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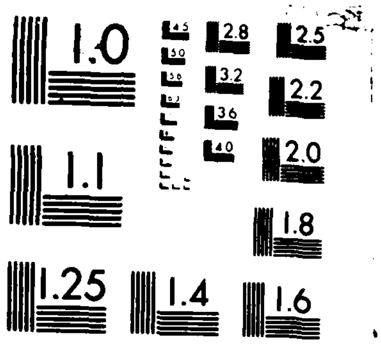

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The purpose of this note is to give a direct proof of some recent important results of E.B. Dynkin and A. Mandelbaum [2]. This also provides immediately the results in [3] with a very simple proof. This is achieved by avoiding the use of Poisson process. Let us set up some notation. Let (L,Σ,μ) be a probability space and (X^k,Σ^k,μ^k) be the k-fold produce probability space. Let $h_k(x_1,\ldots,x_k)$ be a symmetric function of k-variables We call it canonical if $\int h_k(x_1,\ldots,x_{k-1},y)d\mu=0$ for all $x_1,\ldots,x_k \in X^{k-1}$. Let $X_1,\ldots,X_n$ be a i.i.d. X-valued random variable on a probability space with distribution μ.

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ON A LIMIT THEOREM AND INVARIANCE PRINCIPLE FOR SYMMETRIC STATISTICS

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ON A LIMIT THEOREM AND INVARIANCE PRINCIPLE FOR SYMMETRIC STATISTICS

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0. Introduction: The purpose of this note is to give a direct proof of some recent important results of E.B. Dynkin and A. Mandelbaum [2]. This also provides immediately the results in [3] with a very simple proof. This is achieved by avoiding the use of Poisson process. Let us set up some notation. Let (X,Σ,μ) be a probability space and (X^k,Σ^k,μ^k) be the k-fold produce probability space. Let $h_k(x_1,\ldots, x_k)$ be a symmetric function of k-variables. We call it canonical if $\int h_k(x_1,\ldots, x_{k-1},y)d\mu = 0$ for all $x_1,\ldots, x_{k-1} \in X^{k-1}$. Let $X_1,\ldots, X_n$ be a i.i.d. X-valued random variable on a probability space with distribution μ. As in [2], define

$$\begin{aligned}\sigma_k^n(h_k) &= \Sigma_{1\le s_1<\ldots<s_k\le n} h_k(X_{s_1},\ldots, X_{s_k}), \quad \text{for } k\le n \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{for } k>n.\end{aligned}$$

Let $H = \{(h_0,h_1,\ldots):\ h_k \text{ canonical and } \Sigma_{k=1}^{\infty} \frac{1}{k!}\|h_k\|_2^2 < \infty\}$ where h_0 is a constant and $\|\ \|_2$ is the norm in $L^2(X^k,\Sigma^k,\mu^k)$. On H define $\|h\|^2 = \Sigma_{k=0}^{\infty}\|h_k\|_2^2/k!$. H is the so-called exponential (Foch) space of $L_0^2(X,\Sigma,\mu)(\phi \in L^2(X,\Sigma,\mu)$ with $E\phi(X) = 0)$. It is a Hilbert space under coordinate addition, scalar multiplication and $\|\ \|$. For each $\phi \in L_0^2(X,F,\mu)$, $h^{\phi} \in H$ with $h_k^{\phi} = \phi(x_1),\ldots, \phi(x_k)$. It can be easily seen that $sp\{h^{\phi}:\ \phi \in L_0^2(X,F,\mu)\}$ is dense in H. Define for each $h \in H$,

$$(0.1) \qquad Y_n(h) = \sum_{k=0}^{\infty} n^{-k/2}\, \sigma_k^n(h_k).$$

Since $\sigma_k^n(h_k) = 0$ for $k > n$, this is a finite sum. Also, let

$$(0.2) \qquad Y_n^t(h) = \sum_{k=0}^{\infty} n^{-k/2}\, \sigma_k^{[nt]}(h_k).$$

The main purpose is to show directly that $Y_n(h) \xrightarrow{\mathcal{D}} \Sigma_{k=0}^{\infty} \frac{I_k(h_k)}{k!}$ where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $I_k(h_k)$ denotes Ito-Wiener multiple integral of h_k with respect to Gaussian random measure W with $EW(A)W(A') = \mu(A \cap A')$.

In the next section we discuss the convergence of $Y_n^t(h)$. We observe that for $\phi \in L_0^2(X,\Sigma,\mu)$

$$Y_n(h^{\phi}) = \sum_{k=0}^{n} n^{-k/2} \sum_{1 \le s_1 < \dots < s_k \le n} \phi(X_{s_1}) \dots \phi(X_{s_k})$$
$$= \sum_{k=0}^{n} \sum_{1 \le s_1 < \dots < s_k \le n} \frac{\phi(X_{s_1})}{\sqrt{n}} \dots \frac{\phi(X_{s_k})}{\sqrt{n}}$$
$$= \prod_1^n \left(1 + \frac{\phi(X_i)}{\sqrt{n}}\right).$$

Let us observe that for any $\varepsilon > 0$,

$$\sum_j P(|\phi(X_j)| > \sqrt{\varepsilon j}) = \sum_j P(|\phi(X_1)|^2 > \varepsilon j) \le \|\phi\|_2^2 < \infty.$$

Hence by Borel-Cantelli lemma, a.s. (for $j \le n$)

$$|\phi(X_j)| \le \sqrt{\varepsilon j} \le \sqrt{\varepsilon}\sqrt{n} \quad \text{for } j \ge \text{some } N(\omega) \quad (N(\omega) < \infty).$$

But $\prod_1^n \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right) = \prod_1^{N(\omega)} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right) \prod_{N(\omega)}^{n} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right)$ giving for a.s. w, so

$\lim_n Y_n(h^{\phi}) = \lim_n \prod_{N(\omega)}^{n} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right)$. Thus WLOG, we can assume for n large

$\left|\frac{\phi(X_j)}{\sqrt{n}}\right| < 1$ a.s. for all $j \le n$ and $Y_n(h^{\phi}) = \prod_1^n \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right)$. Taking log on both sides and expanding $\log(1+x)$ we have

$$\log \prod_1^n \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right) = \sum_1^n \frac{\phi(X_j)}{\sqrt{n}} - \frac{1}{2} \sum_1^n \frac{\phi(X_j)^2}{n} + \varepsilon_n(\phi)$$

where $\varepsilon_n(\phi) \xrightarrow{P} 0$ by the WLLN and since $\max\left|\frac{\phi(X_j)}{\sqrt{n}}\right| \xrightarrow{P} 0$ by Chebychev's Inequality, i.e. the $(Y_n(h^\phi)) \overset{\mathcal{D}}{\to} \exp[I_1(\phi) - \frac{1}{2}\|\phi\|_2^2$. Using Cramér-Wold device and the above argument we get

0.3 Lemma: For any finite subset $\{\phi_1 \ldots \phi_k\} \subseteq L^2(X,\Sigma,\mu)$

$$(Y_n(h^{\phi_1}),\ldots, Y_n(h^{\phi_k})) \overset{\mathcal{D}}{\to} (\exp(I_1(\phi_1)) - \tfrac{1}{2}\|\phi_1\|_2^2,\ldots, \exp(I_1(\phi_k) - \tfrac{1}{2}\|\phi_k\|_2^2).$$

As a consequence, we get for $\{\phi_i,\ i \in I\}$ a finite subset of $L^2(X,\Sigma,\mu)$ and $\{c_i,\ i \in I\} \subseteq \mathbb{R}$,

$$(0.3)' \quad Y_n(\sum_{i \in I} c_i h^{\phi_i}) \overset{\mathcal{D}}{\to} \sum_{k=0}^{\infty} \frac{I_k([\sum_{i\in I} c_i h^{\phi_i}]_k)}{k!} .$$

We now observe that for $h, h' \in H$,

$$(0.4) \quad E[Y_n(h) - Y_n(h')]^2 = \sum_k \binom{n}{k} n^{-k} \|h_k - h'_k\|^2 \le E\|h - h'\|^2,$$

since $E\sigma_k^n(h_k - h'_k)\sigma_\ell^n(h_\ell - h'_\ell) = \binom{n}{k}\| h_k - h'_k\|^2 \delta_{k\ell}$ by ([2], p. 744). Also,

$$(0.5) \quad E(\sum_{k=0}^{\infty} I_k(h_k)/k! - \sum_{k=0}^{\infty} \frac{I_k(h'_k)}{k!})^2 = \|h - h'\|^2.$$

Thus we get

(0.6) Theorem: For any $h \in H$,

$$Y_n(h) \overset{\mathcal{D}}{\to} W(h) = \sum_{k=0}^{\infty} \frac{I_k(h_k)}{k!}$$

Proof: Let $h \in H$ and $\varepsilon > 0$. Choose $h' = \Sigma_{i\in I} c_i h^{\phi_i}$ such that $\|h - h'\|^2 < \varepsilon/2$. Now consider for $t \in \mathbb{R}$

$$|E(e^{itY_n(h)} - e^{itW(h)})| \le E|e^{itY_n(h)} - e^{itY_n(h')}| + E|e^{itY_n(h')} - e^{itW(h')}|$$

$$+ E|e^{itW(h')} - e^{itW(h)}|.$$

Using Schwartz's Inequality and the fact $|e^{ix} - 1| \le |x|$ we get that the first and third term of the above inequality are dominated by $t^2 E\|h - h'\|^2$ using (0.4) and (0.5). Hence by (0.3)'

$$\overline{\lim}\,|Ee^{itY_n(h)} - Ee^{itW(h)}| \le \varepsilon/2.$$

As ε is arbitrary we get the result.

Finally, we make some observations to be used later.

$$(0.7)\qquad Y_n^t(h^\phi) = \sum_{k=0}^{[nt]} n^{-k/2} \sum_{1\le s_1<\dots<s_k\le[nt]} \phi(X_{s_1})\dots\phi(X_{s_k}) = \prod_{1}^{[nt]} \left(1 + \frac{\phi(X_i)}{\sqrt{n}}\right).$$

Also, $\min(t,s)\mu(A\cap A')$ is a covariance on $[0,\infty)\times\Sigma$ giving that there exists a centered Gaussian process $\underline{W}(t,A)$ with $E\underline{W}(t,A)\underline{W}(s,A') = \min(t,s)\mu(A\cap A')$. Let for $T<\infty$

$$H_T = \{(h_0,h_1,\dots)\in H : \sum_{k=0} T^k \frac{\|h_k\|^2}{k!} < \infty\}.$$

1. <u>Invariance Principle</u>: Let $D[0,T]$, $(T\le\infty)$ be the space of right continuous functions on $[0,T]$ ($[0,\infty)$) with left limits at each $t\le T$. The space $D[0,T]$ is endowed with Skorohod topology [1]. The topology in $D[0,\infty)$ is the one described in Whitt [4]. We note that

$X_{[nt]} = \sum_{1}^{[nt]} \left(\frac{\phi^2(X_i) - E\phi^2}{n}\right)$ has stationary independent increments. So for $\varepsilon > 0$

$$P\left(\sup_{0\le t\le T} |X_{[nt]}| > \varepsilon\right) \le C.P(|X_{[nT]}| \ge \varepsilon) \to 0$$

by the weak law of large numbers. Using this, the arguments preceding Lemma 0.3, invariance principle and Cramér-Wold device we get the following analogue of Lemma 0.3.

<u>Lemma 1.1</u>: $(Y_n^t(h^{\phi_1}),\dots, Y_n^t(h^{\phi_k})) \xrightarrow{\mathcal{D}_{k,T}} (\exp(I_1^t(\phi_j) - \frac{1}{2}t\|\phi_j\|^2),\ j=1,\dots,k)$

where $I^t(\phi_j) = \iint 1_{(0,t]}(u)\phi_j(x)W_k(du,dx)$. Here $\xrightarrow{\mathcal{D}_{k,T}}$ denotes convergence in $D^k[0,T]$ with respect to product topology.

We note that $W(t,A)$ is a Brownian motion for each $A \in \Sigma$. Thus we can choose $I^t(\phi)$ continuous for each ϕ and a martingale in t as $I^t(\phi) = \int \phi(x)W(t,dx)$. We get for $\{c_1, \ldots, c_k\} \subseteq \mathbb{R}$, ($k$ finite),

$$Y^t\left(\sum_{j=1}^{k} c_j h^{\phi_j}\right) \to \sum_{j=1}^{k} c_j \exp\left(I^t(\phi_j) - \frac{1}{2}t\|\phi_j\|^2\right).$$

Let $\phi \in L_0^2(X,\Sigma,\mu)$, $\|\phi\| = 1$, and denote

$$(\phi^k)^t = \phi(x_1)\ldots\phi(x_k)1_{(0,t]}(u_1)\ldots 1_{(0,t]}(u_k).$$

Define $I_k(\phi^k)^t = k!H_k(t,I(\phi))$ where H_k is Hermite polynomial, i.e. $\Sigma_{k=0}^{\infty}\gamma^k H_k(t,x) = \exp(\gamma x - \frac{1}{2}\gamma^2 t)$. For $\phi \in L_0^2(X,\Sigma,\mu)$, $\|\phi\| = 1$, we define for $(h^{\phi})^t = (1,\phi^t,(\phi^2)^t,\ldots)$,

$$W(h^{\phi})^t = \sum_{k=0}^{\infty} \frac{I_k(\phi^k)^t}{k!},$$

and extend it linearly to $(\Sigma c_j (h^{\phi_j})^t)$. It is a martingale. Let $h \in H_T$ $\{h(n)\}$ a sequence in $sp\{(h^{\phi})^t, \phi \text{ in CONS in } L_0^2(X,\Sigma,\mu)\} \subseteq H_T$, then

$$P(\sup_{t\le T}|W^t(h(n) - h(m))| \ge \varepsilon) \le E|W^T(h(m) - h(n))|^2$$

$$= \sum_{k=0}^{\infty} T^k \frac{\|h_k(m) - h_k(n)\|^2}{k!}$$

using Doob's inequality and argument as in (0.5). Define for $h \in H^t$, $W^t(h) = \text{-lim}W^t(h_n)$ where the limit is uniform on compact for $h_n \to h$. Then $W^t(h)$ is right continuous martingale and has the same distribution as $\Sigma_k I_k^t(h_k)/k!$. Now we derive the main theorem of [3].

<u>Theorem 1.2</u>: $Y_n^t(h) \xrightarrow{\mathcal{D}} W^t(h)$ in $D[0,T]$ for $h \in H^T$ for each $T < \infty$.

Proof: Let $h \in H$ and $\varepsilon > 0$, choose $h_k' \in sp\{h^\phi : \phi \in L_0^2(X,\Sigma,\mu)\} \ni h_k \to h$. Now define

$$X_{nk}^\bullet = Y_n^\bullet(h_k'),\ Z_n^\bullet = Y_n^\bullet(h),\ X_k^\bullet = W^\bullet(h_k') \text{ and } X = W^\bullet(h).$$

Then $X_{n,k}^\bullet \xrightarrow{\mathcal{D}} X_k^\bullet$ as $n \to \infty$ in $D[0,T]$ for each $T < \infty$ by Lemma 1.1. Also $X_k^\bullet \xrightarrow{\mathcal{D}} X$ as $n \to \infty$ in $D[0,T]$ for each $T < \infty$. In addition,

$$P(\sup_{0 \le t \le T} |X_{nk}^\bullet - Z_n^\bullet| \ge \varepsilon) \le E|Y_n^T(h - h_k')|^2 \le T\|h - h_k'\|$$

giving $\lim_{k\to\infty} \overline{\lim_n} P(\rho(X_{nk}^\bullet, Z_n^\bullet) \ge \varepsilon) \to 0$ with ρ being the Skorohod metric on $D[0,T]$. This implies by ([1], Thm 4.2, p. 25) that $Z_n^\bullet \xrightarrow{\mathcal{D}} W^\bullet(h)$ in $D[0,T]$ $(T < \infty)$ giving the result.

Remark: In the above arguments we may use an interpolated version of $Y_n^t(h)$ from the beginning and use appropriate version of Donsker's Invariance Principle to conclude above convergence occurs in $D[0,T]$ in sup norm giving $W^t(h)$ continuous.

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