i^N\phi_i^1)(-h_i)| \to 0, \; i = 1,\ldots,p, \tag{5.9}$$

$$N^{1/2}|\phi_{p+1}^1(-h) - (\pi_{p+1}^N\phi_{p+1}^1)(-h-0)| \to 0 \tag{5.10}$$

$$\left|\frac{d}{d\tau}(\phi_i^1 - \pi_i^N\phi_i^1)\right|_{L^2(-h_i,-h_{i-1};\mathbb{R}^n)} \to 0, \; i = 1,\ldots,p, \tag{5.11}$$

and

$$\left|\frac{d}{d\tau}(\phi_{p+1}^1 - \pi_{p+1}^N\phi_{p+1}^1)\right|_{L_g^2(-\infty,-h;\mathbb{R}^n)} \to 0 \tag{5.12}$$

for $\phi \in \tilde{D}_k$ in order to establish $|A^N P^N\phi - A\phi|_g \to 0$ as $N \to \infty$ for $\phi \in \tilde{D}_k$.

It is easy to see that

$$\pi_i^N = \Theta_i\sigma^N\Theta_i^{-1}, \quad i = 1,\ldots,p, \text{ and } \pi_{p+1}^N = \Theta_{p+1}\tilde{\sigma}^N\Theta_{p+1}^{-1}. \tag{5.13}$$

Moreover,

$$|\Theta_i| = \left(\frac{r_i}{2}\right)^{1/2}, \; i = 1,\ldots,p, \quad |\Theta_{p+1}| = \left(\frac{1}{\beta}\right)^{1/2},$$
$$|\Theta_i^{-1}| = \left(\frac{2}{r_i}\right)^{1/2}, \; i = 1,\ldots,p, \quad |\Theta_{p+1}^{-1}| = \beta^{1/2}. \tag{5.14}$$

For $\phi^1 \in L_g^2(-\infty,0;\mathbb{R}^n)$ we put $\chi_i = \Theta_i^{-1}\phi_i^1$, $i = 1,\ldots,p+1$.

<u>Lemma 5.2.</u> a) Let $\phi \in \tilde{D}_k$. Then $\chi_i \in W^{k,2}(-1,1;\mathbb{R}^n)$, $i = 1,\ldots,p$, and $\chi_i^{(j)} = \left(\frac{r_i}{2}\right)^j\Theta_i^{-1}((\phi^1)^{(j)})$, $j = 0,\ldots,k$. Moreover

$$(\phi_i^1 - \pi_i^N\phi_i^1)^{(j)} = \left(\frac{2}{r_i}\right)^j\Theta_i((\chi_i - \sigma^N\chi_i)^{(j)}), \; j = 0,\ldots,k.$$

b) Let $\phi \in \tilde{D}_{2k}$. Then $\chi_{p+1} \in \mathcal{B}_k$ and $\chi_{p+1}^{(j)} = \left(-\frac{1}{\beta}\right)^j\Theta_{p+1}^{-1}((\phi_{p+1}^1)^{(j)})$, $j = 0,\ldots,2k$. Moreover,

$$(\phi^1_{p+1} - \pi^N_{p+1}\phi^1_{p+1})^{(j)} = (-\beta)^j \Theta_{p+1}((x_{p+1} - \tilde{\sigma}^N x_{p+1})^{(j)}),$$

$j = 0,\ldots,2k$. If in addition $\phi \in \tilde{D}_{2k+1}$ then also $\dot{x}_{p+1} \in \mathcal{B}_k$.

c) Let $\phi \in Z$ be such that $\phi^1 \in C(-\infty,0;\mathbb{R}^n)$. Then $x_i \in C(-1,1;\mathbb{R}^n)$, $i = 1,\ldots,p$, $x_{p+1} \in C(0,\infty;\mathbb{R}^n)$ and

$$\phi^1_{i+1}(-h_i) - (\pi^N_{i+1}\phi^1_{i+1})(-h_i-0) = x_{i+1}(1) - (\sigma^N x_{i+1})(1),$$

$$i = 1,\ldots,p-1,$$

$$\phi^1_i(-h_i) - (\pi^N_i\phi^1_i)(-h_i+0) = x_i(-1) - (\sigma^N x_i)(-1),\ i = 1,\ldots,p,$$

$$\phi^1_{p+1}(-h) - (\pi^N_{p+1}\phi^1_{p+1})(-h-0) = x_{p+1}(0) - (\tilde{\sigma}^N x_{p+1})(0).$$

<u>Proof.</u> a) is an easy consequence of the linearity of the functions ζ_i and of (5.13). Similarly we get the formulas for the derivatives of x_{p+1} and of the error $\phi^1_{p+1} - \pi^N_{p+1}\phi^1_{p+1}$ under b). c) is trivial. It remains to prove $x_{p+1} \in \mathcal{B}_k$ for $\phi \in \tilde{D}_{2k}$ (and $\dot{x}_{p+1} \in \mathcal{B}_k$ for $\phi \in \tilde{D}_{2k+1}$). Using $\phi^1 \in C^{2k-1}(-\infty,0;\mathbb{R}^n)$, $(\phi^1)^{(2k-1)}$ locally absolutely continuous on $(-\infty,0]$, $(\phi^1_{p+1})^{(j)} \in L^2_g(-\infty,0;\mathbb{R}^n)$, $j = 0,\ldots,2k$, we get $x_{p+1} \in C^{2k-1}(0,\infty;\mathbb{R}^n)$, $x^{(2k-1)}_{p+1}$ locally absolutely continuous on $[0,\infty)$ and $x^{(j)}_{p+1} \in L^2_w(0,\infty;\mathbb{R}^n)$, $j = 0,\ldots,2k$. For $m = 0,1,2,\ldots$ and $j = 0,\ldots,2k$ we have $(\tau+h)^m(\phi^1_{p+1})^{(j)} \in L^2_g(-\infty,-h;\mathbb{R}^n)$ by Lemma 2.3,b). Therefore $t^m x^{(j)}_{p+1} = (-\beta)^\beta \Theta^{-1}_{p+1}((\tau+h)^m(\phi^1_{p+1})^{(j)}) \in L^2_w(0,\infty;\mathbb{R}^n)$. By Lemma 2.3,c) we have

$$\lim_{\tau\to-\infty} e^{\beta\tau/2}(\tau+h)^m(\phi^1_{p+1})^{(j)}(\tau) = 0$$

for $m = 0,1,2,\ldots$ and $j = 0,\ldots,2k-1$. This and $e^{-t/2}t^m x^{(j)}_{p+1}(t) = (-\beta)^{m-j}e^{\beta h/2}\Theta^{-1}_{p+1}(e^{\beta\tau/2}(\tau+h)^m(\phi^1_{p+1})^{(j)})(t)$ show that $\lim_{t\to\infty} e^{-t/2}t^m x^{(j)}_{p+1}(t) = 0$. This proves $x_{p+1} \in \mathcal{B}_k$. The proof for $\dot{x}_{p+1} \in \mathcal{B}_k$ in case $\phi \in \tilde{D}_{2k+1}$ is analogous ∎

Using Theorem 3.1, a) for $k = 2$ and Lemma 5.2, a) we see that (5.7) is satisfied if $\phi \in \tilde{D}_2$, the rate of convergence being $\frac{1}{N}$). Similarly we obtain (5.8) (by Theorem 3.3 with $k = 2$ and Lemma 5.2, b)) for $\phi \in \tilde{D}_4$ with rate $\frac{1}{N}$, (5.9) (by Theorem 3.1, c) with $k = 2$ and Lemma 5.2, a) and c)) for $\phi \in \tilde{D}_2$ with rate $\frac{1}{N^{1/2}}$, (5.10) (by Theorem 3.5, c) with $k = 1$ and Lemma 5.2, b) and c)) for $\phi \in \tilde{D}_3$ with rate $\frac{1}{N^{1/2}}$, (5.11) (by Theorem 3.1, b) with $k = 2$ and Lemma 5.2, a)) for $\phi \in \tilde{D}_2$ with rate $\frac{1}{N^{1/2}}$ and finally (5.12) (by Theorem 3.5, a) with $\ell = 1$, $k = 1$ and Lemma 5.2, b)) for $\phi \in \tilde{D}_3$ with rate $\frac{1}{N^{1/2}}$). Alltogether we have shown that

$$A^N p^N \phi - A\phi = O\left(\frac{1}{N^{1/2}}\right) \quad \text{for} \quad \phi \in \tilde{D}_4,$$

i.e. (5.6) is established with $\mathcal{D} = \tilde{D}_4$.

Condition (5.5) is an immediate consequence of Lemma 5.2 a) and b) and the completeness of the Legendre polynomials in $L^2(-1,1;\mathbb{R})$ resp. the Laguerre polynomials in $L^2_w(0,\infty;\mathbb{R})$.

Therefore all assumptions of the Trotter-Kato theorem in [14] are verified and the proof of Theorem 5.1 is finished.

Remark. If $B = 0$ then $\phi \in \tilde{D}_4$ can be replaced by $\phi \in \tilde{D}_3$, because in this case the factor N is not present in (5.8) (and in (5.7)) and therefore $\phi \in \tilde{D}_2$ is sufficient for (5.8). Of course, the smoothness requirements on ϕ can be relaxed if one uses interpolation spaces in order to get the estimates of Section 3 also for fractional k.

In case of the scheme given by (4.18) we get for $\phi \in \tilde{D}_k$

$$|A^N p^N \phi - A\phi|_g$$

$$\leq \left(|A+A_0 B|_{L^2_{1/g}} + \left(\frac{N+1}{\beta}\right)^{1/2} |B|_{L^2_{1/g}}\right) |\phi^1 - \phi^N|_{L^2_g} + |\dot{\phi}^1 - \dot{\phi}^N|_{L^2_g}$$

$$+ \left(\frac{N+1}{\beta}\right)^{1/2} |\phi^1(0) - \phi^N(0)| + \left(\frac{N+1}{\beta}\right)^{1/2} \sum_{i=1}^{p} |B_i| |\phi^1(-h_i) - \phi^N(-h_i)| + \sum_{j=1}^{p} |A_i + A_0 B_i| |\phi^1(-h_i) - \phi^N(-h_i)|.$$

Proceeding in a similar way as above we get

$$|A^N p^N \phi - A\phi|_g = O\left(\frac{1}{N^{1/2}}\right) \quad \text{for} \quad \phi \in \tilde{D}_5.$$

The reason for the stronger smoothness requirement in this case is that for terms like $N^{1/2}|\phi^1(-h_i) - \phi^N(-h_i)|$, $i = 1,\ldots,p$, we have to use part b) of Theorem 3.5 instead of part c). If $B_j = 0$, $i = 1,\ldots,p$, then we can replace $\tilde{D}_5$ by $\tilde{D}_3$.

5.3. Approximation of the nonhomogeneous problem

Since (1.1) is linear we only need to consider the case $\phi = 0$ and $f \neq 0$.

Proposition 5.3. Let $z^N(t)$ be the solution of (4.13) and let $z(t) = (y(t),x_t)$, $x(t)$, $y(t)$ being the solution of (1.1) with $\phi = 0$. Then

$$\lim_{N\to\infty} z^N(t) = z(t)$$

uniformly for $t \in [0,\bar{t}]$ and uniformly for f in bounded sets of $L^1(0,\bar{t};\mathbb{R}^n)$, $\bar{t} > 0$.

Proof. The proof is analogous to the proof of the corresponding theorem in [2], using the variation of constants formula (2.6).

5.4. A special case

The scheme presented in this paper has the remarkable property to give the exact solution in special cases. Consider (1.1) with $A_j = B_j = 0$, $j = 1,\ldots,p$, i.e. we have

$$D(x_t) = x(t) - \int_{-\infty}^{0} B(\tau)x(t+\tau)d\tau,$$
$$L(x_t) = A_0x(t) + \int_{-\infty}^{0} A(\tau)x(t+\tau)d\tau. \qquad (5.15)$$

In this case, the schemes defined by (4.11), (4.12) and by (4.18) coincide. Furthermore, the τ-method as described in [7] also yields the same scheme. We put

$$a(\tau) = A(\tau)e^{-\beta\tau}, \quad b(\tau) = B(\tau)e^{-\beta\tau}, \ \tau \leq 0. \qquad (5.16)$$

Note, that (2.1) is equivalent to $a,b \in L_g^2(-\infty,0;\mathbb{R}^{n\times n})$.

Proposition 5.4. In addition to (5.15) assume that

(i) a,b are polynomials of degree $\leq m$

and

(ii) ϕ^1 is a *polynomial of degree* $\leq m$.

Let

$$x^N(t) = w_{00}^N(t) + \sum_{j=0}^{N} B_{1j}w_j^N(t), \quad t \geq 0, \ N = 1,2,\ldots, \qquad (5.17)$$

where $w^N(t) = \mathrm{col}(w_{00}^N(t),w_0^N(t),\ldots,w_N^N(t))$ is the solution of (4.17). Then

$$x^N(t) = x(t), \quad t \geq 0, \quad N = m,m+1,\ldots .$$

Proof. Since for $j \geq m+1$ the polynomials e_j are orthogonal to the columns of a and b in $L_g^2(-\infty,0;\mathbb{R}^n)$ we have $\alpha_j = \beta_j = 0$ for $j \geq m+1$ in (4.19) and (4.20). Thus for any $N \geq m$ the $(m+1)n$-dimensional subspace of z^N spanned by $\hat{e}_{00},\hat{e}_0,\ldots,\hat{e}_m$ is invariant with respect to the system (4.17). Since by (ii) we have $(\phi^0,\phi^1) \in \mathrm{span}\{\hat{e}_{00},\hat{e}_0,\ldots,\hat{e}_m\}$, the coordinates $w_{00}^N(t),w_0^N(t),\ldots,w_m^N(t)$ of the solution $w^N(t)$ of (4.17) do not vary with N, $N \geq m$.

Let $\phi^N(t,\tau) = \sum_{j=0}^{N} e_j(\tau)w_j^N(t)$, $t \geq 0$, $\tau \leq 0$. Then according to Theorem 5.1 and Proposition 5.3

$$\lim_{N\to\infty} w_{00}^N(t) = y(t), \quad \lim_{N\to\infty} \phi^N(t) = x_t \quad \text{in} \quad L_g^2 \tag{5.18}$$

uniformly for t in bounded intervals. Using the definition of B_{1j} and (i) we see that

$$x^N(t) = w_{00}^N(t) + \int_{-\infty}^{0} B(\tau)\phi^N(t,\tau)d\tau = w_{00}^N(t) + \sum_{j=0}^{m} B_{1j}w_j^N(t), \tag{5.19}$$

$t \geq 0$, $N = m,m+1,\ldots$. This shows that

$$x^N(t) = x^m(t), \quad t \geq 0, \quad N \geq m. \tag{5.20}$$

From (5.18), (5.19) and (5.20) we obtain

$$x^m(t) = \lim_{N\to\infty} x^N(t) = y(t) + \int_{-\infty}^{0} B(\tau)x_t(\tau)d\tau = x(t)$$

uniformly for t in bounded intervals ∎

If assumptions (i) and (ii) of Proposition 5.4 are not satisfied we can give an estimate for $x(t) - x^N(t)$.

<u>Proposition 5.5.</u> Consider (1.1) with (5.15). Let π^N be the orthogonal projection $L_g^2(-\infty,0;\mathbb{R}^n) \to Y^N = \operatorname{span}(e_0,\ldots,e_N)$ and put

$$a^N = \pi^N a, \quad b^N = \pi^N b, \quad N = 1,2,\ldots .$$

Then for any $\bar{t} > 0$ there exists a constant c not dependent on N such that

$$|x(t) - x^N(t)| \leq c(|\phi^1 - \pi^N\phi^1| + |a-a^N|_{L_g^2} + |b - b^N|_{L_g^2})$$

for $0 \leq t \leq \bar{t}$, $N = 1,2,\ldots$, where $x^N(t)$ is given by (5.17).

<u>Proof.</u> Let $T_N(t)$, $t \geq 0$, be the solution semigroup generated by the solutions of (1.1) with $f \equiv 0$, $A_j = B_j = 0$, $j = 1,\ldots,p$, and A, B replaced by $e^{\beta\tau}a^N(\tau)$, $e^{\beta\tau}b^N(\tau)$, respectively. For the same equation with f arbitrary we denote the solutions corresponding to initial data $\phi = (\phi^0,\phi^1)$ and $p^N\phi = (\phi^0,\pi^N\phi^1)$ by $x_N(t)$, $y_N(t)$ and $\bar{x}_N(t)$, $\bar{y}_N(t)$, respectively. By Proposition 2.4

$$(y_N(t),(x_N)_t) = T_N(t)\phi + \int_0^t T_N(t-s)(f(s),0)ds, \quad t \geq 0,$$

$$(\bar{y}_N(t),(\bar{x}_N)_t) = T_N(t)p^N\phi + \int_0^t T_N(t-s)(f(s),0)ds, \quad t \geq 0.$$

Proposition 5.4 implies

$$x^N(t) = \bar{x}_N(t) \quad \text{for} \quad t \geq 0.$$

Using the second equation of (1.1) and $\bar{x}_N(t) = \bar{y}_N(t) + \int_{-\infty}^0 e^{\beta\tau}b^N(\tau)\bar{x}_N(t+\tau)d\tau$ we obtain

$$\begin{aligned}
x(t) - x^N(t) &= x(t) - \bar{x}_N(t) \\
&= y(t) - \bar{y}_N(t) + \int_{-\infty}^0 e^{\beta\tau}b(\tau)(x(t+\tau) - \bar{x}_N(t+\tau))d\tau \\
&\quad + \int_{-\infty}^0 e^{\beta\tau}(b(\tau) - b^N(\tau))\bar{x}_N(t+\tau)d\tau \\
&= y(t) - y_N(t) + \int_{-\infty}^0 e^{\beta\tau}b(\tau)(x(t+\tau) - x_N(t+\tau))d\tau \\
&\quad + y_N(t) - \bar{y}_N(t) + \int_{-\infty}^0 e^{\beta\tau}b(\tau)(x_N(t+\tau) - \bar{x}_N(t+\tau))d\tau \\
&\quad + \int_{-\infty}^0 e^{\beta\tau}(b(\tau) - b^N(\tau))\bar{x}_N(t+\tau)d\tau, \quad t \geq 0.
\end{aligned}$$

This implies

$$|x(t) - x^N(t)| \leq \sqrt{2}\,\max(1, |B|_{L^2_{1/g}})\{|(y(t), x_t) - (y_N(t), (x_N)_t)|_Z$$
$$+ |T_N(t)(\phi - p^N\phi)|_Z\} + |b - b^N|_{L^2_g} |(\bar{x}_N)_t|_{L^2_g} \qquad (5.21)$$

for $t \geq 0$, $N = 1,2,\ldots$. The dissipativity estimate (5.4) shows that there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$|T_N(t)| \leq Me^{\omega t}, \quad t \geq 0, \qquad (5.22)$$

for all N. This and the variation of constants formula imply

$$|(\bar{x}_N)_t|_{L^2_g} \leq Me^{\omega t}|\phi|_Z + M\int_0^{\bar{t}} e^{\omega(t-s)}|f(s)|ds \qquad (5.23)$$

for $0 \leq t \leq \bar{t}$ and all N. From Theorem 2.1, c) of [9] we immediately obtain

$$|(y(t), x_t) - (y_N(t), (x_N)_t|_Z \leq \tilde{c}(|a - a^N|_{L^2_g} + |b - b^N|_{L^2_g}),$$

$0 \leq t \leq \bar{t}$, where $\tilde{c}$ is not dependent on N. This together with (5.21) - (5.23) implies the result ∎

6. Numerical results

In this section we discuss some numerical examples which demonstrate the feasability of our scheme. All computations were performed on an IBM 3081 at Brown University using software written in FORTRAN. The integration of the system (4.17) of ordinary differential equations was carried out by an IMSL routine (DVERK) employing the Runge-Kutta-Verner fifth and sixth order method. The coefficients α_j and β_j in the matrix H^N in general were computed using Gauss quadrature formulae [5].

Example 1. This is the equation

$$\dot{x}(t) = x(t) - \int_{-\infty}^{0} (1-\sin\tau)e^{\tau}x(t+\tau)d\tau, \quad t \geq 0,$$

with initial conditions

$$x(0) = 1, \quad x(t) = 0 \quad \text{for} \quad t < 0.$$

Because of the special initial conditions the equation is equivalent to the Volterra integro-differential equation

$$\begin{aligned} &\dot{x}(t) = x(t) - \int_{0}^{t} (1+\sin(t-\tau)e^{-(t-\tau)}x(\tau)d\tau, \quad t \geq 0 \\ &x(0) = 1. \end{aligned} \tag{6.1}$$

Differentiating the equation in (6.1) we see that the solution to (6.1) also satisfies the ordinary differential equation $(D = \frac{d}{dt})$

$$(D^4 + 2D^3 + 2D^2 + D + 1)x(t) = 0, \quad t \geq 0,$$

$$x(0) = 1, \quad \dot{x}(0) = 1, \quad \ddot{x}(0) = 0, \quad \dddot{x}(0) = -1.$$

This equation was used in order to compute the exact solution to problem (6.1).

Since the kernel $A(\tau) = (1-\sin\tau)e^{\tau}$ is oscillatory, the Gauss-Laguerre quadrature formula has difficulties to yield accurate values for the α_j's in (4.19). However, doing the same computations as in the proof of Theorem 3.5, c) one can show that

$$I_k = \int_0^\infty e^{-t} \sin \omega t \, L_k(t)dt, \quad J_k = \int_0^\infty e^{-t} \cos \omega t \, L_k(t)dt,$$

$$k = 0,1,2...$$

satisfy the recursion

$$\begin{pmatrix} I_k \\ J_k \end{pmatrix} = \frac{\omega}{1+\omega^2} \begin{pmatrix} \omega & -1 \\ 1 & \omega \end{pmatrix} \begin{pmatrix} I_{k-1} \\ J_{k-1} \end{pmatrix}, \quad k = 1,2,...,$$

with $I_0 = \frac{\omega}{1+\omega^2}$, $J_0 = \frac{1}{1+\omega^2}$. Using this recursion the computation of the α_j's posed no difficulties. The numerical results are shown in Table 6.1. Note, that for our scheme $w_{00}^N(0) = \phi^0$ for all N. Therefore Table 6.1 does not contain values for $t = 0$.

t	$w_{00}^4(t)$	$w_{00}^8(t)$	$w_{00}^{16}(t)$	$w_{00}^{32}(t)$	$x(t)$
0.2	1.198396925	1.198724502	1.198671451	1.198669250	1.198669247
0.4	1.387642352	1.389698823	1.389419452	1.389413284	1.389413287
0.6	1.559759944	1.565189040	1.564592523	1.564588332	1.564588341
0.8	1.707896224	1.717913392	1.717066793	1.717070745	1.717070745
1.0	1.826226289	1.841347388	1.840439101	1.840451863	1.840451850
1.2	1.909918666	1.929911297	1.929179071	1.929195874	1.929195861
1.4	1.955135808	1.979088201	1.978751749	1.978765590	1.978765589
1.6	1.959053190	1.985499876	1.985714422	1.985719468	1.985718482
1.8	1.919884567	1.946954731	1.947788385	1.947782026	1.947782047
2.0	1.836904312	1.862474344	1.863905575	1.863888825	1.863888839
CPU (sec)	0.018	0.029	0.054	0.126	-

Table 6.1

For this example the assumptions of Proposition 5.5 are satisfied. We have $w_{00}^N(t) = x^N(t)$, $t \geq 0$, because $b = 0$. Observing $\phi^1 = $ we obtain from Proposition 5.5 the estimate

$$|x(t) - w_{00}^N(t)| \leq \text{const.}|a - a^N|_{L_g^2}, \quad 0 \leq t \leq \bar{t}.$$

It is easy to see that $a(-\tau) \in B_k$ for all $k = 1,2,...$ (note that $\beta = 1$). Therefore according to Theorem 3.3 $|a - a^N|_{L_g^2} \leq \frac{\text{const}}{(N+1)^k}$ for all $k = 1.2,...$ (of course with const. depending on k) which means infinite order convergence of $w_{00}^N(t) \to x(t)$ uniformly on compact intervals. This is reflected by Table 6.2 where we show $\Delta^N = \max_{i=0,...,10} |x(0.2i) - w_{00}^N(0.2i)|$ for $N = 4,8,16,32$.

N	Δ^N
4	0.027897480
8	0.001414495
16	0.000016790
32	0.000000021

Table 6.2

The next two examples have their origin in the dynamics of structured populations (see [13]). Let x be the size of individuals in the population, $0 \leq x \leq S$, S being the maximal size. Then a simple model for the evolution of the population density $u(t,x)$ is given by

$$u_t(t,x) + (g(x)u(t,x))_x = -\mu_0 u(t,x), \quad t \geq 0, \ 0 \leq x \leq S,$$
$$u(t,0) = \int_0^S q(x)u(t,x)dx, \quad t > 0,$$
$$u(0,x) = \phi(x), \quad 0 \leq x \leq S.$$

Here $\mu_0 \geq 0$ is a mortality rate (assumed to be constant), g is a growth rate (assumed to be positive on $[0,S)$; for an individual the size changes according to $\frac{dx}{dt} = g(x)$), q is a fecundity rate (assumed to be nonnegative and essentially bounded) and ϕ is the initial size distribution of the population.

By the method of characteristics one can show that the birth rate $\beta(t) = u(t,0)$ satisfies the Volterra equation

$$\beta(t) = \int_0^t a(t-\xi)\beta(\xi)d\xi + h(t), \quad t \geq 0, \qquad (6.2)$$

where

$$a(\xi) = g(0)q(G^{-1}(\xi))e^{-\mu_0\xi}, \quad \xi \geq 0,$$
$$h(t) = e^{-\mu_0 t}\int_0^S q(G^{-1}(G(\xi)+t))\phi(\xi)d\xi, \quad t \geq 0, \qquad (6.3)$$

with

$$G(\xi) = \int_0^\xi \frac{d\sigma}{g(\sigma)}.$$

Assuming that h is locally absolutely continuous on $t \geq 0$ equation (6.2) is of type (1.5).

Example 2. This is equation (6.2) with $\mu_0 = 0.15$, $g(x) = b(S-x)$, $b = 0.0075$, $S = 60$ and

$$q(\tau) = \frac{27}{4S^2}(-\tau^3 + S\tau^2), \quad 0 \leq \tau \leq 60.$$

q satisfies $q(0) = q(S) = q'(0) = q'(\frac{2}{3}S) = 0$, $q(\frac{2}{3}S) = 1$. For this example

$$a(\xi) = \frac{27}{4} bS(1 - e^{-b\xi})^2 e^{-(\mu_0+b)\xi}, \quad \xi \geq 0,$$

and

$$h(t) = \frac{27}{4} e^{-(\mu_0+b)t} \left\{ \int_0^S (1-\frac{\xi}{S})\phi(\xi)d\xi - 2e^{-bt} \int_0^S (1-\frac{\xi}{S})^2 \phi(\xi)d\xi \right.$$
$$\left. + e^{-2bt} \int_0^S (1-\frac{\xi}{S})^3 \phi(\xi)d\xi \right\}, \quad t \geq 0.$$

Since $a(\xi)$ is a linear combination of $e^{-\lambda_j \xi}$, $\lambda_j = \mu_0 + jb$, $j = 1,2,3$, $\beta(t)$ is also solution of an ordinary differential equation $(D = \frac{d}{dt})$:

$$\begin{aligned} &(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)\beta(t) = \frac{27}{2} b^3 S\beta(t), \\ &\beta^{(i)}(0) = f^{(i)}(0), \quad i = 0,1,2. \end{aligned} \tag{6.4}$$

The numerical computations were carried out for $\phi \equiv 1$ on $[0,S]$ and the results are listed in Table 6.3 ($\beta(t)$ was obtained by solving (6.4))

t	$w_{00}^{4}(t)$	$w_{00}^{8}(t)$	$w_{00}^{16}(t)$	$\beta(t)$
1	29.26615192	29.26917027	29.26917041	29.26917041
2	25.38795761	25.39133074	25.39133081	25.39133081
3	22.03843175	22.04086481	22.04086408	22.04086407
4	19.14858020	19.14960995	19.14960983	19.14960983
5	16.65708019	16.65672030	16.65672016	16.65672016
6	14.50980845	14.50831247	14.50831234	14.50831234
7	12.65928231	12.65699712	12.65699703	12.65699703
8	9.68813725	11.06133885	11.06133881	11.06133881
9	8.50035095	9.68528995	9.68528996	9.68528996
10	8.500350949	8.497625723	8.497625775	8.497625775
CPU (sec)	0.021	0.039	0.087	

Table 6.3

With respect to the rate of convergence $w_{00}^{N}(t) \to x(t)$ the same remark are in order as for Example 1.

<u>Example 5.</u> This is (6.2) with $\mu_0 = 0.15$, $g(x) = \frac{b}{S}(S-x)^2$, $b = 0.0075$, $S = 60$. The function q is the same as for Example 3. Simple computations lead to

$$G(x) = \frac{x}{b(S-x)}\ , \quad 0 \le x < S,$$

$$G^{-1}(y) = \frac{bSy}{1+by}\ , \ 0 \le y < \infty,$$

and

$$a(\xi) = \frac{27}{4}\, b^3 S^2 \, \frac{\xi^2}{(1+b\xi)^3}\, e^{-\mu_0 \xi}, \quad \xi \ge 0,$$

$$h(t) = \frac{27S}{4} \int_0^S (S-\xi)\, \frac{[\xi+bt(S-\xi)]^2}{[S+bt(S-\xi)]^3}\, \phi(\xi)d\xi, \ t \ge 0.$$

The Laplace-transform of $a(\xi)$ is no more rational. Therefore the solution $\beta(t)$ of (6.2) in this case does not satisfy an ordinary differential equation. The results of our numerical computations are shown in Table 6.4.

t	$w_{00}^{4}(t)$	$w_{00}^{8}(t)$	$w_{00}^{16}(t)$
1	13.14464011	13.15568691	13.15574424
2	11.40999001	11.42071042	11.42072946
3	9.95006974	9.95575214	9.95572216
4	8.72995239	8.72962527	8.72956630
5	7.71550920	7.71008529	7.71002209
6	6.87480285	6.86594795	6.86589901
7	6.17887223	6.16835637	6.16833122
8	5.60209160	5.59145262	5.59145294
9	5.12224059	5.11264292	5.11266498
10	4.72038561	4.71259466	4.71263182
CPU (sec)	0.054	0.075	0.129

Table 6.4

When the scheme determined by (4.18) is applied to equations satisfying (5.15) the only quantities to be stored are α_j and β_j,

$j = 0,\dots,N$. The approximately linear growth of CPU time observed in the examples indicates that the matrix $[A^N]$ is not stiff.

The next two examples show the advantage (as far as rate of convergence is concerned) using the scheme defined by (4.11), (4.12) over the one defined by (4.18) when (1.1) involves also point delays (see also the remark at the end of Section 5.2).

Example 4. (Example 2.9 in [10]). This is the retarded problem

$$\dot{x}(t) = x(t-1) + 2\int_{-\infty}^{0} se^{s}x(t+s)ds, \quad t \geq 0,$$

$$x(0) = 0, \quad x(t) = -t, \quad t < 0.$$

The exact solution is given by

$$x(t) = \psi(t) = -\frac{1}{4} - \frac{1}{2}t - 2\sin t + \frac{1}{4}e^{-2t}, \quad 0 \leq t \leq 1,$$

$$x(t) = \psi(t) + \frac{1}{4} + \frac{1}{4}(t-1) - \frac{4}{25}\cos(t-1) - \frac{22}{25}\sin(t-1)$$
$$+ \frac{2}{5}(t-1)\cos(t-1) - \frac{4}{5}(t-1)\sin(t-1) - \frac{9}{100}e^{-2(t-1)}$$
$$+ \frac{1}{20}(t-1)e^{-2(t-1)}, \quad 1 < t \leq 2.$$

For this example we used the Legendre-Laguerre scheme and the Laguerre scheme. In the first case we kept $Y_2^N = \mathrm{span}(\hat{e}_{2,0},\hat{e}_{2,1})$ and only increased the dimension of the "Legendre" subspace $Y_1^N = \mathrm{span}(\hat{e}_{1,0},\dots,\hat{e}_{1,N})$, $N = 4,8,16,32$. The results are listed in Table 6.5. As for the other examples Δ is the maximum of the errors at the meshpoints. A comparison of the results for the Legendre-Laguerre scheme and the Laguerre-scheme supports the remark at the end of Section 5.2.

t	$w_{00}^{4,2}(t)$	$w_{00}^{8,2}(t)$	$w_{00}^{16,2}(t)$	$w_{00}^{32,2}(t)$	$w_{00}^{32,Lag}(t)$	$x(t)$
0.2	[illegible]	-0.5799286	-0.5797359	-0.5797637	-0.5792973	-0.5797507
0.4	[illegible]	-1.1166179	-1.1165455	-1.1165130	-1.1181307	-1.1165045
0.6	[illegible]	-1.6034630	-1.6039095	-1.6039821	-1.6043941	-1.6039864
0.8	[illegible]	-2.0350244	-2.0340916	-2.0342144	-2.0284913	-2.0342382
1.0	-2.3977773	-2.3987919	-2.3990503	-2.3990977	-2.3999649	-2.3991082
1.2	-2.7322264	-2.7298785	-2.7300773	-2.7300496	-2.7379325	-2.7300458
1.4	-3.0538812	-3.0545163	-3.0544578	-3.0544525	-3.0559645	-3.0544521
1.6	-3.3585509	-3.3588866	-3.3589146	-3.3589151	-3.3547827	-3.3589153
1.8	-3.6301758	-3.6298989	-3.6298647	-3.6298672	-3.6242206	-3.6298674
2.0	-3.8530714	-3.8533666	-3.8534202	-3.8534271	-3.8500185	-3.8534283
CPU (sec)	0.06	0.14	0.35	1.06	0.10	-
Δ	0.0027073	0.0007862	0.0001366	0.0000238	0.0078867	

Table 6.5

Example 5 (Example 2.11 in [10]). This is the neutral type equation

$$\frac{d}{dt}\,(x(t) + 4\int_{-\infty}^{0} se^{s}x(t+s)ds) = x(t-1),\ t \geq 0,$$

$$x(0) = -4,\ x(t) = 1,\quad t < 0.$$

The true solution is given by

$$x(t) = \psi(t) = \frac{4}{9} - \frac{1}{3}t - \frac{4}{9}e^{-3t},\quad 0 \leq t \leq 1,$$

$$x(t) = \psi(t) + \frac{8}{9} + \frac{13}{27}(t-1) + \frac{1}{18}(t-1)^2 - e^{t-1}$$

$$+ \frac{1}{9}\,e^{-3(t-1)} - \frac{4}{27}(t-1)e^{-3(t-1)},\quad 1 < t \leq 2.$$

The computations for this example were done in the same way as for the previous example. With respect to a comparison between the Legendre-Laguerre-scheme and the Laguerre-scheme the same remarks are in order. The CPU-times and the errors are larger for this example because the equation is of neutral type.

t	$w_{00}^{4,2}(t)$	$w_{00}^{8,2}(t)$	$w_{00}^{16,2}(t)$	$w_{00}^{32,2}(t)$	$w_{00}^{32,\mathrm{Lag}}(t)$	$x(t)$
0.2	0.134278	0.136751	0.133327	0.134192	0.132697	0.133861
0.4	0.165194	0.176987	0.178561	0.177474	0.172196	0.177247
0.6	0.182636	0.168347	0.170816	0.171309	0.182619	0.170978
0.8	0.155815	0.142509	0.136379	0.137287	0.147063	0.137459
1.0	0.043118	0.066564	0.078274	0.083763	0.037833	0.088984
1.2	-0.153099	-0.153280	-0.157691	-0.157101	-0.148588	-0.156977
1.4	-0.409285	-0.416203	-0.414921	-0.414745	-0.400062	-0.414724
1.6	-0.712374	-0.712604	-0.713229	-0.713212	-0.707170	-0.713214
1.8	-1.072318	-1.074507	-1.074089	-1.074137	-1.074970	-1.074146
2.0	-1.517778	-1.517415	-1.517493	-1.517519	-1.525295	-1.517524
CPU (sec)	0.06	0.16	0.51	1.76	0.12	-
Δ	0.045866	0.022420	0.010710	0.005221	0.051151	-

Table 6.6

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