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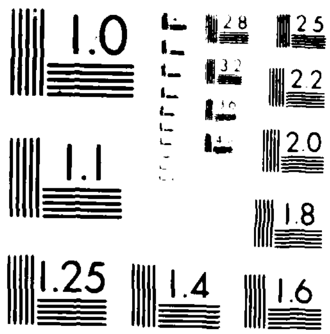
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Abstract

In this paper we study a generalized second order resonance-tunneling ordinary differential equation, $(z + i\tilde{\Gamma})\,\phi''(z) + \alpha\phi'(z) - (\beta_1 - z\beta_2^2)\phi(z) = 0$, where the parameter α is real and positive, $\alpha > 0$, and all other parameters are complex. We have obtained solutions of this equation when the independent variable covers the infinite domain, $-\infty < z < \infty$, and making use of suitable integral representations we have determined their asymptotic expansions. With the aid of their leading terms the two transmission-reflection problems have been solved. Thus, we show that the expected Budden absorption occurs for $\alpha = 2N$, $N = 0, 1, 2, \ldots$. However, in bands centered about $\alpha = 2N + 1$, the modulus of the reflection coefficient is generally larger than unity.

PACS: 03.40.Kf, 02.90.+p, 42.10.-s

1. Introduction

In a brief paper in Physical Review Letters[1] we announced the preparation of a lengthier publication containing rigorous proofs of the outlined results. Thus, the present paper is meant to serve the indicated purpose. It is well known that the propagation of waves through nonuniform media has been the subject of broad interest and study in various areas of basic and applied physics. We wish to emphasize the two physical effects encountered in such cases; namely, the appearance of cutoffs, i.e., points where the local wave-number vanishes $[k(z) = 0]$, and resonance points where short wavelengths develop $[(k(z) \to \infty]$. In general, we expect cutoffs to result in wave reflection, while resonances lead to wave absorption. Because of the seemingly opposite roles played by cutoffs and resonances, it is of considerable importance to investigate model systems in which both effects are present simultaneously and in a sense compete with each other; this is the essence of resonance-tunneling problems. The prototype equation used to approximate physical systems exhibiting such features is Budden's equation.[2] This second order differential equation describes systems in which propagation regions exist on both sides of a resonance, but in which one of the propagation regions also contains a cutoff. An intrinsic property of Budden's equation is the fact that, for waves approaching the resonance $(z = 0)$ from the cutoff side $(z > 0)$, the modulus of the reflection coefficient is less than unity. Such a result is consistent with the intuitive expectation than a fraction of the incident wave is absorbed at the resonance, while yet another portion can tunnel to the propagation region on the other side $(z < 0)$. The converse situation corresponds to a wave approaching the resonance from the left $(z < 0)$ and a portion of which is transmitted to the other side $(z > 0)$, where the cutoff exists, while there is no reflection at all. In neither case is the initial wave energy conserved in Budden's problem as well as in our generalized version.

One important feature of Budden's equation is the absence of a first derivative term. In this paper we have generalized the resonance-tunneling problem by inserting, in the differential equation, a first derivative multiplied by an adjustable parameter α which is real and positive, $\alpha \geq 0$. When we put $\alpha = 0$ we recover Budden's equation. We have further generalized the problem by allowing the presence of dissipation, with the result that all other parameters in the equation, as well as the independent variable, become complex numbers. We have solved the generalized equation in detail and discovered that, as function of the parameter α, the reflection coefficient exhibits an anomalous behavior; namely, that the expected Budden absorption occurs for $\alpha = 2N$, $N = 0, 1, 2, \ldots$, and that in bands centered about $\alpha = 2N + 1$, the modulus of the reflection coefficient can be larger than unity.

Initially, our motivation for the study of the generalized equation arose from a specific problem[3] (electrostatic whistler waves in magnetized plasmas with longitudinal density gradients), but having encountered the extraordinary reflection phenomena described above, we proceeded to carry out a detailed analysis of the solutions of the generalized equation which we present in this paper. Thus, in Section 2 we present the basic differential equation, and then obtain a fundamental set of solutions. In Section 3 we take up the role of Whittaker's equation to study its behavior close to the resonance ($z = 0$) and to investigate, for $\alpha \geq 0$, the small argument leading terms of the solutions of the generalized equation. In Section 4 we set up integral representations for the solutions of our basic differential equation. In Section 5 we compute and tabulate their asymptotic leading terms, which we need in Section 6 to compute the transmission and reflection coefficients.

2. Basic Differential Equation

As explained in the Introduction we obtain our basic differential equation by inserting in Budden's equation a first derivative term. Thus, we have

$$(z + i\hat{\Gamma})\,\phi''(z) + \alpha\phi'(z) - (\beta_1 - z\beta_2{}^2)\phi(z) = 0 \qquad (2.1)$$

which reproduces Eq. (1) of Ref. 1. In (2.1) the parameters $\hat{\Gamma}$, β_1, and β_2 are complex numbers with positive real parts and small positive phase angles, which heuristically account for dissipative processes. In the absense of dissipation, $\hat{\Gamma}$ vanishes and (β_1, β_2) are positive definite numbers. The parameter α in (2.1) which is the coefficient of the first derivative $\phi'(z)$, is real and positive, $0 \leq \alpha < \infty$, and plays a dominant role in the entire development of the present paper. Thus, when $\alpha = 0$ we recover Budden's equation; when $\alpha = 1$ we observe that (2.1) becomes the self-adjoint analogue of the Budden problem, and when α varies over the whole range, $0 \leq \alpha < \infty$, we encounter the periodic variations, as functions of α, of the reflection and transmission coefficients pointed out earlier in Ref. 1 and to be derived rigorously in the sequel. Note that, as a matter of notation, we employ in (2.1) the parameter $\hat{\Gamma}$ in lieu of the parameter ν appearing in Eq. 1 of Ref. 1.

The main purpose of this paper is to present, for the infinite domain, $-\infty < z < \infty$, the analytic and asymptotic properties of the solutions to our basic equation. To this end we introduce in (2.1) the following notation:

$$w = i\xi, \quad \xi = 2\beta_2\,(z + i\hat{\Gamma})\;; \qquad (2.2)$$

$$\hat{\beta}_0 = \beta_0 + i\hat{\Gamma}\beta_2/2, \quad \beta_0 = \beta_1/2\beta_2 \;. \qquad (2.3)$$

We recall, from the definition (2.2), that in the absence of dissipation the parameter β_2 becomes positive definite and the parameter $\tilde{\Gamma}$ vanishes. Hence, in this limit, $\xi = 2\beta_2 z$ is a real number. Further we recall that in the infinite domain, $-\infty < z < \infty$, we have arg $z = 0$ when $z > 0$, and arg $z = \pi$ when $z < 0$, the latter statement being a consequence of causality as embodied in the general definition $\xi = 2\beta_2 (z + i\tilde{\Gamma})$. Thus, in the presence of dissipation both β_2 and $\tilde{\Gamma}$ are complex numbers with positive real parts and (small) positive phase angles; that is $\tilde{\Gamma} = |\tilde{\Gamma}|e^{i\theta_1}$, $\beta_2 = |\beta_2|e^{i\theta_2}$, where $0 < \theta_1, \theta_2 << \pi/2$, from which it follows that, in the general case, arg $\xi \to \theta_2$, $(\pi + \theta_2)$ as $z \to \pm\infty$.

Furthermore, in what follows, we propose to use w as the independent variable and we shall encounter that the multivalued functions that emerge will be expressed in terms of $\pm w = \pm i\xi$. Thus, at this juncture it becomes of interest to ascertain the ranges in arg$(\pm i\xi)$ which correspond to the principal branch of the functions in question. For the purpose we assume initially the absence of dissipation, which means that $\xi = 2\beta_2 z$ is real and covers the infinite domain $-\infty < \xi < \infty$, with arg $\xi = 0$ when $\xi > 0$ and arg $\xi = \pi$ when $\xi < 0$ as shown in Fig. 1, just as in the case of z in the previous paragraph. First, to determine $w = i\xi$ we swing the real axis in the ξ plane through an angle of $\pi/2$ in the counterclockwise direction about the origin $\xi = 0$, which makes $w = i\xi$ coincide with the axis of imaginaries as shown by the dashed line. This procedure shows that the cut in the complex ξ plane must be drawn along the negative imaginary axis, and we deduce that the principal branch for a multivalued function of $w = i\xi$ is given by the range

$$-\pi/2 < \arg (i\xi) \leq 3\pi/2 \quad . \tag{2.4}$$

Then, to determine $-w = -i\xi$ we must now swing the real axis in the ξ plane through an angle of $\pi/2$ in the clockwise direction about the origin of coordinates, which makes $-w = -i\xi$ coincide with the axis of imaginaries as shown by the <u>dotted</u> line. Hence, the cut in the complex ξ plane must again be drawn along the negative axis of imaginaries, exactly as before, except that this time we deduce that the principal branch of a multivalued function of $-w = -i\xi$ is given by the range

$$-\pi/2 \leq \arg\,(-i\xi) < 3\pi/2 \quad , \tag{2.5}$$

which should be contrasted with (2.4). We observe in Fig. 1 that the cut along the negative axis of imaginaries separates the <u>dashed</u> and <u>dotted</u> segments in the lower half plane of $w = i\xi$ and $-w = -i\xi$, respectively, and we note that the corresponding segments in the upper half plane actually coincide with each other.

In the presence of dissipation, as we have already seen, we have $\xi = 2\beta_2(z + i\tilde{\Gamma})$, where β_2 and $\tilde{\Gamma}$ are complex parameters, and asymptotically ($|z| \to \infty$) this definition becomes $\xi \sim 2\beta_2 z$. Confining our attention to positive values of z, $0 < z < \infty$, we compute the complex numbers $w = \pm i\xi$, which are shown in Fig. 1 by two black dots equidistant from the origin and lying on a straight line inclined at an angle $\theta_2 = \arg\beta_2$ with respect to the axis of imaginaries. We observe that the transition from w in the upper half plane to $-w$ in the lower half plane, or vice versa, is achieved by the semicircle drawn to the right, which conforms to the convention

$$(-w)^a = w^a\, e^{i\pi\varepsilon a}, \quad \varepsilon = \mathrm{sgn}\,\mathrm{Re}\{w\}, \tag{2.6}$$

where a is an arbitrary coefficient. In Fig. , the point w is such that $\mathrm{Re}\{w\} < 0$ and $\mathrm{Im}\{w\} > 0$; that is, w is a positive pure imaginary ($\theta_2 = 0$) or else lies on the second quadrant of the complex ζ plane, with corresponding statements for the point $-w$ on the lower half plane.

Thus, in making use of $w = i\zeta$ as the independent variable in (2.1), we obtain, writing $Y(w)$ instead of $\Phi(z)$, the generalized equation

$$w\, Y''(w) + \alpha Y'(w) + (i\tilde{\beta}_0 - w/4)\, Y(w) = 0 \quad , \tag{2.7}$$

which we proceed to solve in terms of confluent hypergeometric functions. Putting $Y(w) = e^{-w/2}y(w)$, it is found from (2.7) that $y(w)$ satisfies the canonical form of Kummer's equation

$$w\, y''(w) + (b - w)\, y'(w) - ay(w) = 0 \tag{2.8}$$

with parameters

$$a = \alpha/2 - i\tilde{\beta}_0,\ b = \alpha,\ b - a = \alpha/2 + i\tilde{\beta}_0 \quad . \tag{2.9}$$

Two linearly independent solutions of (2.8), valid for all possible values of the parameters (a,b) are[4]

$$y_5(w) = U(a,\ b;\ w),\ y_7(w) = e^w\, U(b - a,\ b;\ -w), \tag{2.10}$$

in which $U(a,\ b;\ w)$ is a multivalued solution to Kummer's equation for which the principal branch is usually taken as $-\pi < \arg w \leq \pi$. Consequently, a fundamental set of solutions of (2.7) is given by

$$Y_-(w) = e^{-w/2}y_5(w) = e^{-w/2}\,U(a,\ b;\ w) = e^{-w/2}\,U(\alpha/2 - i\tilde{\beta}_o,\ \alpha;\ w); \qquad (2.11)$$

$$Y_+(w) = e^{-w/2}y_7(w) = e^{w/2}\,U(b - a, b;\ -w) = e^{w/2}\,U(\alpha/2 + i\tilde{\beta}_o,\ \alpha;\ -w), \quad (2.12)$$

in which the latter forms apply specifically to the parameters defined by (2.9). It is noteworthy to point out that these parameters (2.9), and the corresponding solutions (2.11) and (2.12) satisfy the transformation

$$a \leftrightarrow b - a,\ b \leftrightarrow b,\ w \leftrightarrow -w; \qquad (2.13)$$

that is, $Y_-(w)$ as given by (2.11) can be turned into $Y_+(w)$ as given by (2.12), or vice versa, by merely applying the transformation (2.13).

For brevity in writing, it is convenient to introduce from (2.9) the parameters

$$a_\pm = \alpha/2 \pm i\tilde{\beta}_o,\ b = \alpha, \qquad (2.14)$$

with the aid of which the solutions (2.11) and (2.12) become

$$Y_-(w) = e^{-w/2}\,U(a_-,\ \alpha;\ w) \qquad |\arg (w)| < \pi; \qquad (2.15)$$

$$Y_+(w) = e^{w/2}\,U(a_+,\ \alpha;\ -w) \qquad |\arg (-w)| < \pi, \qquad (2.16)$$

which the reader will recognize in Ref. 1 as Eqs. (5) and (4), respectively. It is readily verified that the parameters (2.14) and the solutions (2.15) and (2.16) satisfy the transformation (2.13), i.e., $a_+ \leftrightarrow a_-$, $\alpha \leftrightarrow \alpha$, $w \leftrightarrow -w$, which, with $w = i\zeta$, implies that (2.13) can be condensed into a single reversible transformation $i \leftrightarrow -i$, wherever i appears explicitly.

3. The Role of Whittaker's Equation

In the sequel we shall often find it more convenient to deal with the complex independent variable $\zeta = 2B_2(z + i\tilde{\Gamma})$, as defined by (2.2) than with the independent variable $w = i\zeta$. Accordingly, writing $Z(\zeta)$ instead of $Y(i\zeta)$ in (2.7) yields

$$\zeta Z''(\zeta) + \alpha Z'(\zeta) + (\zeta/4 - \tilde{\beta}_o)\, Z(\zeta) = 0. \tag{3.1}$$

We can show that (3.1) is related to Whittaker's equation by putting

$$Z(\zeta) = \zeta^{-\alpha/2}\, u(\zeta), \tag{3.2}$$

where $u(\zeta)$ satisfies the Helmholtz equation

$$u''(\zeta) + Q(\zeta)\, u(\zeta) = 0, \tag{3.3}$$

in which

$$Q(\zeta) = 1/4 - \tilde{\beta}_o/\zeta + \alpha(2 - \alpha)/4\zeta^2 \tag{3.4}$$

plays the role of an effective potential and is real in the absence of dissipation; that is, when $\tilde{\Gamma} = 0$ in (2.3). It is clear from (3.4) that, when $\alpha = 0$ or $\alpha = 2$, the two cases exhibit identical potentials dominated in the vicinity of the resonance ($\zeta = 0$) by the anti-symmetric term ζ^{-1}. On the other hand, when α differs from these special values, a significant change develops near the resonance. This time, for $0 < \alpha < 2$, it acquires a symmetric term that behaves as ζ^{-2} and for $2 < \alpha < \infty$ the symmetric term becomes $-\zeta^{-2}$. It appears that the presence of the first derivative in (3.1) is responsible for this modification of the potential $Q(\zeta)$ in the vicinity of the resonance and is the principal reason for the anomalous reflection properties mentioned in Ref. 1 and in the Introduction, properties that we demonstrate rigorously in the sequel.

Quite generally we observe that, when $\alpha = 0,2$, the equation $Q(\xi) = 0$ has only one root, namely $\xi_+ = 4\tilde{\beta}_0$, which is the classical cutoff in Budden's equation. When α is different from these two special values, the equation $Q(\xi) = 0$ is a quadratic and has two distinct roots, ξ_- and ξ_+, the intercepts of the corresponding curves. If we assume that $|\tilde{\beta}_0|$ is large or, more precisely, that $|4\tilde{\beta}_0^2| \gg |\alpha(\alpha - 2)|$, we deduce that the intercepts are given approximately by

$$\xi_- \approx -\alpha(\alpha - 2)/4\tilde{\beta}_0; \; \xi_+ \approx 4\tilde{\beta}_0 + \alpha(\alpha - 2)/4\tilde{\beta}_0, \tag{3.5}$$

which satisfy the general conditions for the roots of a quadratic,

$$\xi_- + \xi_+ = 4\tilde{\beta}_0, \; \xi_- \times \xi_+ = \alpha(2 - \alpha), \tag{3.6}$$

and which tell us that, under the present assumptions, the cutoff point at $\xi_+ = 4\tilde{\beta}_0$ (for $\alpha = 0,2$) is shifted only by a relatively small amount as α is changed. Conversely, we notice from (3.5) that the cutoff ξ_- is relatively small but greatly dependent on α. Finally, we observe that, when $\alpha = 1$, which corresponds to the self-adjoint analogue of the Budden problem, ξ_- acquires its maximum value, $1/4\tilde{\beta}_0$, while ξ_+ its mininum, $4\tilde{\beta}_0 - 1/4\tilde{\beta}_0$, statements which imply that $\tilde{\beta}_0$ is assumed real.

To illustrate the essential features discussed in the preceding paragraphs we present in Fig. 2 the curves for $Q(\xi)$, as given by (3.4), assuming for the moment that $\tilde{\beta}_0$ is real. In this way we obtain, putting $\tilde{\beta}_0 \equiv 1$ for convenience, the curves corresponding to $\alpha = 0,2$, $\alpha = 1$, and $\alpha = 3$. We observe that the transition from the case $\alpha = 0,2$, to either the intermediate region $0 < \alpha < 2$ or to the extended domain $2 < \alpha < \infty$, is non-uniform. Finally, we wish to recall that the main thesis of the present paper is the enhanced reflection that we observe in certain periodic bands in α, in which it appears that the vicinity of the resonance at $\xi = 0$ is stimulated to emit waves spontaneously. To the best of our knowledge there is no intuitive way of extracting this periodic behavior from a study of $Q(\xi)$ from the curves of Fig. 2 or analytically from the definition (3.4).

Having studied the behavior of the function $Q(\xi)$, it becomes of interest to examine the small argument leading terms of the solutions (2.15) and (2.16) which we rewrite as

$$Y_{\pm}(i\xi) = e^{\pm i\xi/2} U(a_{\pm}, \alpha; \mp i\xi), \qquad (3.7)$$

where, it is recalled, $a_{\pm} = \alpha/2 \pm i\tilde{\beta}_0$ and $i\xi = 2i\beta_2(z + i\tilde{\Gamma}) = 2i\beta_2 z - 2\beta_2\tilde{\Gamma}$, with $|2\beta_2\tilde{\Gamma}| << 1$. The exponentials in (3.8) thus become

$$e^{\pm i\xi/2} = e^{\pm i\beta_2 z} \cdot e^{\mp\beta_2\tilde{\Gamma}} \xrightarrow[z \to 0]{} e^{\mp\beta_2\tilde{\Gamma}} = 1 \mp \beta_2\tilde{\Gamma} + \ldots, \qquad (3.8)$$

which, because $|\beta_2\tilde{\Gamma}| << 1$, may be equated to unity in a small argument ($z \to 0$) expansion. Consequently, we need only look at the Kummer functions in (3.7). Thus, when α is not zero or an integer (i.e., $\alpha \neq 0,1,2,\ldots$), we express $U(a_{\pm}, \alpha; \mp i\xi)$ as a linear combination of Kummer functions,[6] namely:

$$U(a_{\pm}, \alpha; \mp i\xi) = \frac{\Gamma(1-\alpha)}{\Gamma(a_{\pm} - \alpha + 1)} M(a_{\pm}, \alpha; \mp i\xi)$$

$$+ \frac{\Gamma(\alpha - 1)}{\Gamma(a_{\pm})} (\mp i\xi)^{1-\alpha} M(a_{\pm}-\alpha+1, 2-\alpha; \mp i\xi). \qquad (3.9)$$

To obtain the small agrument expansions for these values of α we replace the Kummer functions on the right of (3.9) by unity, which is their leading term, to obtain

$$U(a_\pm, \alpha; \mp i\xi) \approx \frac{\Gamma(1 - \alpha)}{\Gamma(a_\pm - \alpha + 1)} + \frac{\Gamma(\alpha - 1)}{\Gamma(a_\pm)} (\mp i\xi)^{1-\alpha} , \qquad (3.10)$$

from which we deduce,

$$\text{for } 0 < \alpha < 1: \quad Y_\pm(i\xi) \approx \frac{\Gamma(1 - \alpha)}{\Gamma(a_\pm - \alpha + 1)} ; \qquad (3.11)$$

$$\text{for } 1 < \alpha < \infty: \; Y_\pm(i\xi) \approx \frac{\Gamma(\alpha - 1)}{\Gamma(a_\pm)} (\mp i\xi)^{1-\alpha} , \qquad (3.12)$$

which show that the leading term (3.11) is a constant (independent of ξ), whereas the terms (3.12) behave as reciprocal powers of $(i\xi)$.

When $\alpha = 0,1,2,\ldots$, the formula (3.9) diverges, and we must resort to a limiting process that leads to the so-called logarithmic case.[6] Thus, if we put $\alpha = n + 1$, $n = 1,2,\ldots$, whence $a_\pm = \frac{1}{2}(n + 1) \pm i\tilde{\beta}_0$, the small argument leading term becomes

$$U(a_\pm, n + 1; \mp i\xi) \approx \frac{(n - 1)!}{\Gamma(a_\pm)} (\mp i\xi)^{-n} , \qquad (3.13)$$

from which we deduce, for

$$\alpha = 2, \; n = 1, \; a_\pm = 1 \pm i\tilde{\beta}_0; \; Y_\pm(i\xi) \approx \frac{1}{\Gamma(a_\pm)} (\mp i\xi)^{-1} , \qquad (3.14)$$

in full agreement with (3.12). However, the same formula yields

$$\alpha = 1, \; n = 0, \; a_\pm = \frac{1}{2} \pm i\tilde{\beta}_0; \; Y_\pm(i\xi) \approx - \frac{1}{\Gamma(a_\pm)} \ln (\mp i\xi) ; \qquad (3.15)$$

$$\alpha = 0, \; n = -1, \; a_\pm = \pm i\tilde{\beta}_0; \; Y_\pm(i\xi) \approx \frac{1}{\Gamma(a_\pm)} \left[1 + (\mp i\xi) \ln (\mp i\xi)\right] , \qquad (3.16)$$

which are the logarithmic results mentioned earlier that cannot be obtained from (3.12) and do not blend smoothly with the values obtained for the range $0 < \alpha < 1$.

4. The Method of Integral Representations

To obtain integral representations of the solutions, we introduce the Laplace kernel $K(\xi,t) = e^{i\xi t}$ and apply to (3.1) the standard procedures[7] that abide when all the coefficients of the various terms of the differential equation are at most linear functions of the independent variable. Thus, we assert that the integral representations of the solutions of (3.1) may be written as

$$Z(\xi) = A \int_C e^{i\xi t} \left(t - \frac{1}{2}\right)^{a_+ - 1} \left(t + \frac{1}{2}\right)^{a_- - 1} dt \,, \tag{4.1}$$

where A is an arbitrary normalization constant to be determined a posteriori, a_+ and a_- are the parameters of the problem as defined by (2.14), and C is a suitable path of integration in the complex t plane, which must be chosen to conform with the bilinear concomitant

$$P(\xi,t) = A\left(t - \frac{1}{2}\right)^{a_+} \left(t + \frac{1}{2}\right)^{a_-} e^{i\xi t} \,. \tag{4.2}$$

That is, the contour C must be such that $P(\xi,t)$ returns to its primitive value after t has described a closed circuit, or else the path of integration is drawn between definite limits such that $P(\xi,t)$ vanishes at each limit. In the sequel, we assume that the cuts in the complex t plane are chosen along the real axis; they extend from $t = 1/2$ to ∞, and from $t = -1/2$ to $-\infty$. The branch points at $t = \pm 1/2$ cannot be connected to each other by a single branch cut because the coefficients of the two binomials in the integral (4.1) are different from each other. The reader will readily recognize that (4.1) and (4.2) reproduce Eqs. (8) and (9) of Ref. 1.

We have adopted for convenience the latter procedure (the use of fixed limits of integration) noting that the bilinear concomitant (4.2) vanishes at the branch points $t = \pm 1/2$, provided we have $\text{Re}\{a_\pm\} > 0$, which excludes Budden's problem ($\alpha = 0$) that we treat properly later. We also note that $P \to 0$ as $|t| \to \infty$ along a ray inclined (with respect to the axis of reals) at an angle $(\frac{\pi}{2} - \phi)$, where ϕ satisfies the condition[8]

$$-\frac{\pi}{2} + \phi < \arg \xi < \frac{\pi}{2} + \phi \;, \tag{4.3}$$

that guarantees the convergence of the integral (4.1) as $|t|$ grows without limit; for example, when $\phi = 0$, we have $-\frac{\pi}{2} < \arg \xi < \frac{\pi}{2}$, which includes $\arg \xi = 0$, and when $\phi = \pi$, we obtain $\frac{\pi}{2} < \arg \xi < \frac{3\pi}{2}$, which includes $\arg \xi = \pi$. In fact, in the bilinear concomitant (4.2), we choose $\arg \{i\xi t\} = \pi$, which is tantamount to choosing the path of steepest descents, from which we deduce asymptotically, with $\arg t \sim \frac{\pi}{2} - \phi$, that $\phi = \arg \xi$. Likewise, we observe that (4.3), provided only that the range of ϕ is properly chosen as shown below, guarantees the basic condition

$$0 \leq \arg \xi \leq \pi \;, \tag{4.4}$$

which follows from (2.2) and subsequent discussion in the limit of vanishing dissipation, i.e., $\xi = 2\beta_2 z$ is real.

Accordingly, we define the function $Z_+(\xi)$ by means of the rectilinear integral representation

$$Z_+(\xi) = A \int_{1/2}^{1/2+i\infty e^{-i\phi}} e^{i\xi t} \left(t - \tfrac{1}{2}\right)^{a_+ - 1} \left(t + \tfrac{1}{2}\right)^{a_- - 1} dt \;, \tag{4.5}$$

which starts at $t = \frac{1}{2}$ and proceeds to infinity along a ray inclined at an angle $-\phi$ with respect to the axis of imaginaries, where ϕ satisfies the range

$$-\frac{\pi}{2} < \phi < \frac{3\pi}{2}, \tag{4.6}$$

which contains $\phi = \arg \xi = 0$ and $\phi = \arg \xi = \pi$ in accordance with the requirement (4.4). In addition, we observe from (4.6) that, in varying ϕ about $t = \frac{1}{2}$, the rectilinear path of integration does not encroach upon the other branch point at $t = -\frac{1}{2}$. If ϕ were to be increased beyond the limits stated, it would become necessary to make the path of integration circle the branch point at $t = -\frac{1}{2}$, with an additional contribution to the answer which does not belong in (4.5). Then, if we substitute the specified limits (4.6) into the values of ϕ in (4.3) we find for $\arg \xi$ the permissible range $-\pi < \arg \xi < 2\pi$, which contains symmetrically the basic condition (4.4). Finally, subtracting $\frac{\pi}{2}$ from each term of the preceding set of inequalities we obtain the result $-\frac{3\pi}{2} < \arg(-i\xi) < \frac{3\pi}{2}$, which comprises symmetrically the customary principal branch $-\pi < \arg(-i\xi) \leq \pi$ for the function $U(a_+, \alpha; -i\xi)$ that we associate with (2.16) and (4.5). However, as demonstrated in Fig. 1 and accompanying discussion, the argument range for the principal branch of multivalued functions of $(-i\xi)$ is given by (2.5); that is, $-\pi/2 \leq \arg(-i\xi) < 3\pi/2$, which is the range we shall adopt henceforth for the Kummer function $U(a_+, \alpha; -i\xi)$, and which contains the range $-\pi/2 \leq \arg(-i\xi) \leq \pi/2$ that corresponds to (4.4).

To evaluate the integral (4.5), we change the variable of integration from t to u by putting

$$i\xi t = i\xi(t - \frac{1}{2} + \frac{1}{2}) = i\xi/2 + i\xi\,(t - \frac{1}{2}) = i\xi/2 - u,$$

$$\text{where } -u = e^{i\pi}u\,; \tag{4.7}$$

$$e^{i\pi}u = i\xi(t - \frac{1}{2}),\ t - \frac{1}{2} = e^{i\pi}u/i\xi;\ t + \frac{1}{2} = 1 - \frac{u}{i\xi}\ ,$$

$$dt = (e^{i\pi}/i\xi)du; \tag{4.8}$$

$$t = \frac{1}{2}, \quad u = 0; \; t = \frac{1}{2} + i\infty e^{-i\phi}, \; u = |u|e^{i\beta} \to \infty|\xi|e^{i(\arg\,\xi\, -\, \phi)}, \tag{4.9}$$

which states that the upper limit in (4.5) tends to $\infty e^{i\beta}$, where $\beta = \arg\xi - \phi$ and, according to (4.3), β is an acute angle, positive or negative,

$$-\frac{\pi}{2} < \beta < \frac{\pi}{2}. \tag{4.10}$$

To proceed with the evaluation of the integral, we insert (4.8) and (4.9) into (4.5) to obtain

$$\begin{aligned} Z_+(\xi) &= A\,(e^{i\pi}/i\xi)^{a_+}\, e^{i\xi/2} \int_0^{\infty e^{i\beta}} e^{-u}\, u^{a_+-1}\,(1 - \frac{u}{i\xi})^{a_--1}\, du \\ &= A\Gamma(a_+)e^{i\xi/2}\, U(a_+,\alpha;-i\xi) \\ &\sim A\Gamma(a_+)e^{i\xi/2}(-i\xi)^{-a_+}\; {}_2F_0(a_+,\; 1 - a_-;\; \frac{1}{i\xi}), \end{aligned} \tag{4.11}$$

where the result in the second line stems directly from (2.16) and states that $Z_+(\xi)$ in (4.11) is proportional to $Y_+(i\xi)$; that is, $Z_+(\xi) = A\Gamma(a_+)\, Y_+(i\xi)$, where the constant of proportionality, $A\Gamma(a_+)$, arises from the integral representation itself. The last form in (4.11) is deduced from the first line, where the integral representation makes use of u in (4.7) as the variable of integration. Thus, expanding the binominal $(1 - u/i\xi)^{a_--1}$ into a partial series plus a remainder; interchanging the order of integration and summation, and integrating term by term, we obtain, according to Watson's Lemma,[9] an asymptotic series in the sense of Poincaré, which is condensed in the last form of (4.11). Here, the function ${}_2F_0$ is a generalized hypergeometric series written in Pochhamer's notation as modified by Barnes.[10] It represents symbolically an asymptotic series in reciprocal powers of $(i\xi)$ that must be terminated after a finite number of terms followed by a remainder. It is recalled that in such series, the remainder is of the order of the first discarded term, and that the functions ${}_2F_0$ have unity as the leading term. Comparing the last two forms in (4.11) leads to the asymptotic relation

$$U(a_+,\ \alpha;\ -i\xi) \sim (-i\xi)^{-a_+}\ {}_2F_0(a_+,\ 1 - a_-;\ \frac{1}{i\xi})\ , \tag{4.12}$$

which apparently was first noted by Erdélyi in his definition of the corresponding Kummer function.[11]

Similarly, the companion function $Z_-(\xi)$ is defined by means of the rectilinear integral representation

$$Z_-(\xi) = A \int_{-1/2}^{-1/2+i\infty e^{-i\phi}} e^{i\xi t}\ (t - \frac{1}{2})^{a_+ - 1}\ (t + \frac{1}{2})^{a_- - 1}\ dt\quad , \tag{4.13}$$

which starts at $t = -1/2$ and proceeds to infinity, as before, along a ray inclined at an angle $-\phi$ with respect to the axis of imaginaries, where ϕ satisfies this time the range

$$-\pi < \phi \leq \pi\ , \tag{4.14}$$

which has been so chosen because it contains $\phi = \arg\xi = 0$ and $\phi = \arg\xi = \pi$ in accordance with the requirement (4.4). However, this time, when the rectilinear path of integration is rotated about $t = -1/2$, say from $\phi = \pi$ to $\phi = 0$, it becomes necessary to make the path of integration circle the branch point at $t = 1/2$, giving rise to an additional contribution which is not contained in (4.13) and which is part of the process of analytic continuation, a matter to which we return in Section 6 when we address the transmission-reflection problems that we seek to solve. Proceeding as before, if we substitute the specified limits (4.14) into the values of ϕ in (4.3), we find for arg ξ, the permissible range $-3\pi/2 < \arg \xi \leq 3\pi/2$, which contains the basic condition (4.4). Finally, adding $\pi/2$ to each term of the preceeding set of inequalities, we obtain $-\pi < \arg (i\xi) < 2\pi$, which contains the principal branch $-\frac{\pi}{2} < \arg (i\xi) \leq \frac{3\pi}{2}$ for the function $U(a_-, \alpha; i\xi)$ that we associate with (2.12) and (4.13). Notice that this range contains $\frac{\pi}{2} \leq \arg (i\xi) \leq \frac{3\pi}{2}$, which corresponds to (4.4), and that it agrees with (2.4), as demonstrated in Fig. 1 and accompanying discussion.

To evaluate the integral (4.13) as before, we change the variable of integration from t to u, but this time we put

$$i\xi t = i\xi(t + \frac{1}{2} - \frac{1}{2}) = -i\xi/2 + i\xi(t + \frac{1}{2}) = -i\xi/2 - u,$$

$$\text{where } -u = e^{i\pi}u; \tag{4.15}$$

$$e^{i\pi}u = i\xi(t + \frac{1}{2}),\ t + \frac{1}{2} = e^{i\pi}u/i\xi;\ t - \frac{1}{2} = e^{i\pi}(1 + \frac{u}{i\xi}),$$

$$dt = (e^{i\pi}/i\xi)du; \tag{4.16}$$

$$t = -\frac{1}{2},\ u = 0;\ t = -\frac{1}{2} + i\infty e^{-i\phi},\ u = |u|e^{i\beta} \to \infty|\xi|e^{i(\arg \xi - \phi)}, \tag{4.17}$$

which implies that the upper limit in (4.13) tends to $\infty e^{i\beta}$, where $\beta = \arg \xi - \phi$ is an acute angle, positive or negative. Thus, proceeding as before, we have from (4.13)

$$Z_-(\xi) = Ae^{i\pi(\alpha-1)}\left(\frac{1}{i\xi}\right)^{a_-} e^{-i\xi/2}\int_0^{\infty e^{i\beta}} e^{-u}\, u^{a_- -1}\left(1 + \frac{u}{i\xi}\right)^{a_+ -1} du$$

$$= Ae^{i\pi(\alpha-1)}\,\Gamma(a_-)e^{-i\xi/2}\,U(a_-, \alpha; -i\xi)$$

$$\sim Ae^{i\pi(\alpha-1)}\Gamma(a_-)e^{-i\xi/2}(i\xi)^{-a_-}\,{}_2F_0(a_-, 1 - a_+; -\frac{1}{i\xi}), \tag{4.18}$$

where the expression in the second line stems directly from (2.15) and states that $Z_-(\xi)$ in (4.18) is proportional to $Y_-(i\xi)$, that is, $Z_-(\xi) = Ae^{i\pi(\alpha-1)}\Gamma(a_-)Y_-(i\xi)$, where the constant of proportionality, $Ae^{i\pi(\alpha-1)}\Gamma(a_-)$, arises from the integral representation itself. The last form in (4.18) is deduced from the first line employing the same methods that led to (4.11). Here, again, the function ${}_2F_0$ is an asymptotic series in reciprocal powers of $(i\xi)$, whose leading term is unity. Finally, comparing the last two results in (4.18) leads to the asymptotic relation

$$U(a_-, \alpha; i\xi) \sim (i\xi)^{-a_-}\ {}_2F_0(a_-, 1 - a_+, -\frac{1}{i\xi}) , \tag{4.19}$$

which should be contrasted with (4.12), and which again is due to Erdélyi's formulation.[11]

To determine the constant A appearing in (4.11) and (4.18), we demand that the functions $Z_+(\xi)$ and $Z_-(\xi)$ satisfy the reversible transformation (2.13). That is, in accordance with the last sentence of Section 2, we demand that $\overline{Z_+(\xi)} = Z_-(\xi)$ and $\overline{Z_-(\xi)} = Z_+(\xi)$, where a bar over a symbol signifies that we have imposed the reversible transformation $i \leftrightarrow -i$, wherever i appears explicitly. Thus, since the normalization constant A is arbitrary, we put

$$A = e^{-i\pi(\alpha-1)/2} , \tag{4.20}$$

which, when inserted into (4.11) and (4.18), yields the results

$$Z_+(\xi) = e^{-i\pi(\alpha - 1)/2}\ \Gamma(a_+)e^{i\xi/2}\ U(a_+, \alpha; -i\xi)$$

$$\sim e^{-i\pi(\alpha-1)/2}\ \Gamma(a_+)e^{i\xi/2}(-i\xi)^{-a_+}{}_2F_0(a_+, 1 - a_-; \frac{1}{i\xi}) ; \tag{4.21}$$

$$Z_-(\xi) = e^{i\pi(\alpha - 1)/2}\,\Gamma(a_-)e^{-i\xi/2}\,U(a_-,\ \alpha;\ i\xi)$$

$$\sim e^{i\pi(\alpha-1)/2}\,\Gamma(a_-)e^{-i\xi/2}(i\xi)^{-a_-}\,{}_2F_0(a_-,\ 1 - a_+;\ -\frac{1}{i\xi})\ , \qquad (4.22)$$

that clearly satisfy the reversible transformation $i \leftrightarrow -i$.

When $\alpha = 0$ in (2.1), we recover Budden's equation, and the parameters of the problem become $a_\pm = \pm\, i\tilde{\beta}_o$, which means that the integral representations (4.5) and (4.13) cannot be used because the restriction $\mathrm{Re}\{a_\pm\} > 0$ is being violated. To obviate this difficulty, we make use of the identity[12]

$$U(a,0;w) = wU(1 + a,2;w), \qquad (4.23)$$

which allows us to write down the required solutions for $\alpha = 0$ with the aid of (2.16) and (2.15), respectively. Thus, we have

$$Z_+(\xi) = -\,i\Gamma(1 + i\tilde{\beta}_o)e^{i\xi/2}U(i\tilde{\beta}_o,\ 0;\ -i\xi)$$

$$= -i\Gamma(1 + i\tilde{\beta}_o)e^{i\xi/2}(-i\xi)U(1 + i\tilde{\beta}_o,2;-i\xi)$$

$$\sim -i\Gamma(1 + i\tilde{\beta}_o)e^{i\xi/2}(-i\xi)^{-i\tilde{\beta}_o}\ {}_2F_0(1 + i\tilde{\beta}_o,\ i\tilde{\beta}_o;\ \frac{1}{i\xi})\ ; \qquad (4.24)$$

$$Z_-(\xi) = i\Gamma(1 - i\tilde{\beta}_o)e^{-i\xi/2}U(-i\tilde{\beta}_o,\ 0;\ i\xi)$$

$$= i\Gamma(1 - i\tilde{\beta}_o)e^{-i\xi/2}(i\xi)U(1 - i\tilde{\beta}_o,\ 2;\ i\xi)$$

$$\sim i\Gamma(1 - i\tilde{\beta}_o)e^{-i\xi/2}(i\xi)^{i\tilde{\beta}_o}\ {}_2F_0(1 - i\tilde{\beta}_o,\ -i\tilde{\beta}_o;\ -\frac{1}{i\xi})\ ; \qquad (4.25)$$

which should be contrasted with their counterparts (4.21) and (4.22) valid for $\alpha > 0$, and in which we have made use of the Erdélyi asymptotic relations (4.12) and (4.19), respectively.

The most important results of the preceeding paragraphs are illustrated graphically in Fig. 3. Thus, Figs. 3(a) and 3(b) depict the t plane and emphasize the rectilinear half-ray paths of integration for the functions $Z_+(\xi)$ an $Z_-(\xi)$, respectively, which we have chosen to draw, in the absence of dissipation, parallel to the axis of imaginaries. That is, making sure that in both cases the causality condition (4.4) is preserved. This is accomplished by exhibiting the points $\phi = \arg \xi = 0$ and $\phi = \arg \xi = \pi$ on the paths of integration, and joining them by semicircles drawn to the right of the vertical axis as part of the unit circles which correspond, respectively, to the limits (4.6) and (4.14). Figs. 3(c) and 3(d) represent, respectively, the full argument ranges for $(-i\xi)$ and $(i\xi)$, which are the arguments appearing in $Z_+(\xi)$ and $Z_-(\xi)$. In addition, the portions of the spirals drawn with thick lines represent the respective ranges corresponding to (4.4), i.e., $0 \leq \arg \xi \leq \pi$. In the presence of dissipation, as shown in (2.2) and subsequent discussion, the basic argument range (4.4) becomes, as $|z|$ grows without limit, $\theta_2 \leq \arg \xi \leq \theta_2 + \pi$, where $\theta_2 = \arg \beta_2$. The diagrams of Figs. 3(c) and (3d) can be readily adjusted by simply advancing the thick portions of the spirals by an angle θ_2.

5. Asymptotic Leading Terms

In this section we are concerned with the tabulation of the leading term asymptotic expansions, expressed as functions of $i\xi = 2i\beta_2(z + i\tilde{\Gamma})$ and eventually as functions of the physical independent variable z, $-\infty < z < \infty$, as was done in Ref. 1. The former objective is achieved by replacing the functions ${}_2F_0$ by unity. Thus, from (4.21) and (4.22), we have the asymptotic leading terms with simplified normalization factors:

$$Z_+(\xi) \sim i^{(1-\alpha)}\Gamma(a_+)e^{i\xi/2}(-i\xi)^{-a_+} \; ; \qquad (5.1)$$

$$Z_-(\xi) \sim -i^{(1-\alpha)}\Gamma(a_-)e^{-i\xi/2}(i\xi)^{-a_-} \; , \qquad (5.2)$$

which are valid for the parameters $a_\pm = \frac{\alpha}{2} \pm i\tilde{\beta}_o$ and $\alpha > 0$. In the case of $\alpha = 0$, Budden's problem, we obtain the asymptotic leading terms

$$Z_+(\xi) \sim -i\Gamma(1 + i\tilde{\beta}_o)e^{i\xi/2}(-i\xi)^{-i\tilde{\beta}_o} \; ; \qquad (5.3)$$

$$Z_-(\xi) \sim i\Gamma(1 - i\tilde{\beta}_o)e^{-i\xi/2}(i\xi)^{i\tilde{\beta}_o} \; . \qquad (5.4)$$

In both pairs of equations the adopted notation means that the subscript <u>plus</u> in $Z_+(\xi)$ is associated with progressive waves, whereas the subscript <u>minus</u> in $Z_-(\xi)$ with regressive waves. This wave character of the preceeding solutions is evident asymptotically; i.e., for $|i\xi| >> 1$, where, it is recalled, $\xi = 2\beta_2(z + i\tilde{\Gamma})$ in which β_2 and $\tilde{\Gamma}$ are complex parameters. In the absence of dissipation, β_2 is real and positive and $\tilde{\Gamma}$ vanishes; therefore in this case, $\xi = 2\beta_2 z$ is real.

First, we examine to what extent the presence of dissipation affects the asymptotic leading terms listed before, especially as regards the powers of $(i\xi)$. For the purpose, we assume temporarily that $z = |z|$, which is positive definite, and define from (2.2)

$$\tilde{\xi} \equiv 2\beta_2 |z|, \qquad \eta \equiv 2\beta_2 \tilde{\Gamma} , \tag{5.5}$$

which makes $\tilde{\xi}$ a complex number with positive real part and small positive phase angle; i.e., $\mathrm{Im}\{i\tilde{\xi}\} > 0$ because $\arg \tilde{\xi} = \arg \beta_2$. With this understanding, e.g., the computation of the power $(i\xi)^{-a}$, where a is an arbitrary exponent, real or complex, becomes

$$(i\xi)^{-a} = (i\tilde{\xi} - \eta)^{-a} = (i\tilde{\xi})^{-a} \left(1 - \frac{\eta}{i\tilde{\xi}}\right)^{-a}$$

$$= (i\tilde{\xi})^{-a} \left[1 + \frac{a\eta}{i\tilde{\xi}} + \dots\right] \approx (i\tilde{\xi})^{-a} \tag{5.6}$$

which is valid to lowest order in reciprocal powers of $(i\tilde{\xi})$ because $\eta = 2\beta_2\tilde{\Gamma}$ is a constant. Finally, according to the definitions (2.2) and (5.5), when $z > 0$, we have $\xi = \tilde{\xi}$, and when $z < 0$, then $\xi = -\tilde{\xi}$.

Next, we recall that the confluent hypergeometric functions employed in (4.21) and (4.22) for $\alpha > 0$, as well as their counterparts (4.24) and (4.25) for $\alpha = 0$, are multivalued functions. Their principal branches, as demonstrated in Fig. 1 and accompanying discussion, are given by (2.5) and (2.4), respectively, which accounts for the adoption of the convention (2.6), and which for our purpose we rewrite here as follows:

$$(-w)^c = w^c e^{-i\pi\varepsilon c}, \quad \varepsilon = \mathrm{sgn}\{\mathrm{Im} w\} , \tag{5.7}$$

where c is an arbitrary exponent. We propose to apply this rule to compute the powers $(-i\xi)^{-a_+}$ and $(i\xi)^{-a_-}$ which appear in (5.1) and (5.2). Thus, when $\mathrm{Im}\{i\xi\} > 0$ in (5.1), we have $\varepsilon = 1$ and $(-i\xi)^{-a_+} \sim (-i\tilde{\xi})^{-a_+}$ according to (5.6), whence

$$(-i\tilde{\xi})^{-a_+} = (i\tilde{\xi})^{-a_+} e^{i\pi a_+}; \quad e^{i\pi a_+} = e^{i\pi(\alpha/2 + i\tilde{\beta}_o)} = i^{\alpha} e^{-\pi\tilde{\beta}_o} , \qquad (5.8)$$

and when $\mathrm{Im}\{i\xi\} < 0$ in (5.1), we have simply $(-i\xi)^{-a_+} \sim (i\tilde{\xi})^{-a_+}$. Similarly, when $\mathrm{Im}\{i\xi\} > 0$ in (5.2), we have $(i\xi)^{-a_-} \sim (i\tilde{\xi})^{-a_-}$, and when $\mathrm{Im}\{i\xi\} < 0$, we have $\varepsilon = -1$ and $(i\xi)^{-a_-} \sim (-i\tilde{\xi})^{-a_-}$; i.e.,

$$(-i\tilde{\xi})^{-a_-} = (i\tilde{\xi})^{-a_-} e^{-i\pi a_-}; \quad e^{-i\pi a_-} = e^{-i\pi(\alpha/2 - i\tilde{\beta}_o)} = i^{-\alpha} e^{-\pi\tilde{\beta}_o} . \qquad (5.9)$$

In (5.8) and (5.9), we still have to compute the powers $(i\tilde{\xi})^{-a_\pm}$, where $a_\pm = \frac{\alpha}{2} \pm i\tilde{\beta}_o$. Thus, we have

$$(i\tilde{\xi})^{-a_+} = e^{-a_+ \ell n(i\tilde{\xi})} = e^{-(\alpha/2 + i\tilde{\beta}_o)\ell n(i\tilde{\xi})} = (i\tilde{\xi})^{-\alpha/2} e^{\pi\tilde{\beta}_o/2 - i\tilde{\beta}_o \ell n\tilde{\xi}}; \qquad (5.10)$$

$$(i\tilde{\xi})^{-a_-} = e^{-a_- \ell n(i\tilde{\xi})} = e^{-(\alpha/2 - i\tilde{\beta}_o)\ell n(i\tilde{\xi})} = (i\tilde{\xi})^{-\alpha/2} e^{-\pi\tilde{\beta}_o/2 + i\tilde{\beta}_o \ell n\tilde{\xi}}. \qquad (5.11)$$

The logarithmic terms in the last exponentials in (5.10) and (5.11), when combined eventually with $e^{\pm i\xi/2}$, are asymptotically ($\tilde{\xi} \to \infty$) negligible and they are frequently omitted. Here, however, we retain the logarithmic terms for completness sake.

Hence, combining the results of the two preceding paragraphs with our basic leading term asymptotic expansions (5.1) and (5.2), we may now tabulate the asymptotic values of $Z_+(\xi)$ and $Z_-(\xi)$ for $\mathrm{Im}\{i\xi\} > 0$ and $\mathrm{Im}\{i\xi\} < 0$, which is equivalent to saying $z > 0$ and $z < 0$, respectively. Thus, we obtain the four expressions:

$$\mathrm{Im}\{i\xi\} > 0; \quad Z_+(\xi) \sim i^{(1-\alpha)}\,\Gamma(a_+)[i^{\alpha}e^{-\pi\tilde{\beta}_0/2}](i\tilde{\xi})^{-\alpha/2}e^{i\xi/2-i\tilde{\beta}_0\ln\tilde{\xi}}; \tag{5.12}$$

$$\mathrm{Im}\{i\xi\} < 0; \quad Z_+(\xi) \sim i^{(1-\alpha)}\,\Gamma(a_+)[e^{\pi\tilde{\beta}_0/2}](i\tilde{\xi})^{-\alpha/2}e^{i\xi/2-i\tilde{\beta}_0\ln\tilde{\xi}}; \tag{5.13}$$

$$\mathrm{Im}\{i\xi\} > 0; \quad Z_-(\xi) \sim -i^{(1-\alpha)}\,\Gamma(a_-)[e^{-\pi\tilde{\beta}_0/2}](i\tilde{\xi})^{-\alpha/2}e^{-i\xi/2+i\tilde{\beta}_0\ln\tilde{\xi}}; \tag{5.14}$$

$$\mathrm{Im}\{i\xi\} < 0; \quad Z_-(\xi) \sim -i^{(1-\alpha)}\,\Gamma(a_-)[i^{-\alpha}e^{-3\pi\tilde{\beta}_0/2}](i\tilde{\xi})^{-\alpha/2}e^{-i\xi/2+i\tilde{\beta}_0\ln\tilde{\xi}}, \tag{5.15}$$

in which the first two represent progressive waves and the last two regressive waves. The given limiting forms are valid for all positive values of the parameter α, $0 < \alpha < \infty$. For Budden's problem corresponding to (5.3) and (5.4), we put $\alpha = 0$ everywhere except in the normalization factors were we put $\alpha = 2$ in accordance with (4.23).

To conclude this Section, we extract from the preceeding equations the spatial dependence of the wave functions of unit amplitude, namely:

$$\phi_+(\xi) \sim (i\tilde{\xi})^{-\alpha/2}e^{i\xi/2-i\tilde{\beta}_0\ln\tilde{\xi}} = (i\tilde{\xi})^{-a_+}e^{i\xi/2} \quad ; \tag{5.16}$$

$$\phi_-(\xi) \sim (i\tilde{\xi})^{-\alpha/2}e^{-i\xi/2+i\tilde{\beta}_0\ln\tilde{\xi}} = (i\tilde{\xi})^{-a_-}e^{-i\xi/2} \quad ; \tag{5.17}$$

which equally well may be expressed as functions of the real independent variable, $-\infty < z < \infty$, albeit no longer of unit amplitude; that is:

$$\phi_+(z) \sim (2i\beta_2)^{-a_+}\,|z|^{-a_+}e^{i\beta_2 z-\beta_2\tilde{r}} \quad ; \tag{5.18}$$

$$\phi_-(z) \sim (2i\beta_2)^{-a_-}\,|z|^{-a_-}e^{-i\beta_2 z+\beta_2\tilde{r}} \quad , \tag{5.19}$$

which are the functions that supplant the forms given in Eqs. (7) in Ref. 1. It is important to reiterate, for both pairs of equations, that $a_{\pm} = \alpha/2 \pm i\tilde{\beta}_o$, $\xi = 2\beta_2(z + i\tilde{\Gamma})$, and $\tilde{\xi} = 2\beta_2|z|$, where $\arg \tilde{\xi} = \arg \beta_2 \geq 0$.

Finally, to conform with the nomenclature introduced in Ref. 1, we present in Fig. 4 a diagram indicating the directions of propagation of the four asymptotic leading terms for $Z_{\pm}(\xi)$ and $\mathrm{Im}\{i\xi\} \gtrless 0$, which are associated with the unit amplitude wave functions (5.16) and (5.17). That is, in the spirit of Eqs. (7) of Ref. 1, we have

$$\phi_1(\xi) \sim (i\tilde{\xi})^{-(\alpha/2-i\tilde{\beta}_o)}e^{-i\xi/2} \; ; \tag{5.20a}$$

$$\phi_2(\xi) \sim (i\tilde{\xi})^{-(\alpha/2+i\tilde{\beta}_o)}e^{i\xi/2} \; ; \tag{5.20b}$$

$$\phi_3(\xi) \sim (i\tilde{\xi})^{-(\alpha/2+i\tilde{\beta}_o)}e^{i\xi/2} \; ; \tag{5.20c}$$

$$\phi_4(\xi) \sim (i\tilde{\xi})^{-(\alpha/2-i\tilde{\beta}_o)}e^{-i\xi/2} \; . \tag{5.20d}$$

It is seen from Eqs. (5.20) that, when $\alpha > 0$, there is introduced a decaying envelope of the form $\tilde{\xi}^{-\alpha/2} = (2\beta_o|z|)^{-\alpha/2}$, or conversely, it can be interpreted (asymptotically) as an enhanced swelling of the wave form toward the resonance, which we associate with term ξ^{-2} in the effective potential (3.4). It is worth noting that the swelling disappears when $\alpha = 0$, which is Budden's problem. In addition, it is pertinent at this juncture to emphasize that the exponentials in (5.20), $\exp\{\pm i\xi/2\} = \exp\{\pm i\beta_2 z \mp \beta_2\tilde{\Gamma}\}$, contain the constant attenuation factors $\exp\{\mp\beta_2\tilde{\Gamma}\} = \exp\{\mp\eta/2\}$ in accordance with the definitions (2.2).

The connection that exists between the unit amplitude wave functions listed in (5.20) and the actual asymptotic solutions $Z_{\pm}(\xi)$ that we derived for $\mathrm{Im}\{i\xi\} \gtrless 0$ is as follows: the progressive waves $\phi_2(\xi)$ and $\phi_3(\xi)$ are contained in the functions $Z_+(\xi)$ defined by (5.12) and (5.13), respectively; similarly, the regressive waves $\phi_1(\xi)$ and $\phi_4(\xi)$ are contained in the functions $Z_-(\xi)$ given by (5.14) and (5.15), respectively. The multiplying coefficients in these equations, as we shall see, are of the essence in the computation of the transmission and reflection coefficients that we undertake at once.

6. Computation of Transmission and Reflection Coefficients

The computation of the reflection and transmission coefficients associated with the leading term asymptotic forms for the solutions $Z_{\pm}(\xi)$ of our generalized equation (3.1) is conveniently carried out by resorting to the integral representations that we derived in Section 4, and by applying the method of analytic continuation. This means, with reference to Fig. 4, that we first set up asymptotic transmitted waves of unit amplitude, $\phi_2(\xi)$ for $z > 0$ or $\phi_4(\xi)$ for $z < 0$ [cf. remarks preceeding (5.12)], and then we proceed to seek their analytic continuation as z changes sign. In what follows we shall assume temporarily that $\xi = 2\beta_2 z$ is rigorously real and covers the infinite domain, $-\infty < \xi < \infty$. Accordingly, the four half-ray rectilinear paths of integration defined by (4.5) and (4.13) may be drawn for convenience as strictly vertical in the complex t plane, as shown graphically in Fig. 3.

First we consider the left to right case of Fig. 4 and then making use of (4.5), we draw the diagram of Fig. 5, which illustrates the two rectilinear paths of integration corresponding to $Z_+(\xi > 0)$ with $\phi = 0$ and $Z_+(\xi < 0)$ with $\phi = \pi$, and indicating that their unit amplitude wave functions are $\phi_2(\xi)$ and $\phi_3(\xi)$, respectively. Thus, as stated in the preceding paragraph, we identify the linear path of integration for $Z_+(\xi > 0)$ as the transmitted wave for $z > 0$. The analytic continuation of this solution for $z < 0$, which is the incident wave $Z_+(\xi < 0)$, is obtained by rotating the half-ray path of integration clockwise about the branch point at $t = 1/2$, as indicated by the (directed) semicircle on Fig. 5. This procedure is dictated by the unit circle in Fig. 3(a), which shows that the transition from $\phi = \arg \xi = 0$ to $\phi = \arg z = \pi$ is achieved by means of the thick semicircle to the right of the vertical. Further, this procedure implies an excursion into another sheet of the Riemann surface, which we indicate in both Figs. 3(a) and 5 by drawing the path for $Z_+(\xi > 0)$ as full line, and the path for $Z_+(\xi < 0)$ as dashed line. The transition from the incident wave $\phi_3(\xi)$ to the transmitted wave $\phi_2(\xi)$ is effected by means of an infinitesimal semicircle traversed in the <u>counterclockwise</u> direction about the branch point at $t = 1/2$, which introduces the factor $\exp\{i\pi(\alpha/2 + i\tilde{\beta}_o)\}$ that we identify below as the reflection coefficient.

Hence, $Z_+(\xi)$ is its own analytic continuation as we change the sign of z, and in the left to right case of Fig. 4, there is no reflection, a result also found in the Budden problem ($\alpha = 0$). Thus, the reflection coefficient for $\phi_3(\xi)$ going into $\phi_4(\xi)$ vanishes identically for all α, $R_{34} \equiv 0$. To compute the transmission coefficient T_{32} from $\phi_3(\xi)$ into $\phi_4(\xi)$ quite generally; that is, in the presence of dissipation, we first recall from the discussion that follows (5.20) that the unit amplitude wave functions (5.16) and (5.17) now exhibit the attenuation factors $e^{\pm\eta/2}$, and then we take the ratio of coefficient of $\phi_+(\xi) = \phi_2(\xi)$ in (5.12), the transmitted wave, to the coefficient of $\phi_+(\xi) = \phi_3(\xi)$ in (5.13). Hence, taking due account of the attenuation factors just mentioned, we obtain

$$T_{32} = \frac{i^{(1-\alpha)}\Gamma(a_+)\{i^{\alpha}e^{-\pi\tilde{\beta}_o/2}\}\, e^{-\eta/2}}{i^{(1-\alpha)}\Gamma(a_+)\{e^{\pi\tilde{\beta}_o/2}\}\, e^{-\eta/2}} = \frac{i^{\alpha}e^{-\pi\tilde{\beta}_o/2}}{e^{\pi\tilde{\beta}_o/2}} = i^{\alpha}e^{-\pi\tilde{\beta}_o} \quad , \tag{6.1}$$

which is completely equivalent to the factor already given, and which shows explicitly the periodic dependence on α through the phase factor $i^{\alpha} = e^{i\pi\alpha/2}$. Thus, for Budden's problem ($\alpha = 0$), we recover the well-known result

$$T_{32} = e^{-\pi\tilde{\beta}_o} \quad , \tag{6.2}$$

and for the self-adjoint analogue of Budden's equation ($\alpha = 1$), we obtain

$$T_{32} = ie^{-\pi\tilde{\beta}_0} . \tag{6.3}$$

Notice that all three values of the reflection coefficient T_{32} given above are contained in the factor $\exp\{i\pi(\alpha/2 + i\tilde{\beta}_0)\}$ that arose from the infinitesimal semicircle about $t = 1/2$ when traversed in the counterclockwise direction.

Next, we consider the right to left case of Fig. 4, i.e., the incident wave $\phi_1(\xi)$ approaches resonance from the tunneling side ($z > 0$) and, as we shall see, gives rise to a transmitted wave $\phi_4(\xi)$ for $z < 0$ and a reflected wave $-(1 - e^{2\pi i a_+})\ \phi_2(\xi)$ for $z > 0$. Thus, proceeding as before, we make use of (4.5) and (4.13) to draw the diagra of Fig. 6 illustrating the three rectilinear paths of integration corresponding asymptotically to the incident wave $Z_-(\xi > 0)$ for $\phi = 0$, the transmitted wave $Z_-(\xi < 0)$ for $\phi = \pi$, and the reflected wave $-(1 - e^{2\pi i a_+})\ Z_+(\xi > 0)$ for $\phi = 0$ and $\phi = 2\pi$. We have identified the rectilinear path of integraton for $Z_-(\xi < 0)$ as representing the transmitted wave for $z < 0$. The analytic continuation of this solution for $z > 0$ is obtained by rotating the half-ray path of integration counterclockwise about the branch point at $t = -1/2$ as indicated by the (directed) semicircle in Fig. 5. This procedure is dictated by the unit circle in Fig. 3(b) which shows that the transition from $\phi = \arg \xi = \pi$ to $\phi = \arg \xi = 0$ is achieved by means of the thick semicircle to the right of the vertical. In so doing, however, we cannot ignore the presence of the branch point at $t = 1/2$, with the result that the proposed deformation of the path of integration yields the incident wave $Z_-(\xi > 0)$ from (4.13) with $\phi = 0$, plus the reflected wave from (4.5) with $\phi = 0$ and $\phi = 2\pi$, which we write as

$-(1 - e^{2\pi i a_+}) Z_+(\zeta > 0)$. Here $a_+ = \alpha/2 + i\tilde{\beta}_0$ is the exponent associated with the branch point at $t = 1/2$. We observe that the contour of integration for the reflected wave is represented in Fig. 6 by a solid half-line transversed downwards, an infinitesimal circle around the branch point at $t = 1/2$ traversed in the counterclockwise direction, and a dashed half-line traversed upwards in another sheet of the Riemann surface. We also observe that the transition from the incident wave function $Z_-(\zeta > 0)$ to the transmitted wave function $Z_-(\zeta < 0)$ is effected by means of an infinitesimal semicircle traversed in the clockwise direction about the branch point at $t = -1/2$, and giving rise to the factor $\exp\{-i\pi(\alpha/2 - i\tilde{\beta}_0)\}$. Based on the results of the preceeding paragraph, we identify this factor with T_{14}, the transmission coefficient from $\Phi_1(\zeta)$ into $\Phi_4(\zeta)$; that is,

$$T_{14} = i^{-\alpha} e^{-\pi\tilde{\beta}_0} \ , \tag{6.4}$$

which is the complex conjugate of (6.1) in the absence of dissipation, when $\tilde{\beta}_0$ is real. As before, we consider two cases. For Budden's problem we put $\alpha = 0$ in (6.4) to obtain

$$T_{14} = e^{-\pi\tilde{\beta}_0} \ , \tag{6.5}$$

which is Budden's result and is identical to (6.2). For the self-adjoint analogue of Budden's equation, we put $\alpha = 1$ in (6.4) to yield

$$T_{14} = -ie^{-\pi\tilde{\beta}_0} \ , \tag{6.6}$$

which is the complex conjugate of (6.3) when $\tilde{\beta}_0$ is real.

We have shown that the computation of the transmission coefficients T_{32} and T_{14} emerges quite simply from the contours of integration and does not necessitate a detailed knowledge of the asymptotic solutions given by the integral representations. The same statement, however, does not apply to the computation of the reflection coefficient R_{12}. In fact, we must proceed as we did originally with the computation of the transmission coefficient (6.1) in which we included dissipation and, making use of the conclusions following (5.20), we took cognizance of the fact that the unit amplitude wave functions (5.16) and (5.17) now exhibit the constant attenuation factors $e^{\mp\eta/2}$, respectively. When computing transmission coefficients by taking ratios of the pertinent coefficients in (5.13) - (5.15), we find that these factors play no role because they cancel each other. However, this is no longer the case when computing the reflection coefficient R_{12}. Thus, we must include in addition to the factor $-(1 - e^{2\pi i a_+})$ shown in Fig. 6, the ratio of the coefficient of $\phi_+(\xi) = \phi_2(\xi)$ in (5.12), to the coefficient of $\phi_-(\xi) = \phi_1(\xi)$ in (5.14), and we must take due account of the constant factors $e^{\mp\eta/2}$, which arise in the presence of dissipation. In this manner, we obtain

$$R_{12} = -(1 - e^{2\pi i a_+})\,\frac{i^{(1-\alpha)}\Gamma(a_+)\,[i^{\alpha}e^{-\pi\tilde{\beta}_o/2}]e^{-\eta/2}}{-i^{(1-\alpha)}\Gamma(a_-)\,[e^{-\pi\tilde{\beta}_o/2}]e^{-\eta/2}}$$

$$= i^{-\alpha}\,\frac{\Gamma(\alpha/2+i\tilde{\beta}_o)}{\Gamma(\alpha/2-i\tilde{\beta}_o)}\,[i - e^{i\pi\alpha}e^{-2\pi\tilde{\beta}_o}]e^{-\eta}, \quad (0 < \alpha < \infty)\,, \tag{6.7}$$

which is formally identical to Eq. (10) of Ref. 1, except for the fact that here $\eta = 2\beta_2\tilde{\Gamma}$, whereas previously we wrote ν instead of $\tilde{\Gamma}$. From (6.7), we can deduce the special case corresponding to the self-adjoint analogue of Budden's equation by putting $\alpha = 1$; that is,

$$R_{12} = -i\,\frac{\Gamma(1/2 + i\tilde{\beta}_o)}{\Gamma(1/2 - i\tilde{\beta}_o)}\left[1 + e^{-2\pi\tilde{\beta}_o}\right]e^{-\eta} , \qquad (6.8)$$

in which $|R_{12}| > 1$ unless η is extremely large. Finally, for Budden's case ($\alpha = 0$), we observe that we may put $\alpha = 2$, or else $\alpha = 0$, in all terms of (6.7) to obtain the equivalent forms[2]

$$R_{12} = -\frac{\Gamma(1 + i\tilde{\beta}_o)}{\Gamma(1 - i\tilde{\beta}_o)}\left[1 - e^{-2\pi\tilde{\beta}_o}\right]e^{-\eta} = \frac{\Gamma(i\tilde{\beta}_o)}{\Gamma(-i\tilde{\beta}_o)}\left[1 - e^{-2\pi\tilde{\beta}_o}\right]e^{-\eta} , \qquad (6.9)$$

in accordance with Budden's original result, but now showing the explicit role of dissipation through the complex parameters $\tilde{\beta}_o$ and η. Notice, furthermore, that in (6.9) we have $|R_{12}| < 1$ under all circumstances.

The most unexpected result of the present investigation is the discovery, embodied in Eq. (6.7), that the reflection coefficient R_{12} exhibits an anomalous behavior as a function of α. To study this effect more thoroughly and to simplify the computations, we confine our attention in further remarks to the case of no dissipation; this means that in (6.7) $\tilde{\beta}_o = \beta_o = \beta_1/(2\beta_o)$ is real and $\eta = 2\beta_2\tilde{\Gamma} = 0$. Hence, we deduce that

$$|R_{12}|^2 = 1 + e^{-4\pi\tilde{\beta}_o} - 2\cos(\alpha\pi)e^{-2\pi\beta_o} , \qquad (6.10)$$

which shows that the square of the modulus of the reflection coefficient, $|R_{12}|^2$, oscillates periodically as α is varied, $0 \leq \alpha < \infty$. The smallest reflection occurs for $\alpha = 2N$, $N = 0,1,2,\ldots$, which is min $|R_{12}| = 1 - e^{-2\pi\beta_o}$, whereas the maximum reflection takes place when $\alpha = 2N + 1$, $N = 0,1,2,\ldots$, and attains the value max$|R_{12}| = 1 + e^{-2\pi\beta_o}$, <u>which is greater than unity.</u> This enhancement in the reflection coefficient indicates that there exist bands of values of α within which the resonance spontaneously emits waves. The onset for stimulated emission is obtained from the condition

$$|T_{14}|^2 + |R_{12}|^2 = 0 \quad , \tag{6.11}$$

from which, making use of (6.4) and (6.10), we deduce the equation

$$2\cos(\alpha\pi) = 1 + e^{-2\pi\beta_0} \quad , \tag{6.12}$$

whose roots determine the values of α at the edges of the emission bands. We note further that

$$|R_{12}(\alpha)|^2 = |R_{12}(0)|^2 + 4\sin^2(\alpha\pi/2)e^{-2\pi\beta_0} \quad , \tag{6.13}$$

which indicates that except for the values $\alpha = 2N$, $N = 0,1,2,\ldots$, corresponding to the minimum reflection, the result of putting $\alpha \neq 2N$ is to enhance the reflection coefficient above the Budden value, $|R_{12}(0)|^2$.

In summary, the presence of the first derivative term $\alpha\phi'(z)$, $0 \leq \alpha < \infty$, in the generalized resonance-tunneling equation (2.1) that we have constructed from the well-known Budden's equation, can yield both absorption as well as stimulated emission of waves, depending on the periodic band conditions on α that we established here. It would be interesting and challenging to find examples of physical systems where such a behavior occurs.

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12. Ref. 4, (13.1.7); see also Ref. 5, 6.5(6).

Figure Captions

Figure 1. The lines $\pm i\xi$ in the absence of dissipation and the branch cut along the negative axis of imaginaries. The inclined line through the origin accounts for dissipation when $|z| \to \infty$.

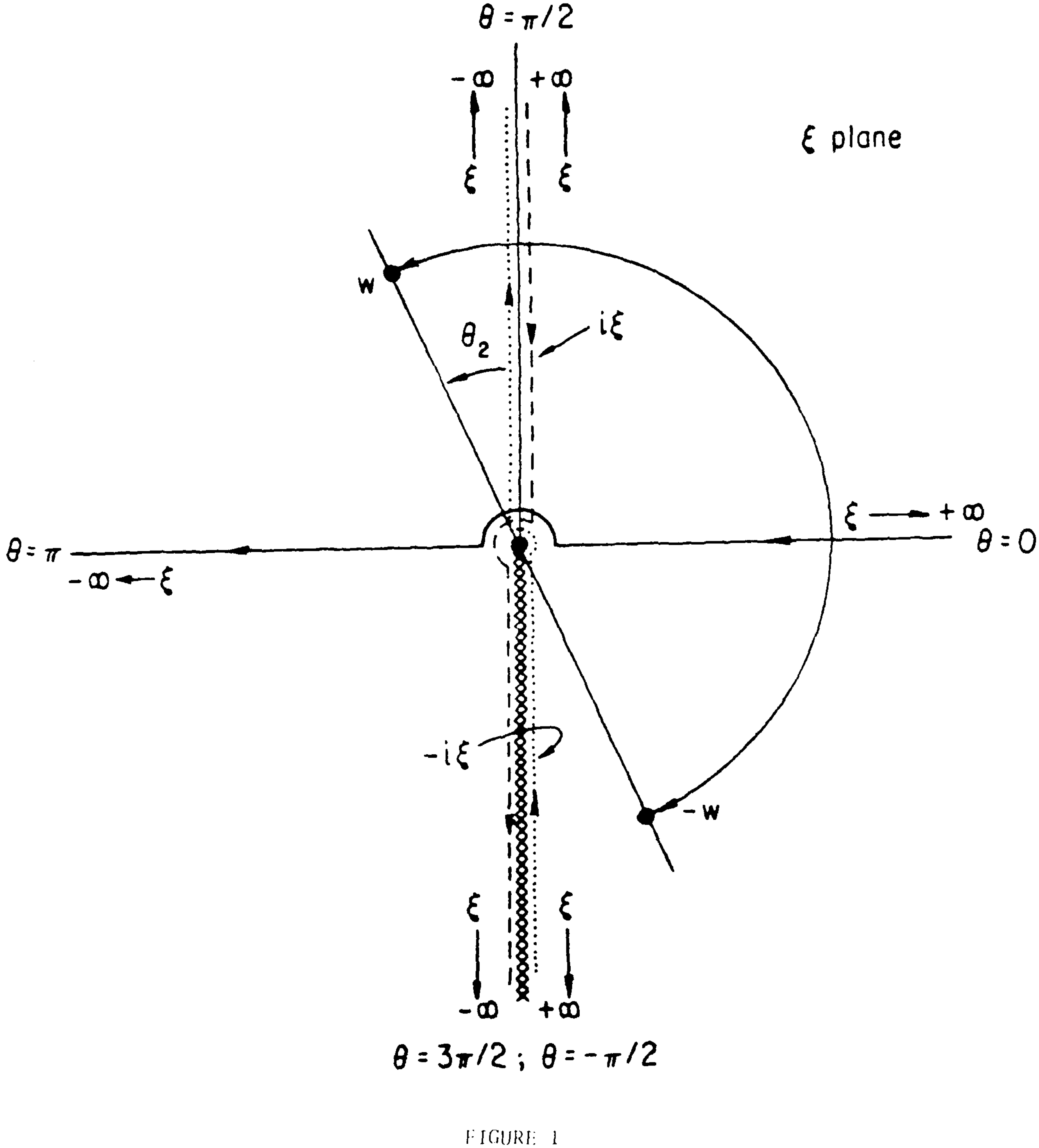
Figure 2. Behavior of the potential $Q(\xi)$, as defined by (3.4), as a function of ξ for $\beta_o = 1$, and for three typical values of α.

Figure 3. The t plane for the integral representations of the functions $Z_+(\xi)$ and $Z_-(\xi)$; the unit circles employed to implement the ranges (4.6) and (4.14) in the angle ϕ, and the corresponding argument ranges for the variables $(\mp i\xi)$.

Figure 4. Qualitative sketch indicating the direction of propagation of the unit amplitude asymptotic wave functions given by Eqs. (5.20), as well as the relative location of resonance and cutoff.

Figure 5. Rectilinear paths of integration for $Z_+(\xi)$ when $\xi > 0$ and $\xi < 0$. An asymptotic unit amplitude wave $\phi_3(\xi)$ approaching the resonance directly from $z < 0$ (Fig. 4) and resulting into an asymptotic transmitted wave $\phi_2(\xi)$ for $z > 0$. There is no reflection in this case for any $\alpha \geq 0$.

Figure 6. Rectilinear paths of integration for $Z_-(\xi)$ when $\xi > 0$ and $\xi < 0$, and the contour of integration about the branch point at $t = 1/2$ involving $Z_+(\xi > 0)$. An asymptotic unit amplitude incident wave $\phi_1(\xi)$ approaching resonance from the tunneling side $z > 0$, gives rise to an asymptotic transmitted wave $\phi_4(\xi)$ for $z < 0$ plus a reflected wave $-(1 - e^{2\pi i a_+})\phi_2(\xi)$ for $z > 0$.



$\theta = \pi/2$
$-\infty$
$+\infty$
ξ
ξ
ξ plane
w
θ_2
$i\xi$
$\xi \longrightarrow +\infty$
$\theta = 0$
$\theta = \pi$
$-\infty \longleftarrow \xi$
$-i\xi$
$-w$
ξ
ξ
$-\infty$
$+\infty$
$\theta = 3\pi/2$; $\theta = -\pi/2$

FIGURE 1

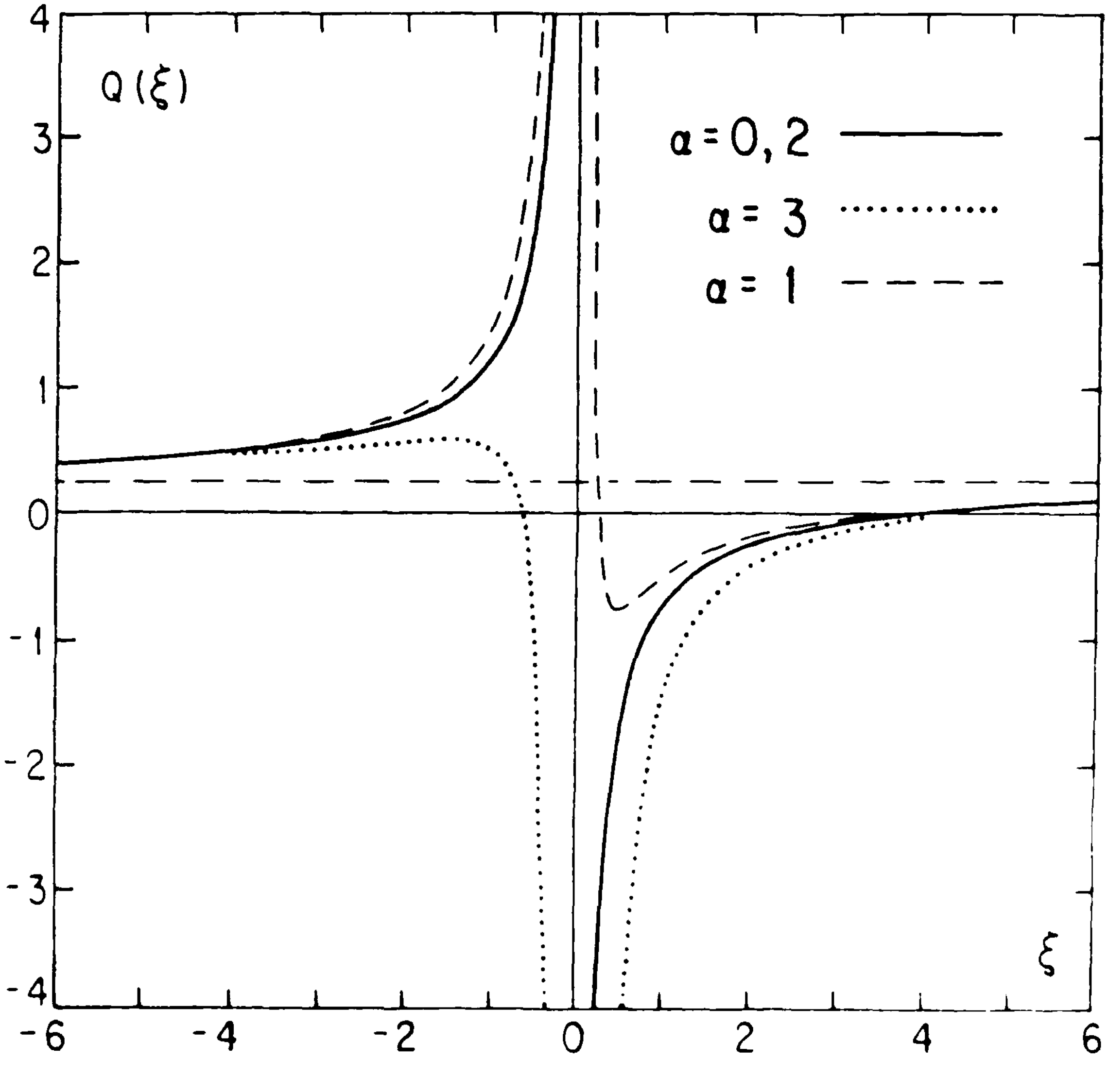
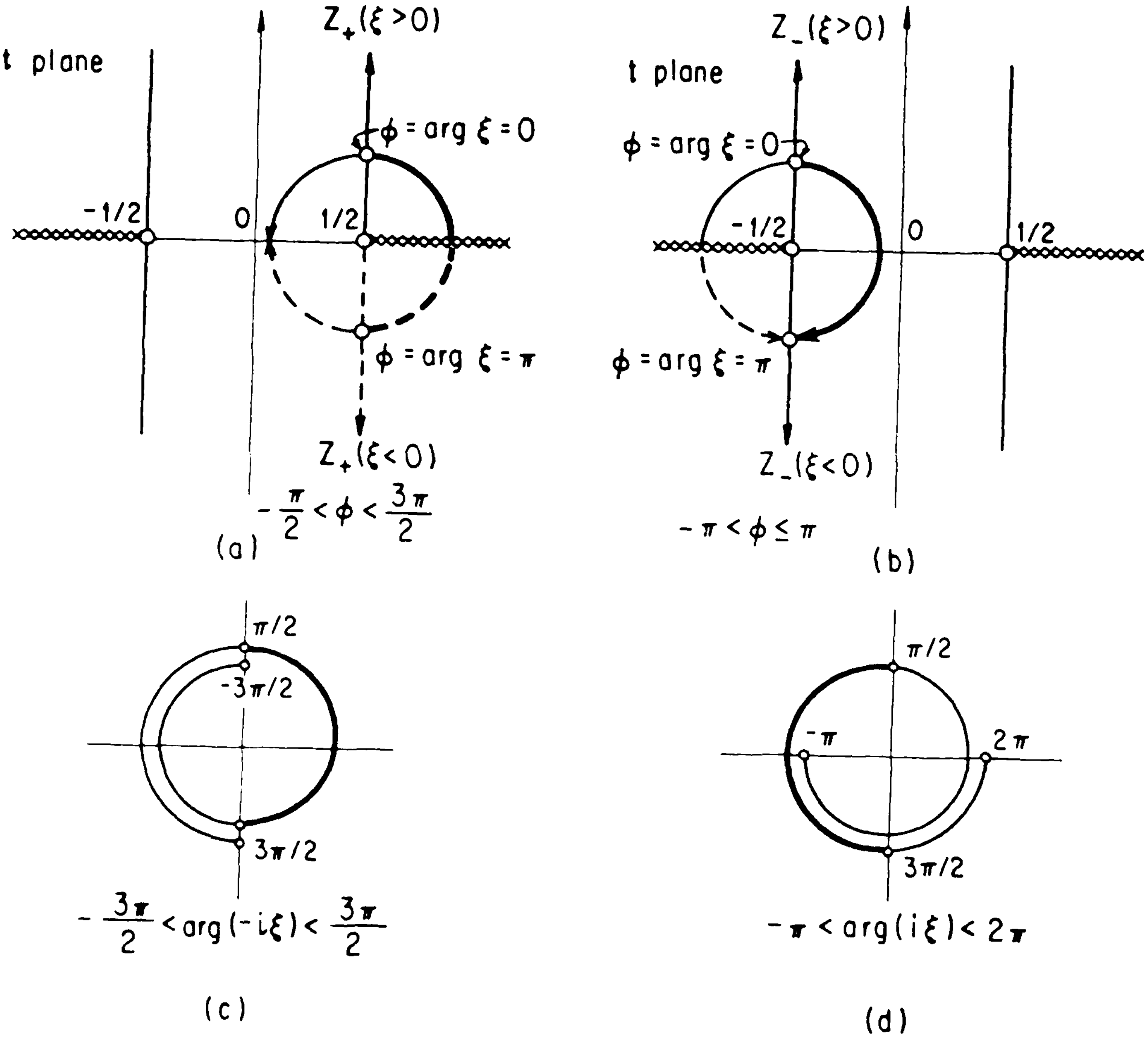


FIGURE 2



t plane
$Z_+(\xi>0)$
$\phi = \arg \xi = 0$
$-1/2$
0
$1/2$
$\phi = \arg \xi = \pi$
$Z_+(\xi<0)$
$-\frac{\pi}{2} < \phi < \frac{3\pi}{2}$
(a)
$Z_-(\xi>0)$
t plane
$\phi = \arg \xi = 0$
$-1/2$
0
$1/2$
$\phi = \arg \xi = \pi$
$Z_-(\xi<0)$
$-\pi < \phi \leq \pi$
(b)
$\pi/2$
$-3\pi/2$
$3\pi/2$
$-\frac{3\pi}{2} < \arg(-i\xi) < \frac{3\pi}{2}$
(c)
$\pi/2$
$-\pi$
2π
$3\pi/2$
$-\pi < \arg(i\xi) < 2\pi$
(d)

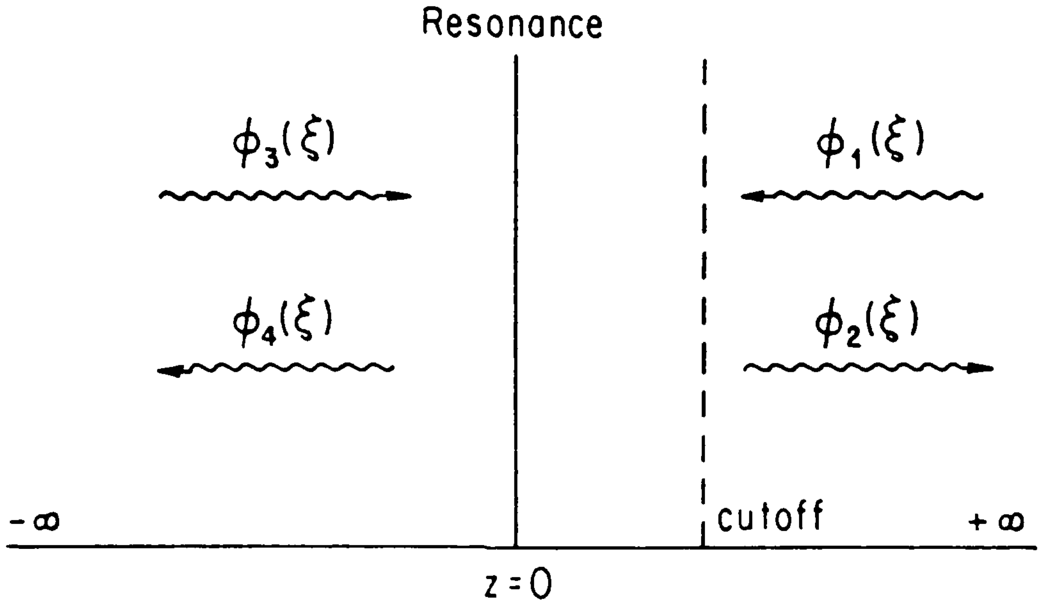

Resonance
$\phi_3(\xi)$
$\phi_1(\xi)$
$\phi_4(\xi)$
$\phi_2(\xi)$
$-\infty$
cutoff
$+\infty$
$z=0$

FIGURE 4

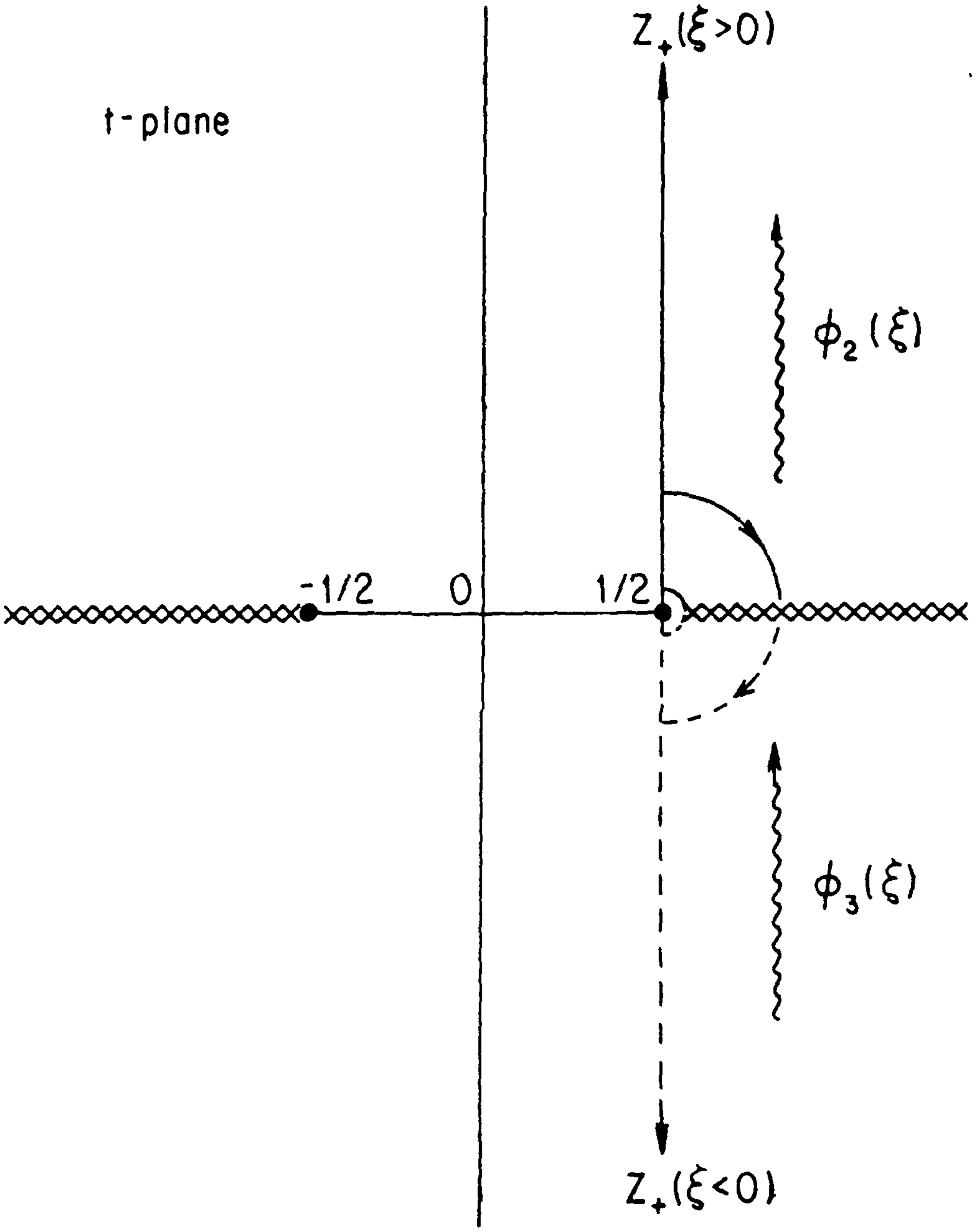


FIGURE 5

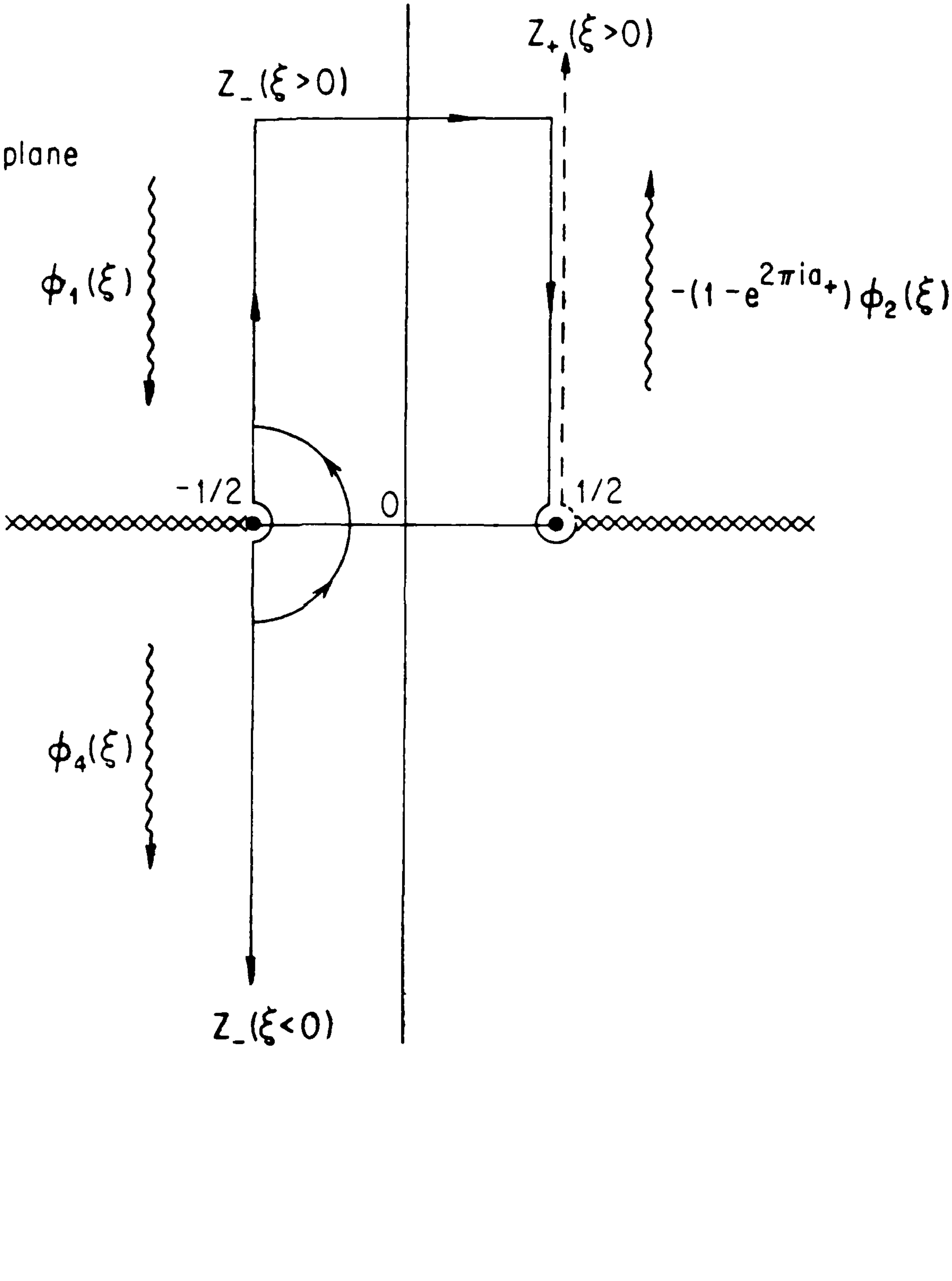


FIGURE 6

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