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DEVELOPMENT OF ADVANCED CONSTITUTIVE MODELS FOR PLAIN
AND REINFORCED CONCRETE(U) S-CUBED LA JOLLA CA
G A HEGENIER ET AL. 01 APR 86 555-R-86-7914

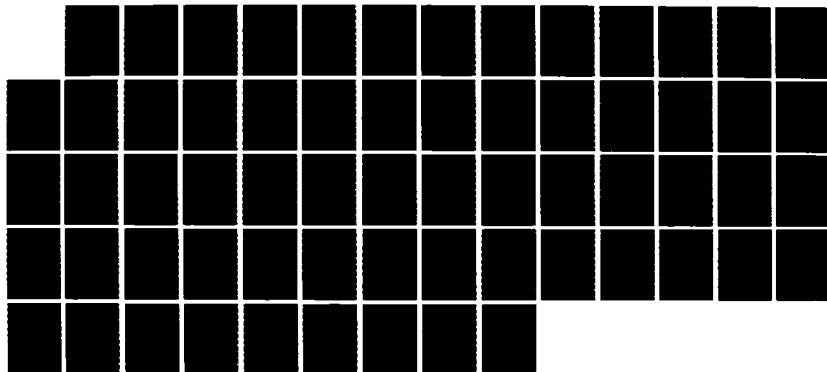
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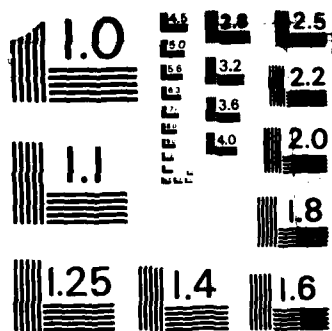
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Toward the end of the research period, exploration of a new method commenced. This procedure, which appears to furnish the desired mathematical simplicity, will be studied further during the forthcoming research period.

(d) Finite Rotations. Some problems, require the proper description of geometric nonlinearities. One such problem concerns dowel action. In a narrow zone adjacent to the crack/slip plane, the dowels that penetrate this plane undergo finite rotations. In order to accommodate this effect, finite rotations were included in the mixture formulation. This phenomenon was modeled using a von Karman-type approximation that is valid for finite but moderate rotations. Details concerning this effort can be found in paper No. 12 of Section 5.

4.4.2 Model Validations

(a) Behavior in Shear (the Dowel Problem). A considerable effort was made during the research period to validate the mixture formulation in shear. For this purpose the dowel problem, which was originally investigated during an earlier research period, was examined again in detail. The results of this examination led to the reformulation of the mixture model to accommodate finite rotations as was noted above. Validations were then re-conducted. These validations consisted of comparisons of simulated versus measured behavior for the problem described in subsection 4.4.2. Details concerning this investigation are contained in paper No. 12 of Section 5. Typical theoretical-experimental comparisons were presented in subsection 4.4.2. These comparisons indicate that the current mixture description furnishes excellent global response simulation capability in the above shear mode.

(b) Behavior in Bending. An attempt was made to predict the stiffness degradation and crack distribution evaluations associated with flexural-type loading of reinforced concrete beams. This was approached by first deriving a "mixture" beam theory from the full 3D mixture model via suitable constraints on the global displacement field. This is analogous to the derivation of beam models, such as the Bernoulli-Euler Theory, from 3D elasticity.

The advantage of a beam-type model over a full 2D or 3D analysis is supposed to be the ease with which one can effect analytical or numerical solutions. However, serious complications were encountered in this case which led to numerical problems. The trouble was traced to the method by which "non-thru" cracks were being incorporated. A new procedure has been proposed and, during the forthcoming research period, another attempt will be made to examine the beam bending example.

5. PUBLICATIONS

The following is a cumulative list of research publications and reports that have been prepared on research performed under the present contract:

Papers

1. Read, H. E., and G. A. Hegemier, "Strain Softening of Rock, Soil and Concrete," *Mechanics of Materials*, 3 (1984), 271.
2. Read, H. E., Discussion of "Hysteretic Endochronic Theory for Sand," by Z. P. Bazant, R. J. Krizek and C.-L. Shieh, *Journal of Engineering Mechanics*, Vol. III(1), Jan. 1985, 103.
3. Hegemier, G. A., H. Murakami, and L. J. Hageman, "On Tension Stiffening in Reinforced Concrete," *Mechanics of Materials*, 4 (2), (1984).
4. Hegemier, G. A., and H. E. Read, "On Deformation and Failure of Brittle Solids: Some Outstanding Issues," *Mechanics of Materials*, 4(1985), 215.
5. Hegemier, G. A., and H. Murakami, "A Nonlinear Theory for Reinforced Concrete," *Proc. Second Symp. on the Interaction of Non-Nuclear Munitions with Structures*, Panama City, FLA (1985).
6. Valanis, K. C., and H. E. Read, "An Endochronic Plasticity Theory for Concrete," *Proc. Second Symp. on the Interaction of Non-Nuclear Munitions with Structures*, Panama City, FLA (1985).
7. Hegemier, G. A., and H. E. Read, "Strain Softening" (Discussion), *Theoretical Foundations for Large-Scale Computations for Nonlinear Material Behavior*, edited by S. Nemat-Nasser, R. J. Asaro and G. A. Hegemier, Martinus Nijhoff, Pub. (1984).
8. Murakami H., and G. A. Hegemier, "On Simulating Steel-Concrete Interaction in Reinforced Concrete. Part I: Theoretical Development," submitted to *Mechanics of Materials*.
9. Hageman, L. J., H. Murakami and G. A. Hegemier, "On Simulating Steel-Concrete Interaction in Reinforced Concrete. Part II: Validation Studies," submitted to *Mechanics of Materials*.

10. Valanis, K. C., and H. E. Read, "An Endochronic Plasticity Theory for Concrete," *Mechanics of Materials*, (1986), to appear.
11. Murakami, H., and H. E. Read, "Endochronic Plasticity: Some Basic Properties of Plastic Flow and Failure," *International Journal of Solids and Structures* (1986), to appear.
12. Murakami, H., and G. A. Hegemier, "A Nonlinear Dowel Action Model," in preparation.
13. Hegemier, G. A., and H. Murakami, "On Simulating the Nonlinear Planar Hysteretic Response of Reinforced Concrete and Concrete Masonry," Third ASCE EMD Specialty Conference on Dynamics of Structures, UCLA, Los Angeles, March 31 - April 2, 1986.

Reports

14. Read, H. E., "Strain Rate Effects in Concrete: A Review of Experimental Methods," S-CUBED Report SSS-R-85-6081, January 1985, 35 pp.
15. Hegemier, G. A., H. E. Read, K. C. Valanis and H. Murakami, "Development of Advanced Constitutive Models for Plain and Reinforced Concrete," S-CUBED Annual Report to AFOSR, SSS-R-85-7150, April 1985.
16. Valanis, K. C., "A Theory of Fracture for Brittle Solids," S-CUBED, La Jolla, California, Informal Report, August 1985.

6. INTERACTIONS

The following is a cumulative list of the interactions (coupling activities) by the contract staff members which have occurred or are schedule to occur during the course of this contract on research done under the contract:

6.1 ORAL PRESENTATIONS AT MEETINGS, CONFERENCES, SEMINARS AND WORKSHOPS

1. Murakami, H., "Some Basic Inelastic Response Features of the New Endochronic Theory," Oral presentation at the 21st Annual Meeting of the Soc. Engrg. Sci., VPI, Blacksburg, VA, October 15, 1984.
2. Hegemier, G. A., "On Deformation and Flow of Brittle Solids," General Lecture presented at the Workshop on Inelastic Deformation and Failure Modes, Northwestern University, Evanston, Illinois, November 18-21, 1984.
3. Valanis, K. C., "An Endochronic Plasticity Theory for Concrete," Oral presentation at the Workshop on Inelastic Deformation and Failure Modes, Northwestern University, Evanston, Illinois, November 18-21, 1984.
4. Read, H. E., "Inelastic Response Characteristics of the New Endochronic Theory with Singular Kernel," Oral presentation at the Workshop on Inelastic Deformation and Failure Modes, Northwestern University, Evanston, Illinois, November 18-21, 1984.
5. Read, H. E., "On Modeling the Dynamic Behavior of Plain Concrete," Oral presentation at DNA Concrete Material Properties Meeting, Terra Tek, Salt Lake City, Utah, March 7, 1984.
6. Hegemier, G. A., "Development of an Advanced Constitutive Model for Reinforced Concrete," Oral presentation at DNA Concrete Material Properties Meeting, Terra Tek, Salt Lake City, Utah, March 7, 1984.
7. Read, H. E., "An Endochronic Plasticity Theory for Concrete," Oral presentation at the 2nd Symp. on the Interaction of Non-Nuclear Munitions with Structures, Panama City, FLA, April 1985.
8. Hegemier, G. A., "A Nonlinear Theory for Reinforced Concrete," Oral presentation at the 2nd Symp. on the Interaction of Non-Nuclear Munitions with Structures, Panama City, FLA, April 1985.

9. Hegemier, G. A., "A Nonlinear Model of Reinforced Concrete," Oral presentation at the U.S.-Japan Cooperative Conference on Earthquake Engineering, Tokyo, Japan, Aug. 1985.
10. Read, H. E., and G. A. Hegemier, "Development of Advanced Constitutive Theories for Plain and Reinforced Concrete," Oral presentation at the Air Force Weapons Laboratory, Civil Engineering Research Seminar, Albuquerque, New Mexico, Sept. 1985.
11. Hegemier, G. A., and H. Murakami, "On Simulating the Nonlinear Planar Hysteretic Response of Reinforced Concrete and Concrete Masonry," 3rd ASCE EMD Specialty Conference on Dynamics of Structures, UCLA, Los Angeles, March 31-April 2, 1986.
12. Read, H. E., "Endochronic Plasticity: Application to Plain Concrete," to be presented at the 10th U.S. National Congress of Applied Mechanics, University of Texas, Austin, Texas, June 1986.
13. Murakami, H., "Basic Flow Properties of Endochronic Plasticity," to be presented at the 10th U.S. National Congress of Applied Mechanics, University of Texas, Austin, Texas, June 1986.
14. Hegemier, G. A., "On Simulating the Nonlinear Planar Hysteretic Response of Reinforced Concrete and Concrete Masonry," to be presented at the US/PRC Workshop on Seismic Resistance of Masonry Structures, China, May 1986.

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APPENDIX A

NEW FAILURE SURFACE FOR THE ENDOCHRONIC CONCRETE MODEL

This appendix describes a new failure surface for concrete and the procedure for introducing it into the endochronic concrete model. Unlike the earlier failure surface used with this model, the new failure representation accounts for the influence of the intermediate principal stress on failure, which is known to be important for concrete, and provides an excellent correlation of various concrete triaxial failure data.

For the purposes of the present discussion, let us recall the expression for deviatoric stress, s , as given by the model i.e.,

$$s = \int_0^{z_s} \rho(z_s - z') \frac{de^p}{dz'} dz' \quad (A-1)$$

where the intrinsic time for shear, z_s , is defined by the expression

$$dz_s = \frac{dz}{F_s} \quad (A-2)$$

Here, F_s , the shear hardening function, is responsible for the characteristics of the failure surface.

In the original endochronic concrete model (Valanis and Read, 1985;1986), F_s was taken to depend on the pressure, σ , in a linear manner. This resulted in a Mohr-Coulomb type failure surface which was independent of the intermediate principal stress and which cut the π -plane in a circle. Despite its simplicity, this simple failure

criterion was evidently adequate for describing the data from Scavuzzo, *et al.*, (1983), which were obtained for stress states that were sufficiently far from failure that no significant cracking occurred. Under these circumstances, it is to be expected that the test results will not be sensitive to the details of the failure surface, and this, in fact, appeared to be the case.

To obtain accurate descriptions of response near failure, a more sophisticated failure criterion is required, specifically one that reflects the influence of the intermediate principal stress. For isotropic media, the failure criterion in its most general form, is expressible in terms of three independent stress invariants. For a convenient representation in principal stress space, it is useful to consider the three stress invariants σ , J_2 and θ where

$$\sigma = \text{tr}(\underline{\sigma}) \quad (\text{A-3})$$

$$J_2 = \frac{1}{2} \underline{s} : \underline{s} \quad (\text{A-4})$$

$$\theta = -\frac{1}{3} \cos^{-1} \left[\frac{3 \sqrt{3} J_3}{2 (J_2)^{3/2}} \right] \quad (\text{A-5})$$

and J_3 is defined as

$$J_3 = \frac{1}{3} \det (\underline{s}) \quad (\text{A-6})$$

The general form of the failure criterion for isotropic media can therefore be written in terms of these stress invariants as

$$\sqrt{J_2} = \phi(\sigma, \theta) \quad (\text{A-7})$$

To determine the relation between the functions F_s and ϕ , consider the case of radial deformation at constant hydrostatic pressure. For this case, we can write

$$d\epsilon^p = n ||d\epsilon^p|| = n d\zeta_s \quad (A-8)$$

so that

$$\bar{\epsilon} = \int_0^{z_s} \rho(z_s - z') \frac{d\zeta_s}{dz'} dz' = n \int_0^{z_s} \rho(z_s - z') \frac{d\zeta_s}{dz'} dz' \quad (A-9)$$

During the initial hydrostatic loading process for $0 \leq z_s \leq z_s^0$, we have $\zeta_s = 0$. Thus, if we set

$$w = z_s - z_s^0 \quad (A-10)$$

it follows that

$$z_s - z' = w + z_s^0 - z' = w - w' \quad (A-11)$$

As a result, Eq. (A-9) can be rewritten in the form

$$\bar{\epsilon} = n \int_0^{z_s} \rho(z_s - z') \frac{d\zeta_s}{dz'} dz' = n \int_0^w \rho(w - w') \frac{d\zeta_s}{dw'} dw' \quad (A-12)$$

It has been shown by Read (1985) that when the hydrostatic hardening function F_H is of the form:

$$F_H = e^{\beta \zeta_H} \quad (A-13)$$

where β is a positive constant, it follows that

$$\frac{d\zeta_s}{dw} = \frac{F_s [2cw + c^2 w^2]^{1/2}}{(1 + cw)} \quad (A-14)$$

where

$$c = \frac{\beta}{k} F_s \quad (A-15)$$

and β , k are material constants. Substitution of Eq. (A-14) into Eq. (A-12) gives

$$\zeta = \eta F_s \int_0^w \rho(w - w') \frac{[2cw' + c^2(w')^2]^{1/2}}{(1 + cw')} dw' \quad (A-16)$$

Using a result given by Valanis and Read (1984), it can be shown that

$$\lim_{w \rightarrow \infty} \int_0^w \rho(w - w') \frac{[2cw' + c^2(w')^2]^{1/2}}{(1 + cw')} dw' = M_\infty \quad (A-17)$$

where

$$M_\infty = \int_0^\infty \rho(x) dx \quad (A-18)$$

Thus, from Eqs. (A-16) and (A-17), we can write

$$\zeta^f = \lim_{w \rightarrow \infty} \zeta = \eta F_s M_\infty \quad (A-19)$$

where the superscript f denotes failure. It then follows that at failure we have

$$\sqrt{J_2} = \frac{1}{\sqrt{2}} F_s M_\infty \quad (A-20)$$

The relation between the functions F_s and ϕ is then obtained by comparing Eqs. (A-7) and (A-20), with the result:

$$F_s = \frac{\sqrt{2} \phi(\sigma, \theta)}{M_\infty} \quad (A-21)$$

Thus, to define F_s , the function $\phi(\sigma, \theta)$ needs to be specified.

There are a number of advanced failure criteria for plain concrete available which have the general form of Eq. (A-7) (Mills and Zimmerman, 1970; Willam and Warnke, 1974; Ottosen, 1977; Lade, 1982; Peyton, 1983; Podgorski, 1985). Considering the large scatter in data between experimenters as well as between different devices (Hegemier and Read, 1986), most of these advanced failure criteria provide a reasonably accurate description of existing concrete failure data. The reader is referred to Ottosen (1977) and Podgorski (1985) for a critical assessment of most of these failure models.

Inasmuch as most of the above failure models are reasonably accurate, and none appear to have a significant advantage in predictive capability or numerical implementation over the others, we adopted for use in the present study the failure criterion that was developed by S-CUBED (Peyton, 1983) during the previous AFOSR program (see Hegemier, *et al*, 1983). A summary of this criterion is given below.

The failure criterion of Peyton (1983) is of the form

$$\frac{J_2}{\tau^2} - a \frac{J_3}{\tau^3} = 1 \quad (A-22)$$

where τ and a are, in general, smooth monotonic functions of pressure σ , i.e.,

$$\begin{aligned} \tau &= \tau(\sigma) \\ a &= a(\sigma) \end{aligned} \quad (A-23)$$

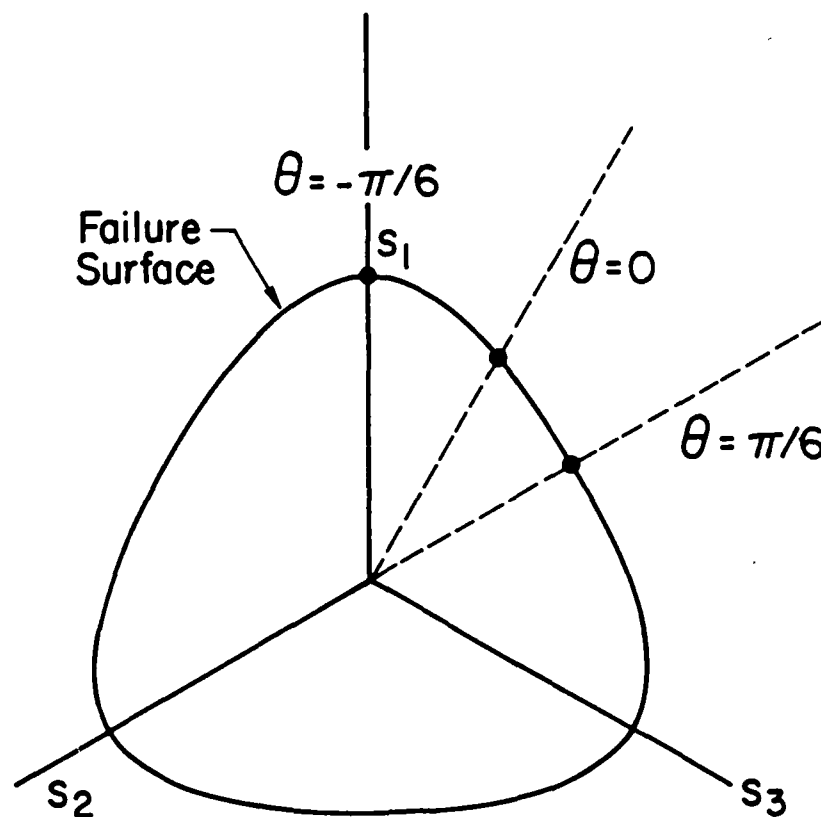
The role of the function $\tau(\sigma)$ is to describe the effect of pressure on the meridians of the failure surface, while the function $a(\sigma)$ determines the manner in which the trace of the failure surface in the π -plane changes with pressure. In order for the trace to be convex, it is necessary that $0 \leq a \leq 1$. When a is a continuous decreasing function of pressure, the trace of the failure surface in the π -plane changes smoothly from a triangle ($a = 1$) to a circle ($a = 0$) with increasing pressure.

In order to express Eq. (A-22) in the form of Eq. (A-7), we introduce the stress invariant J , where

$$J = \frac{3\sqrt{3}}{2} \frac{J_3}{(J_2)^{3/2}} \quad (A-24)$$

Upon defining an angle θ in the π -plane as shown in Figure A.1, it follows that

$$J = -\sin 3\theta \quad (A-25)$$



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Figure A.1 Deviatoric stress plane, showing the manner in which the angle θ is defined.

Equation (A-24) is now solved for J_3 and the result introduced into Eq. (A-22). Thus

$$\frac{J_2}{\tau^2} - a \frac{2J}{3\sqrt{3}} \left(\frac{J_2}{\tau^2} \right)^{3/2} = 1 \quad (\text{A-26})$$

which is of the form

$$x^2 - q x^3 = 1 \quad (\text{A-27})$$

if we set

$$q = \frac{2aJ}{3\sqrt{3}}, \quad x = \sqrt{\frac{J_2}{\tau^2}} \quad (\text{A-28})$$

By making the change of variable $x = 1/y$, Eq. (A-27) assumes the form

$$y^3 - y + 1 = 0 \quad (\text{A-29})$$

A further change of variable, $y = \lambda \sin \theta$, allows Eq. (A-29) to be written as

$$\sin^3 \theta - \frac{1}{\lambda^2} \sin \theta + \frac{q}{\lambda^3} = 0 \quad (\text{A-30})$$

Note that the above equation has the form of the trigonometric identity

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{1}{4} \sin 3\theta = 0 \quad (\text{A-31})$$

if we set

$$\lambda = \frac{2}{\sqrt{3}} ; \quad \sin 3\theta = aJ \quad (\text{A-32})$$

Thus, from Eq. (A-32b), we can write,

$$\theta = \frac{1}{3} [\sin^{-1}(aJ) + 2n\pi] \quad (\text{A-33})$$

and, returning to the original variables, it follows that

$$\sqrt{J_2} = \frac{\sqrt{3} \tau}{2 \sin \left[\frac{1}{3} \sin^{-1} (aJ) + \frac{2\pi}{3} \right]} \quad (\text{A-34})$$

which has the form of Eq. (A-7). To fit this model to a particular set of data, the functions $\tau(\sigma)$ and $a(\sigma)$ need to be specified. A procedure for doing this is described below.

Generally, failure data are available from triaxial compression and triaxial extension tests. Using such data, we define a ratio r as

$$r \equiv \frac{(\sqrt{J_2})_{\text{compression}}}{(\sqrt{J_2})_{\text{extension}}} \quad (\text{A-35})$$

where both values of $\sqrt{J_2}$ are taken at the same pressure. Using Eq. (A-34) with $J = +1$ for compression and $J = -1$ for extension, it is straightforward to show that r has the form:

$$r = \frac{\left(\sin \frac{2\pi}{3} - \frac{1}{3} \sin^{-1} a \right)}{\left(\sin \frac{2\pi}{3} + \frac{1}{3} \sin^{-1} a \right)} \quad (\text{A-36})$$

This expression can be solved for a to give

$$a = \sin \left\{ 3 \tan^{-1} \left(\sqrt{3} \frac{r-1}{r+1} \right) \right\} \quad (\text{A-37})$$

Therefore, by knowing how $(\sqrt{J_2})_{\text{compression}}$ and $(\sqrt{J_2})_{\text{extension}}$ depend upon σ from data, the dependence of a on σ can be determined from Eq. (A-37).

Having determined $a(\sigma)$, and with the variation of $(\sqrt{J_2})_{\text{compr.}}$ with σ known from data, the function $\tau(\sigma)$ can be determined from the following equation

$$\tau(\sigma) = \frac{2}{\sqrt{3}} \sin \left(\frac{1}{3} \sin^{-1} a + \frac{2\pi}{3} \right) (\sqrt{J_2})_{\text{compr.}} \quad (\text{A-38})$$

which is obtained by simply solving Eq. (A-34) for $\tau(\sigma)$ and setting $J = +1$.

The above procedure was used to determine the forms of a and τ for plain concrete from the failure data shown earlier in Figure 3.2. Here, failure data from both triaxial compression and triaxial extension tests on plain concrete conducted by several investigators on samples of several different strengths are depicted.⁴ After

4. Note that the data of Scavuzzo, et al. (1983) increasingly deviate from the other data shown in the figures with increasing pressure.

applying the above methods for determining a and τ from the data, it was found that

$$\begin{aligned} a &= a_0 \\ \tau &= \tau_0 + \beta_0 \sqrt{\sigma - \gamma_0} \end{aligned} \quad (\text{A-39})$$

where

$$\begin{aligned} a_0 &= 0.83 \\ \tau_0 &= -1.913 f'_c \\ \beta_0 &= 2.144 \sqrt{f'_c} \\ \gamma_0 &= -0.896 f'_c \end{aligned} \quad (\text{A-40})$$

The excellent ability of the model, with only four material parameters, to correlate the data was shown earlier in Figure 3.2.

Returning to the endochronic model for plain concrete (Valanis and Read, 1985;1986), we note that the kernel function $\rho(z)$ was taken in the form:

$$\rho = \sum_r A_r e^{-a_r z} \quad (\text{A-41})$$

so that, in view of Eq. (A-18) we have

$$M_\infty = \sum \frac{A_r}{a_r} \quad (\text{A-42})$$

Upon substituting the values for A_2 and a_r given by Valanis and Read (1985;1986) into Eq. (A-18), it follows that

$$M_{\infty} = 17.5 \text{ ksi} \quad (\text{A-43})$$

By combining Eqs. (A-20), (A-34) and (A-43), we obtain the following expression for the shear hardening function in the endochronic model for plain concrete:

$$F_s = \frac{c_o \tau}{\sin \left[\frac{1}{3} \sin^{-1}(\alpha J) + \frac{2\pi}{3} \right]} \quad (\text{A-44})$$

Here, α and τ are given by Eqs. (A-39) and (A-40), and $c_o = 0.07 \text{ ksi}^{-1}$.

APPENDIX B

NEW DAMAGE-CRACKING MODEL FOR CONCRETE

The purpose of this appendix is to describe, in detail, a damage-cracking model that is being developed under the present program for ultimate use in conjunction with the endochronic concrete model. The model was formulated by Valanis (1985) and is in the early stages of development. The discussion given below largely follows that given by Valanis (1985) and does not reflect various topics that are currently under study, which include the addition of (a) plasticity, (b) non-directional damage due to volume compaction, (c) size effects and (d) strain rate dependence.

Formulation of Model

Consider a linearly elastic material which is isotropic in its virgin unstrained state and undergoes small deformation at isothermal conditions. As the deformation proceeds, damage gradually develops, which in turn reduces the material integrity. For sufficiently large deformation, the accumulated damage can lead to complete fracture, in which the material cannot support tensile stress in a direction normal to the crack. We shall formulate this damage-cracking process within the context of irreversible thermodynamics and, for this purpose, we introduce the free energy Ψ per unit volume.⁵ Since fracture leads to a reduction in material integrity, we introduce an integrity tensor ϕ which will be discussed at length later and which is symmetrical such that $\phi = \bar{\phi}$ (the unit tensor), when the material is in its virgin

5. This nomenclature is preferred to free energy density, in view of the subsequent concept of "irreducible" material volume. More will be said about this later on.

(undamaged) state, and $\phi = Q$ when the material has fully failed, i.e., when the material cannot support stress in any direction. We thus set:

$$\Psi = \Psi(\xi, \phi) \quad (B-1)$$

where ξ denotes the strain tensor. In a thermodynamic sense, ϕ now plays the role of an internal variable, in which case we can write the following relations:

$$\sigma = \frac{\partial \Psi}{\partial \xi} \quad , \quad Q = - \frac{\partial \Psi}{\partial \phi} \quad (B-2)$$

where σ is the stress tensor and Q denotes the internal force, dual to ϕ which is driving the fracturing process.

In view of the definition of ϕ , the following relations must hold:

$$\Psi(\xi, Q) = 0 \quad , \quad \Psi(Q, \phi) = 0 \quad , \quad (B-3)$$

the first meaning that a fully failed material cannot contain free energy and the second that an unstrained material must have zero free energy; both conditions are relative to the reference state. Furthermore, the following conditions must also hold:

$$\sigma(\xi, Q) = 0 \quad , \quad \sigma(Q, \phi) = 0 \quad (B-4)$$

$$Q(\xi, Q) = 0 \quad (B-5)$$

Equations (B-4) require that the stress vanish in the fully failed material and that, since the material remains elastic in the damaged state, the stress will vanish at zero strain. Equation (3.5) stipulates that the internal fracture causing force Q must vanish in the fully failed material.

If we now expand Ψ in a Taylor series in ϵ , retain terms no higher than the quadratic, and observe relations (3.3), we obtain the following form of Ψ :

$$\Psi = C_{ijkl} \epsilon_{ij} \epsilon_{kl} \quad (B-6)$$

where $C = C(\phi)$, which by virtue of Eqs. (B-3) may contain no constant terms nor linear terms in ϕ . Since the material is assumed to be isotropic in its virgin state, C may be represented in terms of outer products of the unit tensor δ and ϕ . Furthermore, since C is purely quadratic in ϕ , it has no other representation than

$$C_{ijkl} = \lambda \phi_{ij} \phi_{kl} + 2\mu \phi_{ik} \phi_{jl}, \quad (B-7)$$

in view of the symmetries imposed upon it by the symmetry of ϵ implied in Eq. (B-6). The constants λ and 2μ in Eq. (B-7) must indeed be the Lamé constants of the virgin material, since

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl} \quad (B-8)$$

when $\phi = \delta$. Thus, we can write

$$\Psi = \frac{1}{2} \lambda (\phi_{ij} \epsilon_{ij})^2 + \mu \phi_{ik} \phi_{jl} \epsilon_{ij} \epsilon_{kl}, \quad (B-9)$$

so that, in view of Eq. (B-2a), σ is given by the expression:

$$\sigma_{ij} = \lambda \phi_{ij} \phi_{kl} \epsilon_{kl} + 2\mu \phi_{ik} \phi_{jl} \epsilon_{kl} \quad (B-10)$$

This equation defines the constitutive response of an elastic fracturing material, once we establish how the integrity tensor ϕ is

related to the fracturing process. In particular, an equation is needed which describes the manner in which ϕ evolves with accumulated damage, and this is considered below.

Evolution Equation for ϕ

When the material is not strain rate sensitive, an increment of tensile strain will produce an increment of damage. Specifically, given an increment in strain, $d\epsilon$, let $d\epsilon^a$ be its eigenvalues and n_i^a its eigenvectors. If $d\epsilon$ and ϵ are coaxial, then $d\epsilon^a$ will constitute an increase in tensile strain if $d\epsilon^a > 0$ and $\epsilon^a \geq 0$, where ϵ^a denote the eigenvalues of ϵ . In the more general case where $d\epsilon$ and ϵ are not co-axial, $d\epsilon^a$ is said to be an increase in tensile strain if $d\epsilon^a > 0$ and $\epsilon_{ij}n_i^an_j^a \equiv \epsilon_a^n \geq 0$ (a not summed). Now let $d\phi$ be the change in the integrity tensor ϕ due to the increment $d\epsilon$, and let n_ϕ^a be the eigenvector of $d\phi$. We now specify that $d\phi$ and $d\epsilon$ are co-axial, so that

$$n_\phi^a = n^a \quad (B-11)$$

The physical significance of Eq. (B-11) is examined in the discussion following Eq. (B-14).

To completely characterize the evolution of ϕ , the relation between $d\phi^a$ and $d\epsilon^a$ needs to be specified. For this purpose, we adopt the following expression:

$$d\phi^a = - \left(\phi_n^a \right) d\epsilon^a \quad (B-12)$$

where

$$d\xi^a = \begin{cases} k d\epsilon^a, & \text{for } d\epsilon^a > 0 \text{ and } \epsilon_n^a \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (\text{B-13})$$

Here, k and m are positive constants, and $\phi_n^a = \phi_{ij} n_i^a n_j^a$ (a not summed).

In view of Eq. (B-11), we can therefore write

$$d\phi_{ij} = - \sum_a \left(\phi_n^a \right)^m n_i^a n_j^a d\xi_a \quad (\text{B-14})$$

With this evolution equation for ϕ , the constitutive description of the elastic fracturing solid is now complete, once the material parameters λ , μ , k and m have been specified.

Physical Interpretation of Equations

The physical meaning of the above equations, and their relation to the fracturing process, is best understood by reference to Eq. (B-10), which relates the stress tensor to the strain tensor. Let N^a be the eigenvectors of ϕ and let q and ϵ be referred to a system of coordinates X_a . Then, the components of q and ϵ on the X_a coordinate system can be written as:

$$\bar{\sigma}_{rs} = \sigma_{ij} N_i^r N_j^s \quad (\text{B-15})$$

$$\bar{\epsilon}_{rs} = \epsilon_{ij} N_i^r N_j^s \quad (\text{B-16})$$

In this system of coordinates, ϕ is diagonal, i.e., $\bar{\phi}_{rs} = 0$ for $r \neq s$, and Eq. (B-10) takes the form:

$$\bar{\sigma}_{rs} = \lambda \bar{\phi}_{rs} \bar{\phi}_{mn} \bar{\epsilon}_{mn} + 2\mu \bar{\phi}_{rm} \bar{\phi}_{sn} \bar{\epsilon}_{mn} \quad (\text{B-17})$$

Since $\bar{\phi}$ is diagonal, we can write:

$$\begin{aligned} \bar{\sigma}_{11} &= \lambda \bar{\phi}_{11} \sum_{r=1} \phi_{rr} \bar{\epsilon}_{rr} + \bar{\phi}_{11}^2 \bar{\epsilon}_{11} \\ \bar{\sigma}_{21} &= \bar{\sigma}_{12} = 2\mu \bar{\phi}_{11} \bar{\phi}_{22} \bar{\epsilon}_{12} \end{aligned} \quad (\text{B-18})$$

$$\bar{\phi}_{31} = \bar{\phi}_{13} = 2\mu \bar{\phi}_{11} \bar{\phi}_{22} \bar{\epsilon}_{13}$$

Let us now single out the eigenvector N^1 and examine what happens to the stress when the eigenvalue $\phi^1 (= \bar{\phi}_{11})$ vanishes. Note from Eqs. (B-18) that, if $\bar{\phi}_{11} = 0$, it follows that $\sigma_{11} = \bar{\sigma}_{12} = \bar{\sigma}_{13} = 0$, that is, the solid cannot support stresses on a plane normal to N^1 , on which $\phi^1 \equiv \bar{\phi}_{11} = 0$. The physical meaning of this is that the decrease of ϕ^1 represents damage normal to the plane of N^1 and, as such, it is a measure of the plane microcracks that have developed normal to N^1 and/or the increase in the size of such cracks, so that when $\bar{\phi}_{11} = 0$, a plane crack in the accepted sense has formed across the material element on a plane with normal N^1 , so that the element cannot support tensile or shear stresses on that plane.

On the basis of the above observations, the physical meaning of Eqs. (B-11), (B-12) and (B-13) now becomes clear. An increase in tensile strain $d\epsilon^a$ on a plane with normal n^a causes planar damage $d\phi^a$ on that plane, where $d\phi^a$ is given by Eq. (B-12), in accordance with

the observation that "planar microcracks form perpendicular to the direction of the principal tensile strain."

The constant k in Eq. (B-13) is a material parameter that reflects the fracture resistance or fracture toughness of the material. Eq. (B-12) is representative of processes involving annihilation of populations and catastrophic processes in systems where the increment of annihilation is proportional to the state of integrity, which is given here by the factor $(\phi_n^a)^m$. This form is representative of such systems, even though any monotonically decreasing function $f(\phi_n^a)$, such that $f(0) = 0$, will likely do as well. Finally, we note that the damage tensor D can be expressed in terms of ϕ as follows:

$$D_{ij} = \sum_a \left[1 - (\phi^a)^2 \right] N_i^a N_j^a \quad (B-19)$$

The physical interpretation of the governing equations is now complete.

3.3.3 Application to Some Simple Cases of Homogeneous Deformation

In this section, analytic solutions, based upon the constitutive model described above, are presented for several simple cases of homogeneous deformation including simple tension, simple compression and simple shear. These solutions provide valuable insight into the characteristics of the model and reveal its remarkable predictive capabilities, despite the fact that it involves only four material parameters.

As a preliminary development, consider the case in which the strain field is homogeneous and consists only of principal strains, i.e., $\epsilon_{ij} = 0$ for $i \neq j$. Thus, $d\epsilon$ and ϵ are always coaxial and ϕ is coaxial with ϵ by virtue of Eq. (B-11). As a result, ϕ is diagonal. In view of Eq. (B-10), σ is also diagonal; the principal stresses are given below in terms of the principal strains and the principal values of ϕ :

$$\begin{aligned}\sigma_1 &= \lambda \phi_1 \sum_r \phi_r \epsilon_r + 2\mu \phi_1^2 \epsilon_1 \\ \sigma_2 &= \lambda \phi_2 \sum_r \phi_r \epsilon_r + 2\mu \phi_2^2 \epsilon_2 \\ \sigma_3 &= \lambda \phi_3 \sum_r \phi_r \epsilon_r + 2\mu \phi_3^2 \epsilon_3\end{aligned}\tag{B-20}$$

The appropriate evolution equation for ϕ is obtained from Eq. (B-14) and is given in terms of the principal components of ϕ as follows:

$$d\phi_i = - (\phi_i)^m d\xi_i$$

where

$$d\xi_i = \begin{cases} k d\epsilon_i, & \text{if } d\epsilon_i > 0, \epsilon_i \geq 0 \\ 0, & \text{otherwise} \end{cases}\tag{B-21}$$

We now consider two special cases of the above equations, namely, simple tension and simple compression.

Simple Tension

Consider the case of monotonically increasing simple tension. In this case, $\sigma_2 = \sigma_3 = 0$, and integration of Eqs. (B-21) with $\epsilon_2 = \epsilon_3$ and the initial conditions $\phi_2 = \phi_3 = 1$, shows that $\phi_2 = \phi_3$ always. Hence, we find that

$$\phi_2 \epsilon_2 = - \frac{\lambda}{2(\lambda+\mu)} \epsilon_1 \phi_1 = - \nu \phi_1 \epsilon_1 \quad (\text{B-22})$$

and

$$\sigma_1 = E \phi_1^2 \epsilon_1 \quad (\text{B-23})$$

where

$$E = \frac{(2\mu+3\lambda)\mu}{\lambda+\mu} \quad (\text{B-24})$$

From Eq. (B-21a), the evolution equation for ϕ_1 is

$$d\xi_1 \phi_1^m + d\phi_1 = 0, \quad (\text{B-25})$$

subject to the initial condition $\phi_1(0) = 1$. Thus, the solution is

$$\phi_1 = \begin{cases} e^{-k\epsilon_1} & , \text{ if } m = 1 \\ \left\{ 1 - (1-m)k\epsilon_1 \right\}^{\frac{1}{1-m}} & , \text{ if } m \neq 1 \end{cases} \quad (\text{B-26})$$

Combining Eqs. (B-2.3) and (B-26), 23 obtain

$$\sigma_1 = \begin{cases} e^{-2k\epsilon_1} & , \text{ if } m = 1 \\ E \epsilon_1 \left\{ 1 - (1-m)k\epsilon_1 \right\}^{\frac{2}{1-m}} & , \text{ if } m \neq 1 \end{cases} \quad (\text{B-27})$$

Inspection of Eq. (B-27) reveals that, for $0 \leq m < 1$, σ_1 is not a monotonic function of ϵ_1 , but reaches a maximum and then goes to zero for finite ϵ_1 . For $1 \leq m < 3$, σ_1 again reaches a maximum and then decays monotonically to zero as $\epsilon_1 \rightarrow \infty$. For $m = 3$, σ_1 monotonically increases to an asymptotic constant value as $\epsilon_1 \rightarrow \infty$, while for $3 < m < \infty$, σ_1 increases monotonically with ϵ_1 . The influence of m on the constitutive response is shown in Figure B.1a. Thus, in the range $0 \leq m < 3$, the model exhibits softening in tension in agreement with experimental data.

The model also unloads elasticity with an effective modulus of $E\phi_1^2$ which is diminished by the accumulated damage. Thus, the model has all of the characteristics of a fracturing elastic solid. It follows from Eq. (B-22) that $\epsilon_2 \leq 0$ and hence $\phi_2 = 1$, i.e., there is no damage in the transverse direction to loading.

Simple Compression

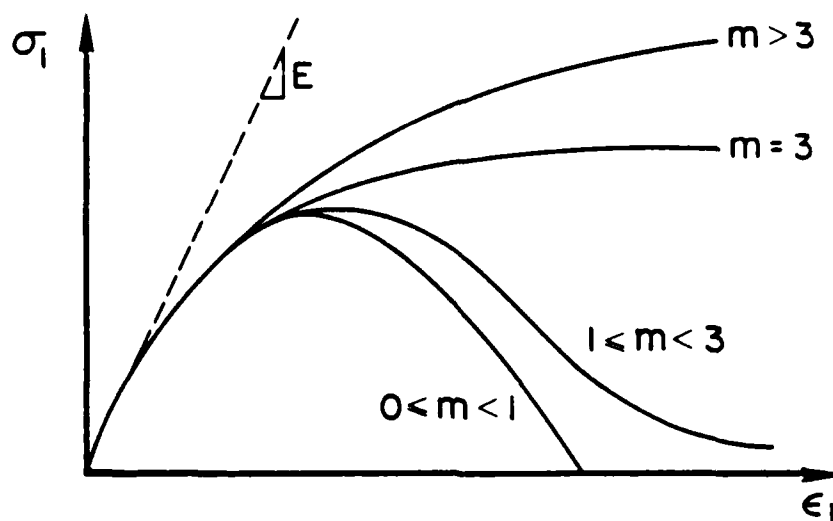
Consider now the case of monotonically increasing uniaxial compression. Here, $\sigma_2 = \sigma_3 = 0$, while σ_1 and ϵ_1 are compressive. Again, Eqs. (B-21) together with $\epsilon_2 = \epsilon_3$, and the initial conditions $\phi_2 = \phi_3 = 1$, lead to the result that $\phi_2 = \phi_3$ always. Similar to the case of simple tension, we find that

$$\phi_2 \epsilon_2 = -\nu \phi_1 \epsilon_1 \quad (B-28)$$

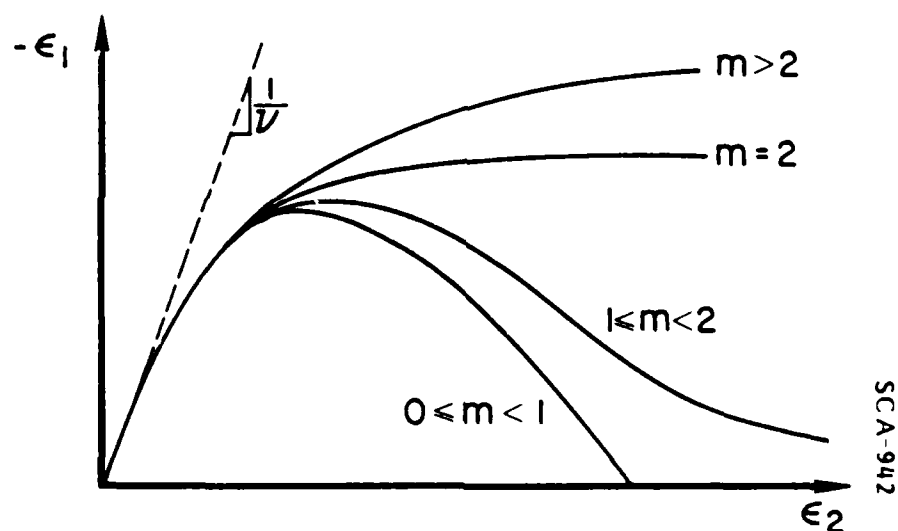
and

$$\sigma_1 = E\phi_1^2 \epsilon_1 \quad (B-29)$$

The damage evolution in this case is, however, entirely different from that for simple tension. Since both $d\epsilon_1$ and ϵ_1 are compressive at all



(a) Stress-strain relation in simple tension



(b) Axial strain vs. lateral strain in simple compression

Figure B.1 Predicted responses for simple tension and simple compression, showing the effect of the parameters m on the resulting behavior.

times, we find from Eq. (B-21) that $d\phi_1 = 0$ and thus $\phi_1 = 1$ always. However, in view of Eq. (B-28), ϵ_2 is now tensile and hence ϕ_2 (and ϕ_3) will increase with the consequence that damage will develop on planes which are parallel to the axis of compression, in accordance with experimental observation; this is the so-called axial splitting mode (Horii and Nemat-Nasser, 1985).

From Eq. (B-21), we can write

$$d\phi_2 + \phi_2^m k d\epsilon_2 = 0 \quad , \quad (B-30)$$

the solution of which is

$$\phi_2 = \begin{cases} e^{-k\epsilon_2} & , m = 1 \\ \left\{ 1 - (1-m)k\epsilon_2 \right\}^{\frac{1}{1-m}} & , m \neq 1 \end{cases} \quad (B-31)$$

Thus, in view of Eq. (B-29) and the condition $\phi_1 = 1$, the axial stress strain relation is

$$\sigma_1 = E \epsilon_1 \quad , \quad (B-32)$$

so that the response in the axial direction is purely elastic.

To find the relation between the strains, we use Eqs. (B-28) and (B-31) together with the fact that $\phi_1 = 1$. It then follows that

$$-\nu\epsilon_1 = \begin{cases} \epsilon_2 e^{-k\epsilon_2} & , m = 1 \\ \epsilon_2 \left\{ 1 - (1-m)k\epsilon_2 \right\}^{\frac{1}{1-m}} & , m \neq 1 \end{cases} \quad (B-33)$$

Here, we may distinguish three different cases:

(a) $0 \leq m < 1$

The strain ϵ_1 is not a monotonic function of ϵ_2 . It reaches a maximum absolute value and then goes to zero for a finite value of ϵ_2 . Specifically,

$$|\epsilon_1|_{\max} = \frac{1}{k\nu} \left(\frac{n}{n+1} \right)^{n+1} \quad (\text{B-34})$$

where $n = 1/(1-m)$.

(b) $1 \leq m < 2$

Again, the strain ϵ_1 is not a monotonic function of ϵ_2 . It reaches a maximum absolute value again given by Eq. (B-34) for $1 < m$ and then goes to zero for infinite ϵ_2 . For $m = 1$

$$|\epsilon_1|_{\max} = \frac{1}{\nu e k}, \quad (\text{B-35})$$

where e denotes the base of the natural logarithm.

(c) $2 \leq m < \infty$

In this case, ϵ_1 is a monotonic function of ϵ_2 and approaches the following limiting condition for increasing lateral strain:

$$\lim_{\epsilon_2 \rightarrow \infty} |\epsilon_1|_{\max} = \begin{cases} \frac{1}{k}, & m = 2 \\ \infty, & m > 2 \end{cases} \quad (\text{B-36})$$

The above cases are depicted graphically in Figure B.1(b).

Collapse of a Block Under Axial Compression

It follows from the above discussion that a block consisting of elastic fracturing material will collapse under axial compression if $0 \leq m \leq 2$, since ϵ_1 has an upper bound given by the expression:

$$|\epsilon_1|_{\max} = \frac{1}{\nu k} \left(\frac{1}{2-m} \right)^{\frac{2-m}{1-m}} \quad (\text{B-37})$$

The limiting cases are

$$|\epsilon_1|_{\max} = \begin{cases} \frac{1}{\nu k e} & , \quad m = 1 \\ \frac{1}{\nu k} & , \quad m = 2 \end{cases} \quad (\text{B-38})$$

The collapse stress may be calculated directly from Eq. (B-32), with the result:

$$|\sigma_1|_{\max} = \frac{E}{\nu k} \left(\frac{1}{2-m} \right)^{\frac{2-m}{1-m}} \quad (\text{B-39})$$

Thus, the theory predicts the collapse of a block under axial compression due to damage on planes parallel to the axis of compression. Again, this is the axial splitting mode.

Ratio of Collapse Stresses in Tension and Compression

At this point, it is of interest to explore the rationality of the above results by comparing the ratio σ^c/σ^t , where σ^c and σ^t are collapse stresses in compression and tension, respectively, obtained analytically with that observed experimentally for concrete, even

though the end-grip conditions in the tests may not be ideal, as assumed in solutions obtained here. Restricting attention to the case $m = 1$, it follows from Eqs. (B-37) and (B-39) that

$$\frac{\sigma^c}{\sigma} = \frac{2}{\nu} \quad (\text{B-40})$$

Since $\nu \approx 0.2$ for concrete, we have

$$\frac{\sigma^c}{\sigma} \approx 10, \quad (\text{B-41})$$

which is close to experimental observation (Raphael, 1984).

Simple Shear

Consider now the case of monotonically increasing shear strain under conditions of simple shear and small strain. The configuration of interest is depicted in Figure

B.2. Inasmuch as the only non-zero strain component is ϵ_{12} , we can write

$$\phi_{kl} \epsilon_{kl} = 2 \phi_{12} \epsilon_{12} \quad (\text{B-42})$$

and

$$\phi_{ik} \phi_{jl} \epsilon_{kl} = (\phi_{i1} \phi_{j2} + \phi_{i2} \phi_{j1}) \epsilon_{12} \quad (\text{B-43})$$

Therefore, in this case, the general constitutive relation

$$\sigma_{ij} = \lambda \phi_{ij} \phi_{kl} \epsilon_{kl} + 2\mu \phi_{ik} \phi_{jl} \epsilon_{kl} \quad (\text{B-44})$$

reduces to the form

$$\sigma_{ij} = 2 \left[\lambda \phi_{ij} \phi_{12} + \mu (\phi_{i1} \phi_{j2} + \phi_{i2} \phi_{j1}) \right] \epsilon_{12} \quad (\text{B-45})$$

Therefore, we can write

$$\sigma_{12} = 2 \left[\lambda \phi_{12}^2 + \mu (\phi_{11} \phi_{22} + \phi_{12}^2) \right] \epsilon_{12} \quad (\text{B-46})$$

$$\sigma_{11} = 2(\lambda + 2\mu) \phi_{11} \phi_{12} \epsilon_{12} \quad (\text{B-47})$$

$$\sigma_{22} = 2(\lambda + 2\mu) \phi_{22} \phi_{12} \epsilon_{12} \quad (\text{B-48})$$

For a small strain, the maximum tensile strain will lie in a direction that makes a 45 degree angle with each of the coordinate axes. Consequently, cracks will develop in the material as shown in Figure B.2. The components of the unit normal n to the crack are $n_1 = n_2 = 1/\sqrt{2}$.

Since there is only one cracking direction, we set $a = 1$ in the evolutionary equation for $d\phi_{ij}$ and write

$$d\phi_{ij} = -k(\phi_n)^m d\epsilon n_i n_j \quad (\text{B-49})$$

where

$$\begin{aligned} \phi_n &= \phi_{ij} n_i n_j \\ d\epsilon &= d\epsilon_{12} \end{aligned} \quad (\text{B-50})$$

Since $n_1 = n_2 = 1/\sqrt{2}$, it follows from Eq. (B-49) that

$$d\phi_{11} = d\phi_{22} = d\phi_{12} \quad (\text{B-51})$$

Hence

$$\int_1^{\phi_{11}} d\phi_{11} = \int_0^{\phi_{12}} d\phi_{12} , \quad (B-52)$$

since $\phi_{ij} = \delta_{ij}$ initially. From Eq. (B-53) we find

$$\phi_{11} = 1 + \phi_{12} , \quad (B-53)$$

and Eq. (B-50a) gives

$$\phi_n = \phi_{11} + \phi_{12} , \quad (B-54)$$

since $\phi_{11} = \phi_{22}$ according to Eq. (B-51).

Substitution of Eq. (B-53) into Eq. (B-54) yields

$$\phi_n = 1 + 2 \phi_{12} \quad (B-55)$$

which, when combined with Eq. (B-49), leads to the result

$$d\phi_{ij} = -k(1 + 2\phi_{12})^m d\epsilon_{12}^{n_i n_j} \quad (B-56)$$

Therefore, since $n_1 = n_2 = 1/\sqrt{2}$, we can write

$$d\phi_{12} = -\frac{k}{2} (1 + 2\phi_{12})^m d\epsilon_{12} \quad (B-57)$$

which can be integrated to give, for $m = 1$:

$$\phi_{12} = \frac{1}{2} \left[e^{-k\epsilon_{12}} - 1 \right] \quad (B-58)$$

and for $m \neq 1$:

$$\phi_{12} = \frac{1}{2} \left\{ \left[\frac{1}{k(m-1)\epsilon_{12}+1} \right]^{\frac{1}{m-1}} - 1 \right\} \quad (B-59)$$

Recalling Eq. (B-53) and the fact that $\phi_{22} = \phi_{11}$, Eq. (B-46) can be placed in the following form:

$$\sigma_{12} = 2 \left[(\lambda + 2\mu) \phi_{12}^2 + 2\mu \phi_{12} + \mu \right] \epsilon_{12} \quad (\text{B-60})$$

Therefore, Eqs. (B-58) to (B-60) describe the relationship between σ_{12} and ϵ_{12} .

From Eqs. (B-58) and (B-59) note that

$$\lim_{\epsilon_{12} \rightarrow \infty} \sigma_{12} = -\frac{1}{2} (\lambda + 2\mu) \epsilon \quad (\text{B-61})$$

which in view of Eq. (B-60) leads to the result:

$$\lim_{\epsilon_{12} \rightarrow \infty} \sigma_{12} = \frac{1}{2} (\lambda + 2\mu) \epsilon_{12} \quad (\text{B-62})$$

Hence, when $\lambda < 2\mu$, the ultimate slope given by Eq. (B-62) will be less than the initial slope, 2μ , as shown in Figure B.3. The case $\lambda > 2\mu$ on the other hand, leads to the ultimate slope being greater than the initial slope, i.e., a hardening, which is difficult to visualize from a physical standpoint. These limiting cases, however, fall outside of the assumption of small strains made here and therefore may not be physically meaningful.

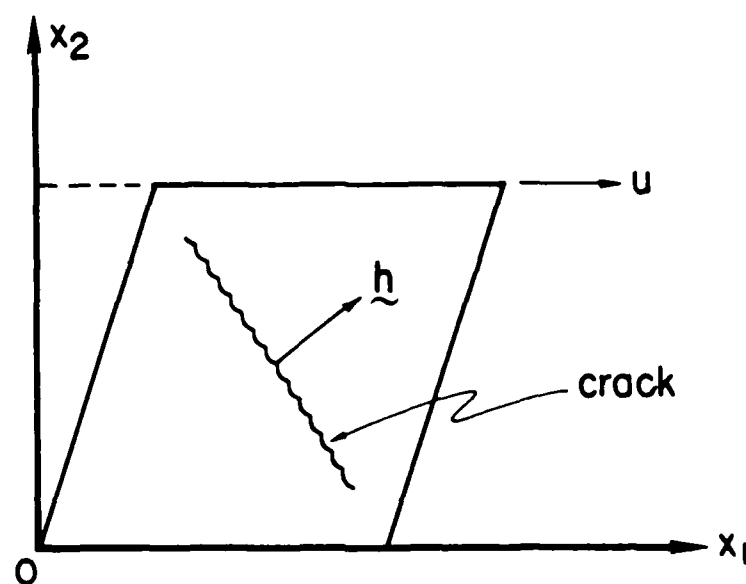
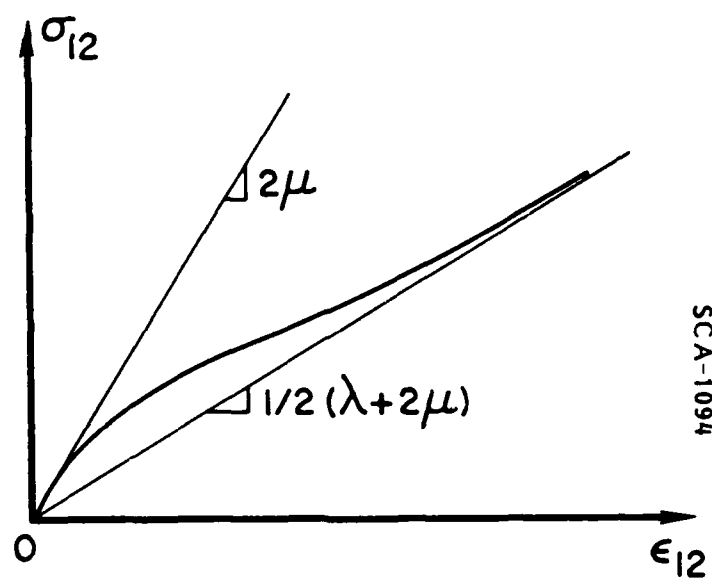


Figure B.2. Simple shear



SCA-1094

Figure B.3. General relationship between σ_{12} and ϵ_{12} for the case in which $\lambda > 2\mu$.

APPENDIX C

LIMITING CONDITION FOR HYDROSTATIC RESPONSE

INTRODUCTION

In Section 4.5, it was pointed out that the limiting condition at high pressures for hydrostatic compression response of concrete is

$$\lim_{\epsilon_p \rightarrow \epsilon_p^*} \sigma = \infty \quad (C-1)$$

where σ denotes the hydrostatic pressure and ϵ_p^* is the limiting volumetric strain at full compaction. It is shown in this appendix that the endochronic model described in Section 4.5 satisfies this condition when the hardening function F_H is taken in the form:

$$F_H = \left(\frac{\epsilon_p^*}{\epsilon_p^* - \epsilon_p} \right)^n \quad (C-2)$$

and appropriate restrictions are placed on $n(>0)$. The following proof of this statement is due to Valanis (1986).

The Model

We consider the following endochronic model of hydrostatic response:

$$\sigma = \int_0^z K(z - z') \frac{d\epsilon_p}{dz'} dz' \quad (C-3)$$

where $K(z)$ is weakly singular and integrable,

$$d\zeta = |d\epsilon_p| \quad (C-4)$$

and

$$dz = \frac{d\zeta}{F_H(\epsilon_p)} \quad (C-5)$$

We consider F_H in the form:

$$F_H = \left(\frac{\epsilon_p^*}{\epsilon_p^* - \epsilon_p} \right)^n, \quad n > 0 \quad (C-6)$$

and, in addition, consider the case where

$$K(z) = \frac{\phi(z)}{z^\lambda}, \quad 0 < \lambda < 1 \quad (C-7)$$

in which $\phi(z)$ is bounded from above and

$$\lim_{z \rightarrow \infty} \phi(z) = 0 \quad (C-8)$$

Analysis

We begin by rewriting Eq. (C-3) in the form

$$\sigma = \int_0^S \frac{\phi[z(\zeta) - z'(\zeta')]}{[z(\zeta) - z'(\zeta')]^\lambda} \frac{d\epsilon_p}{d\zeta'} d\zeta', \quad (C-9)$$

which under monotonic straining conditions becomes

$$\sigma = \int_0^z \frac{\phi[z(\zeta) - z'(\zeta')]}{[z(\zeta) - z'(\zeta')]^\lambda} d\zeta' \quad , \quad (C-10)$$

in view of Eq. (C-4). Since ϕ is bounded from above and is a monotonically decreasing function of its argument, we can write

$$\phi(z - z') \geq \phi(z) \quad (C-11)$$

so that

$$\sigma \geq \phi(z) \int_0^z \frac{d\zeta'}{[z(\zeta) - z'(\zeta')]^\lambda} \quad (C-12)$$

Thus, to show that condition (C-9) is satisfied, it suffices to show that

$$\lim_{\epsilon_p^* \downarrow \epsilon_p} \int_0^z \frac{d\zeta'}{[z(\zeta) - z'(\zeta')]^\lambda} = \infty \quad (C-13)$$

To show this, we use Eqs. (C-4) and (C-5) to obtain the expression:

$$dz = d\zeta \left(\frac{\epsilon_p^* - \zeta}{\epsilon_p^*} \right)^n \quad , \quad (C-14)$$

which may be integrated to give

$$z = \frac{1}{(n+1)(\epsilon_p^*)^n} \left[(\epsilon_p^*)^{n+1} - (\epsilon_p^* - \zeta)^{n+1} \right] \quad , \quad (C-15)$$

where the condition $z(0) = 0$ has been observed. We may therefore write that

$$\int_0^z \frac{d\zeta'}{(z-z')^\lambda} = (n+1)^\lambda (\epsilon_p^*)^{n\lambda} \int_0^S \frac{d\zeta'}{[(\epsilon_p^* - \zeta')]^{n+1} - (\epsilon_p^* - S)^{n+1}}^\lambda \quad (C-16)$$

Upon introducing the transformation

$$y = \epsilon_p^* - \zeta, \quad y' = \epsilon_p^* - \zeta' \quad (C-17)$$

we can rewrite Eq. (C-16) in the form:

$$\int_0^z \frac{d\zeta'}{(z-z')^\lambda} = (n+1)^\lambda (\epsilon_p^*)^{n\lambda} \int_y^{\epsilon_p^*} \frac{dy'}{[(y')^{n+1} - y^{n+1}]^\lambda} \quad (C-18)$$

In view of Eqs. (C-17), it then follows that

$$\epsilon_p^* \lim_{\epsilon_p^*} \int_0^z \frac{d\zeta'}{(z-z')^\lambda} = (n+1)^\lambda (\epsilon_p^*)^{n\lambda} \int_0^{\epsilon_p^*} \frac{dy'}{(y')^{\lambda(n+1)}} \quad (C-19)$$

which can be integrated to give

$$\epsilon_p^* \lim_{\epsilon_p^*} \int_0^z \frac{d\zeta'}{(z-z')^\lambda} = \frac{(n+1)^\lambda (\epsilon_p^*)^{n\lambda}}{[\lambda(n+1) - 1]} (y')^{1-\lambda(n+1)} \Big|_0^{\epsilon_p^*} \quad (C-20)$$

An inspection of the righthand side of this equation reveals that it becomes infinite when

$$\lambda(n+1) - 1 \leq 0 \quad (C-21)$$

or

$$n \geq \frac{1}{\lambda} - 1 \quad (C-22)$$

The condition (C-22) gives the relation between n and λ that must be satisfied by the hardening function (C-6) in order to satisfy the limiting condition (C-1).

APPENDIX D

DESCRIPTION OF CONCRETE UNDER COMPRESSIVE STRESS STATES

1. INTRODUCTION

It is well known that the strength of concrete increases with increasing mean confinement stress. This effect can be observed in the triaxial failure surface of concrete in stress space, Figure D.1.

In addition to strength, the ductility of concrete is observed to increase with increasing mean compressive stress. Typical increases in ductility during standard triaxial tests are shown in Figure D.2. The effect of confinement on ductility is seen to be much greater than that on strength. This can also be deduced from the behavior of reinforced concrete in the presence of confinement steel, Figure D.3.

Analytical and experimental studies (see Nemat-Nasser, *et al.*, [1]) on brittle materials such as concrete suggest that, at low confining stress levels failure occurs by the growth and connection of a narrow zone of microcracks to form one or at most several macrocracks or macrofaults, Figure D.4a,b. The onset of this event (formation of a macrocrack) corresponds to the onset of "strain softening" [2]. In contrast, at higher levels of confining stress microcracks grow in a more uniform manner and a distribution of microcracks evidently develops, Figure D.4c. Although failure may again occur via localization, the formation of this distribution appears to be responsible for the observed increase in ductility.

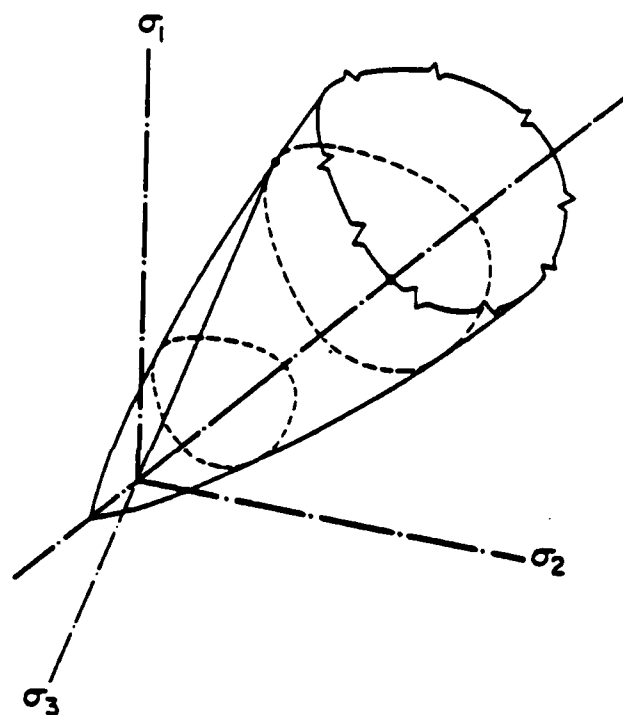


Figure D.1. Triaxial failure surface of concrete.

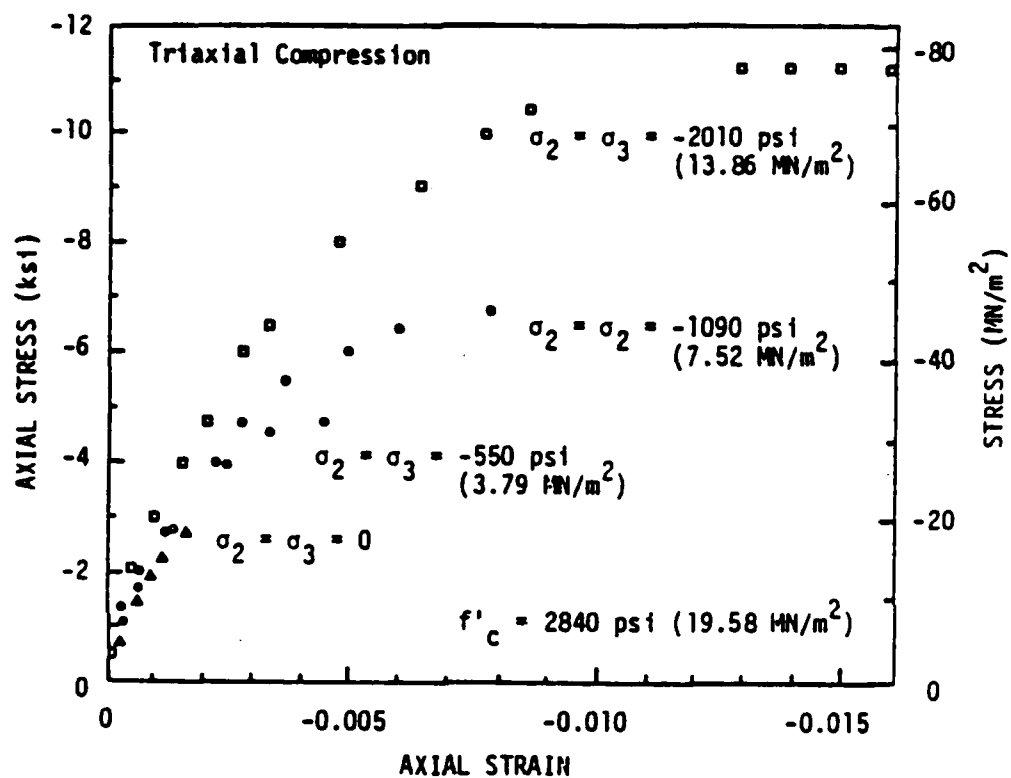


Figure D.2. Influence of confinement on strength and ductility.

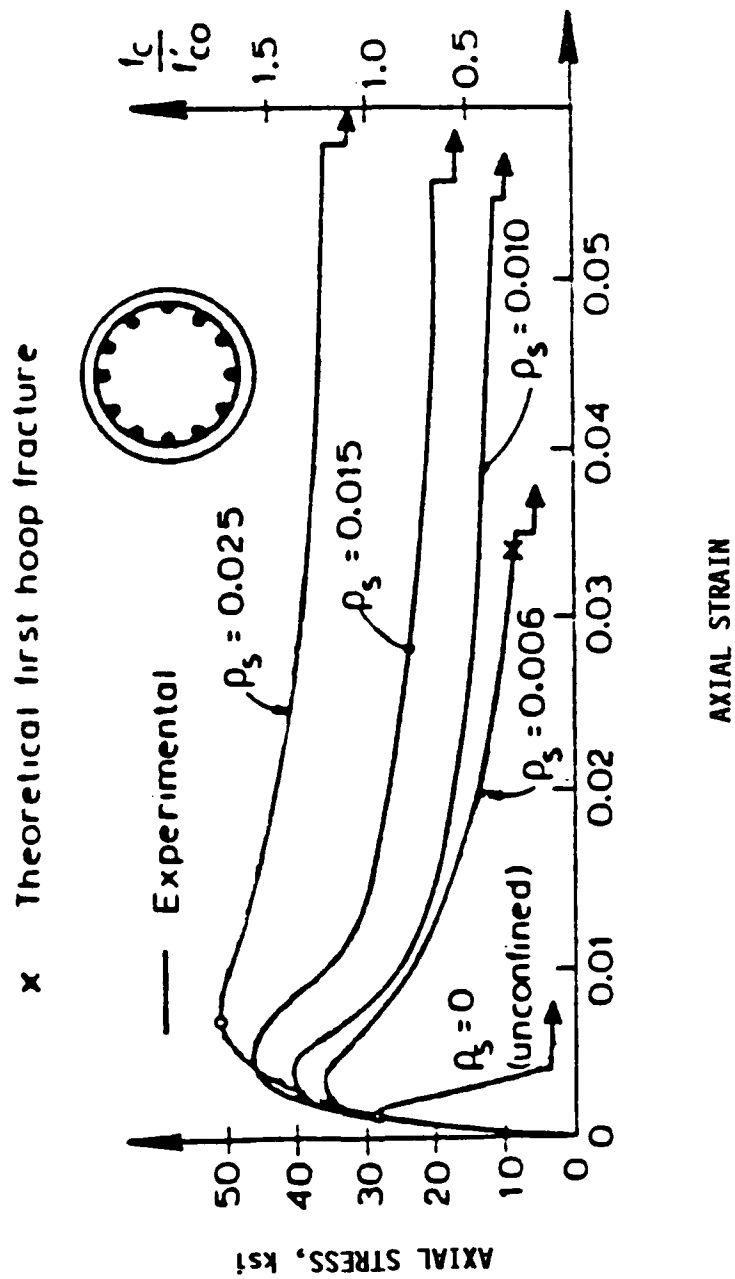


Figure D.3. Behavior of reinforced concrete in compression.

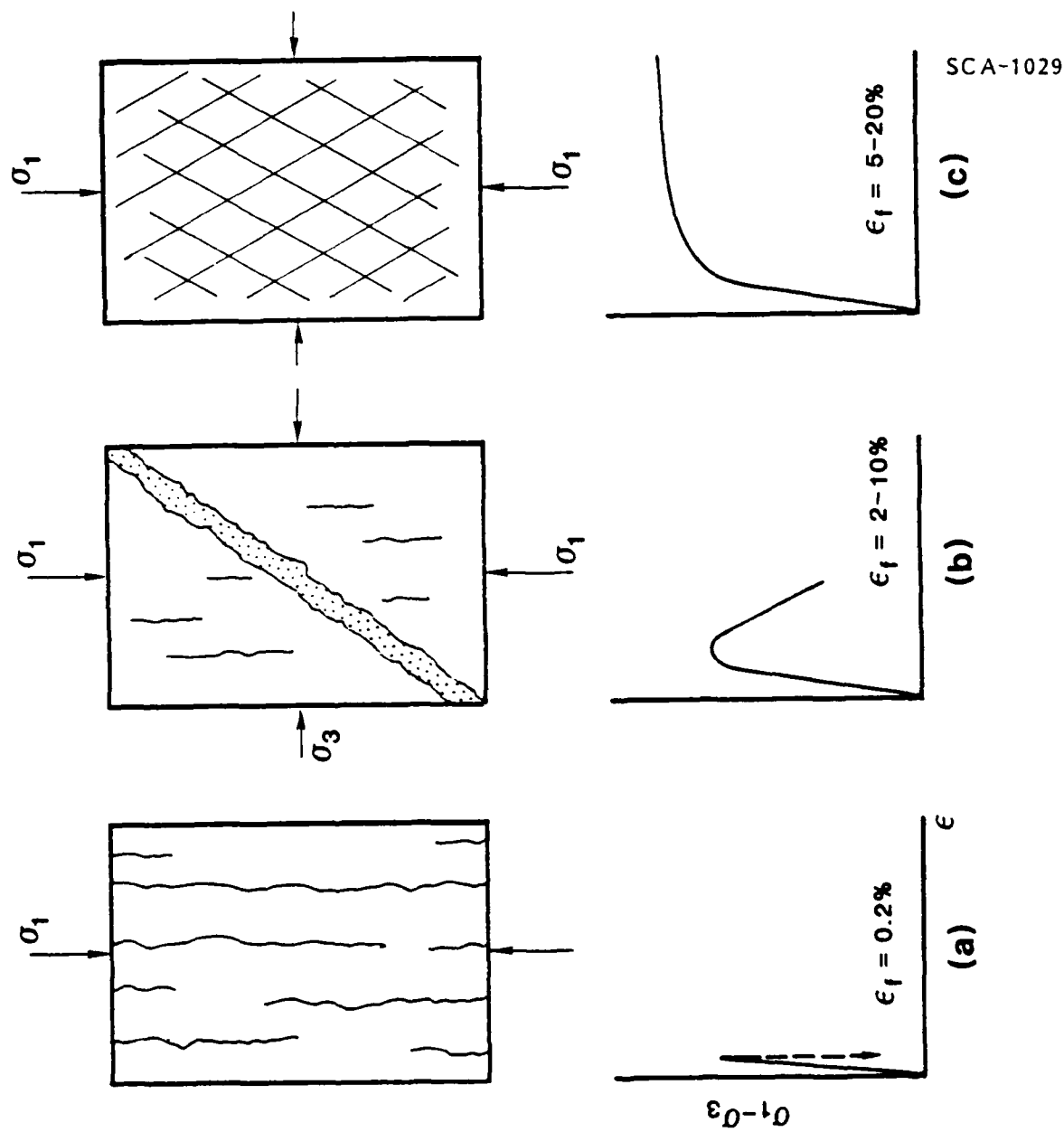


Figure D.4. Behavior of plain concrete in compression.

Below the brittle-ductile transition, slip along the planes that constitute the above distribution is a more plausible explanation of increased ductility than is true plastic flow. Consequently, conventional phenomenological elastoplastic descriptions of concrete under moderate compressive stress states appear to be inappropriate.

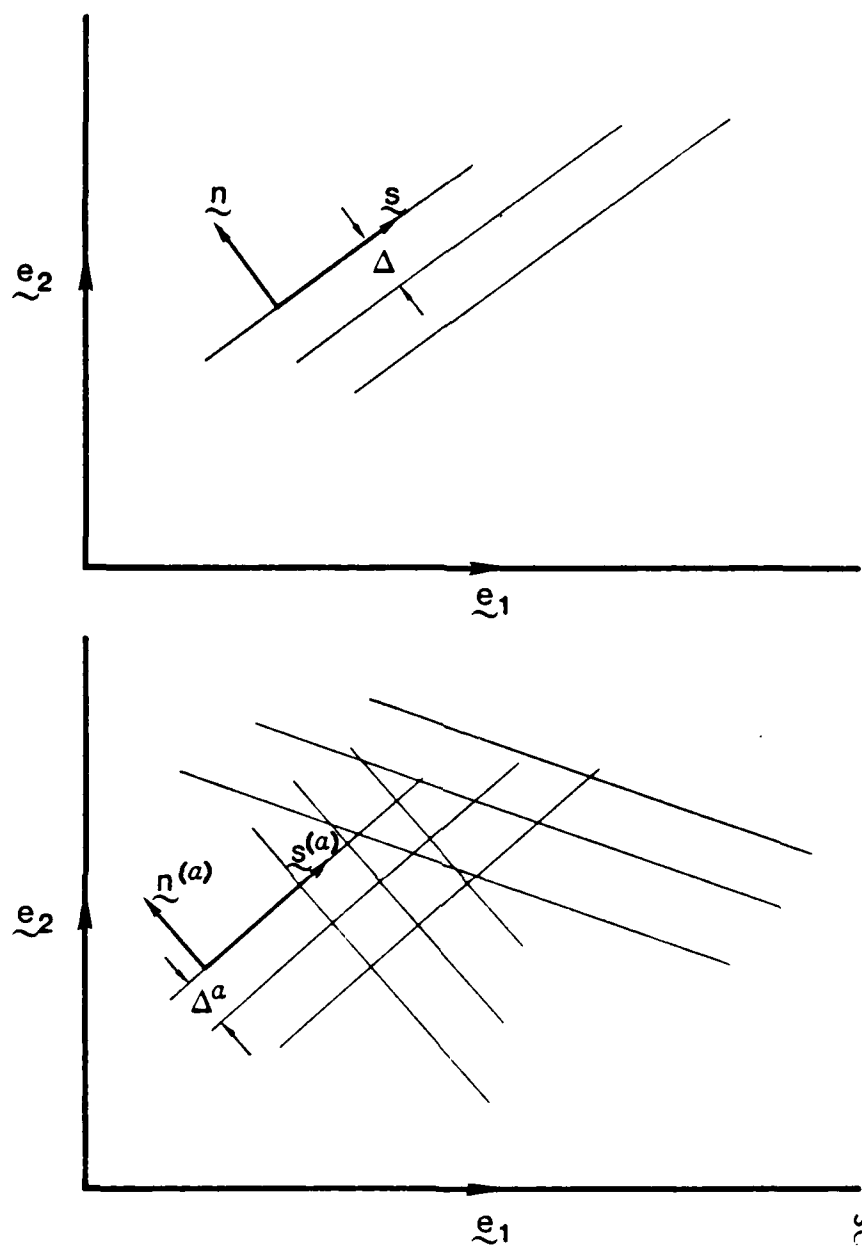
In view of the above discussion, an effort was made to construct a theory of concrete based on a "slip system" concept. A description of progress made to-date on this subject follows.

2. THE SLIP SYSTEM

Slip system models have been used previously to describe both metals and frictional materials. A representative cross-section of such models includes the works of Mandel [3], Spencer [4], Asaro [5], and Nemat-Nasser, *et al.*, [6]. An excellent discussion and in-depth treatment of the subject is given by Nemat-Nasser [7].

Figure D.5 describes the slip concept in the case of a single slip system. Locally $\mathbf{n}$ represents a unit normal to the slip surface and $\mathbf{s}$ is a unit tangent vector which describes the direction of slip. The quantity Δ denotes an effective slip surface spacing. The slip system and its geometry is presumed to form at a certain critical stress state. The collection of these states forms a surface in stress space.

In the sequel, a theory is described for an arbitrary number of slip systems. Associated with the α th system is a normal vector $\mathbf{n}^{(\alpha)}$, tangent vector $\mathbf{s}^{(\alpha)}$, and spacing $\Delta^{(\alpha)}$, Figure D.5, where $\alpha = 1, 2, \dots, N$.



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Figure D.5. Slip system concept.

According to the slip concept, all irreversible deformation (i.e., "plastic slip") is assumed to take place along the slip surfaces. The material between these surfaces will be modeled as elastic.

3. DEFORMATION

Let the relative rate of slip across a slip surface, Figure D.5, be denoted as $[v(s)]$. Further, let $\underline{e}_i$ ($i = 1$ to 3) denote the unit orthogonal base vectors of a rectangular Cartesian reference system. Then, if the rate of deformation tensor

$$d_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) \quad (D-1)$$

where $\underline{x}$ is the velocity vector, is decomposed into elastic and inelastic parts according to

$$d_{ij} = d_{ij}^e + d_{ij}^p \quad (D-2)$$

one obtains

$$d_{ij}^p = \frac{1}{2} (s_i n_j + s_j n_i) \frac{[v(s)]}{\Delta} \quad (D-3)$$

where

$$s_i \equiv \underline{s} \cdot \underline{e}_i, \quad n_i \equiv \underline{n} \cdot \underline{e}_i \quad (D-4)$$

In the case of multiple slip systems, the effects are superposed to give

$$d_{ij}^p = \sum_{a=1}^N \frac{1}{2} \left(s_i^a n_j^a + s_j^a n_i^a \right) \frac{[v(s)]^a}{\Delta^a} \quad (D-5)$$

Since $s^* n = s_i n_i = 0$, it is evident that the inelastic deformation characterized by (D-5) does not involve volume change. There are three plausible ways that inelastic volume change may accompany plastic flow within the context of a slip system (Nemat-Nasser[7]): (a) plastic volumetric expansion may occur normal to the slip plane as slip takes place; (b) uniform isotropic expansion or contraction may accompany slip; and (c) uniform inelastic volumetric deformation may take place independently of slip.

The physical cause of (a) is asperities in the form of aggregate interlock. Let $[v(n)]$ denote the rate of separation of the $\pm$ slip surfaces in the normal direction. Then

$$\sum_{a=1}^N n_i n_j \frac{[v(n)]^a}{\Delta^a}$$

represents the dilatancy due to $[v(n)]$. If the latter is related to the slip $[v(s)]$ according to

$$[v(n)] = \theta_1 [v(s)] \quad (D-6)$$

where θ_1 is a scalar function, then (D-5) can be generalized to give

$$d_{ij}^p = \sum_{a=1}^N \left[\frac{1}{2} \left(s_i^a n_j^a + s_j^a n_i^a \right) + n_i^a n_j^a \theta_1^a \right] \frac{[v(s)]^a}{\Delta^a} \quad (D-7)$$

The physical basis for (b) is the formation of microvoids and microcracks due to slip, while that for (c) is the collapse of voids under hydrostatic compression. These effects can be modeled by adding to (D-7) terms of the form

$$\sum_{a=1}^N \left[s_i^a s_j^a \theta_2^a + \delta_{ij} \theta_3^a \right] \frac{v(s)}{\Delta^a} + \delta_{ij} \theta_3^a$$

Consequently, the inelastic deformation rate tensor becomes

$$d_{ij}^p = \sum_{a=1}^N p_{ij}^a \frac{[v(s)]^a}{\Delta^a} + \dot{\theta} \delta_{ij} \quad (D-8a)$$

where

$$p_{ij}^a \equiv \frac{1}{2} \left(s_i^a n_j^a + s_j^a n_i^a \right) + n_i^a n_j^a \theta_1^a + s_i^a s_j^a \theta_2^a + \delta_{ij} \theta_3^a \quad (D-8b)$$

In addition to the rate of deformation tensor d_{ij}^p , a spin tensor w_{ij}^p is now defined according to (Asaro [5])

$$w_{ij}^p = \sum_{a=1}^N w_{ij}^a \frac{[v(s)]^a}{\Delta^a} \quad (D-9a)$$

where

$$w_{ij}^a \equiv \frac{1}{2} \left(s_i^a n_j^a - s_j^a n_i^a \right) \quad (D-9b)$$

4. INTERFACE BEHAVIOR

The α th slip system is assumed to be activated if the resultant shear stress τ^α has attained a critical value defined by

$$f^\alpha(\tau^\alpha, \sigma^\alpha, p, \epsilon^{a\beta}) = 0 ; \quad \alpha, \beta = 1 \text{ to } N \quad (D-10)$$

where σ^α denotes the normal stress on the slip surface, p is the mean pressure, and $\epsilon^{a\beta}$ are measures of the history of slip. Slip occurs if, in addition to (D-10),

$$\dot{f}^\alpha(\tau^\alpha, \sigma^\alpha, p, \epsilon^{a\beta}) = 0 \quad (D-11)$$

Equation (D-11) can be expressed in the form

$$\dot{\tau}^\alpha + \dot{\sigma}^\alpha \tan \eta_1 + \dot{p} \tan \eta_2 - \sum_{\beta=1}^N h^{a\beta} \frac{[\dot{\epsilon}^{a\beta}]}{\Delta^\beta} = 0 \quad (D-12)$$

where $\tan \eta_1$ and $\tan \eta_2$ are material parameters that characterize, respectively, the effects of normal stress and hydrostatic stress on slip. Here η_1 corresponds to the usual friction. The quantity η_2 reflects the influence of pressure on plastic flow; e.g., in crystal plasticity this term would represent the influence of pressure on the motion of dislocations ℓ . (In two-dimensional problems (D-12) can be re-written in terms of σ^α only.) The parameters $h^{a\beta}$ characterize the manner by which the slip system's resistance changes due to slip. In what follows $h^{a\beta}$ is assumed to be symmetric (but not necessarily positive definite). Finally, the effect of cohesion has been excluded in the above formulation.

Consider now the problem of calculating the time rate of change of τ^a and σ^a in (D-10) - (D-12). Since

$$\tau^a = \underline{\tau}^a \cdot \underline{s}^a = \sigma_{ij} n_i^a s_j^a, \quad \sigma^a = \underline{\tau}^a \cdot \underline{n}^a = \sigma_{ij} n_i^a n_j^a \quad (D-13)$$

it is clear that one must determine $\dot{n}^a$ and $\dot{s}^a$ in order to calculate $\dot{\tau}^a$ and $\dot{\sigma}^a$. The former rates must be objective. Based on the physical processes under consideration, the following rates of change of the unit vectors $\dot{n}^a$ and $\dot{s}^a$ are defined

$$\dot{n}_i^a = w_{ij}^e n_j^a, \quad \dot{s}_i^a = w_{ij}^e s_j^a. \quad (D-14)$$

According to (D-14), the unit vectors are corotational with the elastic deformation of the slip system. With use of (D-14), equations (D-13) furnish

$$\dot{\tau}^a = \underline{\dot{\sigma}}_{ij} n_i^a s_j^a, \quad \dot{\sigma}^a = \underline{\dot{\sigma}}_{ij} n_i^a n_j^a \quad (D-15)$$

where

$$\underline{\dot{\sigma}}_{ij} \equiv \dot{\sigma}_{ij} - w_{ik}^e \sigma_{kj} \sigma_{ki} - w_{jk}^e \sigma_{ki} \quad (D-16)$$

If one defines the quantity q_{ij}^a according to

$$q_{ij}^a \equiv \frac{1}{2} \left(s_i^a n_j^a + s_j^a n_i^a \right) + n_i^a n_j^a \tan \eta_1 + \delta_{ij} \tan \eta_2 \quad (D-17)$$

then (D-12) furnishes the following relation which can be used to calculate $\underline{\dot{\sigma}}_{ij}$:

$$\dot{\sigma}_{ij}^a = \sum_{\beta=1}^N h^{\alpha\beta} \frac{[\dot{v}(s)]^\beta}{\Delta^\beta} \quad (D-18)$$

5. GLOBAL CONSTITUTIVE RELATIONS

In this section the global constitutive relation for σ_{ij} is assembled. For this purpose the elastic part of the slip system deformation is postulated in the form

$$\dot{\sigma}_{ij} = L_{ijkl} d_{kl}^e \quad (D-19)$$

where L_{ijkl} is the instantaneous elastic tensor modulus. If the material between slip planes is isotropic, then

$$L_{ijkl} = G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl} \quad (D-20)$$

It will be assumed that L_{ijkl} is independent of the rate of elastic deformation.

With use of (D-2), (D-8a), and (D-18), the relation (D-19) can be cast in the following form:

$$\dot{\sigma}_{ij} = C_{ijkl} (d_{kl} - \delta_{kl} \dot{\theta}) \quad (D-21)$$

where C_{ijkl} is an "elastic-plastic" modulus defined by

$$C_{ijkl} \equiv L_{ijkl} - \sum_{a,\beta=1}^N \left[L_{ijmn} p_{mn}^a + w_{im}^a \sigma_{mj} + w_{jm}^a \sigma_{mi} \right] M^{\alpha\beta} q_{mn}^{\beta} L_{mnkl} \quad (D-22)$$

where

$$M^{\alpha\beta} \equiv \left(h^{\alpha\beta} + q_{ij}^{\alpha} L_{ijkl} p_{kl}^{\beta} \right)^{-1} \quad (D-23)$$

6. REMARKS

The constitutive relations (D-21) are fully nonlinear and incorporate an arbitrary number of slip systems. Their form is identical, for all practical purposes, to the relations obtained by Nemat-Nasser [7]. The only relevant difference concerns the interpretation of the tensor L_{ijkl} . The definition of this quantity, which may depend on the current stress and the history of deformation, is somewhat vague for brittle materials such as concrete.

The constitutive relations (D-21) include classical plasticity as a special case. However, there is an important difference. It can be easily shown that the current model is such that the stress tensor σ_{ij} is noncoaxial with the inelastic deformation rate tensor d_{ij}^p . This feature furnishes an ability to model localization phenomena.

A number of special case studies using the foregoing results are under current study. These investigations are intended to determine the modeling capabilities and limitations of the slip-system description of concrete behavior.

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