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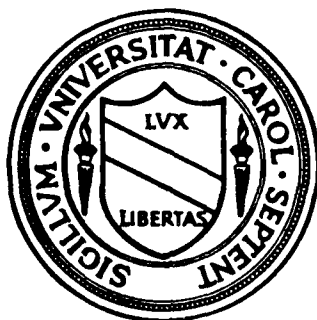
EXTREME VALUES OF QUEUE LENGTHS IN M/G/1 AND GI/M/1 SYSTEMS

by

Richard F. Serfozo

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EXTREME VALUES OF QUEUE LENGTHS
IN M/G/1 and GI/M/1 SYSTEMS

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ABSTRACT

We study the limiting behavior of maximum queue lengths in the M/G/1
and GI/M/1 service systems. When the systems are positive recurrent, the
distributions of their maximum queue lengths, under standard linear nor-
malizations, either do not converge or they converge to degenerate limits.
Consequently, one cannot use classical extreme value theory to characterize
their limiting behavior. We show, however, that by varying the system para-
meters in a certain way as the time interval grows, these maxima do indeed have
three possible limit distributions. Two of them are classical extreme value
distributions and the third one is a new distribution. The latter distribu-
tion is the best one for practical approximations.

1. Introduction

The extreme values of a stochastic process rather than its usual values are often of paramount interest. In a manufacturing plant, ᴌor example, a typical concern is the queue length of parts waiting to be processed at a work station. Small to moderate values of the queue may indicate that the system is operating successfully and fluctuations in the queue are unimportant. On the other hand, large queues may call for extraordinary measures such as the allocation of auxiliary storage space, employee overtime, or rescheduling of production. A natural question is: What is the probability that the queue will exceed a specific critical value in a certain time period? Such questions are the focus of this study. The following paragraphs describe the gist of our results.

Consider an M/ᴊ/1 queueing system in which customers arrive at a single server according to a Poisson process with rate λ and the independent identically distributed service times have a mean μ^{-1}. The queue-length process is positive recurrent, null recurrent, or transient according to whether the traffic intensity $\rho = \lambda/\mu$ is below, equal, or greater than 1, respectively. Let M_n denote the maximum queue length up to the nth time the system becomes empty. We say that M_n <u>has the limit distribution</u> F if there are norming constants $a_n, b_n > 0$ such that

$$\lim_{n \to \infty} P((M_n - a_n)/b_n \leq x) = F(x),$$

for each continuity point of F. If there are no such F, a_n, b_n, we say that M_n <u>does not have a limit distribution.</u>

From the classical extreme value theory, as in Galambos (1978) or Leadbetter et al. (1983), we have the following results:

(1.1) If $\rho = 1$, then $\lim\limits_{n \to \infty} P(M_n/n \le x) = e^{-x^{-1}}$, $x \ge 0$.

(1.2) If $\rho < 1$, then M_n does not have a limit distribution.

Cohen (1969) proved (1.1) and an analogue for GI/M/1 queues as well. Related results appear in Iglehart (1972) for GI/G/1 queues and in Serfozo (1986) for birth and death processes. Other insights on extremes of queues appear in Heathcote (1965) and Heyde (1971). Anderson (1970) observed the anomaly (1.2) and gave bounds on the distribution of M_n. The result (1.2) is due to the property that if M_n is the maximum of n independent integer-valued random variables with distribution F, then M_n does not have a limit distribution when $[F(m) - F(m-1)]/[1 - F(m-1)]$ does not converge to 1; see for instance Theorem 1.7.13 of Leadbetter et al. (1983). The results (1.1) and (1.2) say that the maxima of null recurrent queues have a limit while the maxima of positive recurrent queues do not. This is the reverse of what one would anticipate.

In this paper, we show that maxima of positive recurrent queues are more well behaved than (1.2) suggests. The key idea is that by allowing the basic parameters of the queueing process to vary with n in a certain manner, then their maxima do indeed have the three limiting distributions shown in (2.1). The first and third distributions are the classical Frechet and Gumbel extreme-value distributions, and the other distribution is a new one. We have shown in Serfozo (1986), that the maxima of positive recurrent birth and death processes and M/M/s queues also have these limit distributions. The present analysis differs in that it involves subtle arguments dealing with the convergence of integrals of complex-valued functions. In Section 2 we discuss M/G/1 queues and in Section 3 we discuss GI/M/1 queues.

2. Results for M/G/1 Queues

Property (1.1) encouraged us to investigate the maximum M_n of positive recurrent M/G/1 queues when the arrival rate λ and service distribution G vary with n such that the traffic intensity is nearly 1. This is the basis of the following results.

Consider a sequence of positive recurrent M/G/1 queues indexed by $n = 1, 2, \ldots,$ where the nth process has arrival rate λ_n and service distribution G_n with mean μ_n^{-1}. For the nth process, let M_n denote the maximum queue length up to the nth time the system becomes empty. With no loss in generality, assume that the system is empty at time 0. We shall study the convergence in distribution of $(M_n - a_n)/b_n$ as the traffic intensity $\rho_n = \lambda_n/\mu_n$ tends to 1. Although, our limiting results are for large ρ_n, the limits yield good approximations for the distribution of M_n for small as well as large ρ_n.

We assume that the Laplace transform

$$\gamma_n(z) = \int_0^\infty e^{-zt} dG_n(t),$$

for z complex, is analytic on the region $\text{Re} z > -\bar{x}_n$, where $-\infty \leq -\bar{x}_n < 0$, and that $\gamma_n(x) \to \infty$ for real $x \downarrow -\bar{x}_n$. This assumption is equivalent to $\gamma_n(x)$ being finite in some real interval $|x| < \varepsilon$ and

$$-\bar{x}_n = \inf \{x \leq 0: \gamma_n(x) < \infty\}.$$

One consequence is that all moments of G_n exist. We define

$$\phi_n(z) = \gamma_n(\lambda_n(1-z)) - z, \text{ for } \text{Re} z < 1 + \bar{x}_n/\lambda_n.$$

Lemma 2.2 says that ϕ_n has exactly one zero on the real interval $(1, 1+\bar{x}_n/\lambda_n)$. We let r_n denote this zero. Let G_0 be another such distribution with $\bar{x}_0$, ϕ_0

and r_0 defined similarly.

We will show, under some natural conditions, that if $n(r_n-1) \to c$, then M_n has the limit distribution F_c, where

(2.1)
$$F_0(x) = \exp(-x^{-1}) \qquad \text{when } c = 0$$
$$F_c(x) = \exp[-c/(e^x-1)] \quad x \geq 0, \quad \text{when } 0 < c < \infty$$
$$F_\infty(x) = \exp(-e^{-x}) \qquad \text{when } c = \infty.$$

In particular, we will show that

(2.2)
$$\lim_{n \to \infty} P((M_n-a_n)/b_n \leq x) = F_c(x) \qquad x \in R,$$

where

(2.3)
$$a_n = \begin{cases} \alpha_n \\ \alpha_n \\ \alpha_n + \beta_n \log n(r_n-1) \end{cases} \qquad b_n = \begin{cases} n & c = 0 \\ \beta_n & 0 < c < \infty \\ \beta_n & c = \infty. \end{cases}$$

and

$$\alpha_n = \beta_n \log[(1-\rho_n)/\phi_n'(r_n)] \qquad \beta_n = 1/\log r_n.$$

THEOREM 2.1. Suppose that

(2.4)
$$\lim_{n \to \infty} G_n(x/\lambda_n) = G_0(x)$$

for each G_0 continuity point, $r_0 = 1$, $\varliminf_{n \to \infty} \alpha_n \neq -\infty$, and when $n(r_n - 1) \to 0$

(2.5)
$$\lim_{n \to \infty} (1-\rho_n)/[n(r_n-1)(1+\epsilon)^n] = 0 \qquad \text{for each } \epsilon > 0.$$

Then the possible limit distributions for M_n are F_c, $0 \leq c \leq \infty$. The M_n has limit distribution F_c if and only if $n(r_n-1) \to c$. Appropriate norming constants for the convergence (2.2) are given by (2.3).

REMARKS 2.2 (i) The distributions F_c, $0 \leq c \leq \infty$ are of distinct type: F_c and $F_{c'}$ are of the same type if and only if $c=c'$.

(ii) For any positive recurrent M/G/1 queue, without the artificial dependence on n, Theorem 2.1 yields the approximation

$$(2.6) \qquad P(M_n \log r - \log[(1-\rho)/\phi'(r)] \leq x) \cong \exp[-n(r-1)/(e^x-1)], \quad x \geq 0.$$

There are analogous approximations of the distribution of M_n by F_0 and F_∞ when $n(r-1)$ is very small or very large. However (2.6) is better since the actual value of $n(r-1)$ appears on the right whereas it doesn't for the F_0 and F_∞ approximations, which represent degenerate cases. Furthermore, since $n(r-1)$ appears on the right, (2.6) is accurate when r (or ρ) is not near 1. For the M/M/1 queue, we found that the difference between the two sides of (2.6) is below 0.018 when $n \geq 15$ and below 0.01 when $n \geq 20$ for any ρ in (0,1).

(iii) Assumption (2.4) says that $G_n(\cdot/\lambda_n)$ converges weakly to G_0. This implies that $\gamma_n(\lambda_n z) \to \gamma_0(z)$ and $\phi_n(z) \to \phi_0(z)$ uniformly on compact sets. We use the latter and $r_0 = 1$ to ensure that: (a) $r_n \to 1$, and (b) there is an $\eta > 1$ and N such that $\phi_n(z)$ for $n \geq N$ has no zeros on the annulus $1 < |z| < \eta$ other than at r_n. Theorem 2.1 is true with (2.4) and $r_0 = 1$ replaced by conditions (a) and (b).

(iv) We use assumption $\varliminf_{n \to \infty} \alpha_n \neq -\infty$ along with $r_n \to 1$ to ensure that $\alpha_n + b_n x \to \infty$. Assumption (2.5) is only for the degenerate case $n(r_n-1) \to 0$. These two assumptions are automatically satisfied in natural cases such as those in Corollary 2.4.

To prove Theorem 2.1, we use the following properties of ϕ_n; we suppress the n here for convenience.

LEMMA 2.3. (i) The function $\phi(x)$, for real x, is strictly convex on the interval $(0, 1+\bar{x}/\lambda)$ and

$$(2.7) \qquad \phi(x) \to \infty \quad \text{as } x \uparrow 1 + \bar{x}/\lambda.$$

Furthermore, $\phi(x)$ has exactly two zeros on this interval at 1 and at some $r > 1$.

(ii) The only zeros of $\phi(z)$ on the circles $|z| = 1$ and $|z| = r$ are at 1 and r, respectively, and these zeros are of order 1.

(iii) $\phi(z)$ has no zeros on the annulus $1 < |z| < r$.

Proof. (i) The second derivative of $\phi(x)$ exists and is positive, and so ϕ is strictly convex. Statement (2.7) is obvious when $\bar{x} < \infty$, and, when $\bar{x} = \infty$, it follows by the strict convexity of ϕ and

$$(2.8) \qquad \lim_{x \to 1+\bar{x}/\lambda} \phi'(x) = \lim_{y \to \infty} \int_0^\infty ty e^{ty} dG(t) - 1 = \infty.$$

Clearly $\phi(1) = 0$ and so the strict convexity and (2.7) ensure that there is exactly one other zero of ϕ at some point r in the interval $(1, 1+\bar{x}/\lambda)$.

(ii) Since $\gamma(\lambda(1-x))$ is strictly increasing and convex with fixed points at $x = 1$ and $x = r$ (the zeros of $\phi(x)$), then

$$\gamma(\lambda(1-x)) < 1 \qquad \text{when } 0 < x < 1, \text{ and}$$
$$\gamma(\lambda(1-x)) < r \qquad \text{when } 0 \leq x < r.$$

Then on the circle $|z| = 1$, the only zero of ϕ is at 1, since

$$|\phi(z)| \geq |z| - |\gamma(\lambda(1-z))|$$
$$\geq 1 - \gamma(\lambda(1-\text{Re}z)) > 0 \qquad \text{when } z \neq 1 \text{ (Re}z < 1).$$

Similarly, on the circle $|z| = r$, the only zero of ϕ is at r, since

$$|\phi(z)| \geq r - \gamma(\lambda(1-\text{Re}z)) > 0 \qquad \text{when } z \neq r \text{ (Re}z < r).$$

The zeros of ϕ at 1 and r are of order one since $\phi'(1) = \rho-1 < 0$, and $\phi'(r) > 0$ by strict convexity.

(iii) By Rouche's theorem and

$$|\phi(z) - z| = |\gamma(\lambda(1-z))| \leq \gamma(\lambda(1-|z|)) = \phi(|z|) + |z| < |z| \quad \text{when } 1 < |z| < r,$$

we know that $\phi(z)$ and z have the same number of zeros in $1 < |z| < r$. Thus ϕ has no zeros there.

Proof of Theorem 2.1. We shall first prove that $n(r_n-1) \to c$ implies (2.2). Associated with the nth M/G/1 process, let F_n denote the distribution of the maximum queue length in one busy period (the period between two successive entrances to state 0). From Takacs (1965) and Cohen (1969), we know that

$$(2.9) \qquad F_n(m) = 1 - (2\pi i)^{-1} \int_{C^1} (1-z) z^{-m} \phi_n(z)^{-1} dz / (2\pi i)^{-1} \int_{C^1} z^{-m} \phi_n(z)^{-1} dz,$$

where C^1 is any circle in the z-plane with center at the origin and with radius less than 1, and the complex integrals are over C^1 in the counter-clockwise direction.

Because the queueing process regenerates each time the system becomes empty, then M_n is the maximum of n independent random variables with distribution F_n. Consequently,

$$P((M_n-a_n)/b_n \leq x) = F_n(a_n + b_n x)^n.$$

It is well known that $(1 + \eta_n)^n \to e^\eta$ for any real numbers η_n satisfying $n\eta_n \to \eta$ where $-\infty \leq \eta \leq \infty$. Then to prove (2.2), it suffices to show that

$$(2.10) \qquad \lim_{n\to\infty} n[1 - F_n(a_n + b_n x)] = \begin{cases} x^{-1} & \text{when } c = 0 \\ c/(e^x-1) & 0 < c < \infty \\ e^{-x} & c = \infty. \end{cases}$$

To this end, consider the integrals in (2.9). We shall express them in a more convenient form. Choose circles C_n^1 and C_n in the z-plane with centers

at the origin and radii less than 1 and greater than r_n, respectively, such that ϕ_n has no zeros other than 1 and r_n on the annulus between C_n^1 and C_n. This is possible because of the nature of ϕ_n; see Lemma 2.3. For convenience, we temporarily suppress the subscript n on ϕ_n, C_n^1, C_n, r_n, ρ_n.

From Lemma 2.3, we know that the function $z^{-m}\phi(z)^{-1}$ has poles of order one at 1 and r and is analytic at all other z in the annulus between C^1 and C. The residues of this function at these poles are

$$\lim_{z \to 1} (z-1)z^{-m}\phi(z)^{-1} = 1/\phi'(1) = -1/(1-\rho)$$

$$\lim_{z \to r} (z-r)z^{-m}\phi(z)^{-1} = 1/(r^m\phi'(r)).$$

Then by the residue theorem for complex integration, it follows that

$$(2.11) \qquad (2\pi i)^{-1} \int_C z^{-m}\phi(z)^{-1}dz - (2\pi i)^{-1} \int_{C^1} z^{-m}\phi(z)^{-1}dz$$

$$= -1/(1-\rho) + 1/(r^m\phi'(r)) \quad .$$

Similarly,

$$(2.12) \qquad (2\pi i)^{-1} \int_C (1-z)z^{-m}\phi(z)^{-1}dz - (2\pi i)^{-1} \int_{C^1} (1-z)z^{-m}\phi(z)^{-1}dz$$

$$= (1-r)/(r^m\phi'(r)),$$

where the last term is the residue of the integrand of these integrals at r; the residue at 1 is 0.

Combining (2.9), (2.11), (2.12) and letting m_n denote the integer part of $a_n + b_n x$, we have

$$n[1-F_n(a_n+b_n x)] = [n(1-r)/(r^{m_n}\phi'(r)) - n(2\pi i)^{-1}\int_C (1-z)z^{-m_n}\phi(z)^{-1}dz]$$

$$/[-1/(1-\rho) + r^{-m_n}\phi'(r)^{-1} - (2\pi i)^{-1}\int_C z^{-m_n}\phi(z)^{-1}dz].$$

That is,

$$(2.13) \qquad n[1-F_n(a_n+b_nx)] = (1+\delta_n)/\{(\zeta_n-1)/[n(r_n-1)] + \varepsilon_n\}$$

where, returning the subscript n to ϕ_n, r_n, ρ_n, C_n,

$$\zeta_n = r_n^{m_n} \phi_n'(r_n)/(1-\rho_n)$$

$$\delta_n = \zeta_n(1-\rho_n)[n(r_n-1)]^{-1}(2\pi i)^{-1} \int_{C_n} (1-z)z^{-m_n}\phi_n(z)^{-1}dz$$

$$\varepsilon_n = \zeta_n(1-\rho_n)[n(r_n-1)]^{-1}(2\pi i)^{-1} \int_{C_n} z^{-m_n}\phi_n(z)^{-1}dz.$$

We first note that $n(r_n - 1) \to c$ implies

$$(2.14) \qquad \lim_{n\to\infty} (\zeta_n-1)/[n(r_n-1)] = \begin{cases} x & c = 0 \\ (e^x-1)/c & 0 < c < \infty \\ e^x & c = \infty. \end{cases}$$

This follows since

$$(2.15) \qquad \zeta_n = \exp\{[a_n+b_nx + 0(1)]\log r_n - \alpha_n/\beta_n\}$$

$$= \begin{cases} 1 + xn(r_n-1) + o(1) & c = 0 \\ e^{x+o(1)} & 0 < c < \infty \\ n(r_n-1)e^{x+o(1)} & c = \infty. \end{cases}$$

For case $c = 0$, use $e^u = 1+u+o(u)$ as $u \to 0$ and $(v - 1)^{-1}\log v \to 1$ as $v \downarrow 1$.

We now show that δ_n and ε_n converge to 0. Choose η in the interval $(1, 1+\bar{x}_0)$ such that ϕ_0 has no zeros on the annulus $1 < |z| < \eta$. This is possible due to the nature of ϕ_0. Since $\phi_n(z) \to \phi_0(z)$ uniformly on compact sets, there is a positive integer N such that $\phi_n(z)$, for $n \geq N$, has no zeros on $1 < |z| < \eta$ other than at r_n. Therefore, we can take each C_n, for $n \geq N$, to be the circle with radius η. Letting $B_n = \sup\{|\phi_n(z)^{-1}|: |z| = \eta\}$ and

using (2.15) we have

$$|\delta_n'| \le \zeta_n(1-\rho_n)[n(r_n-1)]^{-1}(2\pi)^{-1}\int\limits_{|z|=\eta} |1-z|\ |z|^{-m_n}|\phi_n(z)|^{-1}dz$$

$$\le \begin{cases} O(1)B_n(1-\rho_n)/[n(r_n-1)\eta^{nx+\alpha_n+O(1)}] & c = 0 \\ O(1)B_n(1-\rho_n)/\eta^{m_n} & 0 < c \le \infty. \end{cases}$$

Applying $B_n \to B_0 < \infty$, assumption (2.5), and $m_n \to \infty$ to these expressions, it follows that $\delta_n \to 0$. A similar argument yields $\varepsilon_n \to 0$.

Using these limit statements for ζ_n, δ_n and ε_n in (2.13) implies (2.10), which in turn yields (2.2). Thus, we have shown that $n(r_n-1) \to c$ implies (2.2).

To prove the converse, suppose M_n has the limit distribution F_c. Let $n'(r_{n'}-1)$ be any convergent subsequence of $n(r_n-1)$ on the compact set $[0,\infty]$ and let $c' = \lim\limits_{k\to\infty} n'(r_{n'}-1)$. Then, from what we just proved, $M_{n'}$ has the limit distribution $F_{c'}$. But $M_{n'}$, as a subsequence of M_n, also has the limit distribution F_c. Then by Khinchine's Theorem (see for instance Theorem 1.2.3 of Leadbetter et al. (1983)), the distributions $F_{c'}$ and F_c are of the same type. Consequently $c'=c$. Thus, any convergent subsequence of $n(r_n-1)$ must converge to c, and hence $n(r_n-1) \to c$. A similar argument shows that F_c, $0 \le c \le \infty$, are the only possible limit distributions of M_n.

Two examples of Theorem 2.1 are as follows.

COROLLARY 2.4. The assertions of Theorem 2.1 are true under the single hypothesis $\rho_n \to 1$ when (i) the service times are constant or (ii) each G_n is a gamma distribution with scale parameter η_n and order k.

Proof. It suffices to verify the hypotheses of Theorem 2.1 for the two cases. First, suppose the service times are constant. Then $\gamma_n(z) = \exp(-z/\mu_n)$ and

$\phi_n(z) = \exp[-\rho_n(1-z)]-z$ for each complex z ($\bar{x}_n = \infty$). The zero r_n of ϕ_n on $(1,\infty)$ is given by $(r_n-1)^{-1}\log r_n = \rho_n$. Assumption (2.4) is satisfied since $\rho_n \to 1$ implies that $\gamma_n(\lambda_n z) = \exp(-\rho_n z) \to \gamma_0(z) = \exp(-z)$. Clearly $r_n \to r_0 = 1$. Now, using $\phi_n'(z) = \rho_n[\phi_n(z) + z] - 1$ and $\phi_n'(r_n) = \rho_n r_n - 1$, we can write

$$\alpha_n = \{\log[(r_n-1) - \log r_n] - \log[r_n \log r_n - (r_n-1)]\}/\log r_n.$$

Then by four applications of L'Hospital's rule and $r_n \to 1$, it follows that $\alpha_n \to -1$. Also, assumption (2.5) is satisfied, since an application of L'Hospital's rule gives

$$(1-\rho_n)/(r_n-1) = [(r_n-1) - \log r_n]/(r_n-1)^2 \to 1/2.$$

Next, assume that G_n is a gamma distribution with Laplace transform

$$\gamma_n(z) = (1 + \eta_n^{-1}z)^{-k}, \qquad \text{Re}\,z \geq -\bar{x}_n = -\eta_n.$$

Then $\phi_n(z) = [1+(\rho_n/k)(1-z)]^{-k} - z$, $\text{Re}\,z \leq 1 + \eta_n/\lambda_n$, and its zero r_n is one of the k+1 solutions of $1-r_n[1-(\lambda_n/\eta_n)(r_n-1)]^k=0$. Assumption (2.4) is satisfied since $\rho_n=k\lambda_n/\eta_n \to 1$ implies that $\gamma_n(\lambda_n z) \to \gamma_0(z) = (1+k^{-1}z)^{-k}$. Clearly $r_n \to r_0$ and $r_0=1$ since $\phi_0'(1) = \rho_0-1 = 0$. Using

$$\phi_n'(z) = \rho_n(\phi_n(z)+z)/[1 + \rho_n/k)(1 - z)] - 1 \quad \text{and} \quad \phi_n'(r_n) = \rho_n r_n^{1+k^{-1}} - 1,$$

we can write

$$\alpha_n = \{\log[(r_n-1) - k(1-r^{-1/k})] - \log[k(r^{1+1/k}-r) - (r-1)]\}/\log r_n.$$

From five applications of L'Hopital's rule, it follows that $\alpha_n \to -1$. Finally, assumption (2.5) is satisfied since an application of L'Hospital's rule gives

$$(1-\rho_n)/(r_n-1) = [(r_n-1) - k(1-r_n^{-1/k})] / (r_n-1)^2 \to (1+1/k)/2.$$

3. Results for GI/M/1 Queues

In this section, we show that the limiting behavior of extreme values of GI/M/1 queues is analagous to that for M/G/1 queues. Consider a sequence of positive recurrent GI/M/1 queueing systems indexed by $n \neq 1, 2, \ldots$ For the nth system, let G_n denote the distribution of the times between arrivals and let μ_n^{-1} denote the mean of the exponential service times. Assuming the system is empty at time 0, let M_n denote the maximum queue length up to the nth time the system becomes empty.

Suppose that the Laplace transform $\gamma_n(z)$ of G_n has the same form as in Section 2, and let

$$\psi_n(z) = \gamma_n(\mu_n(1 - z)) - z \quad \text{for Re} z < 1 + \bar{x}_n/\mu_n.$$

An easy check shows that ψ_n has the same properties as ϕ_n in Lemma 2.2 with μ_n in place of λ_n and r_n and 1 reversed. In particular, $\psi_n(x)$ has exactly two zeros on $(0, 1+\bar{x}_n/\mu_n)$ at 1 and at some $r_n < 1$, and $\psi_n(z)$ has no zeros in the annulus $r_n < |z| < 1$.

For the following result, we use the notation in (2.2) and (2.3) with $n(r_n-1)$ replaced by $n(1-\rho_n)$ and

$$(3.1) \qquad \alpha_n = \beta_n \log[-(1-\rho_n)/\psi_n'(r_n)] \qquad \beta_n = -1/\log r_n.$$

THEOREM 3.1. Suppose $G_n(\cdot/\mu_n)$ converges weakly to G_0 with $r_0 = 1$, and $\varliminf_{n \to \infty} \alpha_n \neq \infty$. Then the possible limit distributions for M_n are F_c, $0 \leq c \leq \infty$. The M_n has limit distribution F_c if and only if $n(1-\rho_n) \to c$. Appropriate norming constants for the convergence (2.2) are given by (2.3) and (3.1).

Proof. This follows by a proof paralleling that of Theorem 2.1. Here the maximum queue length in a busy cycle has the distribution

(3.2) $$1 - F_n(m) = 2\pi i / \int_C z^{-m} \phi_n(z)^{-1} dz \quad m = 0, 1, \ldots,$$

where C is a circle in the z-plane with center at the origin an. r ius less than r_n; see Cohen (1969). Choose circles C_n^1 and C_n in the z-plane with centers at the origin and radii less than r_n and greater than 1, respectively, such that $\psi_n(z)$ has no zeros other than r_n and 1 on the annulus between C_n^1 and C_n. Then by the residue theorem

$$(2\pi i)^{-1} \int_{C_n^1} z^{-m} \psi_n(z)^{-1} dz - (2\pi i)^{-1} \int_{C^n} z^{-m} \psi_n(z)^{-1} dz = \psi_n'(1)^{-1} + r_n^{-m} \psi_n'(r_n)^{-1}.$$

Using this, (3.2) and $\psi_n'(1)^{-1} = \rho_n^{-1} - 1$, we have

(3.3) $$n[1 - F_n(a_n + b_n x)] = \{(\zeta_n - \rho_n)/[n(1 - \rho_n)] + \varepsilon_n\}^{-1},$$

where

$$\zeta_n = -r_n^{-m_n}(1 - \rho_n)/\psi_n'(r_n)$$

$$\varepsilon_n = (2\pi i n)^{-1} \int_{C_n} z^{-m_n} \psi_n(z)^{-1} dz$$

and m_n is the integer part of $a_n + b_n x$. Arguing as in the proof of Theorem 2.1, one can show that $n(1 - \rho_n) \to c$ implies (2.10) and hence (2.2). The rest of the proof is the same as before.

REMARKS 3.2. Analogues of Remarks 2.2 and Corollary 2.4 apply to Theorem 3.1. Note that since (3.3) is simpler than (2.13), Theorem 3.1 does not need an assumption like (2.5).

4. Acknowledgement

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