UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD-A177 223

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFIT/CI/NR 87-28D	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Scattering By A Cavity-Backed Baffled Membrane		5. TYPE OF REPORT & PERIOD COVERED THESIS/DISSERTATION
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) William Paul Baker		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS AFIT STUDENT AT: Northwestern University		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS AFIT/NR WPAFB OH 45433-6583		12. REPORT DATE 1987
		13. NUMBER OF PAGES 85
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

DTIC ELECTE FEB 25 1987 S D A

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

APPROVED FOR PUBLIC RELEASE: IAW AFR 190-1

LYNN E. WOLAVER 27 JAN 87
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AFIT/NR

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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SCATTERING BY A CAVITY-BACKED BAFFLED MEMBRANE

BY

WILLIAM P. BAKER

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The solution to the time harmonic problem is approximated using the method of matched asymptotic expansion. The solution obtained is uniformly valid in the frequency domain and reveals the nature of the coupling between the membrane and the cavity. Special consideration is given to the possible coalesing modes. The backscattered farfield amplitude is determined and results are illustrated for a specific example.

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28

NORTHWESTERN UNIVERISTY

SCATTERING BY A CAVITY-BACKED BAFFLED MEMBRANE

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFULLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Applied Mathematics

By

William Paul Baker

Evanston, Illinois

June 1987

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Dedicated to My Parents

Burl and Betty Baker

ACKNOWLEDGEMENTS

This work was hardly a solitary effort. In the past three years at Northwestern there were many people whose contributions will be remembered. I wish to give them partial recognition here.

To begin with, my studies at NU would never have been possible, were it not for the sponsorship of the U.S. Air Force. I am grateful for the support provided by the Air Force Institute of Technology and its program managers.

Next, I must thank my advisor, Professor Gregory A. Kriegsmann, for his support and understanding, for listening to my complaints and for giving me that frequent "push" in the right direction. Additionally, I thank Professors William L. Kath and Edward L. Reiss for their time as members of my committee.

For contributing to the atmosphere of a united struggle, I thank my fellow students in applied mathematics. A partial, but surely incomplete, list would be as follows: Bill, Jeff, Son, Dave, John, Glenn, Jerry, Clyde, Leslie, Larry, Peter, Michel, Kirk, Pat, Ned, Ann, Kathy, Adam, Dave. I must give special recognition to my long term office mates; Bill Grayhack, Jeff Angell and Son Tu. They listened to numerous stories that began "I remember flying to...", and seldom complained.

The typing for this document would still be in process were it not for Judy and Crystal. Thank you, Judy not only for your typing skills but also for listening to my almost daily homeowner's tale of woe. And Crystal, as you type this, please accept my most sincere thanks for putting up with those numerous revisions.

I also thank my St. Martin's family for their love and support. One of my more important respites was the Wednesday evening choir rehearsal and, of course, the post rehearsal at Jake's. For this, I thank Bill and all the choir members. To the tenor section; Art, Dan and Birch, thanks for letting me add one more voice to complete that "four-part" unison. I especially thank Ann and Tom who not only provided me with a place to lay my weary body, but also shared my joys and frustrations in the last few months.

Finally and most importantly, I thank my wife, Nancy, and children, Michael and Christina, for their sacrifice and understanding. Michael, thank you for your judicious use of the "practice peddle" and your unique understanding of "why the sky is blue". Christina, thank you for our time together on "wackey" Wednesday and for knowing when I could take a break to study geography with you. Nancy, you're the most marvelous women I know; I love you.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Introduction

Problems of scattering by an object are of interest in many fields of applied sciences; such as, electromagnetics, elastic theory, and acoustics. In electromagnetics, waves are scattered from dielectric materials or seen as radar signals from conducting materials. Elastic waves are used to detect defects in materials as in nondestructive testing or to probe the earth for natural resources. In acoustics, sound waves are scattered as aerodynamic noise or from objects in the ocean by sonar.

The scattering of acoustic waves from a surface have been investigated for many years [1]. When the surface is flexible, it can become a source for scattered waves. Surfaces of finite extent admit the possibility of constructive interference or resonance at discrete frequencies (eigenfrequencies). This resonance phenomenon is physically observed [2] and is the object of mathematical study.

The study of resonance scattering can be either numerical or analytical. Numerical methods include variational techniques; such as, Galerkin's method, integral equation techniques; such as boundary integral methods, or transition matrix method [3], finite difference techniques using various numerical radiation conditions (see e.g. Ref. 4) and finite element techniques [5]. Analytical methods include separation of variables and the use of free space Green's functions, or approximation techniques using asymptotic methods. Problems with identifiable small or large parameters are ideally suited for asymptotic methods.

Leppington [6] considered the fluid loading as a small parameter and used it to approximate a solution to the problem of scattering sound from finite membranes near resonance. Kriegsmann et. al. [7] used the method of matched asymptotic expansions to obtain a uniformly valid approximation to the general time harmonic problem. They later solved the pulse scattering problem [8] for the same complex structure. Ahluwalia et. al. [9] considered the same problem with frequency, instead of fluid loading, as his small parameter.

We analyze the scattering properties of a thin, tightly stretched membrane set in an acoustically rigid infinite plane. Above the plane is an acoustic fluid while below the plane and the membrane is a cavity also filled with an acoustic fluid. The resonance phenomena is enhanced with the additional finite structures of the cavity. This problem was considered in part by Abrahams [10] where he extend the analysis of Leppington. Our approach, however, is an extension of the more general approximation given by Kriegsmann et. al.. We also consider the possibility of coalescing modes of the cavity and the membrane.

In Chapter 1 we analyze the scattering from this membrane-cavity structure by a time harmonic incident wave. The method of matched asymptotic expansions is used in the frequency domain under the light fluid loading assumption. The additional resonance effects due to the cavity are analyzed and compared to those results for no cavity [7]. An asymptotic expansion uniform in frequency is given and used to determine the farfield scattered potential. Finally, we show results for a specific scattering structure.

Chapter 2 examines the effect of a pulse scattering from the membrane-cavity structure. The problem is reduced to an infinite coupled system of algebraic equations using the Laplace transform. The solution to this system is then approximated using a generalized asymptotic series. Inverting the Laplace transform produced an asymptotic approximation uniformly valid in time. The solution is analyzed for a "spiked" pulse and physical interpretations are given. Special attention is given to the situation where an eigen-mode of the cavity coalesces with that of the membrane. Lastly, a specific geometry is examined and numerical results are given.

Each chapter is self contained and equations are numbered sequencially starting from one at the beginning of the chapter.

Chapter 1

Acoustic Scattering by Baffled Cavity-Backed Membranes

1.0 Introduction

A thin, tightly stretched membrane is set into an otherwise acoustically rigid, infinite plane, z = 0. The membrane region and its boundary are denoted by M and B, respectively. The cavity backing the membrane is bounded by an acoustically rigid surface S that is attached to the plane along B, and lies beneath the membrane, as shown in Figure 1. This surface and the membrane form the boundary of the closed cavity region Ω in the lower half space. The cavity is filled with a homogeneous acoustic fluid whose density and sound speed are denoted by ρ_b and C_b, respectively. The upper half-space contains another homogeneous fluid with density ρ_a and sound speed C_a. This baffled, cavity-backed membrane is a simple model of an element of a hollow structure, such as the bay of an air-flight or submerged vehicle, whose exterior surface is more flexible than its interior.

In Ref. 7 we studied the scattering of a time harmonic wave propagating in the upper half-space that is incident on the plane z = 0 when the cavity was absent, that is, the lower half-space contains a vacuum (we refer to this scatterer as the vacuum-backed membrane). The total field in the upper half-space is the sum of the incident, the specularly reflected, and the scattered waves. The incident wave scatters from the rigid part of the plane only as a specularly reflected wave. However, the field scattered by the membrane is more complex since the membrane deflects and oscillates in response to the incident field. It consists of the specularly reflected wave and a scattered wave. In particular, when the frequency of the incident field is far from a resonant (natural) frequency of the membrane in a vacuum, the reactions of the fluids on the membrane are small. However, when the incident field is close to a resonant frequency, the fluids and the membrane strongly couple, and the amplitude of the membrane's motion and of the scattered field are "large".

In this paper we study the effect of the cavity on the scattering of the incident wave. In particular, we analyze the effects of the additional resonant frequencies that correspond to the cavity when the membrane is replaced by an acoustically rigid surface. We observe that the cavity-backed membrane radiation problem where the cavity surface is driven by a prescribed normal motion, is "equivalent" to the scattering problem investigated in this paper.

We employ, as in Ref. 7, the method of matched asymptotic expansions to obtain a uniform asymptotic approximation of the scattered wave as the parameter $\varepsilon \to 0$. The small parameter ε is defined as the ratio of ρ_a to the membrane density, where ρ_a/ρ_b is assumed to be $O(1)$ as $\varepsilon \to 0$. Leppington[6] introduced the idea of solving scattering problems for flexible surfaces using this small parameter. His work was extended by Abrahams[10] to study the present problem. However, these authors employed different methods of analysis.

The outer expansion of the method of matched asympotic expansions, which is $O(\epsilon)$, gives the scattered wave away from the resonances. The inner expansions, which are $O(1)$, give the scattered waves at and near the resonance. The inner and outer expansions are then combined to obtain the composite, or uniformly valid asymptotic approximation to the scattered field. This approximation is expressed in terms of the in-vacuo normal modes of the membrane and the normal modes of the cavity when M is an acoustically rigid surface. The inner expansions are related to the classical resonant scattering approximation, for those problems which can be solved explicity (see Ref. 2 and references given there for a discussion of resonant scattering theory and its applications). A virtue of our method is that the leading term in the asymptotic expansion reveals the precise structure of the solution and its dependence on the incident frequency without requiring an explicit representation, e.g., by partial waves. However, for its validity ϵ must be small, or equivalently the membrane must be heavy compared to the fluids. Futhermore, the results of our analysis suggest efficient numerical methods for evaluating the scattered fields, as we have previously discussed in Ref. 7.

We apply the method in Section 1.5 to the specific problem of the normal incidence of a plane wave on a baffled circular membrane, where the cavity is an acoustically rigid cylinder of the same radius as the membrane, and of finite depth. Then, the required resonant frequencies and modes of the membrane and of the cavity are known explicity. The solid curve in Figure 3 gives the back scattered, far-field amplitude as a function of a dimensonless wave number k of the incident wave, for typical values of the physical and geometrical properties of the problem for which the resonant frequenties are simple. The crosses and circles on the k-axis locate the resonant frequencies of the membrane and cavity, respectively. The bandwidths of the response near the membrane resonant frequencies are much larger ($O(\varepsilon)$) than the bandwidths near the cavity resonant frequencies, which are $O(\varepsilon^2)$. This variation in bandwidths thickness characterizes the presence of a cavity. When k is away from the resonant frequenices, the scattered amplitude is $O(\varepsilon)$ so that the total far-field is essentially the sum of the incident and specularly reflected waves, which are both $O(1)$. However, when k is near a resonant frequency the backscattered field is $O(1)$, i.e. is of the same magnitude as the incident and specularly reflected fields.

The dashed curve in Figure 3 gives the backscattered, far-field amplitude for the vacuum-backed membrane, using the results of Ref. 7. Thus, we observe that one of the main effects of the cavity is to substantially increase the number of peaks in the response, and hence the cavity-backed membrane is a more "suseptible" scatterer than the vacuum-backed membrane. In addition. the minimum resonant frequency of the cavity-backed membrane may be less than the minimum resonant frequency of the equivalent vaccum-backed membrane. This implies that the low frequency scattering regime may be narrowed by the presence of a cavity. Finally, we observe that the cavity detunes the response of the membrane near the membrane resonant frequencies.

Figure 4 represents the backscattered, far-field amplitude when the physical and geometrical parameters are selected so that the lowest membrane and the lowest cavity resonant frquencies coincide (at $k = k_1 = 4$). The interaction produced at this "double resonance" is a resonant "packet" of overall bandwidth $O(\varepsilon^{1/2})$, which is much wider than the bandwidths ($O(\varepsilon)$) of either of the constituent peaks that form the packet. There is a near-null in the amplitude that occurs near $k = k_1$. At the simple cavity resonant frequency at $k = 9$, the peak is small, so that the scattered response amplitude is $O(\varepsilon)$. This is another feature of the response of the cavity-backed membrane that does not occur for the vaccum-backed membrane.

Section 1.1 of the paper contains the mathematical formulation of the scattering problem. We develop our method in Section 1.2 and 1.3 for resonant frequencies which are simple and distinct. That is, there is only one normal mode of the membrane and of the cavity corresponding to each of its resonant frequencies. Moreover, the membrane and cavity resonant frequencies are distinct. In Section

1.4 we analyze the case where simple resonant frequencies of the membrane and of the cavity coalesce.

1.1 Formulation

We assume that the incident, reflected, and scattered fields, the membranes motion, and the cavity field are proportional to exp(-iωt), where ω is the circular frequency of the incident wave. This time factor is omitted in the subsequent analysis. Dimensionless space variables $\underline{x} = (x,y,z)$ are defined by dividing the dimensional variables by a characteristic length L of the membrane, such as its maximum "diameter". Then the acoustic velocity potentials, $\Phi^a(\underline{x})$ and $\Phi^b(\underline{x})$ for the half-space $z > 0$ and for the cavity respectively, satisfy the Helmholtz equations:

$$\Delta_0\Phi^a + k^2\Phi^a = 0, \qquad z > 0; \tag{1}$$

$$\Delta_0\Phi^b + k^2c_0^2\Phi^b = 0, \qquad \underline{x}\varepsilon\Omega. \tag{2}$$

The dimensionless wave number k is defined by $k = \omega L/C_a$, $c_0 = C_a/C_b$, and Δ_0 the Laplacian in $\underline{x}$. The acoustic pressures P_a and P_b are related to these potentials by

$$P_a = i\omega\rho_a\Phi^a, \qquad P_b = i\omega\rho_b\Phi^b. \tag{3}$$

The equation of motion for the lateral deflection w(x,y) of the membrane is,

$$\Delta w + k^2c^2w = \frac{L^2}{T}\{P_a(x,y,0) - P_b(x,y,0)\}, \qquad (x,y)\varepsilon M \tag{4}$$

Here, Δ is the Laplacian in x and y, $c = C_a/C_m$, where, $C_m = (T/\rho_m)^{1/2}$, ρ_m is the density per unit area of the membrane, and T is the tension applied to the membrane. The difference in the acoustic pressures $P_a(x,y,0) - P_b(x,y,0)$ acts as a driving force on the membrane.

Since the cavity surface S and the plane $z = 0$ outside of the M are acoustically rigid, we have the conditions

$$\phi_z^a(x,y,0) = 0, \quad (x,y)\varepsilon M, \tag{5}$$

$$\phi_\nu^b(\underset{\sim}{x}) = 0, \quad \underset{\sim}{x}\varepsilon S, \tag{6}$$

where subscripts denote partial differentiation and ν denotes differentiation with respects to the outward normal to S. In addition, the acoustic potentials and membrane motions are coupled by the requirement that all vertical velocities are continuous on the membrane's surface. This gives,

$$\phi_z^a(x,y,o) = \phi_z^b(x,y,0) = - i\omega w(x,y), \quad (x,y)\varepsilon M. \tag{7}$$

We denote the incident acoustic field with upper half-space by ϕ^I. It is a solution of (1). Then we express the total acoustic field in $z > 0$ as

$$\phi^a(x,y,z;k) = \phi^I(x,y,z;k) + \phi^I(x,y,-z;k) + \phi(x,y,z;k) \tag{8}$$

where $\phi^I(x,y,-z;k)$ is the specularly reflected field from $z = 0$ and ϕ is the field scattered by the membrane-cavity structure. By inserting (8) into (1), (3), (4), (5), and (7), we obtain upon eliminating w,

$$\Delta_0\phi + k^2\phi = 0, \quad z > 0 \tag{9}$$

$$\phi_z(x,y,0) = 0, \quad (x,y)\notin M \tag{10}$$

$$(\Delta + k^2c^2)\phi_z(x,y,0) = \varepsilon^2k^2c^2[2\Phi^I(x,y,0;k) + \phi(x,y,0;k) - \chi\Phi^b(x,y,0;k)], \; (x,y)\varepsilon M \tag{11}$$

$$\Phi_z^b(x,y,0;k) = \phi_z(x,y,0;k), \quad (x,y)\varepsilon M. \tag{12}$$

The dimensionless parameter ε in (11), which is defined by $\varepsilon = (\rho_a/\rho_m)L$, is the ratio of the volume densities of the fluid and the membrane. For many acoustic fluids and membrane materials ε is small; e.g. for air and aluminum $\varepsilon \sim 5 \times 10^{-4}$. The dimensionless parameter χ in (11) is defined, by $\chi = \rho_b/\rho_a$.

In summary the scattering problem for the baffled, cavity-backed membrane is to determine the scattering potential $\phi(\underline{x})$ in the upper half-space that satisfies the Sommerfeld radiation as $r = |\underline{x}| \to \infty$, and to determine the cavity potential $\Phi^b(\underline{x})$ in Ω such that they satisfy the coupled problem consisting of (2), (6) and (9)-(12). When this problem is solved to membrane motion is determined from (7) as

$$w(x,y) = \frac{i}{\omega L}\, \phi_z(x,y,0). \tag{13}$$

This formulation of the scattering problem is now simplified by employing the Green's function $g(\underset{\sim}{x}|\underset{\sim}{\xi};k)$ of the Helmholtz equation (9) for the upper half-space that satisfies the boundary condition $\phi_z(x,y,0) = 0$ for all x and y and the radiation condition as $|\underset{\sim}{x}| \to \infty$. Then the scattered potential is given by

$$\phi(\underset{\sim}{x}) = G(\underset{\sim}{x};k)v = \int\int_M g(\underset{\sim}{x}|\xi,\eta,0;k)v(\xi,\eta)d\xi\, d\eta, \tag{14}$$

where $\underset{\sim}{\xi}$ is the vector with components (ξ,η,ζ), g is given by

$$g(\underset{\sim}{x}|\underset{\sim}{\xi};k) = -(e^{ik|\underset{\sim}{x}-\underset{\sim}{\xi}|})/(2\pi|\underset{\sim}{x}-\underset{\sim}{\xi}|), \tag{15}$$

and we have employed the notation

$$v(x,y) = \phi_z(x,y,0) \quad \text{for}\,(x,y)\varepsilon M. \tag{16}$$

Thus, the scattering problem is reduced to

$$\Delta\, v + k^2c^2v = \varepsilon k^2c^2[G(k)v + 2\Phi^I(x,y,0;k) - \chi\Phi^b(x,y,0;k)],\ (x,y)\varepsilon M, \tag{17a}$$

$$v(x,y) = 0, \quad \text{on } B, \tag{17b}$$

$$\Phi^b_z(x,y,0) = v, \quad (x,y)\varepsilon M. \tag{17c}$$

where Φ^b satisfies the cavity Helmholtz equation (2) and boundary condtion (6), and B is the boundary curve of M. The boundary condition (17b) is obtained by continuity from (10) and (16), and (17c) follows from (12) and (16). The integral operator G(k) in (17a) is defined by

$$G(k) = G(x,y,0;k). \tag{17d}$$

It is proportional to the acoustic "back" pressure of the fluid, in $z > 0$, acting on the membrane's surface. Likewise, the term $\phi^b(x,y,0;k)$ in (17a) is proportional to the back pressure of the cavity fluid.

We observe that if the terms Gv and ϕ^b are omitted from (17a), the resulting boundary value problem is for the oscillations of a membrane in a vacuum driven by a force proportional to the incident field on the membrane. The dimensionless resonant frequencies of the free oscillations of the membrane and the corresponding modes are proportional to the eigenvalues $k_n c$ and the corresponding eigenfunctions θ_n of

$$\Delta v + k^2 c^2 v = 0, \quad v = 0 \quad \text{on } B. \tag{18}$$

Consequently, for the driven membrane with $k = k_n$, $n = 1, 2, \ldots$ resonance occurs and the membrane's amplitude is unbounded if $\phi^I(x,y,0)$ is not orthogonal to θ_n. However, for the scattering problem with $k = k_n$ the back pressure terms Gv and ϕ^b in (17a) restrain resonance, and the amplitude is bounded. Moreover, if ε is small, as we assume, then the back pressures are small and a "near" resonance occurs.

Similary, a near resonance occurs when k is near k_m^b, $m =$ 1,2,... where the dimensionless resonant frequencies k_m^b of the cavity's free oscillations. Its corresponding modes are proportional to the eigenvalues $k_m^b c_0$ and the corresponding eigenfunctions ψ_m of :

$$\Delta_0 \psi + k^2 c_0^2 \psi = 0, \quad \underset{\sim}{x} \varepsilon \Omega;$$

$$\psi_z(x,y,0) = 0, \quad (x,y)\varepsilon M; \quad \psi_\nu = 0, \ \underset{\sim}{x} \varepsilon S \tag{19}$$

In this paper we employ as in Ref. 7, the method of matched asymptotic expansions to obtain an asymptotic approximation of the solution of the scattering problem (2), (6) and (17) that is uniformly valid in k. The corresponding acoustic field is then obtained from (14). For k bounded away from k_n and k_m^b (nonresonance) we obtain a regular peturbation expansion in ε of the solution (the outer expansion). As either $k \to k_n$ or $k \to k_m^b$ this outer expansion becomes singular, as we demonstrate. If k approaches k_n and k_n is distinct from k_m^b, m = 1,2..., then we construct a second asymptotic expansion of the solution (an inner expansion) that is valid for k near k_n. Similarly, another inner expansion is obtained when k is near k_m^b where, k_m^b and k_n, n = 1,2,..., are distinct. The inner and outer expansions are then combined to form the composite expansion[11] which is the desired uniform asymptotic expansion. The case when k_n and k_m^b coalesce for some m and n is considered separately. We should mention that in the traditional applications of the method of matched asymptotic expansions, see e.g., Ref. 11, the singularity, or boundary layer, occurs as the independent variables approach critical values. In the present application the singularity in the expansion occurs when the parameter k approaches a critical value.

1.2. The Uniform Asymptotic Expansion

A. The Outer Expansion

For k bounded away from k_n and k_n^b, n = 1,2,3,..., we seek an asymptotic expansion of the solution of the scattering problem in the form

$$v = \sum_{j=0}^{\infty} v_j(x,y,k)\varepsilon^j \tag{20a}$$

$$\Phi^b = \sum_{j=0}^{\infty} \Phi^j(x,y,z,k)\varepsilon^j \tag{20b}$$

The coefficients v_j and Φ^j are determined by inserting (20) into (2), (6), and (17) and equating coefficients of the same powers of ε in the result. Thus, we obtain for j = 0, 1,...,

$$\Delta v_j + k^2c^2v_j = k^2c^2[G(k)v_{j-1} - \chi\Phi^{j-1}(x,y,0,k) + 2\delta_{1j}\Phi^I(x,y,0,k)] \tag{21a}$$

$$v_j = 0, \quad \text{on } B \tag{21b}$$

and

$$\Delta_0\Phi^j + k^2c_0^2\Phi^j = 0, \ \underset{\sim}{x}\varepsilon\Omega, \tag{22a}$$

$$\Phi_z^j(x,y,0,k) = v_j, \ (x,y)\varepsilon M, \quad \Phi_\nu^j = 0, \ \underset{\sim}{x}\varepsilon S, \tag{22b}$$

where $v_{-1} = \Phi^{-1} = 0$ and δ_{j1} is the Kronecker delta function. Since $k \neq k_n$, we deduce from (21) with j = 0 that $v_0 = 0$. Inserting this into (22) with j = 0 and noting that $k \neq k_n^b$ we deduce that $\Phi^0 = 0$.

Then by substituting these results into (21) with j = 1, and solving the resulting problem for v_1 by an expansion in the eigenfunctions θ_n of (18) we get,

$$v_1 = 2k^2c^2L(k)\phi^I = 2k^2c^2\left[\frac{1}{c^2}\sum_{j=1}^{\infty}\frac{\beta_j(k)}{k^2 - k_j^2}\theta_j(x,y)\right]. \tag{23a}$$

Here the coefficients β_j are defined by

$$\beta_j(k) = \langle\phi^I(x,y,0,k),\ \theta_j(x,y)\rangle, \tag{23b}$$

and we have used the notation

$$\langle f,g\rangle = \int\int_M f(x,y)\ g(x,y)dxdy \tag{23c}$$

for any two functions defined on M. The solution operator, L(k) is defined in (23a).

Inserting the result (23) into (22) with j = 1, and assuming that the eigenfunction ψ_j of (19) are complete, we deduce that

$$\dot{\phi}^1 = L_b(k)v_1 = \frac{1}{c_0^2}\sum_{i=1}^{\infty}\frac{\gamma_i(k)}{(k_i^b)^2 - k^2}\psi_i(\underset{\sim}{x}) \tag{24a}$$

where the coefficients γ_i are defined by

$$\gamma_i(k) = \langle\psi_i(x,y,0),\ v_1\rangle, \tag{24b}$$

and the solution operator $L_b(k)$ is defined in (24a). Similarly, we obtain from (21) with j = 2 that,

$$v_2 = 2(kc)^4\{L(k)[G(k)L(k)\phi^I] - \chi L(k)[L_b(k)L(k)\phi^I]\}. \tag{25}$$

It is clear from (23a) that v_1 is unbounded as $k \to k_j$, $j = 1,2,\ldots$, and consequently by (24) ϕ^1 is unbounded in the same limit. It follows from (24a) that ϕ^1 is also unbounded as $k \to k_m^b$, $m = 1,2,\ldots$, and hence from (25) v_2 is unbounded in the same limit. Thus, the outer expansion

$$v = \varepsilon v_1 + \varepsilon^2 v_2 + O(\varepsilon^3) \tag{26a}$$

$$\phi^b = \varepsilon\phi^1 + O(\varepsilon^2) \tag{26b}$$

is invalid for k at and near the resonant frequencies, k_n of the membrane and k_n^b and the cavity. Of course, if $\beta_j(k_j) = 0$ or $\gamma_i(k_i^b) = 0$, then the expansion (25) may still be valid. The outer expansions (26) is $O(\varepsilon)$ as $\varepsilon \to 0$ for k bounded away from these resonant frequencies.

We observe from (23a) and (25) that when the outer expansion is valid, the cavity fluid does not affect the leading term in this expansion but influences the $O(\varepsilon^2)$ term. Thus, the leading order and for k bounded away from the membrane and cavity resonant frequencies, the cavity backed membrane responds the same as the vacuum backed membrane.

B. The Inner Expansion near k_n

We obtain an asymptotic expansion of the solution (2), (6) and (17) that is valid for k at and near these resonant frequencies by first defining the stretched parameter α as

$$k = k_n(1 + \alpha\varepsilon) \tag{27}$$

for each fixed value of $n = 1,2,3,\ldots$. Then by inserting (26) into (2), (6), and (17), we seek a second asymptotic expansion of the solution in the form

$$v = \sum_{j=0}^{\infty} V_j(x,y,\alpha)\varepsilon^j \tag{28a}$$

$$\phi^b = \sum_{j=0}^{\infty} U^j(\underset{\sim}{x},\alpha)\varepsilon^j. \tag{28b}$$

Then we find, for example, that V_0, U^0, V_1 are solutions of the boundary value problems

$$\Delta V_0 \; k_n^2c^2V_0 = 0, \; (x,y)\varepsilon M, \; V_0 = 0 \quad \text{on } B; \tag{29a}$$

$$\Delta_0 U^0 + k_n^2c_0^2U^0 = 0, \quad \underset{\sim}{x}\varepsilon\Omega, \; U_z^0(x,y,0) = V_0, \; (x,y)\varepsilon M,$$

$$\text{and } U_\nu^0\,(\underset{\sim}{x}) = 0, \; \underset{\sim}{x}\varepsilon S; \tag{29b}$$

$$\Delta V_1 + k_n^2c^2V_1 = R_1 = k_n^2c^2[-2\alpha V_0 + G(k_n)V_0 + 2\phi^I(x,y,0;k_n)$$
$$-\chi U^0(x,y,0;k_n)], \; (x,y)\varepsilon M$$

$$V_1 = 0, \quad \text{on } B. \tag{29c}$$

The coefficients V_2, V_3,... and U^1, U^2,... satisfy similar inhomogeneous boundary value problems.

Assuming than $k_n c$ is a simple eigenvalue of (18) with eigenfunction θ_n, the solution of (29a) is

$$V_0 = A_n \theta_n(x,y) \tag{30}$$

where the constant amplitude A_n is to be determined. Assuming that $k_n c_0$ is not an eigenvalue of (19), the solution of (29b) is

$$U^0(\underline{x}) = A_n L_b(k_n)\theta_n \tag{31}$$

where the operator L_b is defined in (24a). Inserting (31) with z= 0 and (30) into (29c) we deduce that V_1 satisfies an inhomogeneous eigenvalue problem. Thus, R_1 in (29c) must satisfy the solvability condition that it is orthogonal to every solution of the homogeneous problem to (29a), i.e.

$$\langle R_1, \theta_n \rangle = 0. \tag{32}$$

By inserting R_1 into (32) and determining the required integrals, we find, after some calculation, that (32) gives a linear algebraic equation for A_n. Its solution is

$$A_n = \frac{2\beta_n(k_n)}{2\alpha + a_n}, \tag{33}$$

where the a_n are defined by

$$a_n = - \langle G(k_n)\theta_n, \theta_n \rangle + \chi S_n, \tag{34a}$$

and the S_n are given by

$$S_n = \langle L_b(k_n)\theta_n|_{z=0}, \theta_n\rangle. \tag{34b}$$

Thus, the inner expansion valid near k_n, is given by

$$v = A_n\psi_n(x,y) + O(\varepsilon) \tag{35a}$$

$$\Phi_b = A_n L_b(k_n)\theta_n + O(\varepsilon), \tag{35b}$$

and they are O(1) as $\varepsilon \to 0$.

C. <u>The Inner Expansion near k_n^b</u>

We now obtain an asymptotic expansion of the solution of (2), (6), and (17) that is valid at and near these frequencies. First, we define for each n = 1,2..., a new stretched parameter β by

$$k = k_n^b[1 + \ell\varepsilon + \beta\varepsilon^2] \tag{36}$$

where as we will show, see (42), the parameter ℓ is determined by the inner expansion and $\ell \neq 0$. Consequently the stretching is with respect to an $O(\varepsilon)$ shift of k_n^b. Then we seek an asymptotic expansion of the solution in the form,

$$v = \sum_{j=0}^{\infty} X_j(x,y,\beta)\varepsilon^j \tag{37a}$$

$$\Phi^b = \sum_{j=-1}^{\infty} W^j(\underline{x},\beta)\varepsilon^j. \tag{37b}$$

The leading term in the expansion (37b) for Φ^b must be $O(\varepsilon^{-1})$ in order to satisfy the matching conditions for the outer and inner expansions for k near k_n^b; see Sub-Section D. Then we find, by substituting (36) and (37) into (2), (6), and (17), that W^{-1}, W^{0}, W^{1}, X_0 and X_1 are solutions of the boundary value problems:

$$H_b(k_n^b)W^{-1} = \Delta_0 W^{-1} + (k_n^b c_0)^2 W^{-1} = 0, \ \underset{\sim}{x}\varepsilon\Omega,$$

$$W_z^{-1}(x,y,0) = 0, \ (x,y)\varepsilon M, \quad W_\nu^{-1}(\underset{\sim}{x}) = 0, \ \underset{\sim}{x}\varepsilon S; \tag{38a}$$

$$H(k_n^b)\ X_0 = \Delta\ X_0 + (k_n^b c)^2\ X_0 = Q_0 = -\chi(k_n^b c)^2\ W^{-1}(x,y,0),$$

$$(x,y)\varepsilon M, \quad X_0 = 0, \quad \text{on } B; \tag{38b}$$

$$H_b(k_n^b)W^0 = P_0 = -2\ell(k_n^b c_0)^2 W^{-1}, \ \underset{\sim}{x}\varepsilon\Omega,$$

$$W_z^0(x,y,0) = X_0, \ (x,y)\varepsilon M, \quad W_\nu^0(\underset{\sim}{x}) = 0, \ \underset{\sim}{x}\varepsilon S; \tag{38c}$$

$$H(k_n^b)\ X_1 = Q_1 = -2\ell(k_n^b c)^2\ X_0 + (k_n^b c)^2\{2\Phi^I(x,y,0)$$

$$+\ G(k_n^b)\ X_0 - \chi W^0(x,y,0)\}, \ (x,y)\varepsilon M,$$

$$X_1 = 0, \quad \text{on } B; \tag{38d}$$

$$H_b(k_n^b)W^1 = P_1 = -(k_n^b c_0)^2\ \{2\ell W^0 + (2\beta+\ell^2)W^{-1}\}, \ \underset{\sim}{x}\varepsilon\Omega,$$

$$W_z^1\ (x,y,0) = X_1, \ (x,y)\varepsilon M, \quad W_\nu^1\ (\underset{\sim}{x}) = 0, \quad \underset{\sim}{x}\varepsilon S \tag{38e}$$

The coefficients X_2, X_3,... and W_2, W_3,... satisfy similar inhomogeneous boundary value problems.

Assuming that $k_n^b c_0$ is a simple eigenvalue of (19) with eigenfunction ψ_n, the solution of (38a) is

$$W^{-1} = B_n \psi_n \tag{39}$$

where the constant amplitude B_n is to be determined. Assuming again that $k_n^b c_0$ is not an eigenvalue of (18) the solution of (38b) is

$$X_0 = L(k_n^b)Q_0. \qquad (40)$$

Inserting this result into (38c) we deduce that W^0 satisfies an inhomogeneous eigenvalue problem. Thus, P_0 and X_0 in (38c) must satisfy the solvability condition

$$\langle\psi_n(x,y,0),\ X_0\rangle = [P_0,\psi_n] = \iiint_\Omega P_0(x,y,z)\psi_n(x,y,z)dxdydz \qquad (41)$$

This relationship is obtained by first multiplying the differential equation in (38c) by ψ_n and then integrating the result over the region Ω. The result then follows by applying the standard Green identity, (19) and the boundary conditions in (38c).

Inserting (39) into the definitions of P_0 and Q_0, given in (38c) and (38b) and into (40), and then using these results in (41) we find

$$\ell = \frac{\chi}{2}\left(\frac{c}{c_0}\right)^2 \langle\psi_n,\ L(k_n^b)\psi_n\rangle\big|_{z=0}. \qquad (42)$$

This value of ℓ is necessary to insure that B_n is not zero. However, the value of B_n is unknown at this stage of the analysis and therefore so are W^{-1} and X_0.

The solution of (38c) for W^0 is obtained by using eigenfunction expansion of the generalized Green's function. Thus gives,

$$W^0 = D_n\psi_n + \hat{L}_b(k_n^b)P_0, \quad (43)$$

where the hat (^) on L_b denotes the omission of the i = n term in its definition (24a). The constant D_n is to be determined. We next insert (43) and (40) into (38d) and solve for X_1, to get

$$X_1 = L(k_n^b)Q_1. \quad (44)$$

Finally, we insert (39), and (42)-(44) into (38e) and obtain an inhomogeneous eigenvalue problem for W^1, as we did to determine W^0. The solvability condition gives a linear algebraic equation for B_n, where the terms involving D_n vanish. The solution of this equation is,

$$B_n = \frac{-2\langle\psi_n, L(k_n^b)\phi^I\rangle}{\left(\frac{c_0}{c}\right)^2 (2\beta + \ell^2) + 2\ell\chi\, T_n - \chi\, b_n}, \quad (45)$$

where the T_n and b_n are defined by

$$T_n = (k_n^b c)^2 \langle\psi_n,\ L(k_n^b)L(k_n^b)\psi_n\rangle, \quad (46)$$

$$b_n = (k_n^b c)^2 \langle\psi_n, L(k_n^b)G(k_n^b)L(k_n^b)\psi_n\rangle \quad (47)$$

and the functions in the < > are evaluated at z = 0; see the definition (23c).

Thus, the inner expansion valid near k_n^b, is given by,

$$\phi^b = \frac{1}{\varepsilon} B_n\psi_n + O(1) \quad (48a)$$

$$v = -\chi(k_n^b c)^2 B_n L(k_n^b)\psi_n + O(\varepsilon) \quad (48b)$$

We observe that $v = O(1)$ as $\varepsilon \to 0$ and $\Phi^b = O(\varepsilon^{-1})$ as $\varepsilon \to 0$. That is, the field inside the cavity is an order of magnitude larger than the membrane's amplitude and the scattered field in $z > 0$. This is a result of the membrane cover on the cavity which diminishes the radiation of a energy from the cavity back into the far field. This suggests that a nonlinear acoustic theory, may be necessary to describe the cavity field when k is near k_n^b.

D. Matching Conditions

Omitting all details, we can show, as in Ref. 1, that the matching conditions of the method of matched asymptotic expansions are identically satisfied for k in the overlap interval near k_n. In this interval $|\alpha| \to \infty$ as $\varepsilon \to 0$, and moreover, in this interval the outer expansion (26a) in terms of the inner parameter is given by

$$v = [B_n(k_n)/\alpha][1 + O(1/\alpha)]\theta_n. \quad (49)$$

Similarly, the matching conditions for κ near the shifted cavity resonant frequencies K_n^b, where K_n^b are defined by

$$K_n^b = k_n^b(1+\ell\varepsilon), \quad (50)$$

are identically satisfied. In addition, in the overlap interval near K_n^b the expansion (26a) in terms of the inner parameter β is given by

$$v \sim -\frac{\chi}{\beta}\left(\frac{c}{c_0}\right)^2 (k_n^b c)^2 \langle\psi_n, L(k_n^b)\phi^I\rangle\, L(k_n^b)\psi_n(x,y,0)[1 + O(1/\beta)]$$

$$+ 2\varepsilon(k_n^b c)^2\, L(k_n^b)\phi^I. \qquad (51)$$

E. The Uniform Expansion

The composite expansion of the method of matched asymptotic expansions provides the desired uniform asymptotic expansion of the solution for k in an interval about either k_n or k_n^b. It is given by the sum of the inner and outer expansion minus the outer expansion in the inner parameter in the overlap interval. In the interval about k_n the composite expansion is to lowest order,

$$v \sim -\frac{\beta_n(k_n)a_n\theta_n}{\alpha(2\alpha + \alpha_n)} + 2\varepsilon k^2 \sum_{j=1}^{\infty} \frac{\beta_j(k)\theta_j}{k^2 - k_j^2}, \qquad (52a)$$

where α equals $(k - k_n)/\varepsilon k_n$.

Similarly, the composite expansion in an interval about K_n^b is, to lowest order,

$$v \sim q\,(x,y,\beta,\varepsilon) = 2\varepsilon(kc)^2\, L(k)\phi^I + \varepsilon^2 v_2$$

$$-\left(\frac{c}{c_0}\right)^2 \frac{\chi(k_n^b c)^2\langle\psi_n, L(k_n^b)\phi^I\rangle}{\beta\left[2\left(\frac{c_0}{c}\right)^2\beta + \Gamma_n\right]}\, \Gamma_n L(k_n^b)\psi_n(x,y,0) \qquad (52b)$$

where v_2 is given by (25), Γ_n is defined by

$$\Gamma_n = \left(\frac{c_0}{c}\right)^2 \ell^2 + \ell\chi\, T_n - \chi\, b_n, \tag{53}$$

and T_n and b_n are defined in (47). In obtaining (52b) the second term in (51) is absent because it is cancelled by a term in X_1. (See equations (44) and (38d)).

The uniform expansion, in a neighborhood of k_n, for the scattered acoustic potential is obtained from (52a), (14), and (16) as

$$\phi \sim \sum_{j=1}^{\infty} \left(- \frac{\beta_n(k_n)a_n\delta_{nj}}{\gamma(2\alpha + a_n)} + \frac{2\varepsilon k^2\beta_j(k)}{k^2 - k_j^2}\right)G(\underline{x};k)\theta_j \tag{54a}$$

where G(x;k) is defined in (14). Similarly the uniform expansion for ϕ in a neighborhood of K_n^b is found by combining (52b), (14), and (16). It is given by

$$\phi \sim G(\underline{x};k)q \tag{54b}$$

where q is defined in (52b). In the farfield, where $r = |\underline{x}| \to \infty$, we find by the standard application of the law of cosines and the binomial theorem to $|\underline{x} - \underline{\xi}|$ in (15) that (54) and (54b) become, respectively,

$$\phi \sim \left[\sum_{j=1}^{\infty} \left(- \frac{\beta_n(k_n)a_n\delta_{nj}}{\alpha(2\alpha + a_n)} + \frac{2\varepsilon k^2\beta_j(k)}{k^2 - k_j^2}\right)F_j(k,\hat{r})\right] \frac{e^{ikr}}{r}, \tag{55a}$$

$$\phi \sim H(k,\hat{r}) \frac{e^{ikr}}{r}. \tag{55b}$$

The directivity factors F_ℓ, $\ell = 1,2,\ldots$ and $H(k,\hat{r})$ are defined by

$$F_\ell(k,\hat{r}) = -\frac{1}{2\pi}\iint_M e^{-ik\hat{r}\cdot\gamma}\theta_\ell(\xi,\eta)d\xi\, d\eta, \tag{56a}$$

$$H(k,\hat{r}) = -\frac{1}{2\pi}\iint_M e^{-ik\hat{r}\cdot\gamma}q(\xi,\eta)d\xi\, d\eta. \tag{56b}$$

Here, $\hat{r} = \underset{\sim}{x}/|\underset{\sim}{x}|$ is the unit vector in the vector of observation, and $\gamma = (\xi,\eta,0)$. Thus in the farfield the composite expansion for the acoustic potential is reduced to a spherically outgoing wave whose amplitude is given by the sum in (55a) for k near k_n or the factor H defined in (56b) for k near K_n^b. The square of the absolute value of this sum, and $|H(k,\hat{r})|^2$ are the differential cross sections of the membrane corresponding to the composite expansions (55a) and (55b), respectively. The F_ℓ are the Fourier transforms of the modes θ_ℓ and H is the Fourier transform of q with respect to the observation direction.

1.3. Interpretation of the Results

The inner and outer expansions can be recovered from the uniform expansions (52), (54), and (55) by taking appropriate limits in these equations. Thus, if $k - k_n = O(1)$ as $\varepsilon \to 0$, so that $\alpha = O(1/\varepsilon)$ and $\beta = O(1/\varepsilon^2)$, then the first term in (52d) and the second and third terms in (52b) are $O(\varepsilon^2)$. Consequently the second term in (52a) and the first in (52b) dominate and both reduce to the leading term in the outer expansion (26a). Similarly the far field potentials both reduce to

$$\phi \sim (2\varepsilon k^2 \sum_{j=1}^{\infty} \frac{\beta_j(k)F_j(k,\hat{r})}{k^2 - k_j^2}) \frac{e^{ikr}}{r}. \tag{57}$$

Thus the displacement v and the scattered potential ϕ are $O(\varepsilon)$ when the incident frequency is bounded away from all of the resonant frequencies of the membrane and cavity. That is, the acoustic potential is given essentially by the sum of the incident and specularly reflected waves because the acoustic fluid density is much smaller than the membrane density. This qualitative behavior has already been observed[2] in other scattering problems that can be solved explicity, e.g., by partial wave expansions and from their subsequent numerical evaluations.

However, when the incident frequency approaches a membrane's resonant frequency, i.e., when $k = k_n(1 + \alpha\varepsilon)$ for $\alpha = O(1)$ as $\varepsilon \to 0$, the second term in (52a) is O(1). It combines with the first term to yield the inner result (35a). Similarly, the farfield expression (55a) for the scattered potential is reduced for k near k_n to

$$\phi \sim A_n F_n(k_n,\hat{r})(e^{ik_n r}/r). \tag{58a}$$

The "shift" in the cavity resonant frequencies from k_n^b to K_n^b is caused by the presence of the acoustic fluid in the region z > 0 and the membrane. The first is demonstrated by the factor χ in (42) and the second by the inner product there. When k approaches K_n^b, i.e. when $k = k_n^b(1+\varepsilon\ell+\varepsilon^2\beta)$ for $\beta = O(1)$ as $\varepsilon \to 0$, the term $\varepsilon^2 v_2$ is O(1). It combines with the third term to yield the inner result (48b). Similarly, the farfield expression (55b) for the scattered potential is reduced for k near K_n^b to

$$\phi \sim -\chi\ B_n\ D(k_n^b,\hat{r})\frac{e^{ik_n^b r}}{r} \tag{58b}$$

where the directivity factor D is defined by

$$D(k_n^b,\hat{r}) = (k_n^b)^2 \sum_{j=1}^{\infty} \frac{\langle\psi_n,\theta_j\rangle}{k_j^2 - (k_n^b)^2} F_j(k_n^b,\hat{r}). \tag{58c}$$

Both (58a) and (58b) are O(1/ε) larger than (57) and are of the same order as the incident and specularly reflected waves. Thus, the scattered potential contribute to the lowest order approximation only when k is near a resonant frequency of either the membrane or cavity.

The coefficient of the outgoing spherical wave in (58a) is the produce of the amplitude A_n and the directivity factor F_n which gives the radiation pattern of the membrane for k near k_n. Futhermore, $|A_nF_n|^2$ is the differential cross section of the scattered acoustic potential for k near k_n and $|F_n(k_n,\hat{r})|^2$ is the differential cross section of the farfield scattered acoustic potential ϕ_n produced by the membrane vibrating with frequency k_n and mode θ_n. The amplitude A_n contains information about the coupling between the acoustic medium and the membrane, which we now describe.

The parameter a_n defined in (34) is the sum of two terms, which have the following physical interpretations. From the definition of L_b in (24a) it is clear from (34b) that S_n is the inner product of θ_n and the corresponding cavity potential Φ_n^b. It then follows, as we show in Appendix A, that S_n is real and given by,

$$S_n = \frac{1}{c_0^2} \sum_{m=1}^{\infty} \frac{\langle\theta_n,\psi_m(x,y,0)\rangle^2}{(k_m^b)^2 - k_n^2} \qquad (59)$$

The first term in (34a) was analyzed in Reference [1]. By combining this result with (59), we obtain

$$a_n = R + i\,I \qquad (60a)$$

where R and I are defined by

$$R = \frac{\chi}{c_0^2} \sum_{m=1}^{\infty} \frac{\langle \theta_n, \psi_m(x,y,0)\rangle^2}{(k_m^b)^2 - k_n^2} + \iiint_{z>0} \{|\nabla_0\phi_n|^2 - k_n^2|\phi_n|^2\} dxdydz \qquad (60b)$$

$$I = k_n \int_0^{\pi} \int_0^{\pi/2} |F_n(k_n,\hat{r})|^2 \sin \phi' \, d\phi' \, d\phi' \geq 0. \qquad (60c)$$

Here I/k_n is the total cross section of ϕ_n, and the second term in (60b) is twice the corresponding dimensionless Lagrangian. The total acoustic scattering cross section for k near k_n can therefore be written as

$$\sigma_T = (|A_n|^2/k_n)I. \qquad (61)$$

Combining (33) and (60a) we obtain

$$|A_n|^2 = \frac{4|\beta_n(k_n)|^2}{(2\alpha + R)^2 + I^2}. \qquad (62)$$

The square of the modulus of A_n, is sketched as a function of α in Fig. 2. Since the maximum occurs at $\alpha = -R/2$, (62) implies that a maximum occurs for k shifted from k_n. In Ref. 1 we showed that the second term in (60b) is always positive. However, we are unable to determine the sign of S_n given in (59), for arbitrary membrane and cavity geometries. Similarly, the sign of the detuning parameter is undetermined, but its dependence upon the fluid densities, χ, is explicit. It follows from (60c) that I, which is the bandwidth of $|A_n|$, is independent of χ and hence of the fluid within the cavity. To evaluate $|A_n|^2$ it is assumed that k_n, θ_n, and ψ_n are known explicity or by numerical computation. Then the integrals that define β_n, R, and I must be similarly determined.

For k near k_n^b the coefficient of the outgoing spherical wave, as given in (58b), is the product of χ, β_n, and the directivity factor $D(k_n^b,\hat{r})$, which is composed of an infinite series of terms involving the directivity factors F_j. The infinite sum is present because the cavity mode ψ_n induces a membrane motion which is expressed as an eigenfunction expansion in the membrane modes θ_j. This accounts for the inner products $\langle\psi_n,\theta_j\rangle$ in (58b) the factor χ in (58b) and its definition (12b) indicate that the scattered field increases as the density of cavity fluid becomes larger. The amplitude B_n contains information about the coupling between the acoustic media and the membrane, which we now describe.

In Appendix B we show that T_n and ℓ given by (46) and (42), respectively, can be expressed as,

$$T_n = \left(\frac{k_n^b}{c}\right)^2 \sum_{j=1}^{\infty} \frac{\langle\theta_j,\psi_n\rangle^2}{[(k_n^b)^2 - k_j^2]^2} > 0, \qquad (63)$$

$$\ell = \frac{\chi}{2c_0^2} \sum_{j=1}^{\infty} \frac{\langle\psi_n,\theta_j\rangle^2}{(k_n^b)^2 - k_j^2}, \qquad (64$$

where ℓ is a real quantity whose sign depends on the membrane and cavity eigenfrequencies and modes, and hence on their geometry. Finally, in Appendix C we show that the parameter b_n, defined in (47) is given by

$$b_n = -R_b - i\, I_b \tag{65a}$$

where R_b and I_b are defined by

$$R_b = (k_n^b c)^2 \iiint_{z>0} \{|\nabla\phi_n^b|^2 - (k_n^b)^2|\phi_n^b|^2\}dxdydz \tag{65b}$$

$$I_b = \frac{c^2}{k_n^b} \int_0^{\pi/2}\int_0^{\pi} |D(k_n^b,\hat{r})|^2 \sin\phi' d\phi' d\theta'. \tag{65c}$$

and D is given in (58c). The constant R_b is proportional to the dimensionless Lagrangian of ϕ_n^b. This potential is caused by the membrane motion $L(k_n^b\, \psi_n)$ which is induced by the potential ψ_n. The total acoustic cross section for k near K_n^b is

$$\sigma_T^b = \frac{\chi^2}{c^2} |B_n|^2\, k_n^b\, I_b \tag{66}$$

Combining (45) and (63)-(65) we obtain

$$|B_n|^2 = \frac{4|\langle\psi_n, L(k^b [illegible] {}^I\rangle|^2}{[\frac{2c_0^2}{c^2}\beta + \frac{c_0^2}{c^2}\ell^2 + 2\ell\chi\, T_n + \chi R_b]^2 + \chi^2 I_b^2} \tag{67}$$

The graph of $|B_n|^2$ as a function of β is similar to the sketch shown in Figure 2 for $|A_n|^2$, and is omitted. We observe from (67) that $|B_n|^2$ has a maximum when

$$\beta = -\tilde{R} = -[2\ell\chi\, T_n + \chi R_b + \frac{c_0^2}{c^2}\ell^2](c^2/c_0^2)/2 \tag{68}$$

whose sign is difficult to determine. The bandwidth of $|B_n|^2$ is I_b.

We now present some observations about the spectrum of the fluid-elastic system. First, if k is near k_n and $\Phi^I = 0$ then the solvability condition (32) gives

$$A_n(2\alpha + a_n) = 0 \tag{69}$$

A nonzero solution of this equation requires $\alpha = - a_n/2$. From (27) and (60) if follows that the fluid-elastic "resonant" frequencies, which are complex, are given by

$$k - k_n[(1 - \frac{\varepsilon}{2} R) - \frac{i\varepsilon}{2} I], \tag{70}$$

to order $O(\varepsilon)$. The negative imaginary part in (70) corresponds to damping because $e^{-i\omega t}$ is the assumed time dependence. Since the decay rate is proportional to I, this parameter measures the ability of the membrane vibrating at frequency k_n, to convert motion into acoustic energy. The effect of the cavity's fluid is contained only in R which shifts the eigenvalue's real part. Similarly, if k is near k_n^b and $\Phi^I = 0$, then

$$k \sim k_n^b[1 - \varepsilon\ell + \varepsilon^2\tilde{R}) - \frac{i\varepsilon^2\chi}{2} (\frac{c}{c_0})^2 I_b] \tag{71}$$

to $O(\varepsilon^2)$. The decay rate for these fluid-elastic resonant frequencies is $O(\varepsilon^2)$, while the decay rate for the fluid-elastic resonant frequencies given in (70) is $O(\varepsilon)$. This is a consequence of the weak coupling of the cavity fluid to the membrane which itself is weakly coupled to the acoustic fluid in $z > 0$.

1.4. The Coalescence of a Membrane and Cavity Eigenvalue

The inner expansions obtained in Sections 1.2B and C are valid for k near k_n and k_m^b, respectively, for $k_n \neq k_m^b$. We now obtain an asymptotic expansion of the solution (2), (6), and (17) that is valid at and near a frequency where $k_n = k_m^b$ for some n and m, and where k_n and k_m^b are simple membrane and cavity resonant frequencies. The analysis is similar to those in and Section 1.2C, and in Section IV of Ref. 7, so all details are omitted.

The relevant small parameter is now $\sqrt{\varepsilon}$ and we obtain an asymptotic expansion of the solution of (2), (6) and (17) in powers of $\sqrt{\varepsilon}$. To leading order we get

$$v = \tilde{A}_n \theta_n + O(\sqrt{\varepsilon}), \quad \Phi^b = \frac{\tilde{B}_m}{\sqrt{\varepsilon}} \psi_m + O(1) \tag{72a}$$

where $\tilde{A}_n$ and $\tilde{B}_m$ are as given by,

$$\tilde{A}_n = \frac{2\beta_n(k_n)}{[4\tilde{\beta} - 2\tilde{\gamma}^2 - 2\hat{\ell} + \hat{a}_n]}, \quad \tilde{B}_m = -\frac{2\tilde{\gamma}}{\chi\langle\psi_m, \theta_n\rangle} \tilde{A}_n \tag{72b}$$

Here: $\beta_n(k_n)$ is given by (23b); $\hat{a}_n$ is given by (34) with a carrot [14] over the operator L_b in (34b); $\hat{\ell}$ is given by (42) with a carrot over the operator L; and $\tilde{\beta}$ is the new stretched variable which is defined by the stretching transformation,

$$k = k_n[1 + \tilde{\gamma}\sqrt{\varepsilon} + \tilde{\beta}\varepsilon], \tag{73a}$$

where $\tilde{\gamma}$ is given by

$$\tilde{\gamma} = \pm \left|\langle\psi_m, \theta_n\rangle\right| \sqrt{\chi}/(2k_n c_o). \tag{73b}$$

We observe that (72) represents two inner expansions corresponding to the two signs in (73b).

The expansions are valid in order $O(\varepsilon)$ neighborhoods of $K_n^{\pm} = k_n(1 \pm \sqrt{\varepsilon}|\tilde{\gamma}|)$. The two expansions occur because the degenerate resonant frequency $k_n = k_m^b$ is "split" by the membrane-cavity interaction.

For either sign in (73b), the inner expansion for the scattered acoustic potential corresponding to (72) is obtained from (14)-(16) and (72) as,

$$\phi = \tilde{A}_n G(\underset{\sim}{x};\ k_n)\theta_n + O(\varepsilon). \tag{74}$$

In the far field as $r \to \infty$ this reduces to

$$\phi = \tilde{A}_n\ F_n(k_n,\hat{r})\ \frac{e^{ik_n r}}{r} + O(\varepsilon). \tag{75}$$

The physics of the membrane-cavity interaction are contained again in the coefficient $\tilde{A}_n$. Using results from Ref. 7 and the arguments used in Appendix A we find that

$$\hat{a}_n = \hat{R} + i\ I \tag{76}$$

where $\hat{R}$ is given by (60b) with the term m = n omitted in the sum and I is given by (60c). Similarly, by using the arguements in Appendix

C, we find that $\hat{\ell}$ is given by (64) with the term j = n omitted in the sum. Thus, we deduce that

$$|\tilde{A}_n|^2 = \frac{4|\beta_n(k_n)|^2}{(4\tilde{\beta} - 2\tilde{\gamma}^2 - 2\hat{\ell} + \hat{R})^2 + I^2}. \tag{77}$$

The graph of this function is qualitatively the same as that of A_n shown in Figure 2. The parameter I is again the bandwidth but now $|\hat{A}_n|^2$ achieves its maximum at $\tilde{\beta}_{max}$ which is defined by

$$\tilde{\beta}_{max} = (2\hat{\ell} + 2\tilde{\gamma}^2 - \hat{R})/4. \tag{78}$$

The total cross section is now given by (61) with $|\tilde{A}_n|^2$ from (77) replacing $|A_n|^2$. Finally, when $\Phi^I = 0$ we can deduce information about the complex resonant frequencies of the fluid-elastic structure. Omitting all the details because they are similar to those described in Section 1.3, we find that

$$k^{\pm} = k_n[1 \pm \sqrt{\varepsilon}|\tilde{\gamma}| - \varepsilon(\hat{R} - 2\tilde{\gamma}^2 - 2\hat{\ell})) - \frac{i\varepsilon}{2} I]. \tag{79}$$

Thus, as we mentioned earlier, the degenerate resonant frequency k_n has been "split" into k^+ and k^-.

1.5 Normal Incidence of a Plane Wave on a Cylinder Backed Circular Membrane

The normally incident plane wave is

$$\Phi^I(\underline{x},k) = e^{-ikz}. \tag{80}$$

The cross section of the cylindrical cavity is circular with the same radius as the membrane. The depth of the cavity is d. Then the membrane displacement, the cavity potential, and the scattered acoustic potential vary only with z and the cylindrical radius $\rho = \sqrt{x^2 + y^2}$. The axisymmetric resonant frequencies and modes of the unit circular membrane are,

$$k_n = \frac{\lambda_n}{c}, \quad \theta_n = \frac{J_0(k_n c_\rho)}{\sqrt{\pi}\, J_0(\lambda_n)}; \tag{81}$$

and of the circular cylindrical cavity are,

$$k_n^b = \frac{1}{c_0}\{z_{1\ell}^2 + (\frac{m\pi}{d})^2\}^{\frac{1}{2}},$$

$$\psi_n = \sqrt{\frac{2}{\pi d}}\, \frac{J_0(z_{1\ell}\rho)}{J_0(z_{1\ell})} \cos(\frac{m\pi}{d}z). \tag{82}$$

Here, λ_n is the n^{th} zero of J_0, and $z_{1\ell}$ is the ℓ^{th} zero of $J_1(z)$. The subscript n in (82) is chosen so that the resonant frequencies form a non-decreasing sequence.

We have selected the cavities depth so that the k_n^b in (82) are simple for $1 \leq n \leq 20$. Since the resonant frequencies of the membrane are also simple, the results of the previous sections apply.

We have numerically evaluated (60), (56a), and (55a) to obtain the amplitude of the scattered potential in the farfield. It is uniformly valid in any interval not containing cavity resonant frequencies. Also, we have numerically evaluated (55b) and (56b) using (52b). These farfield results are valid in any interval not containing membrane resonant frequencies. Finally, we have added these results and removed their common parts to obtain a representation of the farfield that is uniformly valid in k. the resulting magnitude of the amplitude in the backscattered direction, $\hat{r} = (0,0,1)$ is shown by the solid curve in Figure 3, where we have denoted the membrane and cavity resonant frequencies by x's and O's, respectively. The dashed curve represents the result for the vacuum-backed cavity ($\rho_b = \chi = 0$).

We observe that the bandwidth of the response is order $O(\varepsilon)$ near membrane resonant frequencies and order $O(\varepsilon^2)$ near cavity resonant frequencies. The amplitude are $O(1)$ near these frequencies and they are $O(\varepsilon)$ away from these frequencies. The cavity has detuned the membrane response as can be seen by the shift between the solid and the dashed curves. Moreover, this detuning can either increase or decrease the frequency of maximum response. This is because S_n, defined in (59), can take on either positive or negative values, depending on the index of the resonant frequency.

The results presented in Fig. 4 are for the same parameters as in Fig. 3, except that $c_0 = 0.7967$. For this value of c_0 the first cavity resonant frequency and the first membrane resonant frequency coalesce at $k = k_1$. Since the remaining cavity resonant frequencies are simple for $2 \leq n \leq 20$, the results of Section 1.4 are applicable. We have obtained an expansion that is valid in an order $O(\sqrt{\varepsilon})$ neighborhood about k_1 from (72). Combining this result with the outer expansion in the same manner as before we have obtained a uniformly valid approximation to $\hat{A}_n$ given by (72b). Inserting this into (75) we obtain the amplitude of the scattered field as $r \to \infty$. We have plotted the magnitude of this amplitude, in the backscattered direction, in Figure 4. As can be seen, the total bandwidth of the response about k_1 is now order $O(\sqrt{\varepsilon})$. Within this bandwidth packet, the interaction of the cavity and membrane now produces two sharp peaks, each of order $O(\varepsilon)$ wide. The magnitude of these peaks is still order $O(1)$. However, there is a local minimum in the amplitude of order $O(\varepsilon)$ near $k = k_1$.

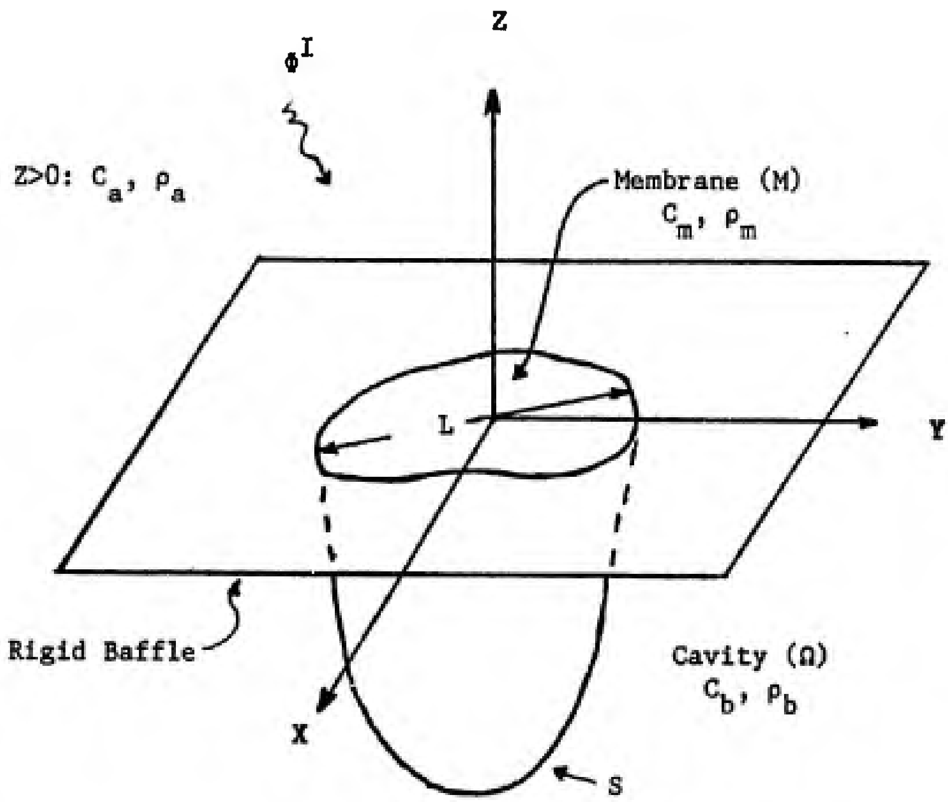


Figure 1: Geometry of the scattering problem.

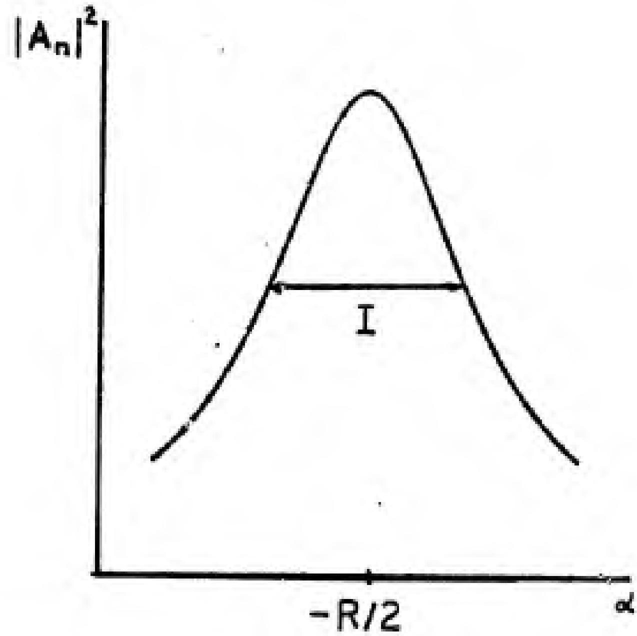


Figure 2: A typical graph of $|A_n|^2$ for a simple membrane eigenvalue. It is graphed with respect to the detuning parameter, α.

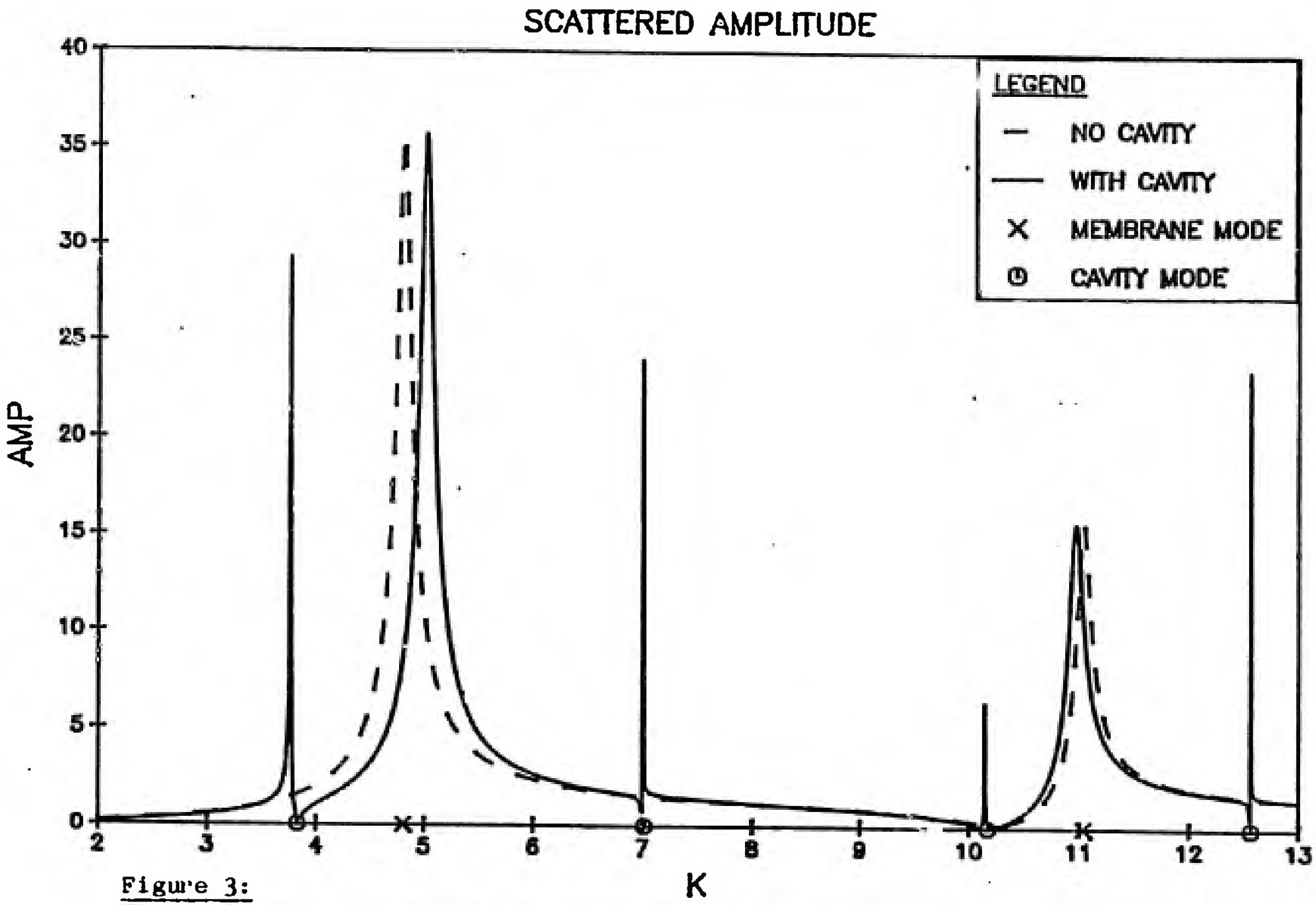


Figure 3:
The solid curve represents the magnitude of the backscatterd, farfield amplitude. The dashed curve represents the same quantity for a vacuum-backed membrane. The parameters are $c = .5$, $c_0 = 1$, $\chi = 2$, $\varepsilon = .1$, and $d = .25$.

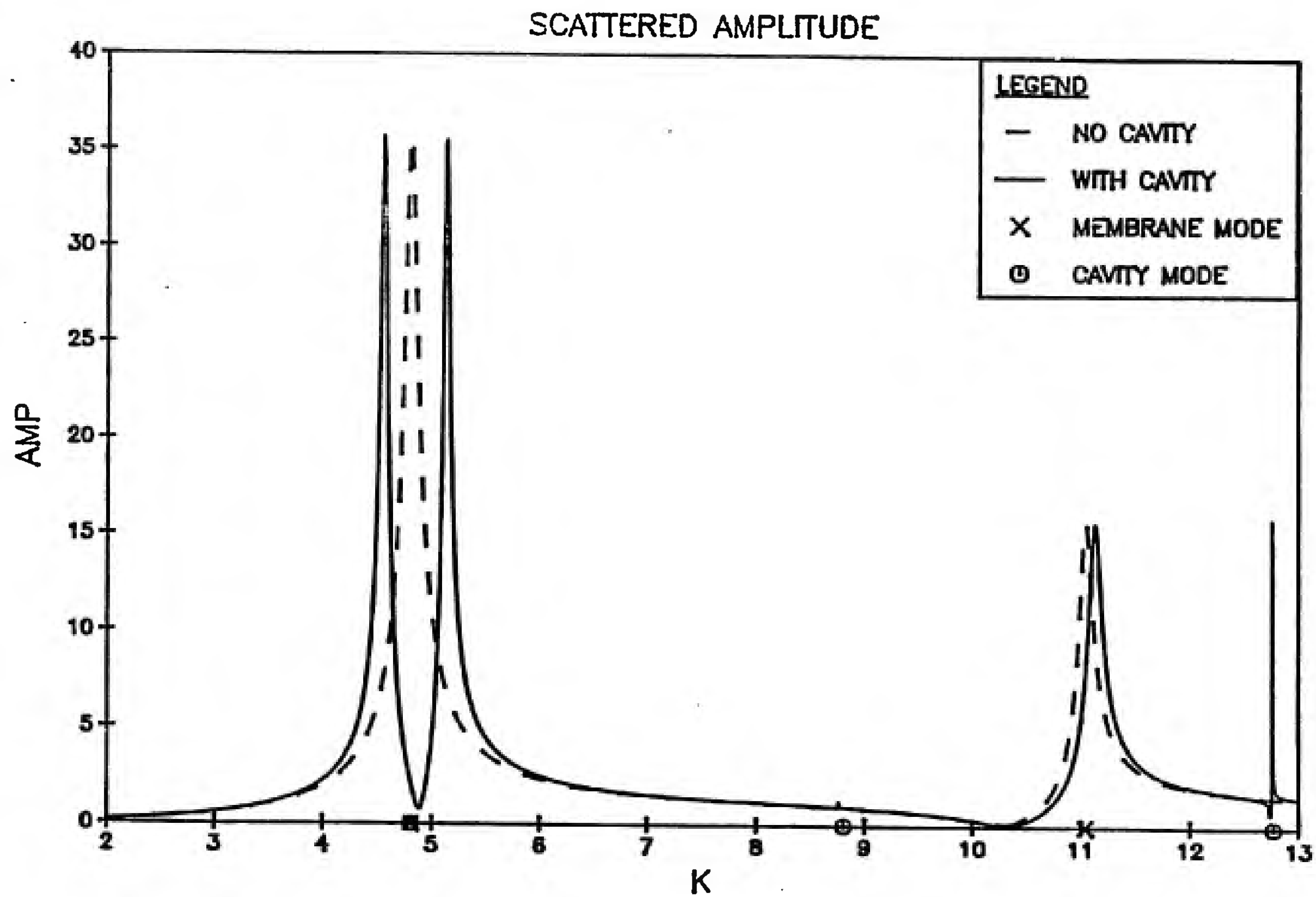


Figure 4: Same as Figure 3, with the first membrane eigenfrequency coalesced with the first cavity eigenfrequency. The parameters are now $c = .5$, $c_0 = .7967$, $\chi = 2$, $\varepsilon = .1$, and $d = .25$.

Chapter 2

Pulse Scattering by Baffled Cavity-Backed Membranes

2.0 Introduction

A thin tightly stretched membrane M is set in an acoustically rigid infinite plane z = 0. The upper half space is a homogeneous acoustic fluid whose density and sound speed are ρ_a and c_a, respectively. An acoustically rigid surface is placed in the lower half plane directly beneath the membrane so that the surface and M together form a cavity. (See Figure 1.) The interior of this cavity is filled with another acoustic fluid whose density and sound speed are ρ_b and c_b, respectively. A pulse, which satisfies the acoustic wave equation in the upper half-space, is incident on this structure.

The scattered field depends on the pulse's structure and the geometric and physical properties of the cavity and the membrane. A pulse with a very narrow bandwidth, such that its spectral content is free of any of the in vacuo natural frequencies of both the membrane and the cavity, will be reflected as though the entire plane z = 0 is rigid. However, if the spectral content of the pulse contains one or more of these natural frequencies, a "near resonance" condition can exist between the pulse and the membrane-cavity structure. This produces a secondary scattered field in addition to the rigidly reflected pulse. The secondary field is a result of the "ringing" of the membrane which is caused by the complicated interaction of the fluid-structure system. It possesses in certain situations a "beating" structure as well as two decay rates. (See Sections 2.3 and 2,4).

Since membrane (and plate) theories are valid essentially only for the "lower" modes of vibration, we assume that the pulse's bandwidth is restricted so that it can only excite the first few modes of the membrane. It may be necessary to consider the flexible region M of the plane as a three-dimensional elastic body if broader bandwidth pulses are considered.

In this paper we develop asymptotic approximations to the scattered field, the membrane displacement, and the cavity pressure, as the parameter $\varepsilon \to 0$. The small parameter ε is defined as the ratio of the fluid and membrane densities, where we have assumed that ρ_a and ρ_b are similar in size. The analysis given in this paper is an extension of a previous work on baffled membranes backed by a vacuum [8]. The modifications required are due to the presence of the cavity.

We shall now briefly outline the remainder of the paper. The problem is formulated in Section 2.1 as an integrodifferential equation which describes the lateral motion of the membrane. The problem is then solved in Section 2.2 by first expanding the membrane's motion in its in vacuo normal modes, which we assume are known either analytically or numerically. The time dependent coefficients in this expansion satisfy an infinite, coupled system of ordinary, integrodifferential equations. They are reduced to a coupled system of algebraic equations by applying the Laplace transform. An asymptotic expansion of the solution of this algebraic system is obtained in the small parameter ϵ, which is obtained under a light fluid loading assumption. Then, by inverting the asymptotic expansion of the Laplace transform, we obtain an integral representation for the membrane's motion. From this result we deduce the scattered field and the pressure within the cavity. In Section 2.3 we specialize the results of Section 2.2 by considering a "spiked" incident pulse. The physical interpretations of these results are also given with special attention payed to the situation where an in vacuo eigenfrequency of the membrane coalesceswith an eigenmode of the cavity. Finally, in Section 2.4 we apply the results of Section 2.3 to the specific physical example of a circular membrane backed by a circular, cylindrical cavity.

2.1 Formulation

In dimensionless variables $x = (x,y,z)$ and t, the acoustic pressures $P(x,t)$ and $P_b(x,t)$ satisfy their respective wave equations

$$\Delta_0 P = \frac{\partial^2 P}{\partial t^2} \quad , \quad z > 0 \tag{1a}$$

$$\Delta_0 P_b = c_0^2 \; \frac{\partial^2 P_b}{\partial t} \quad , \quad x \quad \Omega \tag{1b}$$

where Δ_0 is the three-dimensional Laplacian, $c_0 = c_a/c_b$, and c_a and c_b are the sound speeds in $z > 0$ and the cavity respectively. The dimensionless space variables are obtained by scaling with respect to a characteristic length L of the membrane. The dimensionless time t is obtained by scaling with respect to L/c_a.

The equation of motion for the dimensionless lateral deflection $w(x,y,t)$ of the membrane, which lies in the region M of the plane $z = 0$, is given by

$$\Delta w - c^2 \frac{\partial^2 w}{\partial t^2} = \varepsilon c^2 [P(x,y,0,t) - \chi P_b(x,y,0,t)] \; , \; (x,y) \quad M \tag{2}$$

where Δ is the Laplacian in x and y. In addition we have introduced the notation

$$c = c_a/c_m \; , \; \varepsilon = \frac{\rho_a}{\rho_m} L \; , \; c_m = (T/\rho_m)^{1/2} \; , \; \chi = \rho_b/\rho_a \tag{3}$$

where T and ρ_m are the membrane's tension and density per unit area, and ρ_a and ρ_b are the densities in $z > 0$ and the cavity, respectively. The scale factors for the membrane displacement and the acoustic pressure are related by $L/(\rho c_a^2)$ and the parameter χ is assumed to be an order one quantity. The difference between the acoustic pressures P and χP_b acts as a driving force on the membrane.

Since the plane $z = 0$ is acoustically rigid outside of M, we have the condition

$$P_z(x,y,0,t) = 0 \quad , \quad (x,y) \notin M \quad . \tag{4a}$$

Similarly, since the surface S is rigid we have

$$\frac{\partial}{\partial \nu} P_b = 0 \quad , \quad \underset{\sim}{x} \in S \tag{4b}$$

where $\frac{\partial}{\partial \nu}$ denotes an outward normal derivative on S. The requirement that the acoustic and membrane velocities are continuous on the membrane's surface gives the conditions

$$P_z(x,y,0,t) = - w_{tt}(x,y,t) \quad , \quad (x,y) \in M \tag{5a}$$

$$P_{b,z}(x,y,0,t) = - w_{tt}(x,y,t) \quad , \quad (x,y) \in M \quad . \tag{5b}$$

The acoustic pulse which is incident on the plane $z = 0$ is denoted by $P^I(\underset{\sim}{x},t)$. It is a solution of the wave equation (1a). If the entire plane was rigid, then the incident pulse would be reflected as the pulse $P^R(\underset{\sim}{x},t) = P^I(x,y,-z,t)$, which is also a solution of (1a). Thus, we express the total acoustic pressure in $z > 0$ as

$$P(\mathbf{x},t) = P^I(\mathbf{x},t) + P^R(\mathbf{x},t) + p(\mathbf{x},t) \quad , \tag{6}$$

where $p(\mathbf{x},t)$ is the scattered pressure field that is caused by the presence of the cavity and membrane. By inserting (6) into (1a) and (4a), and (5a), we find that the scattered field satisfies the following problem:

$$\Delta_0 p = p_{tt} \quad , \quad z > 0 \tag{7a}$$

$$p_z(x,y,0,t) = \begin{cases} 0 \quad , & (x,y) \notin M \\ -w_{tt} \quad , & (x,y) \in M \end{cases} \tag{7b}$$

$$\Delta w - c^2 w_{tt} = \varepsilon c^2 [2P^I(x,y,0,t) + p(x,y,0,t) - \chi P_b(x,y,0,t)]$$

$$(x,y) \in M \quad , \tag{7c}$$

$$w(x,y,t) = 0 \quad \text{on} \quad B \quad , \tag{7d}$$

where B is the boundary of M and P_b satisfies (1b), (4b), and (5b). To complete the formulation of the scattering problem, we impose the quiescent initial conditions

$$p(\mathbf{x},0) = p_t(\mathbf{x},0) = 0 \quad , \quad z > 0 \quad , \tag{7e}$$

$$w(x,y,0) = w_t(x,y,0) = 0 \quad , \quad (x,y) \in M \quad , \tag{7f}$$

$$P_b(\mathbf{x},0) = P_{b,t}(\mathbf{x},0) = 0 \quad , \quad \mathbf{x} \in \Omega \tag{7g}$$

and the outgoing wave condition

$$p_r + p_t \to 0 \quad \text{as} \quad r \to \infty \quad , \tag{7h}$$

where $r = |\underset{\sim}{x}|$. Equations (7e) and (7g) imply that the incident pulse reaches the membrane at $t = 0$.

To simplify the analysis of the scattering problem (7), we now reformulate it as a problem for w. Thus we first employ the adjoint Green's functions $G(\underset{\sim}{x},t|\underset{\sim}{x}',t')$ for the half space $z > 0$ which is given by

$$G(\underset{\sim}{x},t|\underset{\sim}{x}',t') = \frac{\delta(t'-t-R)}{4\pi R} + \frac{\delta(t'-t-R_1)}{4\pi R_1} , \tag{8a}$$

where δ is the Dirac delta function and R and R_1 are defined by

$$R = |\underset{\sim}{x}-\underset{\sim}{x}'| \quad , \quad R_1 = [(x-x')^2 + (y-y')^2 + (z+z')^2]^{1/2} \quad . \tag{8b}$$

The function G satisfies

$$\Delta_0 G - G_{tt} = - \delta(t-t')\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad , \quad z > 0 \quad ;$$

$$G_z = 0 \quad , \quad \text{for } z = 0 \quad ; \quad G = G_t = 0 \quad , \quad \text{for } t' < t \quad . \tag{9}$$

By combining Eqs. (7a) and (9), integrating the result over the four-dimensional region, $0 < t < t'$ and $z > 0$, applying the appropriate divergence theorem, and making use of (7b) and (8a), we obtain

$$p(\underset{\sim}{x},t) = \frac{1}{2\pi} \iint_M \frac{w_{tt}(x',y',t-q)}{q} H(t-q)dx'dy' \quad , \tag{10a}$$

where q is defined by

$$q(\underset{\sim}{x};\underset{\sim}{x}') = [(x-x')^2 + (y-y')^2 + z^2]^{1/2} \quad . \tag{10b}$$

Next, we employ the adjoint Green's function $G_b(x,t|x',t')$ for the cavity Ω [12] which is given by

$$G_b(x,t|x',t') = \frac{1}{c_0^2} H(t'-t) \sum_{n=1}^{\infty} \frac{\sin \lambda_n(t'-t)\phi_n(x)\phi_n(x')}{\lambda_n} \tag{11a}$$

where H is the Heaviside function. The functions ϕ_n and the numbers $(c_0\lambda_n)^2$ are respectively the orthonormal eigenfunctions and eigenvalues of the cavity. They satisfy

$$\left.\begin{aligned} &\Delta_0\phi_n + (c_0\lambda_n)^2\phi_n = 0 \quad , \quad x \in \Omega \\ &\frac{\partial\phi_n}{\partial\nu} = 0 \quad , \quad x \in S \\ &\frac{\partial\phi_n}{\partial z} = 0 \quad , \quad x \in M \quad . \end{aligned}\right\} \tag{11b}$$

By using an argument similar to the one used to relate p and w_{tt}, we find that the solution to (1b), (4b), and (5b) is given by

$$P_b(x,t) = -\frac{1}{c_0^2} \sum_{n=1}^{\infty} b_n(t)\phi_n(x) \tag{12a}$$

where

$$b_n(t) = \frac{1}{\lambda_n} \int_0^t \sin \lambda_n(t-t') < w_{tt},\phi_n > dt' \tag{12b}$$

and

$$<w_{tt},\phi_n> = \iint_M w_{tt}(x',y',t')\phi_n(x',y',0)dx'dy' \quad . \tag{12c}$$

Inserting (10a) and (12) into (7c), (7d), and (7f) gives the required integrodifferential equation problem for w as

$$\Delta w - c^2 w_{tt} = \epsilon c^2 \Big(2P^I(x,y,0,t) + \frac{1}{2\pi} \iint_M \frac{w_{tt}(x',y',t-q_0)}{q_0} H(t-q_0)dx'dy' + \frac{\chi}{c_0^2} \sum_{n=1}^{\infty} b_n(t)\phi_n(x,y,0) \Big) \quad \text{for} \quad (x,y) \in M \ , \qquad (13a)$$

$$w(x,y,t) = 0 \quad \text{for} \quad (x,y) \in B \ , \qquad (13b)$$

$$w(x,y,0) = w_t(x,y,0) = 0 \ , \qquad (13c)$$

where H is the Heaviside function and the "cylindrical" radius q_0 is defined by

$$q_0(x,y;x',y') = [(x-x')^2 + (y-y')^2]^{1/2} \ . \qquad (13d)$$

The integral operator in (13a) is proportional to back pressure exerted by the fluid, in $z > 0$, on the membrane while the infinite sum is the corresponding back pressure of the cavity's fluid. Once (13a) is solved for w(x,y,t), the scattered acoustic pressure is given by (10a) and the pressure within the cavity is obtained from (12).

2.2 The Solution of the Integrodifferential Equation Problem

We solve (13) by the eigenfunction expansion method; that is, we seek solutions of (13) in the form

$$w(x,y,t) = \sum_{n=1}^{\infty} w_n(t)\psi_n(x,y) \quad , \tag{14}$$

where the $\psi_n(x,y)$ are the orthonormal, in vacuo eigenfunctions of the membrane. They satisfy

$$\Delta\psi_n + c^2k_n^2\psi_n = 0 \quad , \quad (x,y) \in M \quad , \tag{15a}$$

$$\psi_n = 0 \quad , \quad (x,y) \in B \quad , \tag{15b}$$

where $\mu_n^2 = k_n^2c^2$ is the eigenvalue associated with ψ_n. By virtue of the orthonormality of the eigenfunctions, the modal amplitudes $w_n(t)$, which are given by

$$w_n(t) = \langle\psi_n,w\rangle \tag{16}$$

satisfy the following infinite, coupled, system of integrodifferential equations:

$$Lw_n = 2\varepsilon g_n(t) - \frac{\varepsilon}{2\pi} \sum_{m=1}^{\infty} \iiiint_{M\times M} \psi_n(x',y')\psi_n(x,y)$$

$$\times \frac{H(t-q_0)}{q_0} \frac{d^2w_m}{dt^2}(t-q_0)dx\, dy\, dx'\, dy'$$

$$- \frac{\varepsilon\chi}{c_0^2} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} E_{mn}E_{m\ell} \int_0^t \sin\lambda_n(t-\tau) \frac{d^2}{d\tau^2} w_\ell(\tau)d\tau$$

$$n = 1,2,3,\cdots \quad . \tag{17a}$$

The operator L, the coefficients g_n, and the matrix elements E_{mn} are defined by

$$Lw_n = \frac{d^2 w_n}{dt^2} + k_n^2 w_n \tag{17b}$$

$$g_n(t) = - \langle \psi_n , P^I \rangle \tag{17c}$$

$$E_{mn} = \langle \phi_m , \psi_n \rangle \quad . \tag{17d}$$

From (7f) and (16) it follows that the $w_n(t)$ satisfy the initial conditions

$$w_n(0) = \frac{dw_n(0)}{dt} = 0 \quad , \quad n = 1,2,3,\cdots \quad . \tag{18}$$

The functions $g_n(t)$ are the coefficients of the eigenfunction expansion of the incident pulse evaluated on z = 0.

To solve the system (17) and (18), we first take its Laplace transform. This gives the following infinite, coupled systems of algebraic equations for the Laplace transform $\hat{w}_n$ of w_n:

$$D_n(s,\varepsilon)\hat{w}_n(s,\varepsilon) = [s^2 + k_n^2 + \varepsilon a_{nn}(s) + \frac{\varepsilon\chi}{c_0^2} q_n(s)]\hat{w}_n(s,\varepsilon)$$

$$= 2\varepsilon\hat{g}_n(s) - \varepsilon \sum_{m=1}^{\infty}{}' a_{mn}(s)\hat{w}_m(s;\varepsilon)$$

$$- \frac{\varepsilon\chi}{c_0^2} \sum_{m=1}{}' \left[\sum_{\ell=1}^{\infty} E_{\ell n}E_{\ell m} \frac{s^2}{s^2+\lambda_\ell^2} \right] \hat{w}_m(s,\varepsilon) \quad . \tag{19}$$

Here, s is the transform variable, $\hat{g}_n$ is the transform of $g_n(t)$, the $a_{mn}(s)$ are defined by the fourfold integrals

$$a_{mn}(s) = \frac{s^2}{2\pi} \iiiint\limits_{M\times M} \psi_m(x,y)\psi_n(x',y') \frac{e^{-sq_0}}{q_0} \, dx\, dy\, dx'dy' \qquad (20a)$$

and the $q_n(s)$ are defined by

$$q_n(s) = \sum_{m=1}^{\infty} E_{mn}^2 \frac{s^2}{s^2+\lambda_m^2} \quad . \qquad (20b)$$

The prime on the sum in (19) signifies the omission of the m = n term.

To obtain an asymptotic expansion of the solution of (19) as $\varepsilon \to 0$ that leads to an asymptotic expansion of the solution of (17) and (18) which is uniformly valid in t as $t \to \infty$, we first observe that $\hat{w}_n(s;\varepsilon)$ is $O(\varepsilon)$ since the right side of (19) is $O(\varepsilon)$. Then we seek a solution in the form

$$\hat{w}_n(s;\varepsilon) = \sum_{j=1}^{\infty} \hat{w}_n^{(j)}(s;\varepsilon) \quad , \qquad (21)$$

where the $w^{(j)}$ form an asymptotic sequence [11] in j for each n. In particular, $w_n^{(j)} = O(\varepsilon^j)$ as $\varepsilon \to 0$. Thus, we obtain from (19) the asymptotic approximation

$$\hat{w}_n = 2\hat{g}_n(s)\varepsilon/D_n(s;\varepsilon) + O(\varepsilon^2) \quad . \qquad (22)$$

It is not possible to solve (19) by a regular perturbation expansion in ε since this leads to an expansion for $w_n(t)$ that is unbounded as $t \to \infty$, as we can demonstrate. The zeros of $D_n(s;\varepsilon)$ are approximations to the complex eigenfrequencies of the fluid-structure system [8].

Furthermore, we wish to emphasize that the asymptotic approximation (22) is valid only when μ_n^2 is a simple, in vacuo eigenvalue of the membrane. This is because the $O(\varepsilon^2)$ term becomes $O(\varepsilon)$ when μ_n^2 is not simple. The modification required to handle the case of a multiple eigenvalue will not be discussed here, as it is identical to the one used in reference 8, where no cavity was present in the problem.

We now assume that the in vacuo eigenvalues of the membrane are simple or that the corresponding $\hat{g}_n$ for multiple eigenvalues are zero or negligibly small. Then (22) and the convolution theorem imply that

$$w_n(t;\varepsilon) = 2\varepsilon \int_0^t g_n(\xi) d_n(t-\xi) d\xi + O(\varepsilon^2) \quad , \tag{23}$$

where $d_n(t)$ is the inverse Laplace transform of $D_n^{-1}(s;\varepsilon)$. To determine this function we first obtain the zeros of the nonlinear equation $D_n(s;\varepsilon) = 0$. Using a regular asymptotic expansion as $\varepsilon \to 0$

we find that the roots of this equation break up into two sets. The first contains the two complex numbers

$$s_1 = - \varepsilon\gamma_n + i\Gamma_n + O(\varepsilon^2) \tag{24a}$$

$$s_2 = s_1^* \tag{24b}$$

where * denotes complex conjugation and

$$\gamma_n = \frac{I_n(k_n)}{2k_n} \tag{24c}$$

$$\Gamma_n = k_n + \frac{\varepsilon}{2k_n}\left[R_n - \frac{\chi}{c_0^2}\sum_{m=1}^{\infty}\frac{k_n^2 E_{mn}^2}{\lambda_m^2 - k_n^2}\right] . \tag{24d}$$

The quantities R_n and I_n are the real and imaginary parts of $a_{nn}(-ik_n)$, respectively. As in the analysis in Appendix A of Ref. 7 we can show that

$$R_n = - k_n^2 \iiint_{z>0} (|\nabla\phi_n|^2 - k_n^2|\phi_n|^2)dx\, dy\, dz \quad ,$$

(25a)

$$I_n(k_n) = k_n^3 \int_0^{\pi}\int_0^{\pi/2} |F_n(k_n,\hat{r})|^2 \sin\phi' d\phi' d\Theta > 0 \quad , \tag{25b}$$

where the directivity factors $F_n(k_n,\hat{r})$ and the scattered acoustic potentials $\phi_n(x)$ are produced by the membrane vibrating with frequency μ_n and mode $\psi_n(x,y)$. They are given, respectively, by

$$F_n(k,\hat{r}) = - \frac{1}{2\pi}\iint_M e^{-ik\,\hat{r}\cdot\gamma}\psi_n(\xi,\eta)d\xi\, d\eta \quad , \tag{26a}$$

$$\phi_n(x) = -\frac{1}{2\pi}\iint_M \frac{e^{ik_n q(x;x')}}{q(x;x')}\psi_n(x',y')dx'dy' \quad , \tag{26b}$$

where $\hat{r} = x/|x|$ is the unit vector in the observation direction and Υ is the vector with components $(\xi,\eta,0)$. The F_n are the Fourier transforms of the modes ψ_n with respect to the observation direction. In addition, I_n/k_n^3 is the total cross section of ϕ_n and R_n/k_n^2 is twice the corresponding dimensionless Lagrangian.

The second set of roots contains an infinite number of elements which are given by

$$z_m^1 = iH_m - \varepsilon^2 h_m \tag{27a}$$

$$z_m^2 = (z_m^1)^* \quad , \quad m = 1,2,3,\cdots \tag{27b}$$

where h_m and H_m are defined by

$$H_m = \lambda_m + \frac{\chi\varepsilon E_{mn}^2}{2c_0^2}\cdot\frac{\lambda_m}{\lambda_m^2-k_n^2} + O(\varepsilon^2) \tag{27c}$$

$$h_m = \frac{\chi E_{mn}^2}{2c_0^2}\ \frac{\lambda_m}{(\lambda_m^2-k_n^2)^2} I_n(\lambda_m) + O(\varepsilon) \quad . \tag{27d}$$

These roots are perturbations of the cavity's eigenvalues whereas the ones given in (24) are perturbations of the membrane's in vacuo eigenfrequencies. They are valid when the eigenvalues of the membrane, in vacuo, and the eigenvalues of the cavity are distinct, so that the term $\lambda_m^2 - k_n^2$ in the denominators of (24) and (27) never vanishes. The roots given by (24) and (27) are the complex eigenfrequencies of the fluid-structure system [13].

If an in vacuo eigenvalue of the membrane and a cavity eigenvalue coalesce, say $k_n = \lambda_\ell$, then (27) is valid, except for $m = \ell$, and (24) is invalid. To rectify this, we seek a new asymptotic expansion, for the coalesced root of $D_n(s,\varepsilon) = 0$, as a power series in $\sqrt{\varepsilon}$. This choice is motivated by the analysis in Reference 13 and gives the four roots

$$s_1 = -\frac{\varepsilon}{2}\gamma_n + i\Omega_n^+ + O(\varepsilon) \tag{28a}$$

$$s_2 = -\frac{\varepsilon}{2}\gamma_n + i\Omega_n^- + O(\varepsilon) \tag{28b}$$

$$s_3 = s_1^* \quad , \quad s_4 = s_2^* \tag{28c}$$

where γ_n is given by (24c) and

$$\Omega_n^{\pm} = k_n \pm \sqrt{\varepsilon}\,\hat{\Gamma}_n \quad , \quad \hat{\Gamma}_n = \frac{\sqrt{\chi}}{2c_0} E_{\ell n} \quad . \tag{28d}$$

It follows from (24) and (27) by using standard residue calculus that the $d_n(t)$ are given by

$$d_n(t) = H(t)\left[\left(e^{-\varepsilon\gamma_n t}\right)\frac{\sin(\Gamma_n t)}{k_n} - \frac{\varepsilon\chi}{c_0^2}\sum_{m=1}^{\infty}\frac{\lambda_m^2 E_{mn}^2}{(\lambda_m^2-k_n^2)^2}\, e^{-\varepsilon^2 h_m t}\right.$$

$$\left.\times\frac{\sin(H_m t)}{\lambda_m} + O(\varepsilon e^{-\varepsilon\gamma_n t})\right] \tag{29}$$

when the cavity and membrane (in vacuo) eigenfrequencies are distinct. Similarly, if $\lambda_\ell = k_n$, then (27) and (28) imply

$$d_n(t) = H(t)\left[\frac{e}{2}^{-\frac{\varepsilon}{2}\gamma_n t}\left(\frac{\sin(\Omega_n^+ t)}{\Omega_n^+} + \frac{\sin(\Omega_n^- t)}{\Omega_n^-}\right) - \frac{\varepsilon\chi}{c_0^2}\sum_{\substack{m=1\\ m\neq\ell}}^{\infty} \frac{\lambda_m^2 E_{mn}^2}{(\lambda_m^2 - k_n^2)^2} \times e^{-\varepsilon^2 h_m t}\,\frac{\sin(H_m t)}{\lambda_m} + O(\sqrt{\varepsilon}\, e^{-\varepsilon\gamma_n t})\right] \qquad (30)$$

Combining either (29) or (30) with (23), (17c) and (14), and interchanging the order of summation and integration, we find that the membrane's displacement is given by

$$w(x,y,t,\varepsilon) = -2\varepsilon \int_M\!\!\int\!\!\int_0^t P^I(\alpha,\beta,0,\xi) \times K(x,y;\alpha,\beta;t-\xi)\,d\xi\, d\alpha\, d\beta + O(\varepsilon^2) \quad , \qquad (31a)$$

where the kernel K is defined by

$$K(x,y;\alpha,\beta;t) = \sum_{n=1}^{\infty} \psi_n(x,y)\psi_n(\alpha,\beta)d_n(t) \quad . \qquad (31b)$$

The scattered pressure is then obtained by inserting (31) into (10) and the pressure in the cavity is then deduced by combining (31) and (12).

2.3 The Normally Incident, Plane, Spiked Pulse

In this section we evaluate (31), the induced cavity pressure, and the corresponding far field, scattered pressure, and we physically interpret these results for a simple incident pulse. This pulse is given by

$$P^I(x,y,z,t) = \delta(z + t) \quad , \quad t > 0 \quad , \tag{32}$$

so that it "touches" the membrane at $t = 0$. Inserting into (31) and performing the t integration, we find that

$$w(x,y,t;\varepsilon) = -2\varepsilon \iint_M K(x,y;\alpha,\beta;t)d\alpha\, d\beta + O(\varepsilon^2) \quad . \tag{33}$$

Thus, the integral of the kernel (31b) over the membrane M is the response of the membrane to the pulse (32). Combining (31b) with (33), we obtain

$$w(x,y,t;\varepsilon) = -\,2\varepsilon \sum_{n=1}^{\infty} \langle\psi_n,1\rangle d_n(t)\psi_n(x,y) + O(\varepsilon^2) \quad , \tag{34}$$

where the inner product $\langle\psi_n,1\rangle$ is defined in (12c).

When the in vacuo eigenfrequencies of the membrane are distinct from the cavity eigenmodes we may insert (29) into (34) to obtain the membrane's motion. It is evident that w is the superposition of modes whose amplitudes evolve according to (29). The first term in (29) is dominant for bounded times. Thus, the amplitudes oscillate at a radian frequency of Γ_n which is a slight perturbation of the in vacuo result, k_n. The correction is given in (24d) where the term proportional to R_n is caused by the fluid in $z > 0$ and the term proportional to χ is due to the cavity's fluid. In fact when $\chi = 0$, the result reduces to the one presented in Reference 8. The amplitude also decays on a slow time scale, because of the exponential, $\exp(-\epsilon\gamma_n t)$, which is caused by the weak radiation of energy into the fluid in $z > 0$. However for large times of the order $O(\frac{1}{\epsilon} \ln(1/\epsilon))$, the second term in (29) is the dominant one. The amplitudes then are order $O(\epsilon)$ and are given by a superposition of oscillatory functions. Each of these is damped on a very slow time scale, because of the exponential terms $\exp(-\epsilon^2 h_m t)$, and oscillates at a radian frequency H_m, which is a slight perturbation of the cavity eigenfrequency λ_m. The damping rates are proportional to ϵ^2 rather than ϵ because the cavity is weakly coupled to the membrane which in turn is weakly coupled to the surrounding acoustic fluid. The damping rates are also proportional to E_{mn}^2 which is the square of the projection of ϕ_m onto the ψ_n and is a measure of how efficiently a cavity mode couples to a membrane mode. Finally, we note that the membrane modes are dominant for order $O(1)$ times and the cavity modes are dominant for large times.

If an in vacuo eigenvalue of the membrane and an eigenvalue of the cavity coalesce ($k_n = \lambda_\ell$), then we can insert (30) into (34) to obtain the membrane's motion. For large times of order $O(\frac{1}{\epsilon} \ln(1/\epsilon))$ the second term in (30) is domiant and the physical interpretation of the results is the same as described above. However, for order O(1) times the first term dominates and the results are slightly different. First, the damping rate is $\frac{1}{2}\varepsilon\gamma_n$ so that the decay takes twice as long. This is caused by the mechanism of energy being transferred efficiently between the membrane and cavity but inefficiently to the ambient fluid, $z > 0$. That is, the energy is stored in the mechanical structure for a longer time. Secondly, the mode not only oscillates at the in vacuo eigenfrequency k_n but its amplitude is slowly modulated by $\cos(\sqrt{\varepsilon}\,\hat{\Gamma}_n)t$, as can be seen by expanding $\Omega_n^{\pm}$. The radian frequency $\sqrt{\varepsilon}\,\hat{\Gamma}_n$ depends upon χ and the modal coupling factor $E_{\ell n}$ as seen in (28d).

The pressure within the cavity is obtained by combining (34), (29), and (12) when the eigenmodes of the cavity are distinct from the in vacuo eigenfrequencies of the membrane. Omitting the details we find that

$$P_b(x,t) = \frac{2\varepsilon}{c_0^2} H(t) \sum_{j=1}^{\infty} \left[A_j e^{-\varepsilon^2 h_j t} \sin(H_j t) + B_j(t) \right] \phi_j(x) \tag{35a}$$

$$A_j = \lambda_j \sum_{m=1}^{\infty} \frac{E_{jm}\langle\psi_m, 1\rangle}{\lambda_j^2 - k_m^2} \tag{35b}$$

$$B_j(t) = -\sum_{m=1}^{\infty} \frac{k_m E_{jm}\langle\psi_m, 1\rangle}{\lambda_j^2 - k_m^2} e^{-\varepsilon\gamma_m t} \sin(\Gamma_m t) \quad . \tag{35c}$$

Thus, P_b is an order $O(\varepsilon)$ pressure which is given by a superposition of cavity modes. The modal amplitude $B_\ell(t)$ is a complicated function given by the infinite sum in (35c). Each term of this sum decays slowly as a function of εt and oscillates at a radian frequency Γ_n. The other amplitude in (35a) oscillates at the radian frequency H_ℓ and decays on a slower time scale. This term will dominate $B_\ell(t)$ at large times.

When a cavity mode λ_ℓ coalesces with an in vacuo eigenfrequency k_n then the pressure P_b is found by combining (34), (30), and (12). Omitting all the details we find that

$$P_b(x,t) = \bar{P}_b(x,t) + \frac{\sqrt{\varepsilon}}{c_0^2\hat{\Gamma}_n} \cos(k_n t)\sin(\sqrt{\varepsilon}\,\hat{\Gamma}_n t)e^{-\frac{\varepsilon\gamma_n t}{2}} \tag{36}$$

where $\bar{P}_b$ is given by (35), interpreted as a double sum, with the single term $j = \ell$, $m = n$ omitted. The most striking feature here is the size of the second term in (36). It is order $O(1/\sqrt{\varepsilon})$ larger than the first and indicates the strong tendency of the system to store energy by exchanging it between the cavity and membrane. This is again born-out by the decay rate $\frac{\varepsilon\gamma_n}{2}$ which is one half the value of the other modes. The second term also gives rise to a "beating" response due to the slow modulation of $\cos(k_n t)$ by $\sin(\sqrt{\varepsilon}\,\hat{\Gamma}_n t)$. This phenomena also occurs in simple mechanical systems where two spring-mass oscillators of the same frequency are weakly coupled together.

The scattered pressure is obtained by combining (10), (34), and (29) when there is no coalescence of cavity and membrane (in vacuo) eigenmodes. Using the law of cosines to expand q in (10b) for $r >> 1$ and (26a) this result simplifies to

$$p \sim -\frac{\varepsilon}{\pi}\left(\sum_{n=1}^{\infty} \langle\psi_n,1\rangle^2\right)\frac{\delta(t-r)}{r} - 2\varepsilon H(t-r)\sum_{j=1}^{\infty} k_j\langle\psi_j,1\rangle\frac{p_j(t-r,\hat{r},\varepsilon)}{r} + \frac{2\varepsilon^2}{c_0^2}H(t-r)\sum_{j=1}^{\infty}\langle\psi_j,1\rangle\frac{q_j(t-r,\hat{r},\varepsilon)}{r} \tag{37a}$$

where

$$p_j(t-r,\hat{r},\varepsilon) = e^{-\varepsilon\gamma_j(t-r)}\left|F_j(k_j,\hat{r})\right|\sin(\Gamma_j(t-r)-\theta_j) \tag{37b}$$

and

$$q_j(t-r,\hat{r},\varepsilon) = \sum_{m=1}^{\infty}\frac{\lambda_m^3}{(\lambda_m^2-k_j^2)^2}e^{-\varepsilon^2 h_m(t-r)}\left|F_j(\lambda_m,\hat{r})\right|\sin(H_m(t-r)-\phi_m) \quad . \tag{37c}$$

The angles θ_j and ϕ_m are the phases of $F_j(k_j,\hat{r})$ and $F_j(\lambda_m,\hat{r})$, respectively.

The total pressure at a point in the farfield consists first of the reflected spike $P^R = \delta(t-z)$ that passes the point at the instant

$t = z$. It is then followed by the smaller-amplitude $[O(\varepsilon)]$ scattered pressure (37) that arrives at $t = r \geq z$. This wave is composed of three parts. The first is an outgoing spherical spike that corresponds to the first term in (37a). It only acts at the instant $t = r$. The second component is the superposition of outgoing spherical pulses that decay slowly as functions of $t - r$ and whose angular behavior is described by the directivity factors $F_j(k_j,\hat{r})$. It acts, for all $t > r$, and has a decay rate of $\varepsilon\gamma_j$. For large times of order $O(\frac{1}{\varepsilon} \ln \frac{1}{\varepsilon})$ the third term in (37a) dominates and again shows the accumulative effect of the very weak coupling between the cavity and the exterior fluid.

Finally, when an in vacuo eigenfrequency k_n of the membrane and a cavity mode λ_ℓ coalesce, the scattered pressure is obtained by combining (10), (34), and (30). In the farfield this result becomes

$$p = \tilde{p} + \varepsilon k_n |F_n(k_n,\hat{r})| \langle\psi_n, 1\rangle \frac{H(t-r)}{r} e^{-\frac{\varepsilon\gamma_n}{2}(t-r)}$$

$$\times \cos[\sqrt{\varepsilon}\hat{\Gamma}_n(t-r)]\sin[k_n(t-r)\theta_n] \qquad (38)$$

where θ_n is the phase of $F_n(k_n,\hat{r})$. The function $\tilde{p}$ is given by (37) with the $j = n$ term omitted from the second sum and the term $j = n$, $m = \ell$ omitted from the third. Thus, most of the remarks made after (37) are still valid except now, the second term in (38) needs a word of explanation. It is an outward going spherical wave which is damped at a rate of $\frac{\varepsilon}{2}\gamma_n$ which is one half the value of the other modes. Moreover, it is slowly modulated by the cosine term which gives rise to the same beating phenomenon as experienced by the cavity pressure and membrane motion.

2.4 A Numerical Example

We shall now illustrate some of the results of the previous section for a circular membrane backed by a cylindrical cavity. The cross section of the cavity is identical to that of the membrane and the depth of the cavity is d. Since the incident pulse (32) depends only spatially upon z, the circular symmetry of the structure implies that the membrane displacement, cavity pressure, and scattered pressure vary only with z and the cylindrical radius $\rho = \sqrt{x^2 + y^2}$. The axisymmetric eigenvalues and eigenfrequencies of the unit circular membrane are

$$k_n = \frac{x_n}{c}, \quad \psi_n = \frac{J_0\,(k_n c\rho)}{\sqrt{\pi}\,J_0'(x_n)} \tag{39}$$

where x_n is the n^{th} root of $J_0(\lambda) = 0$. Similarly, the axisymmetric eigenvalues and eigenfunctions of the circular cavity are

$$\lambda_n = \frac{1}{c_0}\left[z_{1\ell}^2 + \left(\frac{m\pi}{d}\right)^2\right]^{1/2} \tag{40a}$$

$$\phi_n = \sqrt{\frac{2}{\pi d}}\;\frac{J_0(z_{1\ell}\rho)}{J_0(z_1\ell)}\;\cos\left(\frac{m\pi}{d}\,z\right) \tag{40b}$$

where $z_{1\ell}$ is the ℓ^{th} zero of $J_1(z) = 0$. The integer n in (40b) is chosen so that the eigenvalues form an increasing sequence.

We have adjusted tne depth of the cavity to insure that the eigenvalues λ_n are simple for $1 \leq n \leq 20$ and that $\lambda_n \neq k_m$. Then, since the eigenvalues of the in vacuo membrane are also simple, the results of sections 2.2 and 2.3 are valid. Specifically, we have taken $c = 0.5$, $c_0 = 1.0$, $\chi = 2$, $d = 0.25$ and $\varepsilon = 0.1$.

We shall now exhibit the graph of $w(0,0,t)$ which is representative of the results described in Section 2.3. Accordingly, we have combined (17d), (24c)-(26d), (29), (39)-(40), and (33) and have numerically performed the required integrations and summations. Omitting all the details of these numerical considerations, our result is shown in Figures (5a) and (5b) as a solid curve. The dashed graph is $\tilde{w}(0,0,t)$, the membrane displacement without the cavity, and is given for comparison. It is found by setting $\chi = 0$ in (24d) and (29). There are several observations to be made which illustrate the results of the preceeding section. First, the two wave forms are initially in phase but after a short time $w(0,0,t)$ leads $\tilde{w}(0,0,t)$. This is because of the additional term in (24d) which is proportional to χ. The two functions then go in and out of phase as t increases. Secondly, $w(0,0,t)$ is an oscillatory function which undergoes a slow decay as t increases. This ringing phenomenon is caused by the excitement of the membrane modes by the incident delta function. Finally, for large enough time, $t \sim 30$, the second term in (29) becomes dominant. This decays at a much slower rate $(\varepsilon^2\gamma_{\pi})$ as can be seen by direct comparison with $\tilde{w}(0,0,t)$.

In the next numerical example we hose the same parameters as before except that $c_0 = 0.7967$. The value of c_0 was selected to coalesce the first cavity eigenmode with the first in vacuo eigenfrequency of the membrane. The remaining cavity and in vacuo membrane eigenmodes are simple and do not coalesce for $2 \leq n \leq 20$. We have performed the same numerical calculations as before with (30) replacing (29). The results of our effort are shown in Figures (6a) and (6b) as a solid curve. In addition to the previous remarks there are two striking features of this graph. First, the decay rate is initially one half as large as the previous case. Secondly, the beating phenomenon is quite pronounced for $t \geq 30$.

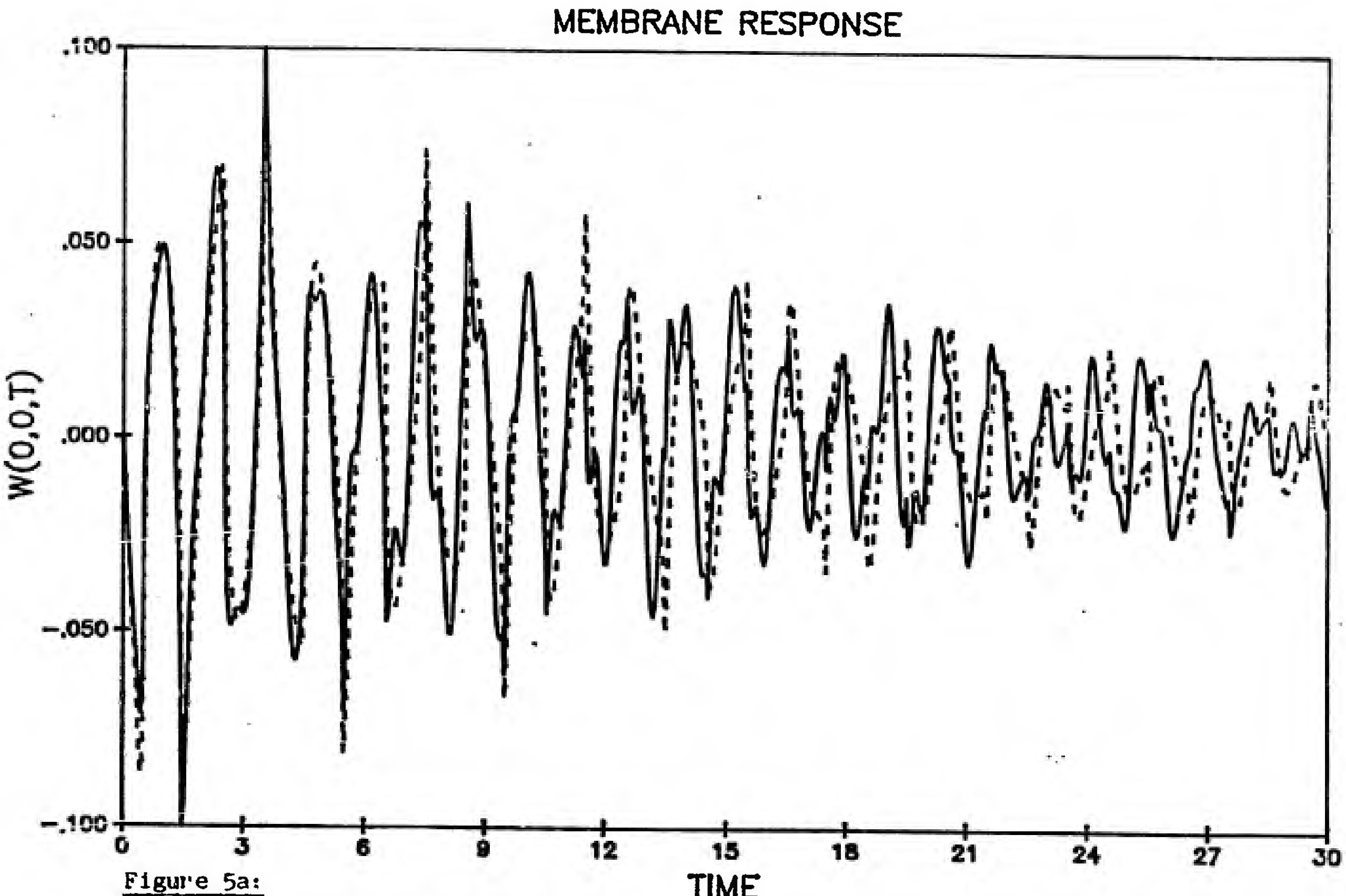
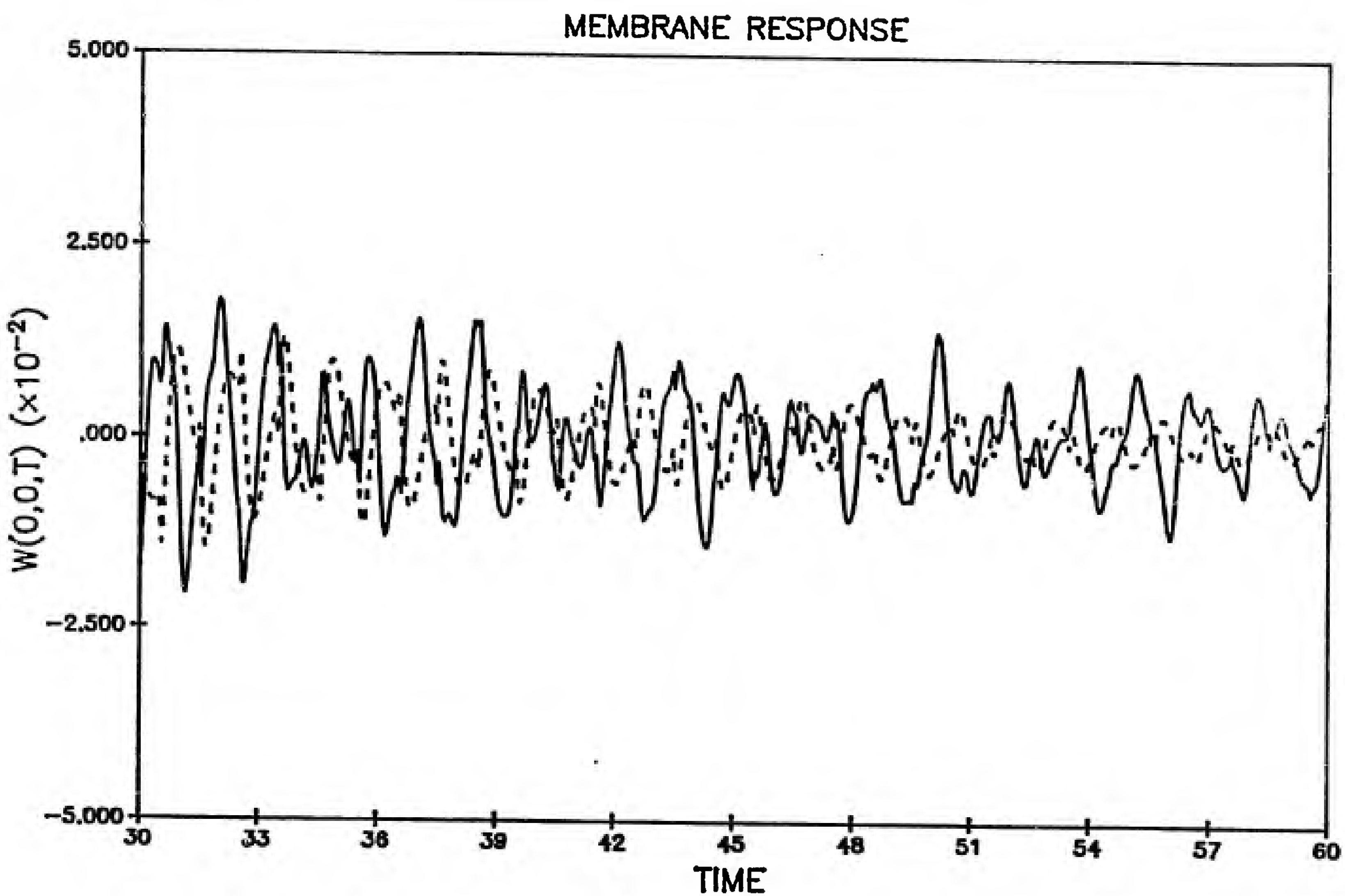


Figure 5a:
The membrane displacement at $x = y = 0$ as a function of time is shown as the solid curve. The dashed curve is the same displacement in the absence of the cavity. The parameters are $c = .5$, $c_0 = 1$, $\chi = 2$, $d = .25$, $\varepsilon = .1$, $0 \le t \le 30$.



<u>Figure 5b:</u> The same as figure 5a for $t \geq 30$ except there is now a scale change.

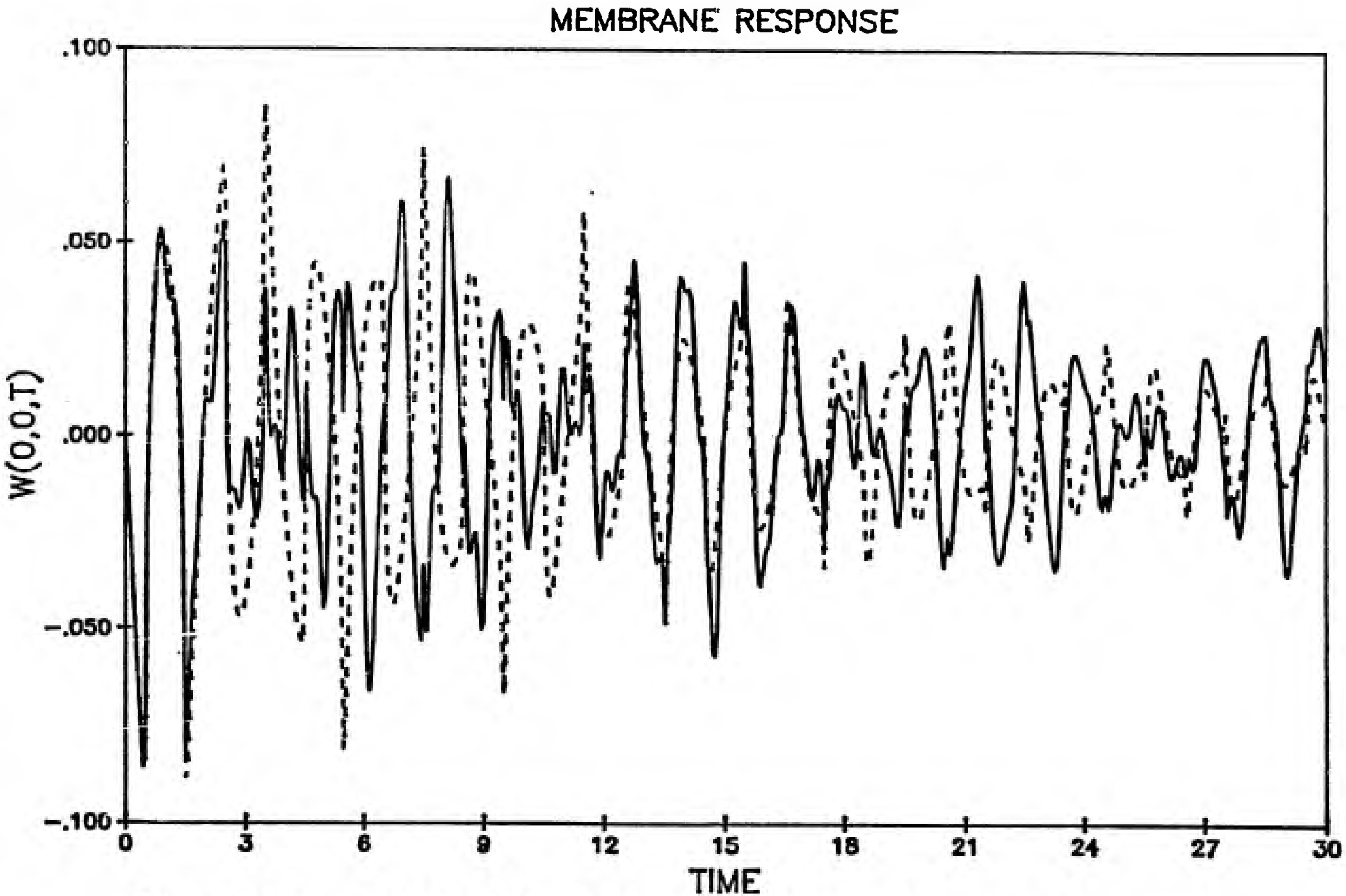


Figure 6a: Same as figure 5a except now c_0 = .7967 and there is a coalescence of modes.

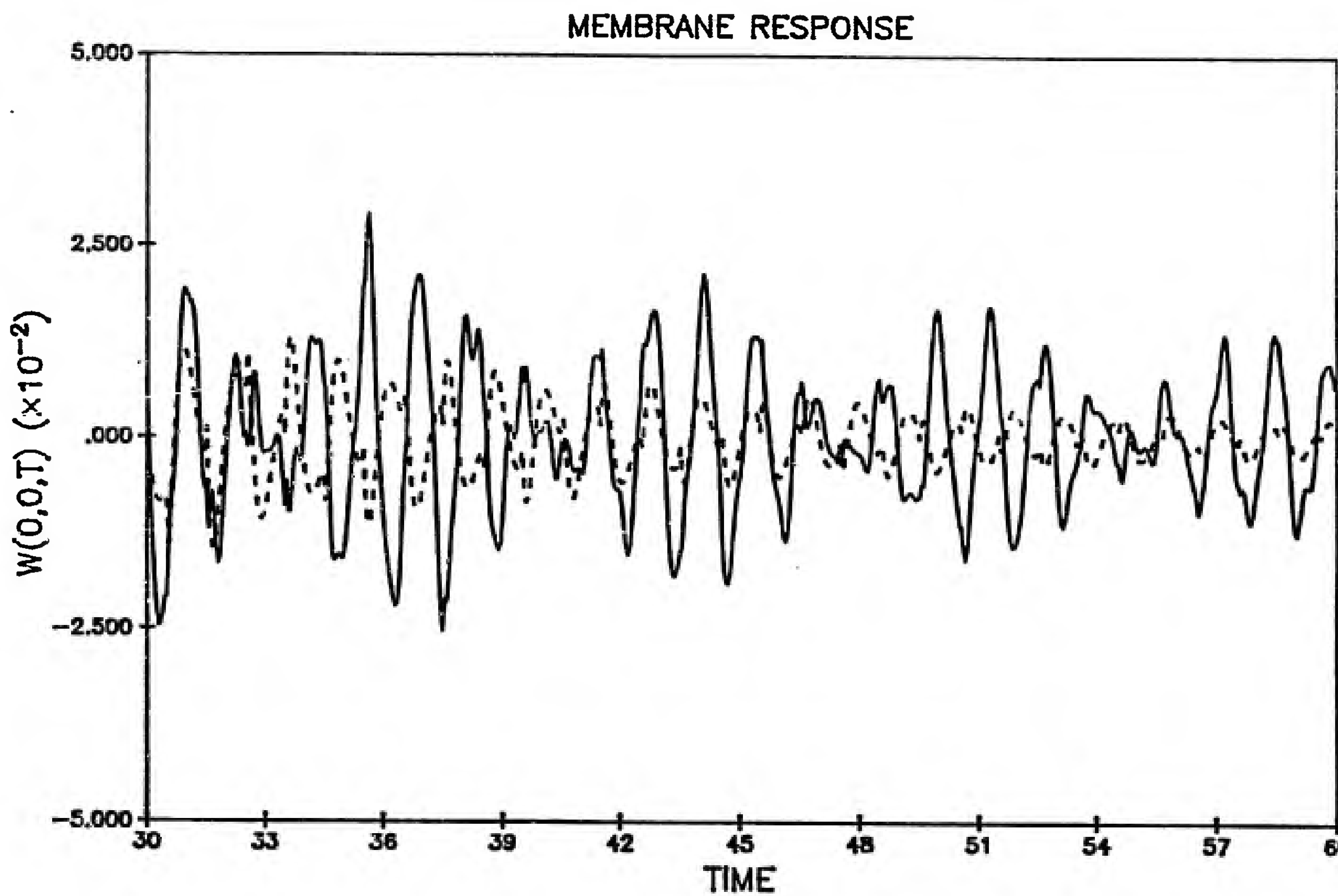


Figure 6b: Same as figure 6a with a scale shift and $t \geq 30$.

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Appendix A

From the definition of L_b in (24) it is clear that

$$L_b(k_n)\theta_n = \frac{1}{c_0^2} \sum_{m=1}^{\infty} \frac{\langle \psi_m, \theta_n \rangle}{(k_m^b)^2 - k_n^2} \psi_m . \tag{A1}$$

Inserting this result into the definition of S_n in (34b) and interchanging the order of summation and integration, we obtain (59).

Appendix B

From the definition of L in (23) we deduce that

$$L\,(k_n^b)\psi_n = \frac{1}{c^2} \sum_{j=1}^{\infty} \frac{\langle \theta_j, \psi_n \rangle}{(k_n^b)^2 - k_j^2}\, \theta_j . \tag{B1}$$

Applying the operator $L(k_n^b)$ to this expression we find that

$$L(k_n^b)\, L(k_n^b)\psi_n = \frac{1}{c^4} \sum_{\ell=1}^{\infty} \frac{\langle \theta_\ell, \psi_n \rangle}{[(k_n^b)^2 - k_\ell^2]^2}\, \theta_\ell \tag{B2}$$

where we have used $\langle \theta_\ell, \theta_j \rangle = \delta_{ij}$. Finally, we combine (B2) with (46) to deduce (63).

Inserting the formula (B1) into the definition of ℓ in (42) we deduce (64).

Appendix C

Let w_1 be the membrane displacement caused by the cavity eigenmode ψ_n. That is, $w_1(x,y)$ satisfies

$$H(k_n^b)w_1 = \psi_n, \quad w_1 = 0 \quad \text{on } B \tag{C1}$$

and hence is given by

$$w_1 = L(k_n^b)\,\psi_n. \tag{C2}$$

Next, let ϕ_n^b be the acoustic pressure generated by w_1. Then ϕ_n^b satisifes

$$\nabla_0^2\phi_n^b + (k_n^b)^2\,\phi_n = 0$$

$$\frac{\partial\phi_1}{\partial z} = w_1, \quad (x,y)\varepsilon M$$

$$\frac{\partial\phi_1}{\partial z} = 0, \quad (x,y)\varepsilon M. \tag{C3}$$

From (14)-(16) we deduce that

$$\phi_n^b = G(k_n^b)w_1. \tag{C4}$$

Finally we let w_2 be the membrane motion induced by ϕ_n^b, i.e.,

$$H(k_n^b)w_2 = \phi_n^b, \quad w_2 = 0 \quad \text{on } B \tag{C5}$$

The w_2 is given by $L(k_n^b)\phi_n^b$. Combining this result with (C4) and (C2) we obtain

$$w_2 = L(k_n^b)\ G(k_n^b)\ L(k_n^b)\psi_n. \tag{C6}$$

Multiplying (C5) by w_1 and (C1) by w_2 and subtracting the result we find

$$\nabla \cdot (w_1 \nabla w_2 - w_2 \nabla w_1) = w_1 \phi_n^b - \psi_n w_2. \tag{C7}$$

Integrating this result over M, using the divergence theorem, applying the boundary conditions on B, using (C6), and noting the definition of b_n in (47), we arrive at

$$\frac{b_n}{(k_n^b c)^2} = \int\int_M \phi_n^b w_1 dxdy. \tag{C8}$$

The integral in (C8) is the complex power generated by the membrane displacement w_1. Hence, using the same arguement as presented in Reference [8] we obtained (65).

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