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THE HELMHOLTZ THEOREM

BY

A. V. HERSHEY

September 1986

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THE HELMHOLTZ THEOREM

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TABLE OF CONTENTS

ABSTRACT

The Helmholtz theorem is rederived with rigorous vector analysis. The theorem is valid everywhere within any arbitrary mathematical boundary. Applications of the theorem to hydrodynamics are discussed.

INTRODUCTION

One of the theorems of Helmholtz states that a vector field with moderate restrictions on continuity can be expressed as the sum of the gradient of a scalar potential and the curl of a vector potential. This theorem has been known for a long time, and is derived in texts[1-3] on vector analysis. An elegant proof was prepared as part of a course which was taught in 1958 at the Naval Proving Ground. Publication of the proof at this time is justified by the impending application of the Helmholtz theorem to the computation of flow around a ship.

The existence of the scalar and vector potentials is proven by the derivation of formulae which express the potentials. Only vector identities are used in the derivation. The theorem is valid within any closed mathematical boundary.

The algebra and the calculus of scalars, vectors, and tensors are the subjects of many texts. Proofs are both geometrical constructions and component manipulations. The most relevant theorems which are background for the derivation are presented herewith. Formulations are stated in Gibbs notation. Symbolic expressions are invariant with respect to the choice of coordinate system.

RIEMANN INTEGRATION

Let a number of variables vary over a range of integration. Let the range be divided into elements such that the increment of each variable in any element is less than ϵ. An analytic function of the variables can be approximated within each element by a Taylor series expansion. The true integral of the function within each element differs from the trapezoidal integral by infinitesimals of higher order than ϵ. The true integral reduces to the trapezoidal integral in the limit as ϵ goes to zero. The structure of each element is immaterial as long as the elements fill the range of integration. This is known as Riemann integration[4].

VECTORS

A vector is a quantity with magnitude and direction. The magnitude and the direction are invariants of space.

The scalar product of two vectors **a** and **b** is given by the equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \tag{1}$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the magnitudes of the vectors and θ is the angle between them.

The vector product of two vectors **a** and **b** is given by the equation

$$\mathbf{a}\times\mathbf{b} = \mathbf{n}\,|\mathbf{a}||\mathbf{b}| \sin\theta \tag{2}$$

where **n** is a unit vector normal to the plane of **a** and **b**, $|\mathbf{a}|$ and $|\mathbf{b}|$ are the magnitudes of the vectors, and θ is the angle between them. The direction of **n** is such that the direction of **b** is obtained from the direction of **a** by a right-handed rotation about the vector **n**.

The scalar-vector product of three vectors is given by the equation

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{n})\,|\mathbf{b}||\mathbf{c}| \sin\theta \tag{3}$$

where **n** is a unit vector normal to the plane of **b** and **c**, while θ is the angle between **b** and **c**. The scalar product $\mathbf{a}\cdot\mathbf{n}$ is the separation of two parallelograms of area $\mathbf{b}\times\mathbf{c}$.

Thus the scalar–vector product $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ is the volume of a parallelepiped with edges **a**, **b**, **c**.

The triple vector product of three vectors is given by the equation

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{n})\,|\mathbf{b}||\mathbf{c}|\sin\theta \tag{4}$$

where **n** is a unit vector normal to the plane of **b** and **c**, while θ is the angle between **b** and **c**. The vector product $\mathbf{a}\times\mathbf{n}$ has a magnitude equal to the projection of **a** on the plane of **b** and **c**, while it has a direction perpendicular to the projection of **a** on the plane of **b** and **c**. Planes which contain the tip of **a** and are perpendicular to **b** and **c** intersect **b** and **c** at the projections of **a** on the directions of **b** and **c**. Lines which contain the origin of **a** and are parallel to the planes complete a parallelogram whose sides are the components of **a** in directions perpendicular to **b** and **c**. After rotation through a right angle the components of **a** become the components of $\mathbf{a}\times\mathbf{n}$ in directions parallel to **b** and **c**. The triple vector product is given by the equation

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\mathbf{a}\cdot\mathbf{c} - \mathbf{c}\mathbf{a}\cdot\mathbf{b} \tag{5}$$

which resolves the triple vector product into components in the plane of **b** and **c**.

A tensor is an operator which converts one set of vectors into another set in accordance with a linear transformation. The tensor has a matrix whose elements are the coefficients in the linear transformation. Typical tensors are rotations and deformations.

The tensor product of three vectors is given by the equation

$$\mathbf{ab}\cdot\mathbf{c} = (\mathbf{c}\cdot\mathbf{b})\,\mathbf{a} \tag{6}$$

where **c** is transformed into a vector in the direction of **a** by a special tensor with a matrix whose elements are the products of the components of **a** and **b**. Any letter could be used to represent the tensor, but the dyadic notation **ab** is more useful.

Let **i**, **j**, **k** be orthogonal unit vectors in the directions of increasing coordinates x, y, z in a right–handed Cartesian coordinate system. Then the vectors have the properties

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 \tag{7}$$

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0 \qquad \mathbf{i}\times\mathbf{j} = \mathbf{k} \quad \mathbf{j}\times\mathbf{k} = \mathbf{i} \quad \mathbf{k}\times\mathbf{i} = \mathbf{j} \tag{8}$$

Any vector can be resolved into components along the unit vectors.

All nine pairs of vectors among **i**, **j**, **k** form a fundamental set of unit dyadics. Any tensor can be resolved into components along the unit dyadics. The identity tensor **I** is defined by the equation

$$\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} \tag{9}$$

The identity tensor transforms every vector into itself.

GRADIENTS

The gradient is an operator which gives the differential change of a function for a differential displacement in space. The gradient of a function A is expressed by the equation

$$\nabla A = \mathbf{i}\frac{\partial A}{\partial x} + \mathbf{j}\frac{\partial A}{\partial y} + \mathbf{k}\frac{\partial A}{\partial z} \tag{10}$$

where the function A may be a scalar, vector, or tensor.

The divergence of a function A is expressed by the equation

$$\nabla\cdot\mathbf{A} = \mathbf{i}\cdot\frac{\partial \mathbf{A}}{\partial x} + \mathbf{j}\cdot\frac{\partial \mathbf{A}}{\partial y} + \mathbf{k}\cdot\frac{\partial \mathbf{A}}{\partial z} \tag{11}$$

where the function A may be a vector or tensor.

The curl of a function A is expressed by the equation

$$\nabla\times\mathbf{A} = \mathbf{i}\times\frac{\partial \mathbf{A}}{\partial x} + \mathbf{j}\times\frac{\partial \mathbf{A}}{\partial y} + \mathbf{k}\times\frac{\partial \mathbf{A}}{\partial z} \tag{12}$$

where the function A may be a vector or tensor.

The partial derivatives of a continuous function are independent of the order of differentiation. Thus the function A satisfies the identities $\nabla\times\nabla\mathbf{A} = 0$ and $\nabla\cdot\nabla\times\mathbf{A} = 0$.

The Laplacian of a function A is expressed by the equation

$$\nabla\cdot\nabla\mathbf{A} = \frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2} \tag{13}$$

where the function A may be a scalar, vector, or tensor.

Laplace's equation is $\nabla\cdot\nabla\mathbf{A} = 0$. An important solution of Laplace's equation is given by the equation

$$\frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \tag{14}$$

which may be verified by substitution in the Laplacian.

GAUSS THEOREM

Let a volume of space be enclosed by a boundary surface. Integration of the divergence throughout the volume is expressed by the equation

$$\int \nabla\cdot\mathbf{A}\,d\tau = \int\int\int \mathbf{i}\cdot\frac{\partial \mathbf{A}}{\partial x}\,dxdydz + \int\int\int \mathbf{j}\cdot\frac{\partial \mathbf{A}}{\partial y}\,dxdydz + \int\int\int \mathbf{k}\cdot\frac{\partial \mathbf{A}}{\partial z}\,dxdydz \tag{15}$$

where $d\tau$ is a differential element of volume. Partial integration leads to the equation

$$\int \nabla\cdot\mathbf{A}\,d\tau = \int\int \Big[\mathbf{i}\cdot\mathbf{A}\Big]_{(1)}^{(2)} dydz + \int\int \Big[\mathbf{j}\cdot\mathbf{A}\Big]_{(1)}^{(2)} dzdx + \int\int \Big[\mathbf{k}\cdot\mathbf{A}\Big]_{(1)}^{(2)} dxdy \tag{16}$$

where subscript (1) and superscript (2) indicate the difference between values of an integral at the opposite ends of the line of integration. The surface elements

$$dydz \qquad\qquad dzdx \qquad\qquad dxdy \tag{17}$$

can be replaced in the surface integration by the parallelograms which are the projections of a surface element ds on each coordinate plane. Thus

$$\int \nabla\cdot\mathbf{A}\,d\tau = \int d\mathbf{s}\cdot\mathbf{A} \tag{18}$$

where $d\tau$ is the differential volume element and ds is the differential surface element. The vector ds is directed outward from the volume of integration. This is known as the Gauss theorem.

STOKES THEOREM

Let a surface be bounded by a closed circuit. Let u, v be right-handed coordinates in the surface. A vector surface element is defined by the equation

$$ds = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du dv \tag{19}$$

where r is the position vector of the element with coordinates u, v. Application of the triple vector product transformation leads to the equation

$$\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \cdot \nabla \times \mathbf{A} = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times (\nabla \times \mathbf{A}) = \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{A}}{\partial u} - \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{A}}{\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial \mathbf{r}}{\partial v} \cdot \mathbf{A}\right) - \frac{\partial}{\partial v}\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \mathbf{A}\right) \tag{20}$$

Then integration is given by the equation

$$\int ds \cdot \nabla \times \mathbf{A} = \int \left[\frac{\partial \mathbf{r}}{\partial v} \cdot \mathbf{A}\right]_{(1)}^{(2)} dv - \int \left[\frac{\partial \mathbf{r}}{\partial u} \cdot \mathbf{A}\right]_{(1)}^{(2)} du = \oint \frac{\partial \mathbf{r}}{\partial u} \cdot \mathbf{A}\, du + \oint \frac{\partial \mathbf{r}}{\partial v} \cdot \mathbf{A}\, dv \tag{21}$$

where subscript (1) and superscript (2) indicate the difference between values of an integral at opposite ends of the line of integration. Thus

$$\int ds \cdot \nabla \times \mathbf{A} = \oint d\mathbf{r} \cdot \mathbf{A} \tag{22}$$

where ds is the differential surface element and dr is the differential circuit displacement. Circuit integration is right-handed relative to the surface normal. This is known as the Stokes theorem.

HELMHOLTZ THEOREM

Let a volume be enclosed by a surface and let a point within the volume be enclosed by a sphere of radius ϵ. In the space between the two surfaces a vector a satisfies the Green's theorem which is expressed by the equation

$$\int \nabla \cdot (\nabla U\, \mathbf{a})\, d\tau = \int \nabla \cdot \nabla U\, \mathbf{a}\, d\tau + \int \nabla U \cdot \nabla \mathbf{a}\, d\tau = \int ds \cdot \nabla U\, \mathbf{a} \tag{23}$$

Let U be given by the equation

$$U = \frac{1}{r} \qquad \nabla \cdot \nabla U = 0 \tag{24}$$

where r is the distance of the point of integration from the center of the sphere. Within the sphere the integral

$$\int ds \cdot \nabla\left(\frac{1}{r}\right) = 4\pi \qquad (r = \epsilon) \tag{25}$$

is the solid angle of the sphere. Therefore the value of a at the center of the sphere is given by the equation

$$\mathbf{a} = \frac{1}{4\pi} \int \nabla\left(\frac{1}{r}\right) \cdot \nabla \mathbf{a}\, d\tau - \frac{1}{4\pi} \int ds \cdot \nabla\left(\frac{1}{r}\right) \mathbf{a} \tag{26}$$

Transformations of the integrals are made with the aid of the following identities.

$$\nabla\cdot\left\{\mathbf{a}\nabla\left(\frac{1}{r}\right)\right\} = (\nabla\cdot\mathbf{a})\nabla\left(\frac{1}{r}\right) + \mathbf{a}\cdot\nabla\nabla\left(\frac{1}{r}\right) \tag{27}$$

$$\nabla\cdot\left\{\mathbf{I}\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)\right\} = \nabla\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right) + \nabla\nabla\left(\frac{1}{r}\right)\cdot\mathbf{a} \tag{28}$$

Inasmuch as $\nabla\nabla(\frac{1}{r})$ is a symmetric tensor, it may be cancelled to give the equation

$$\int(\nabla\cdot\mathbf{a})\nabla\left(\frac{1}{r}\right)d\tau - \int\nabla\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)d\tau = \int d\mathbf{s}\cdot\mathbf{a}\nabla\left(\frac{1}{r}\right) - \int d\mathbf{s}\,\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right) \tag{29}$$

Transformations of the integrals are made with the aid of the following identities.

$$\nabla\left(\frac{1}{r}\right)\times(\mathbf{a}\times d\mathbf{s}) = \mathbf{a}\nabla\left(\frac{1}{r}\right)\cdot d\mathbf{s} - d\mathbf{s}\,\nabla\left(\frac{1}{r}\right)\cdot\mathbf{a} \tag{30}$$

$$\nabla\left(\frac{1}{r}\right)\times(\nabla\times\mathbf{a}) = \nabla\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right) - \nabla\left(\frac{1}{r}\right)\cdot\nabla\mathbf{a} \tag{31}$$

Therefore, the value of a at the center of the sphere is given by the equation

$$\mathbf{a} = \frac{1}{4\pi}\int\nabla\left(\frac{1}{r}\right)\nabla\cdot\mathbf{a}\,d\tau - \frac{1}{4\pi}\int\nabla\left(\frac{1}{r}\right)\times\nabla\times\mathbf{a}\,d\tau - \frac{1}{4\pi}\int\nabla\left(\frac{1}{r}\right)\mathbf{a}\cdot d\mathbf{s} - \frac{1}{4\pi}\int\nabla\left(\frac{1}{r}\right)\times(\mathbf{a}\times d\mathbf{s}) \tag{32}$$

Let the potentials φ and A be defined by the equations

$$\varphi = \frac{1}{4\pi}\int\frac{1}{r}\nabla\cdot\mathbf{a}\,d\tau - \frac{1}{4\pi}\int\frac{1}{r}\mathbf{a}\cdot d\mathbf{s} \tag{33}$$

$$\mathbf{A} = \frac{1}{4\pi}\int\frac{1}{r}\nabla\times\mathbf{a}\,d\tau + \frac{1}{4\pi}\int\frac{1}{r}\mathbf{a}\times d\mathbf{s} \tag{34}$$

where r is now the distance of the center of the sphere from the point of integration. Then the vector a is given by the equation

$$\mathbf{a} = -\nabla\varphi + \nabla\times\mathbf{A} \tag{35}$$

This is known as the Helmholtz theorem.

HYDRODYNAMICS

Let a solid body be moving at constant speed U through a fluid which was initially at rest. A Cartesian coordinate system can be set up with origin fixed in the body. Let i be in the direction of motion, let j be in the direction to the right, and let k be in the direction downward. Then relative to this coordinate system the flow is backward. The flow consists of two parts. The first part is a uniform flow with a velocity which is constant everywhere, and the second part is a local flow with a velocity which diminishes rapidly with distance. Let there be a nonslip boundary condition at the surface of the body. Then the uniform flow has a velocity $-U\mathbf{i}$ everywhere, while the local flow has a velocity $+U\mathbf{i}$ at the surface of the body. The velocity v in the local flow satisfies the Helmholtz theorem

$$\mathbf{v} = \frac{1}{4\pi}\int\nabla\left(\frac{1}{r}\right)\nabla\cdot\mathbf{v}\,d\tau - \frac{1}{4\pi}\int\nabla\left(\frac{1}{r}\right)\times\nabla\times\mathbf{v}\,d\tau - \frac{1}{4\pi}\int\nabla\left(\frac{1}{r}\right)\mathbf{v}\cdot d\mathbf{s} - \frac{1}{4\pi}\int\nabla\left(\frac{1}{r}\right)\times(\mathbf{v}\times d\mathbf{s}) \tag{36}$$

which gives the velocity at a field point in terms of velocities at all integration points. Volume integrals are evaluated throughout the fluid. Surface integrals are evaluated over the body, but vanish over the surface at infinity because of the diminution of velocity with distance.

Insofar as the fluid is incompressible the divergence $\nabla \cdot \mathbf{v}$ is zero and the first integral vanishes everywhere. Next to the body there is a boundary layer. Within the boundary layer vorticity is diffusing outward. Outside the boundary layer the curl $\nabla \times \mathbf{v}$ is zero and the second integral is limited to the boundary layer.

In the limiting case of a thin boundary layer, let the volume element be a prism which straddles the boundary layer. Let the dimensions of the prism be ϵ perpendicular to the boundary, δ in the direction across the flow, and λ in the direction along the flow. Then the volume element and the surface element are given by the equations

$$d\tau = \epsilon\delta\lambda \qquad\qquad d\mathbf{s} = \delta\lambda\mathbf{n} \tag{37}$$

where $\mathbf{n}$ is a unit vector in the inward direction. Application of the Stokes theorem to a rectangle of width ϵ and length λ leads to the equivalence

$$\nabla \times \mathbf{v}\, d\tau = -\, \mathbf{v} \times d\mathbf{s} \tag{38}$$

For a field point far outside the boundary layer the second integral cancels the fourth integral, but for a field point inside the boundary layer the sign of $\nabla(\frac{1}{r})$ is reversed and both integrals contribute to the velocity. The predominant integral outside the boundary layer is the third integral, which expresses the velocity as the integral over the body of radial flow from a continuous source distribution with the source density

$$\frac{U}{4\pi}\, \mathbf{i} \cdot \mathbf{n} \tag{39}$$

where $\mathbf{n}$ is a unit vector directed outward from the body. Evaluation of the integrals is a quadrature which is complicated by the presence of an inverse square in the integrands. The inverse square introduces a spike in each integrand and the spike cannot be integrated by summation over finite intervals. A possible technique for integration through a spike will be the subject of a future report.

The boundary layer flows into the wake behind the body. The structure of the wake depends upon the separation of flow in the boundary layer. The body may have a wave train. If the body is self-propelled, there is flow into the propeller. The uniform flow is disturbed by these local patterns of flow. The Helmholtz theorem may by applied to any or all of the patterns which disturb the uniform flow.

A Rankine ovoid is that streamline of zero stream function which is generated by the ideal flow around a point source and a point sink. The NSRDC[5] made a model with the same profile as the Rankine ovoid and towed the model in the towing tank. The real model had a boundary layer and wake which would be absent from the ideal flow around the source and sink.

The velocity $\mathbf{v}$ for the ideal flow is given by the equation

$$\mathbf{v} = -\, U\mathbf{i} - \nabla\varphi \tag{40}$$

where φ is the potential of the source and sink. The surface normal at the boundary is given by the equation

$$\mathbf{n} = \frac{\mathbf{b} \times \mathbf{v}}{|\mathbf{b} \times \mathbf{v}|} \tag{41}$$

where b is the binormal transverse to the streamline. For the nonslip boundary condition with a thin boundary layer the real flux is just equal to the ideal flux as indicated by the equation

$$U\mathbf{i}\cdot\mathbf{n} = -\nabla\varphi\cdot\mathbf{n} \tag{42}$$

Thus the normal component of velocity is the same for both ideal flow and real flow. It is well known that two solutions of Laplace's equation which have the same normal gradients at a boundary surface are identical to within an additive constant. The real flow outside the boundary is the ideal flow from the source and sink.

The NSRDC[5] measured the elevation in the wave train in the towing tank, and a computing program at NSWC[6] was used to compute the wave elevations for the point sources. The model was towed with a twisted cable and the boundary layer was fully turbulent. The forward motion in the boundary layer met partially the boundary conditions, and less source distribution was required to complete the boundary conditions. The experimental elevations were a few percent less than the computed elevations. Turbulence in the wake was revealed by photographs of smoke traces[7].

DISCUSSION

Among the texts which have been inspected only two had derivations of the Helmholtz theorem, and they derived formulae for an infinite space, or for a field which was finite within a boundary and zero outside the boundary. In this report enough background material has been included to define notation and to clarify concepts.

The gradient of velocity satisfies the identity

$$\nabla\mathbf{v} = \tfrac{1}{2}(\nabla\mathbf{v} - \nabla'\mathbf{v}) + \tfrac{1}{2}(\nabla\mathbf{v} + \nabla'\mathbf{v}) \tag{43}$$

where $\nabla'\mathbf{v}$ is the transpose of $\nabla\mathbf{v}$. The antisymmetric part is the rate of rotation and the symmetric part is the rate of strain. The differential change of velocity $d\mathbf{v}$ for a differential displacement $d\mathbf{r}$ is given by the equation

$$d\mathbf{v} = d\mathbf{r}\cdot\nabla\mathbf{v} \tag{44}$$

Application of the triple vector product rule to the antisymmetric part gives the equation

$$d\mathbf{r}\cdot\tfrac{1}{2}(\nabla\mathbf{v} - \nabla'\mathbf{v}) = \tfrac{1}{2}(\nabla\times\mathbf{v})\times d\mathbf{r} \tag{45}$$

Thus the angular velocity ω is given by the equation

$$\omega = \tfrac{1}{2}\nabla\times\mathbf{v} \tag{46}$$

It has become fashionable among hydrodynamicists to use omega for the curl itself, but this is a deplorable deviation from standard notation. The use of omega for angular velocity is pre–empted by its universal use in classical and quantum physics. It would be better to use gamma as defined by the equation

$$\gamma = \nabla\times\mathbf{v} \tag{47}$$

where γ stands for the density of circulation.

CONCLUSION

The Helmholtz theorem is a useful law to which the flow in any flow field must conform within any mathematical boundary.

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