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The operating characteristics, namely detection probability vs. false alarm probability, are derived and evaluated for a multi-channel input to an or-ing and selection processor with pre- and post-averaging. The input signal-to-noise ratio, the number of input channels, the amount of pre-averaging prior to or-ing, and the amount of post-averaging after selection can be varied in order to investigate the tradeoffs between the various parameters. A wide variety of results are plotted, along with simulation results that verify the analysis.					
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Operating Characteristics of an Or-ing And Selection Processor With Pre- and Post-Averaging

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Preface

This research was conducted under NUSC Project No. B12012, Subproject No. S0219-AS, Passive System Engineering, Principal Investigator Bruce Spear (Code 031), Program Manager CAPT Van Metre, NAVSEA (PMS-409). Also, this research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, Applications of Statistical Communication Theory to Acoustic Signal Processing, Principal Investigator, Dr. Albert H. Nuttall (Code 3314), sponsored by the NUSC In-House Independent Research Program, Program Manager Mr. W. R. Hunt, Director of Navy Laboratories (SPAWAR 05).

The Technical Reviewer for this report was Robert Garber (Code 211).

When this report was originally published as TR 6929 on 4 May 1983, it contained information proprietary to Bell Laboratories. Its distribution status has now been revised and the material is being published for public release as TR 6929A, dated 7 January 1987, to reflect the fact that U. S. Patent Number 4622519 has been assigned to AT&T Bell Laboratories, regarding this selection processor with pre- and post-averaging.

Reviewed and Approved: 7 January 1987

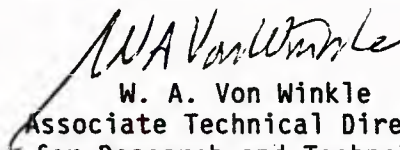

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LIST OF SYMBOLS

N	number of input channels
K	amount of pre-averaging
M	amount of post-averaging
h	hypothesis number: 0 or 1
H_0	signal absent hypothesis
H_1	signal present hypothesis
$x_n(t)$	channel n input at time t
$f_x^{(h)}(f)$	input characteristic function
$a_n(t)$	pre-averager n output at time t
$y(t)$	output of Greatest-Of
$\ell(t)$	channel number of largest input to Greatest-Of, at time t
$v(t)$	selector output at time t
$w(t)$	post-averager output at time t ; decision variable
Λ	threshold value
P_D	detection probability
P_F	false alarm probability
$\mu_x^{(h)}$	mean of input under hypothesis h
σ_x^2	variance of input under both hypotheses
d	input deflection: $(\mu_x^{(1)} - \mu_x^{(0)})/\sigma_x$
γ	modified threshold; (C-12)
I_ℓ	effective number of independent samples; (A-9)-(A-12)
I_2	integer part of I_ℓ
α_2, β_2	parameters associated with I_ℓ and I_2 : (C-16) and (B-3)
Q_a	probability that signal channel dominates Greatest-Of on one trial; (C-5)-(C-7)
T_t	total number of trials in simulation
w_k	decision variable for k -th trial

OPERATING CHARACTERISTICS OF AN OR-ING AND SELECTION PROCESSOR WITH PRE- AND POST-AVERAGING

INTRODUCTION

The analysis of systems with or-ing has been the subject of considerable interest [1-8]. These analyses have been in terms of a second-moment theory and have usually made timely use of the Central Limit Theorem at various points in the system, such as an accumulator output. A notable exception to this approach is the exact analysis in [9], where the detection and false alarm probabilities for a system with quantizers, or-ing, and an accumulator were determined exactly. There, the characteristic function of the decision variable was determined exactly via a fast Fourier transform to yield threshold-crossing probabilities.

Here, we attempt to continue this theme of getting exact detection and false alarm probabilities for an or-ing system with pre- and post-averaging. However, the analytical situation is greatly complicated by the presence of the pre- and post-averagers: the pre-averager creates statistical dependencies between its successive output time samples that the post-averager must operate on. For the signal present hypothesis H_1 , we thus encounter the classical problem of determining the output distribution function of a linear system subject to non-Gaussian and non-white noise; since this is an unsolved problem, we must instead revert to determining an approximation for the characteristic function and/or cumulative distribution function of the decision variable. On the other hand, for the signal absent hypothesis H_0 , the characteristic function of the decision variable can be determined in closed form. Thus we have a mixed situation, where part of the analysis is exact and part approximate.

The technique utilized here is not a second-moment approach, although one part of the analysis does match moments through second-order, in order to determine the effective number of independent samples at the or-ing input. The utility of this combined approximate analysis is borne out by simulation which corroborates the technique in a wide variety of examples evaluated numerically.

The inclusion of or-ing in a system in the usual fashion necessitates an increase in the decision threshold, if the false alarm probability is to be kept the same as when there was no or-ing, i.e., one input channel. The amount of threshold increase required is a function of the number of input channels, N , and must therefore be re-calculated every time N is changed. Also, this increased threshold leads to decreased detection probabilities, which is undesirable.

Here we will investigate a novel or-ing and selection processor proposed* by Bell Laboratories which uses a threshold completely independent of the number of input channels, N , and yet also realizes a false alarm probability independent of N . The required threshold does, of course, depend on the required false alarm probability and the statistics of the noise-only on one input channel, in addition to the amount of post-averaging; however, it is independent of N and the amount of pre-averaging.

Although the new processor utilizes a threshold independent of N , it still suffers a loss in detection probability relative to no or-ing, because a succession of decisions must be reached after the pre-averager (i.e., in the or-ing device) but before the post-averager. That is, the nonlinear operations are not deferred until after all the linear signal-enhancing operations are completed, but in fact, occur at an intermediate point in the processor. How well the new processor performs in comparison with a standard or-ing system with pre- and post-averaging is not known; the latter is currently under investigation and will be reported on separately.

*U.S. Patent Number 4622519 has been assigned to AT&T Bell Laboratories, regarding this selection processor with pre- and post-averaging.

DESCRIPTION OF OR-ING AND SELECTION PROCESSOR

The system of interest is depicted in figure 1. There are N input channels, and the input on the n -th channel at discrete time t is $x_n(t)$. The inputs $\{x_n(t)\}$ are stationary and statistically independent of each other, from channel-to-channel, and from one time instant to any other within the same channel. Furthermore, the channels without a signal in them are identically distributed, with an arbitrary characteristic function $f_x^{(0)}(\xi)$, where the superscript denotes hypothesis H_0 , signal absent from all channels. For hypothesis H_1 , the signal is present in one and only one (unknown) channel, and remains in that channel for the total observation time leading to a particular decision at the system output $w(t)$; this total time is $K+M$ samples for the system of figure 1. The characteristic function of the signal-bearing input channel is $f_x^{(1)}(\xi)$. It is not presumed that the signal and noise are additive at the inputs $\{x_n(t)\}$; all that is needed are the input characteristic functions $f_x^{(h)}$ for hypotheses $h=0$ and 1 . (Later, for numerical evaluation purposes, we will specialize the results to Gaussian inputs.)

The inputs $\{x_n(t)\}$ are all delayed by one unit of time (for reasons to be seen below) and then subjected to pre-averaging of K samples; that is, the pre-averager outputs are

$$a_n(t) = \sum_{k=1}^K x_n(t-k) \quad \text{for } 1 \leq n \leq N \quad . \quad (1)$$

Due to the unit delay, $a_n(t)$ is the sum of the past K samples of input $x_n(t)$ on channel n . The purpose of this pre-averaging is to enhance the signal (if present), prior to making a nonlinear decision in the or-ing procedure.

The greatest-of device has two outputs. The first is the actual largest input it is subjected to at time t :

$$y(t) = \max \{a_1(t), \dots, a_N(t)\} \equiv \max_{1 \leq n \leq N} \{a_n(t)\} \equiv a_{\lambda(t)}(t) \quad . \quad (2)$$

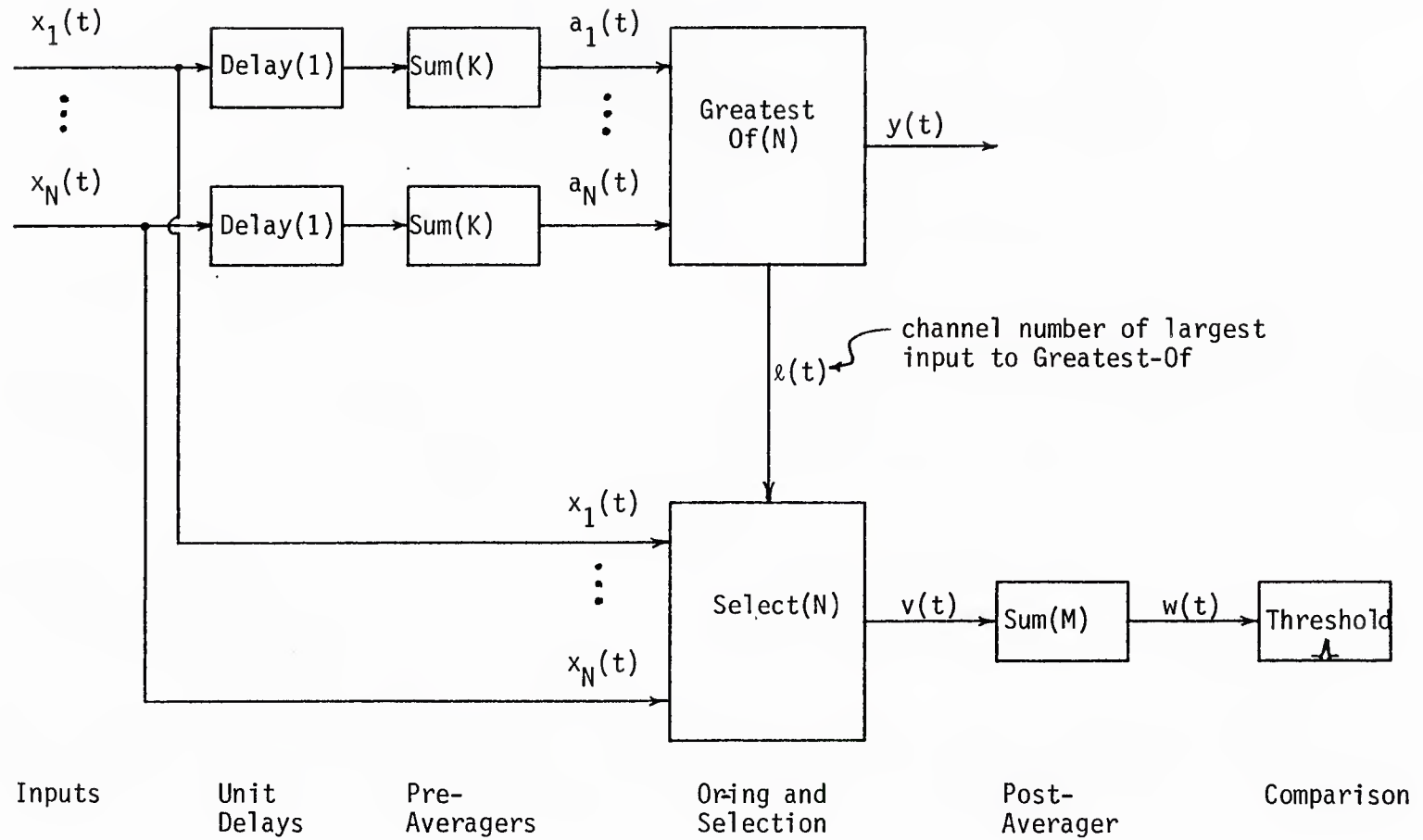


Figure 1. Or-ing and Selection Processor with Pre- and Post-Averaging

The other output is $\ell(t)$, as indicated in (2); it is the channel number at time t of the largest input to the greatest-of device. Although $y(t)$ is discarded, $\ell(t)$ is used in the selection procedure.

The channel number output, $\ell(t)$, is used to make a selection of one of the N inputs at time t , not $t-1$. That is, the selector output is

$$v(t) = x_{\ell(t)}(t) \equiv \tilde{x}(t; \ell(t)), \quad (3)$$

which is one ^{of} the current input samples.

Finally, the post-averager output is the sum of M consecutive selector outputs:

$$w(t) = \sum_{j=0}^{M-1} v(t-j) = \sum_{j=0}^{M-1} \tilde{x}(t-j; \ell(t-j)) \quad . \quad (4)$$

This quantity is called the decision variable; it is compared with a threshold Λ for decisions about signal absence vs signal presence in some input. It is important to observe that although the decision variable only employs M samples, the "oldest" of these samples itself depends on the previous K input samples, by way of the pre-averaging. Thus if the processor is to function well under hypothesis H_1 , the signal must be present at the input for a total of $K+M$ samples, not just M . This feature must be kept in mind when comparing this processor with others, in terms of the detection probability and false alarm probabilities; both processors must be allowed $K+M$ samples for a fair comparison.

We recall from (1) that the pre-averager outputs at time t were unable to utilize the current input samples $\{x_n(t)\}$, for $1 \leq n \leq N$. Since the system inputs are all independent at different times, the selection procedure will always yield a random variable independent of those used in the or-ing. For H_0 , this means that post-averager output $w(t)$ in (4) is the sum of M independent, identically distributed, random variables, each with characteristic function $f_x^{(0)}(\mathfrak{F})$, regardless of the particular sequence of channel values $\ell(t)$, $\ell(t-1)$, ..., $\ell(t-M+1)$, that occurred at the selector

input. Thus the statistics of $w(t)$ under H_0 are independent of N , the number of input channels, but depend solely on $f_x^{(0)}(\mathbf{f})$ and M . This enables the setting of a threshold which realizes a fixed false alarm probability, regardless of the value of N .

Under H_1 , although the post-averager output is the sum of M statistically independent random variables*, some of the selector outputs are drawn from the actual signal channel (if the pre-averager is functioning well), but some are noise-only channel inputs. The number of correct signal-channel selections can be anywhere in the range $(0, M)$; this greatly complicates the analysis for the detection probability, and necessitates some approximations.

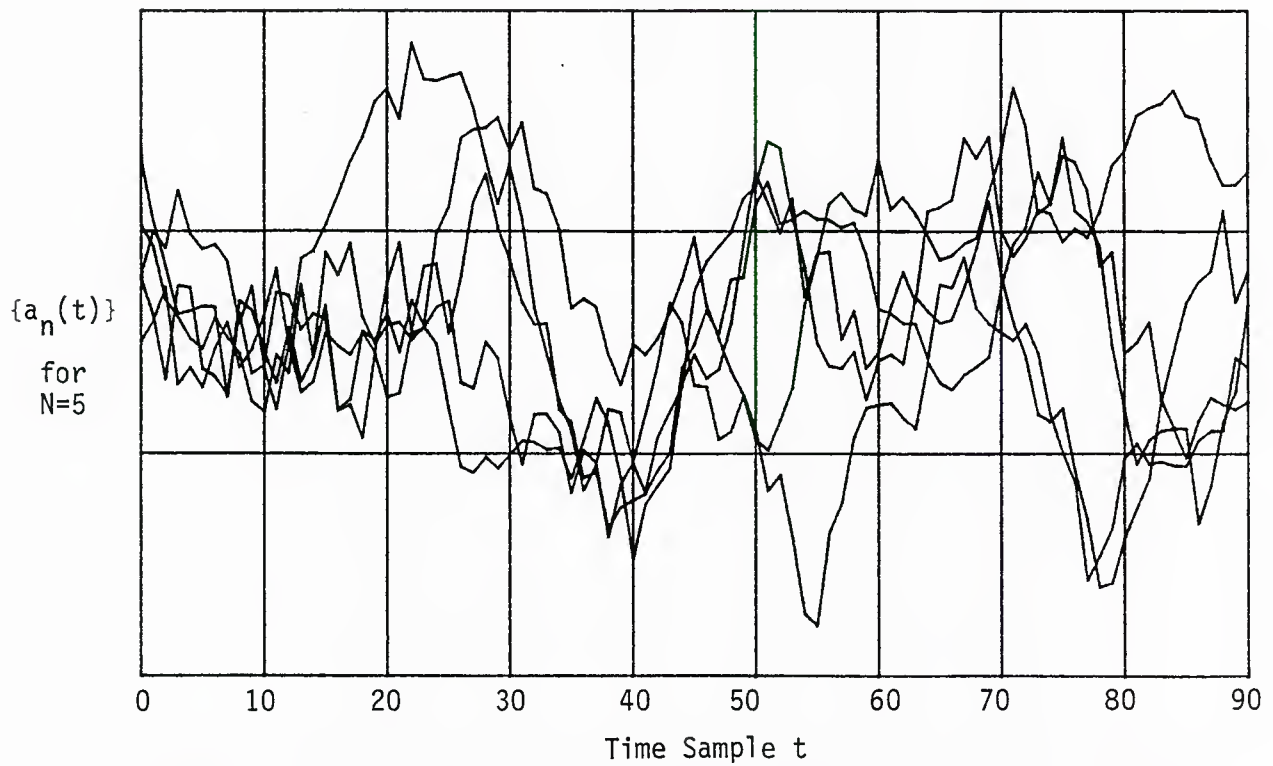
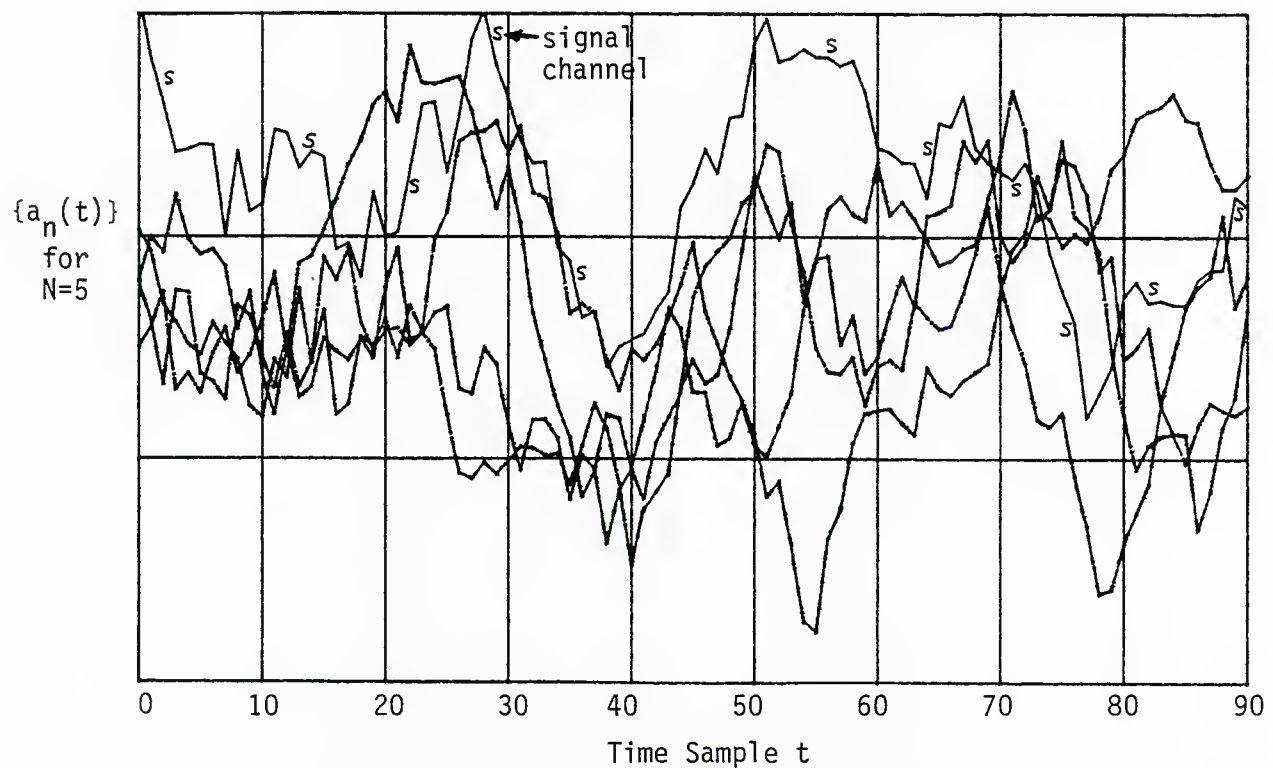
Behavior of Pre-Averager Outputs

Since the memory of the pre-averagers is K samples, the inputs to the greatest-of device cannot "forget" the past very quickly. What this means is that if one system input is momentarily very noisy, it continues to adversely affect or-ing decisions for a considerable time. Thus, channel-number output $l(t)$ can remain at the same value for several consecutive times, even if the signal is absent. A sample plot of this behavior is depicted in figure 2 for $K=10$ and $N=5$. Thus, for example, one particular channel pre-averager output dominates from sample $t=13$ to $t=27$.

When the signal is added, the situation improves, in that the signal channel dominates most often; see figure 3. However, there are still extended regions, such as $(16, 26)$ and $(68, 90)$, where one of the noise channels dominates. The effects of these extended regions of dominance must be accounted for in any analysis of performance.

There are conflicting requirements at work in the processor of figure 1. One indicates that K should be increased so that good decisions can be reached in the greatest-of. On the other hand, larger K can cause prolonged dominance, as indicated above; and it subtracts from the time, M , available for post-averaging, if total observation time $K+M$ is fixed or limited. Since it is also desirable to increase M , for better quality decisions at the system

* when conditioned upon a particular sequence of $l(t)$ values; see (A-1), (A-5), and (A-6).

Figure 2. Outputs of Pre-Averagers for $K=10$, $N=5$ Channels, Noise-OnlyFigure 3. Outputs of Pre-Averagers for $K=10$, $N=5$ Channels, Signal Present

output, it is conceivable that there is an optimum division between pre- and post-averaging times, if $K+M$ is fixed. This behavior can be extracted from the operating characteristics, when plotted for a wide variety of values of K and M . Of course, there remains the dependency of any conclusions on N and the input statistics.

RESULTS

Derivation of detection probability P_D and false alarm probability P_F is conducted in a series of appendices, under various degrees of generality. The only one we consider numerically here is that of Gaussian inputs, for which the input characteristic functions are

$$f_x^{(h)}(\xi) = \exp(i\xi u_x^{(h)} - \frac{1}{2}\xi^2 \sigma_x^2) \quad \text{for } h = 0 \text{ and } 1. \quad (5)$$

Thus the two hypotheses have different means $u_x^{(h)}$, but the same variance σ_x^2 , for the system inputs. We define the system input deflection

$$d \equiv \frac{u_x^{(1)} - u_x^{(0)}}{\sigma_x}. \quad (6)$$

The detection probability and false alarm probability are given in (C-19) and (C-20), where the various parameters are also explained. In the derivations, it was necessary to define the number of independent samples at a pre-averager output in an interval of length M , and then to approximate characteristic functions raised to fractional powers. The details are lengthy and are reserved for the appendices for the interested reader.

A variety of cases are plotted in the following figures. Values of the parameters have been chosen as indicated in table 1 for figures 4 - 35. We have looked at all possible combinations of these choices, for a total of $4 \times 4 \times 2 = 32$ cases.

N, Number of Input Channels	K, Amount of Pre-Averaging	M, Amount of Post-Averaging
2	1	32
4	2	64
8	4	
16	8	

Table 1. Values of N, K, M Considered for Figures 4 - 35

Superposed on most figures is one simulation run for a representative value of d , equal to one utilized in the analysis; for example, figure 4 contains a simulation result for $d = .8$. These (jagged) simulation curves closely overlay the (smooth) analysis curves except in the region of rare events, namely where the detection probability is near unity ($1 - P_D \ll 1$); the random discrepancy in this latter region is due to the relatively small and limited number of trials employed in the simulation to estimate P_D , namely 1000 trials. There is no need to simulate the false alarm case at all, since we can calculate the false alarm probability exactly in any region and for any parameter values.

For the smaller values of N , the characteristics are nearly equally spaced; however, as N increases, the curves get more crowded around the $d=0$ curve. This is a manifestation of small signal suppression, brought about by the nonlinear action of the or-ing operation. The plots in figures 4 - 35 are also generally fairly straight; this is due to the considerable amount of post-averaging used, namely $M = 32$ or 64 .

When M is decreased, as in figures 36 - 39, the operating characteristics develop a significant knee, where the detection probability temporarily saturates as the false alarm probability increases. However, this undesirable behavior generally occurs for the larger values of false alarm probability, where the system will not be operated. The superposed simulation results confirm the presence of this knee in the operating characteristics.

SUMMARY

The detection and false alarm probabilities have been derived for the or-ing and selection processor with pre- and post-averaging and specialized to Gaussian inputs. Plots of the operating characteristics for a variety of choices of number of input channels, amount of pre-averaging, amount of post-averaging, and input signal strength are given. Superposed on most of the analytical results is a simulation result for each case, which corroborates the approximations which were necessary for a tractable analysis. Other cases of interest to the reader may be obtained by means of the program listed in appendix D in BASIC for the Hewlett Packard 9845 Desk Calculator.

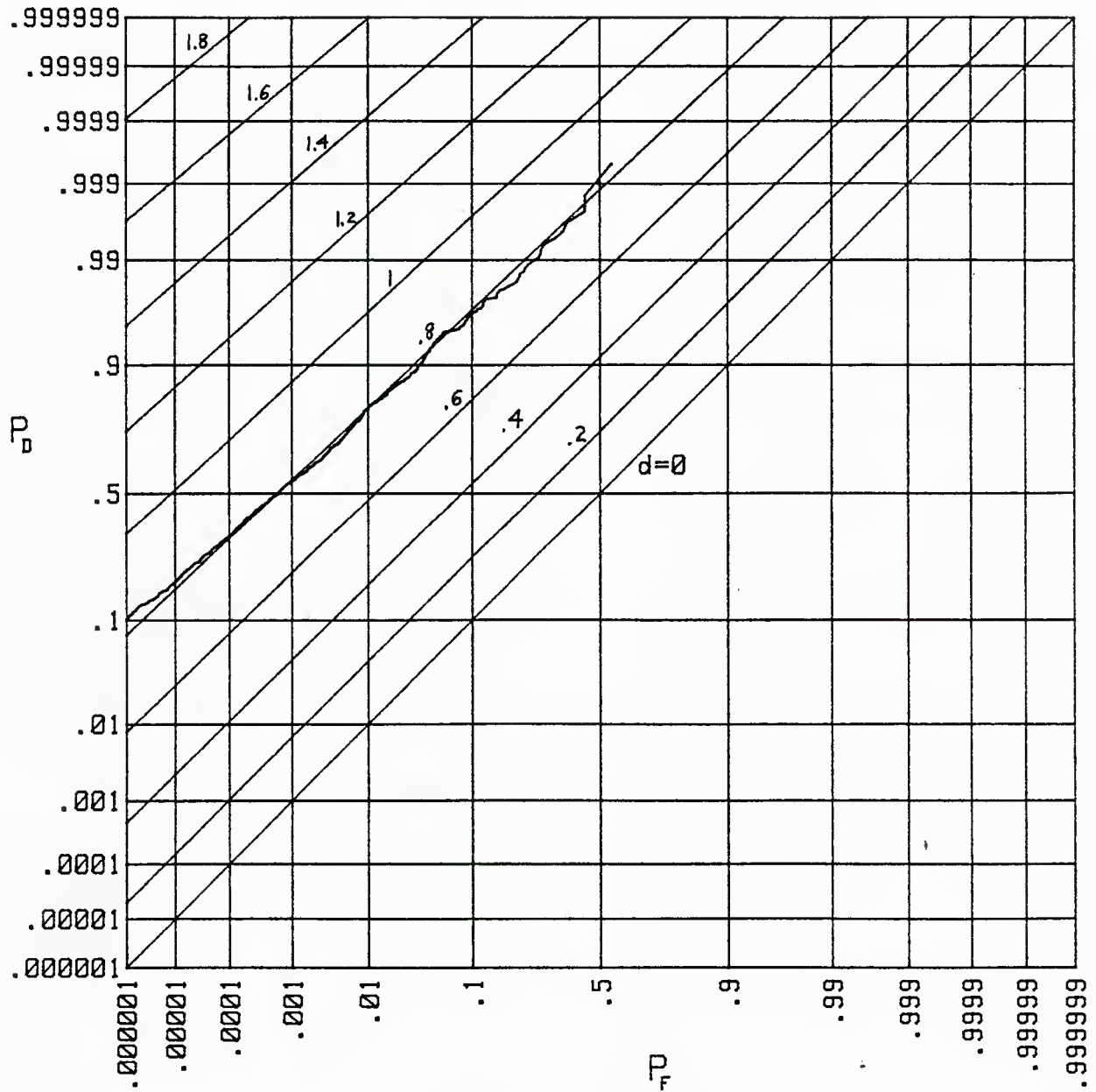


Figure 4. Operating Characteristics for $N=2$, $K=1$, $M=32$

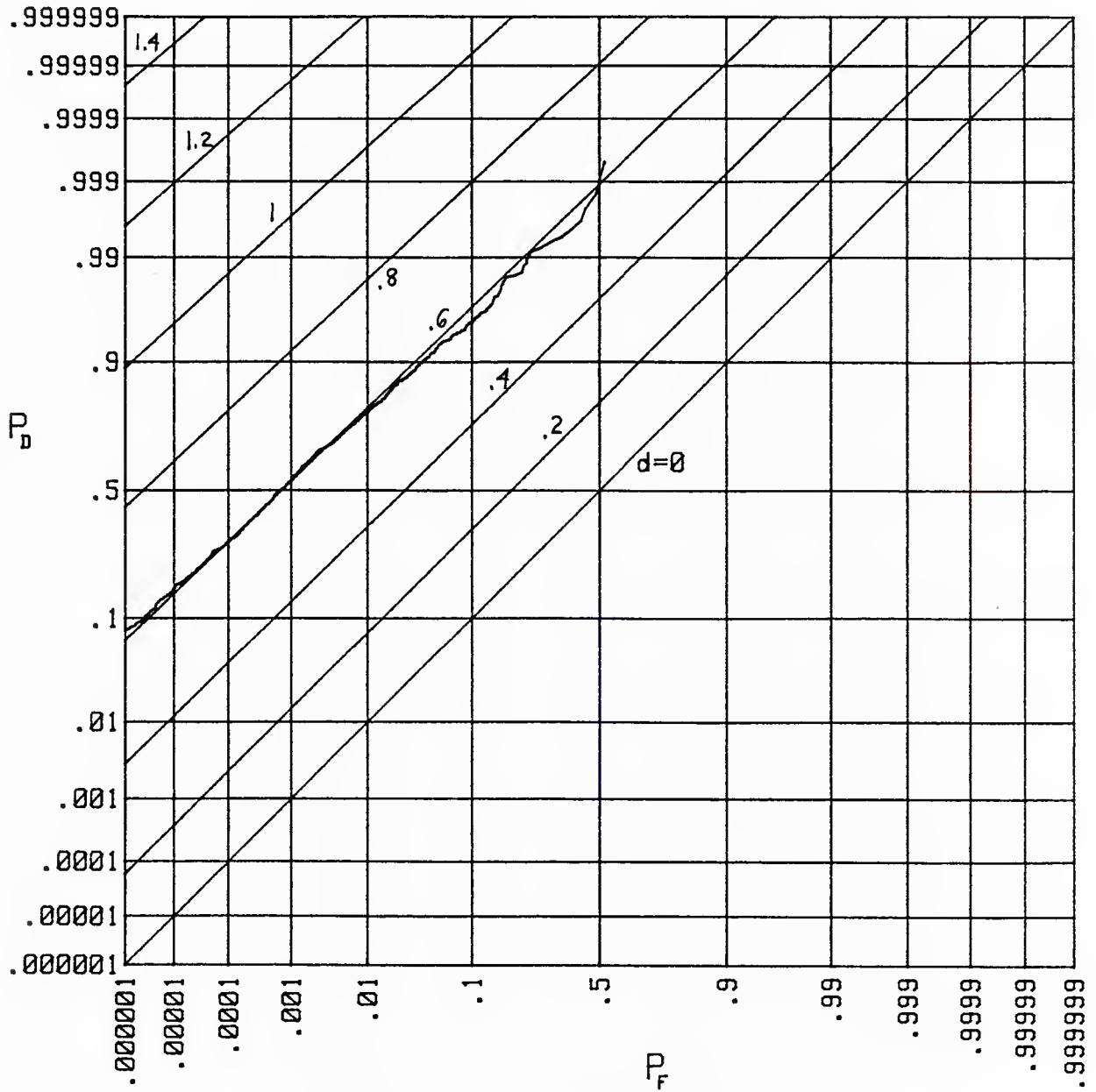


Figure 5. Operating Characteristics for $N=2$, $K=1$, $M=64$

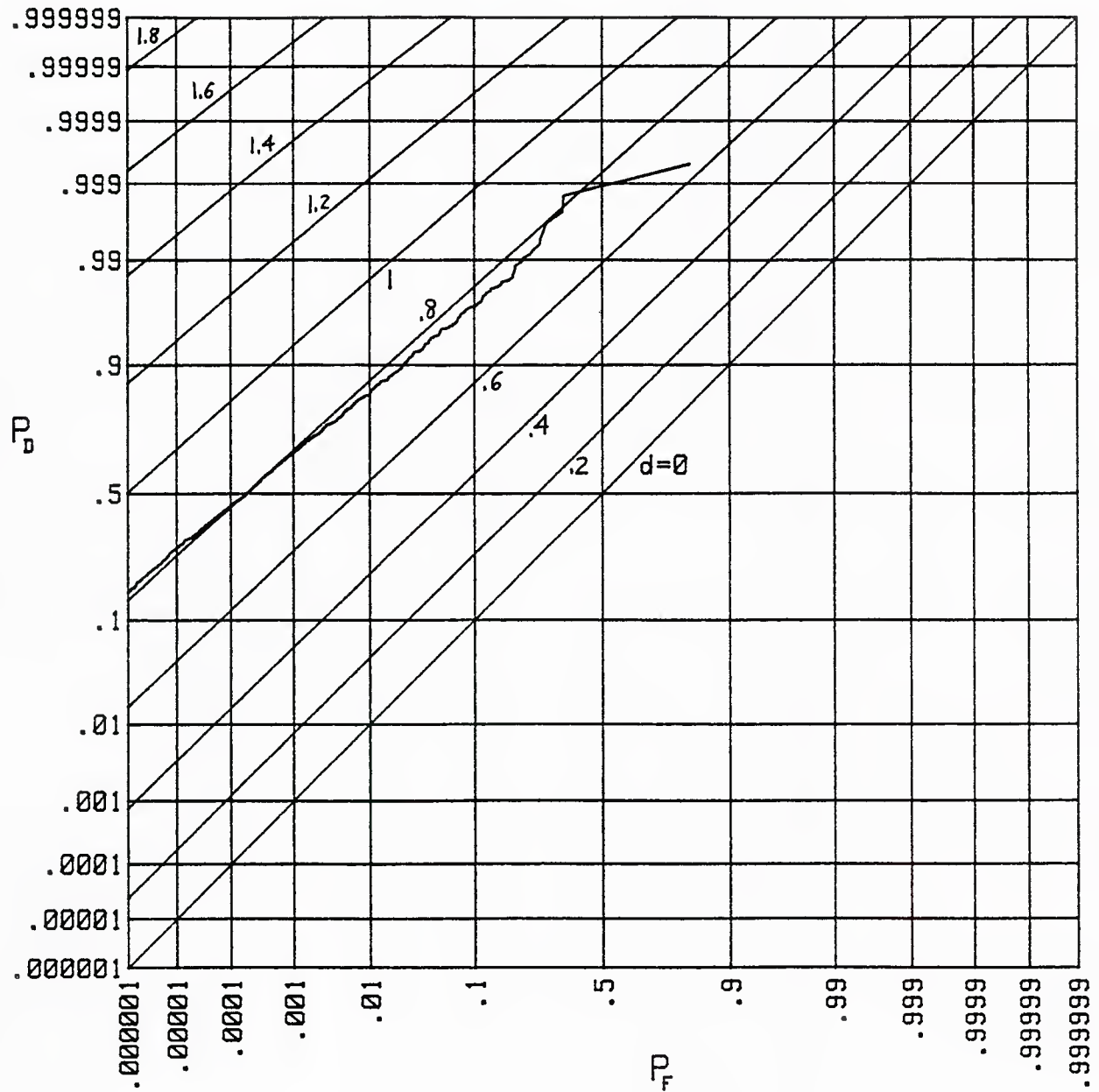


Figure 6. Operating Characteristics for $N=2$, $K=2$, $M=32$

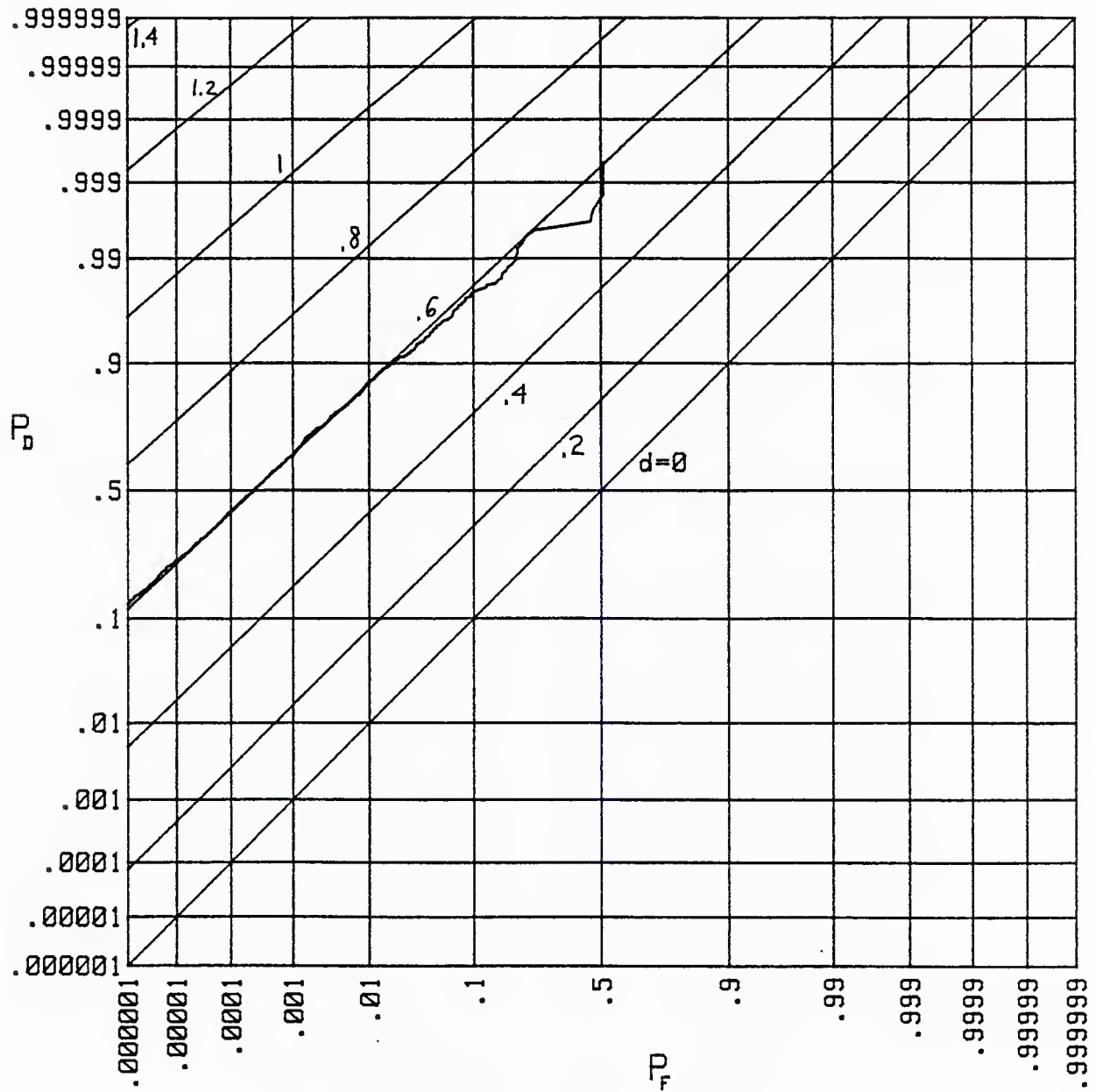


Figure 7. Operating Characteristics for $N=2$, $K=2$, $M=64$

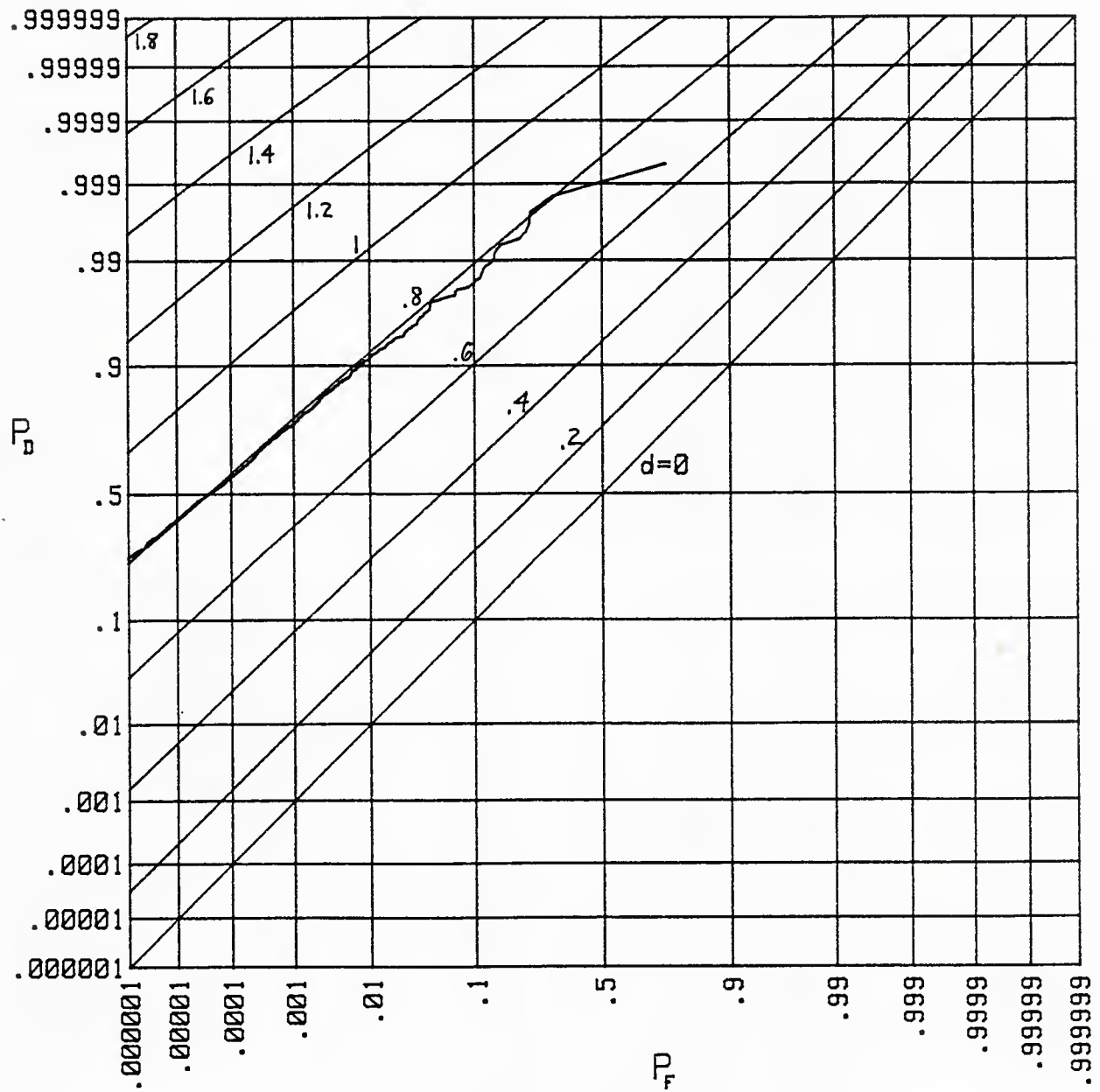


Figure 8. Operating Characteristics for $N=2$, $K=4$, $M=32$

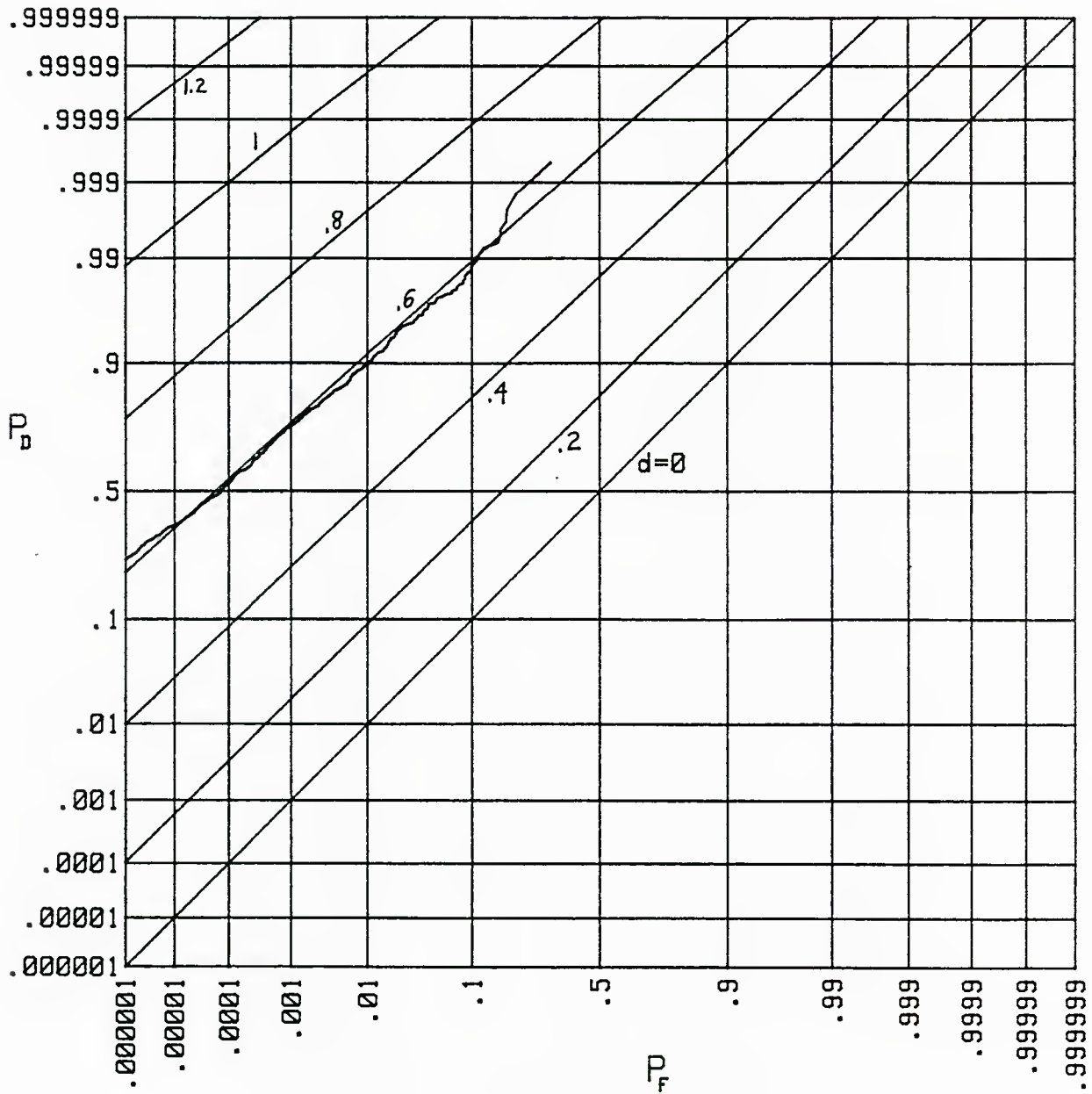


Figure 9. Operating Characteristics for $N=2$, $K=4$, $M=64$

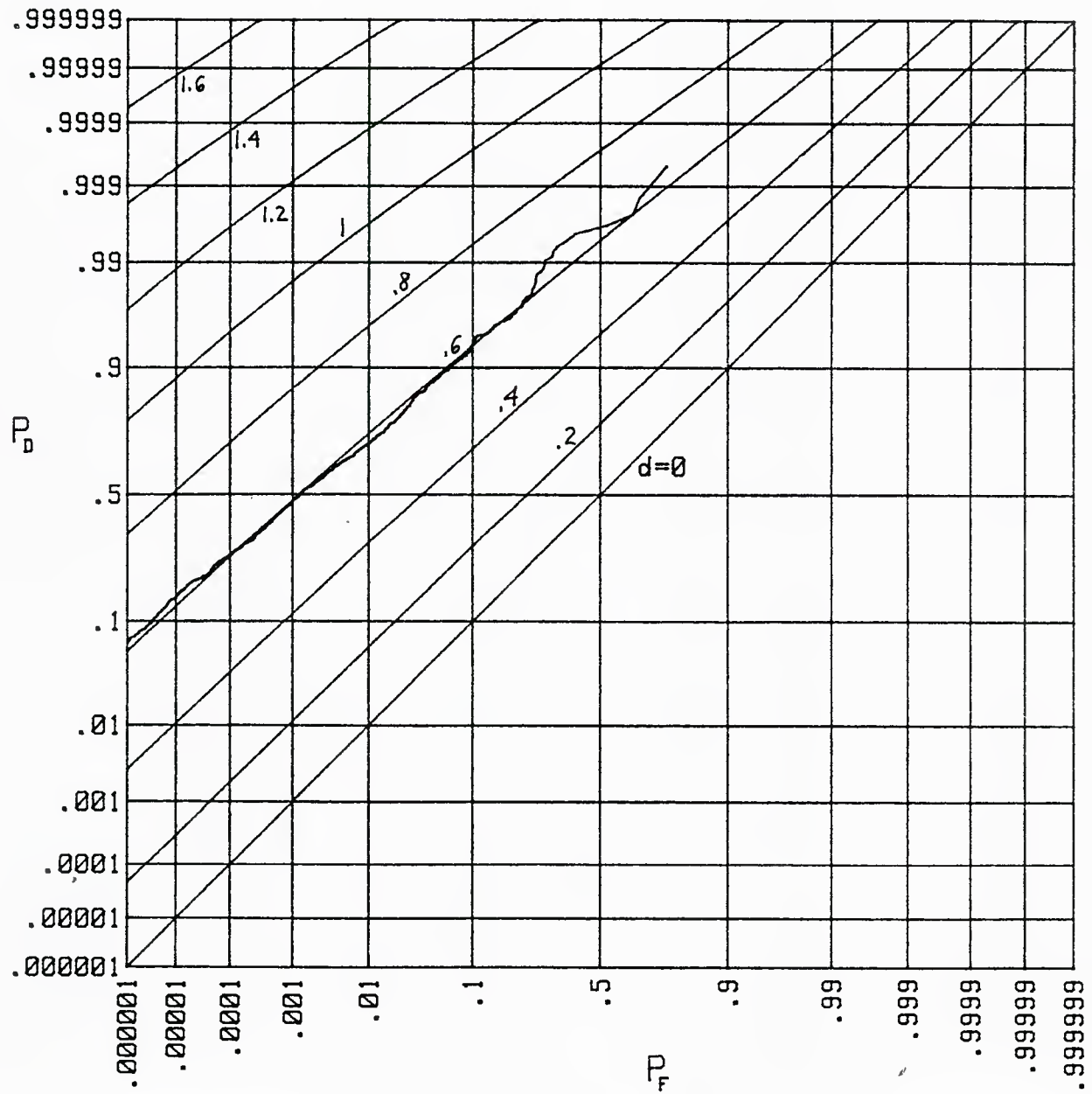
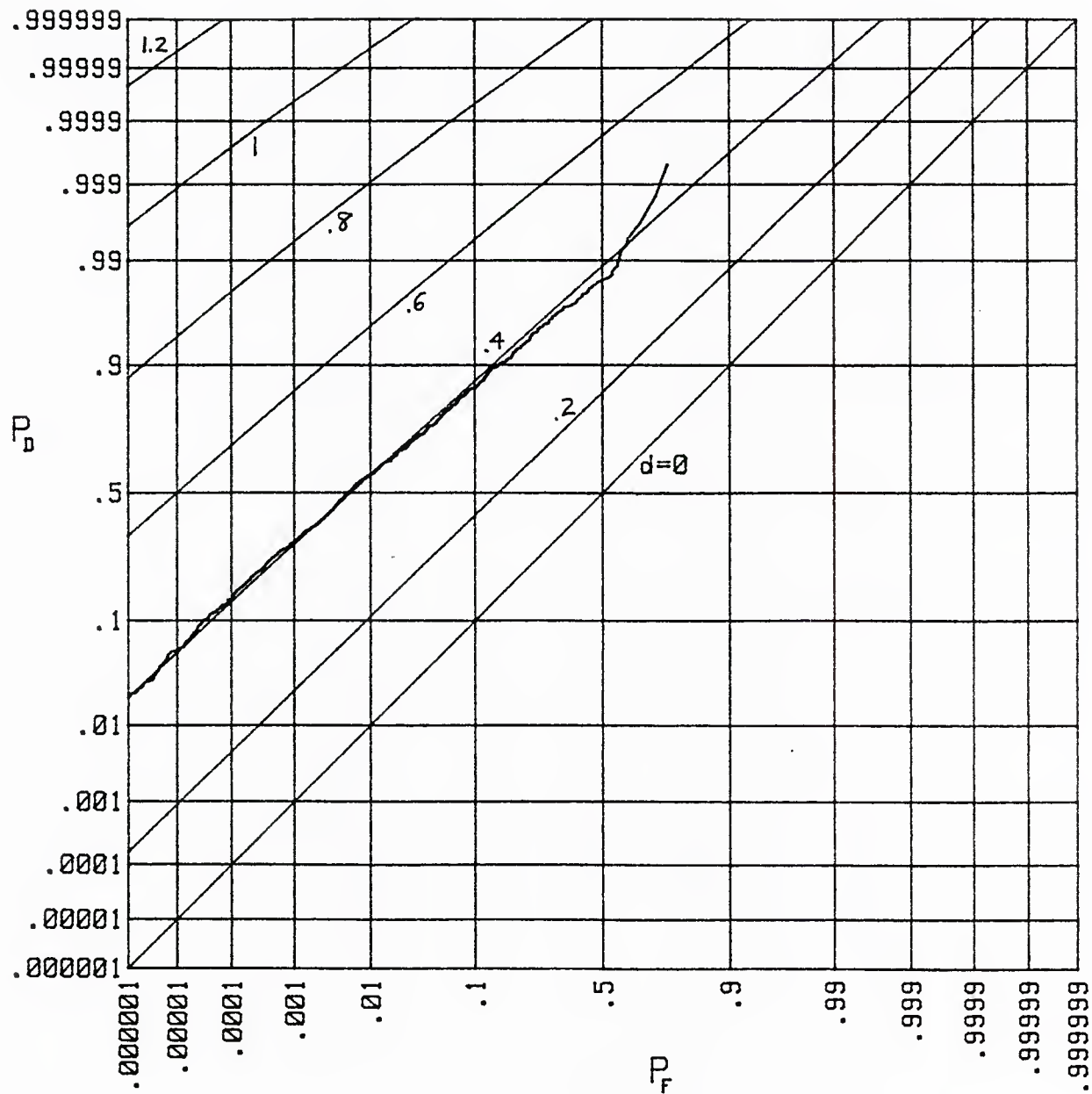


Figure 10. Operating Characteristics for $N=2, K=8, M=32$

Figure 11. Operating Characteristics for $N=2$, $K=8$, $M=64$

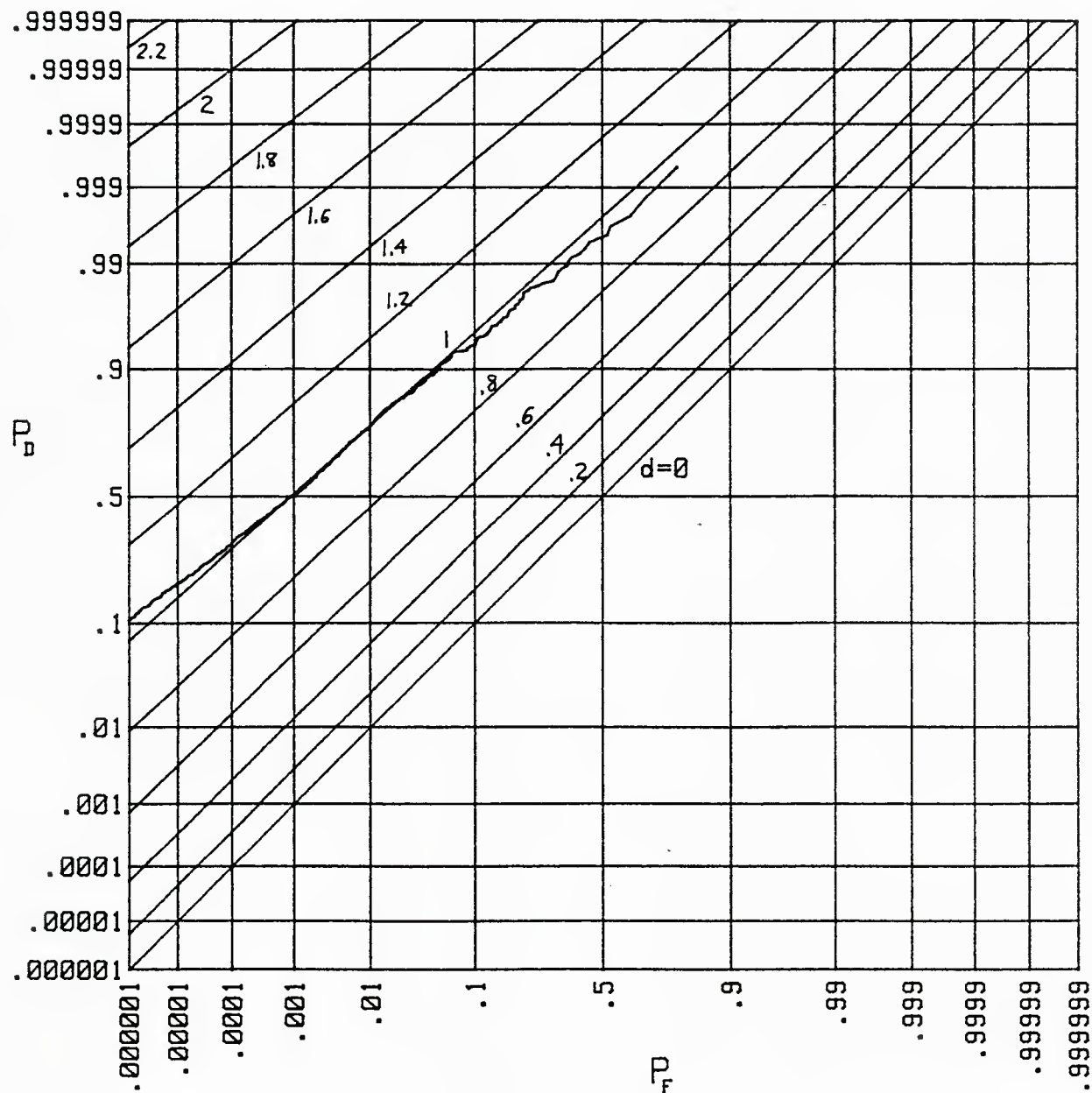


Figure 12. Operating Characteristics for $N=4$, $K=1$, $M=32$

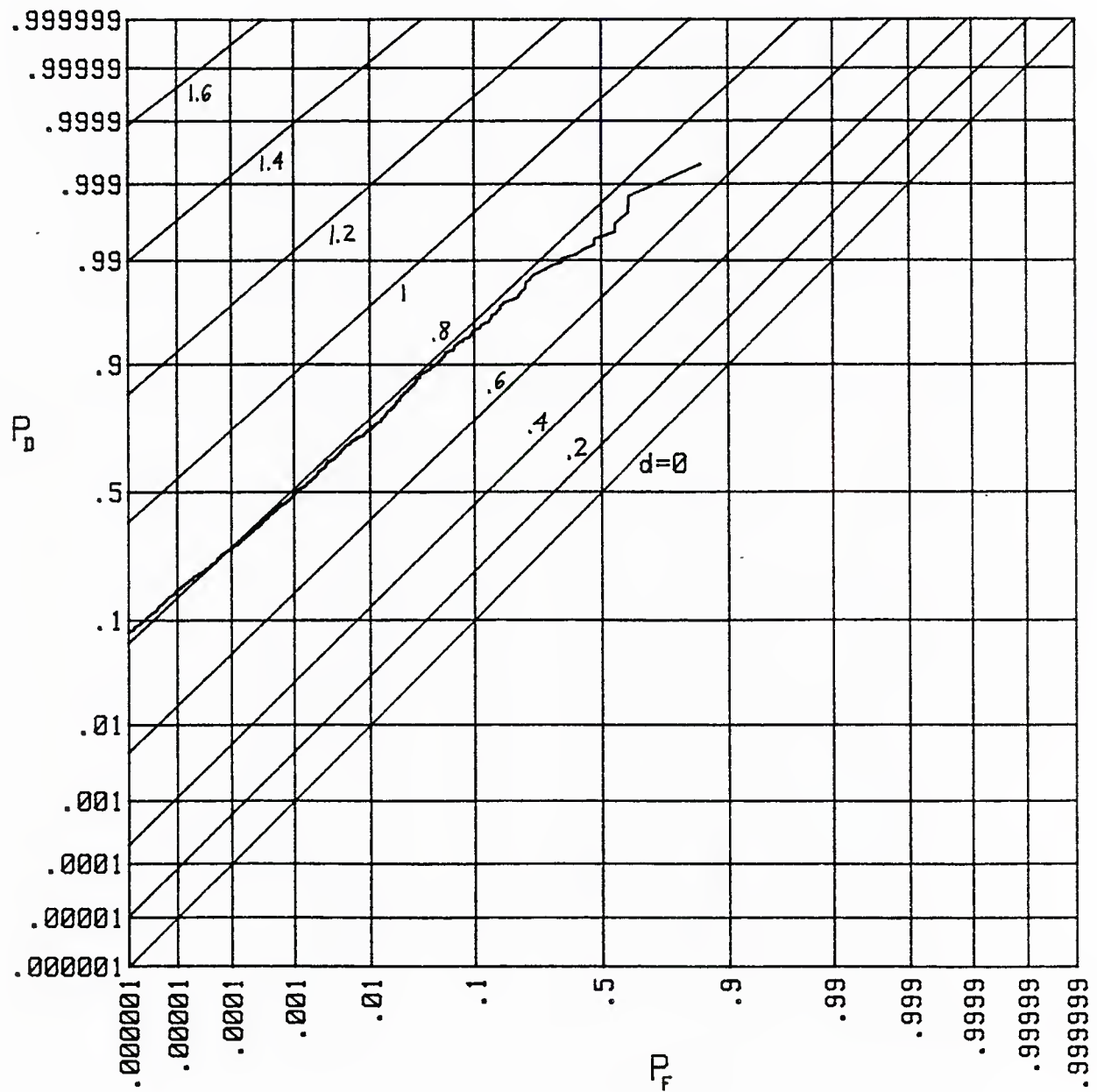


Figure 13. Operating Characteristics for $N=4$, $K=1$, $M=64$

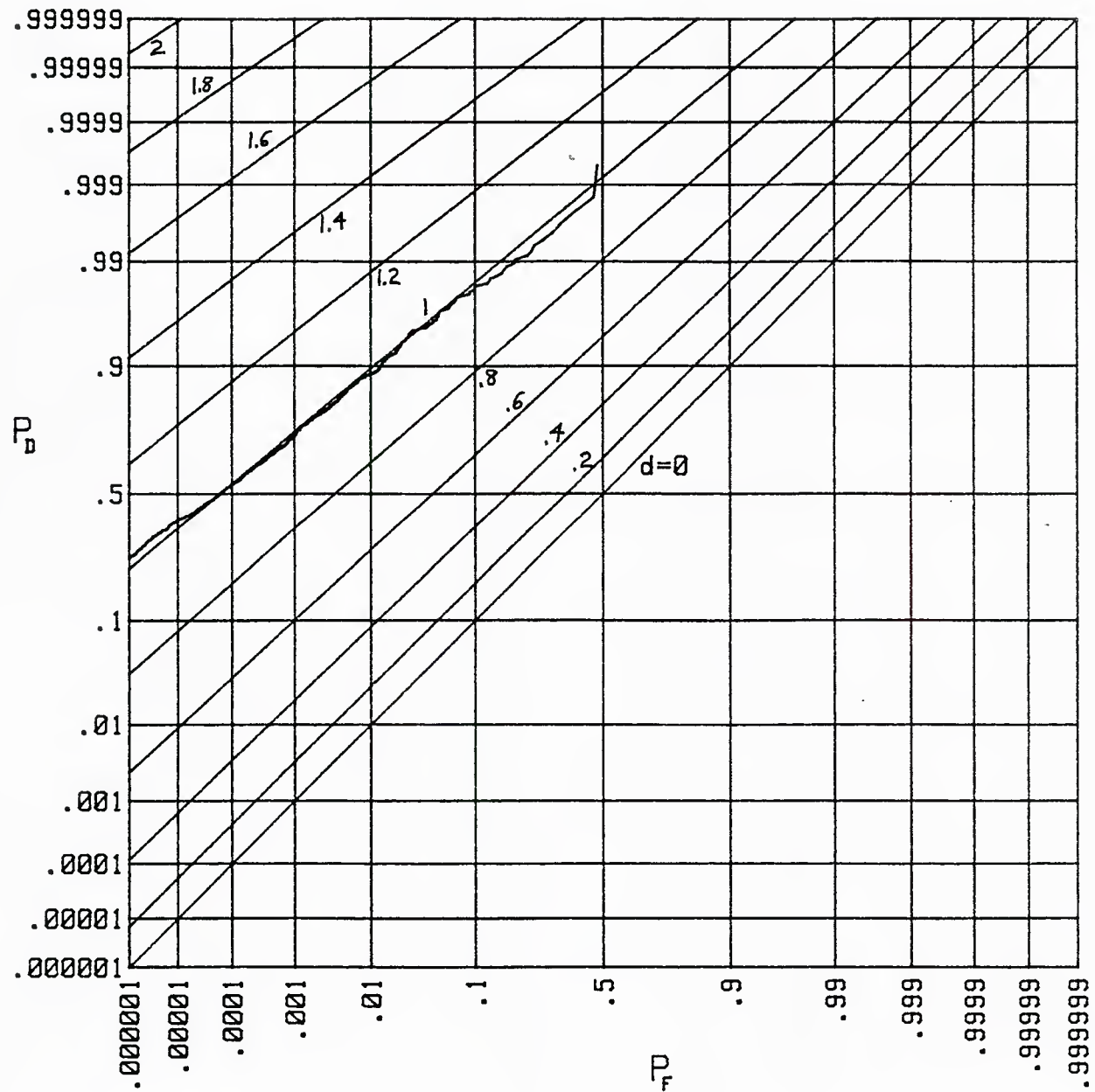


Figure 14. Operating Characteristics for $N=4$, $K=2$, $M=32$

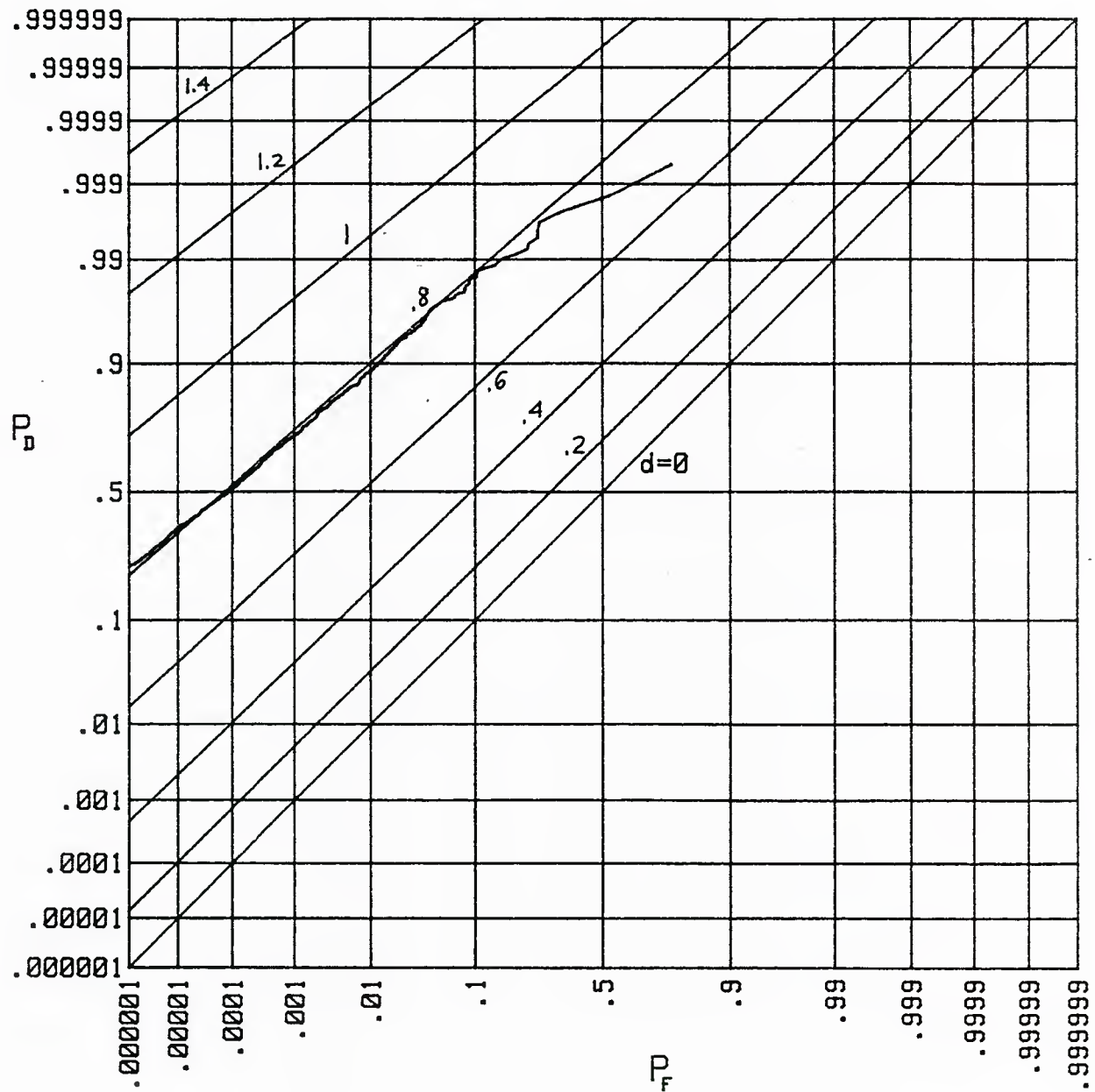


Figure 15. Operating Characteristics for $N=4$, $K=2$, $M=64$

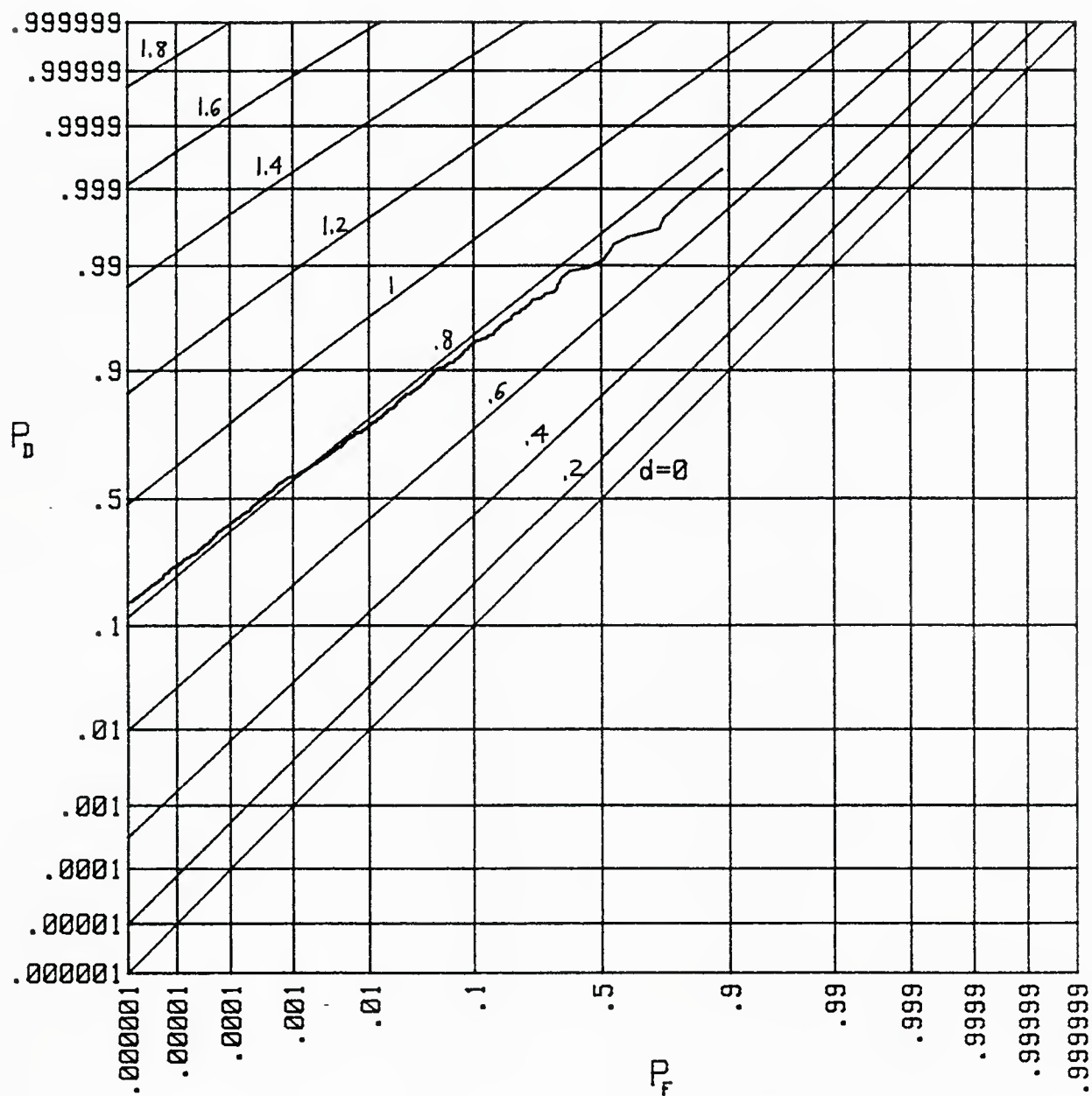
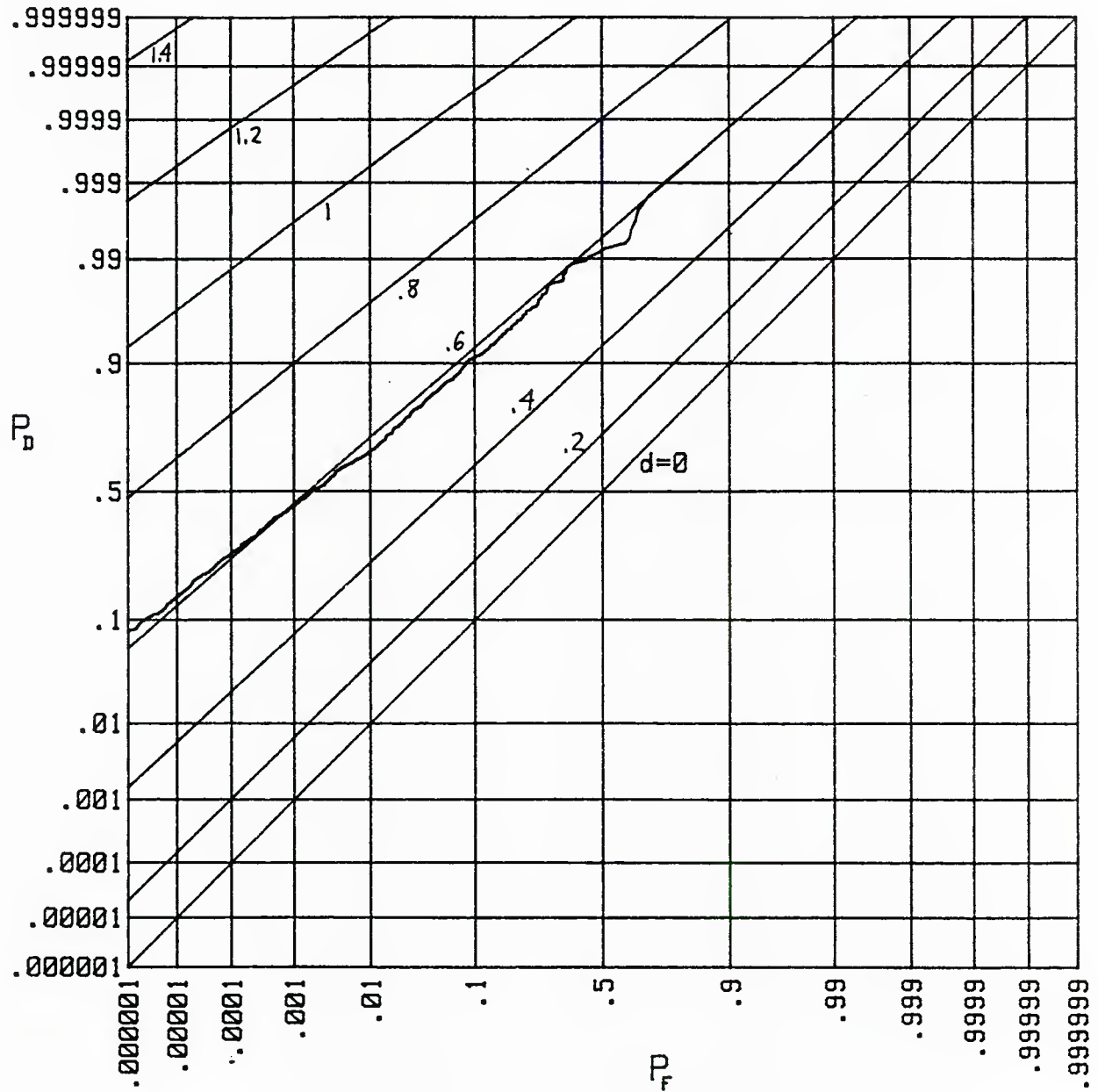


Figure 16. Operating Characteristics for $N=4$, $K=4$, $M=32$

Figure 17. Operating Characteristics for $N=4$, $K=4$, $M=64$

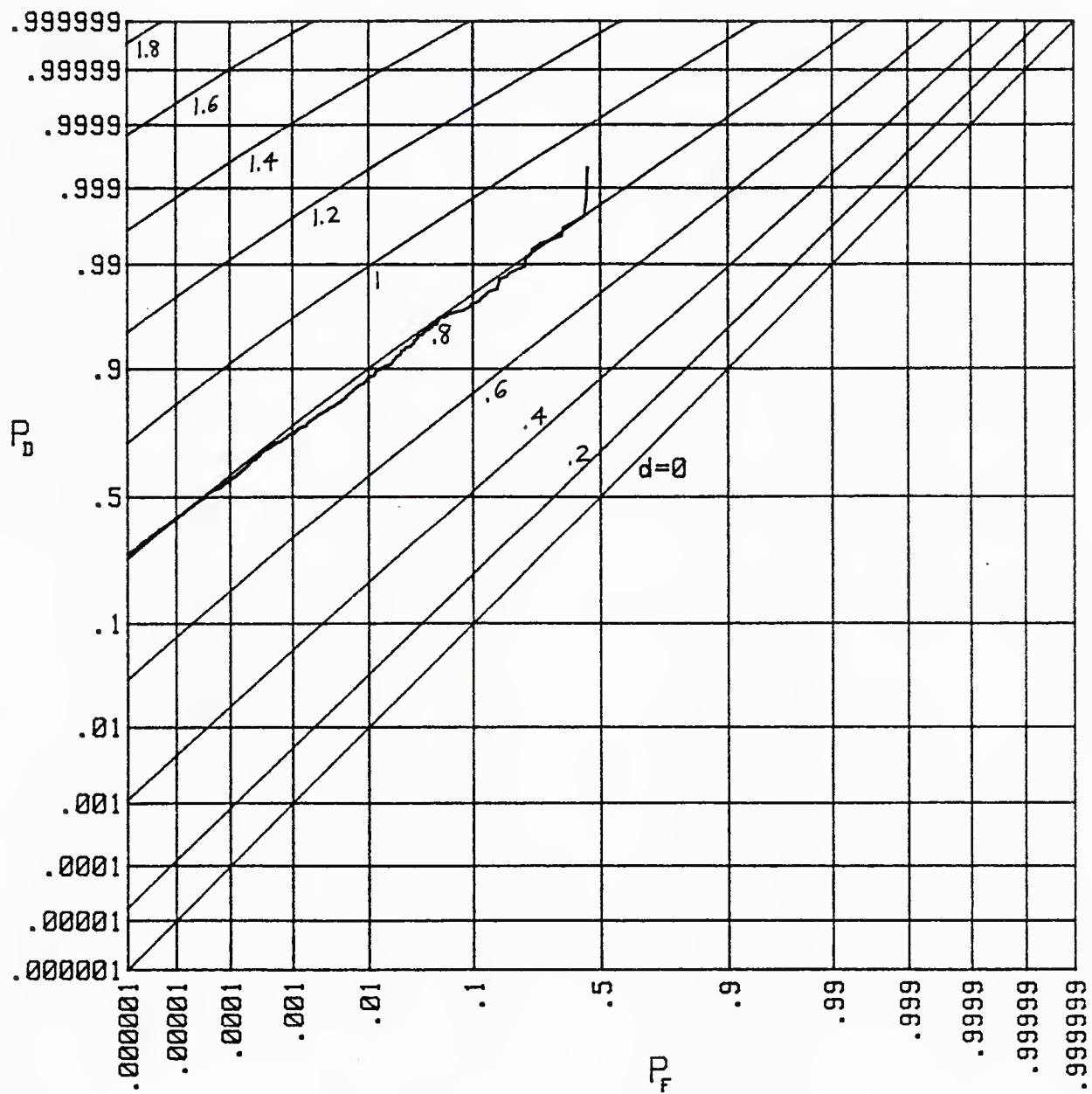


Figure 18. Operating Characteristics for $N=4$, $K=8$, $M=32$

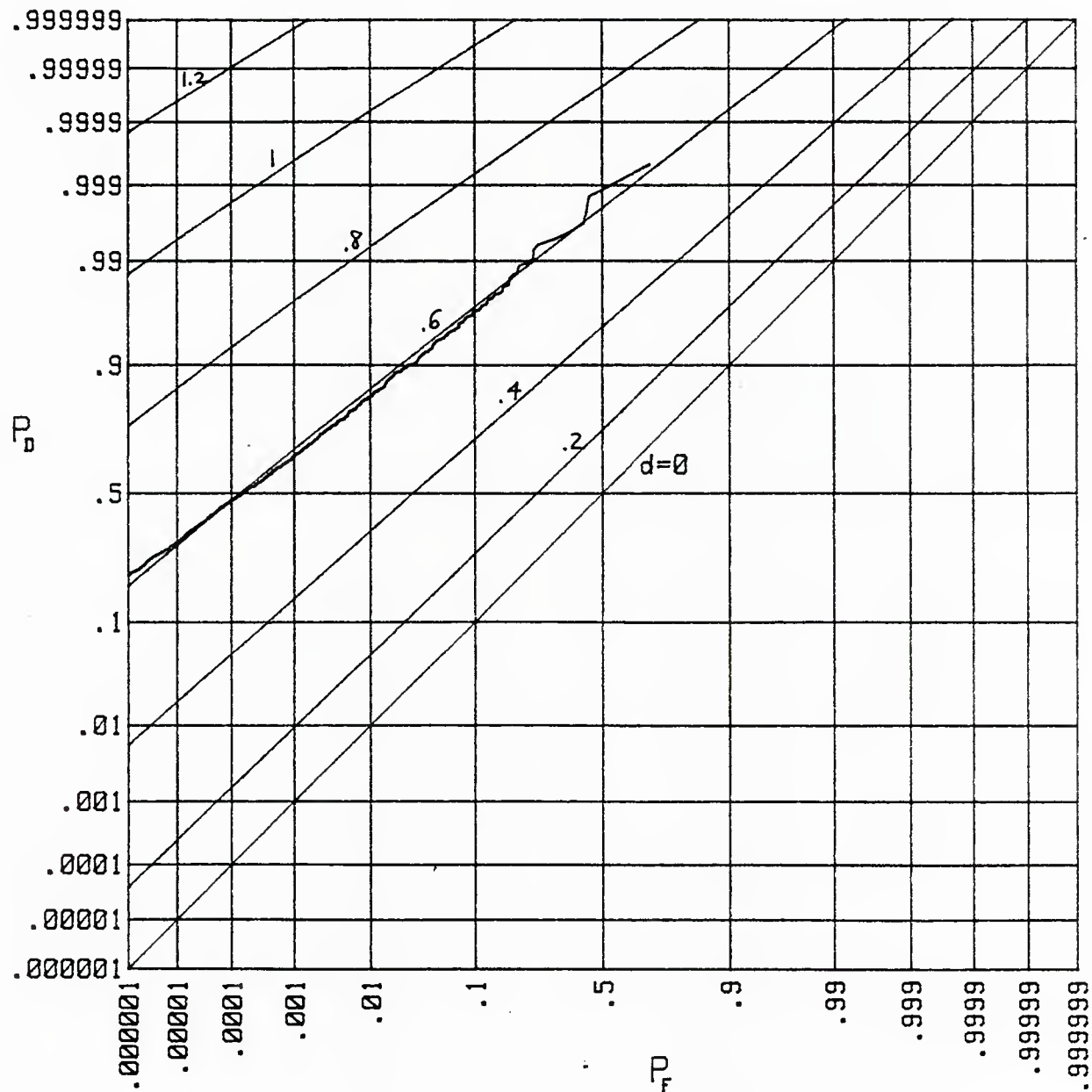


Figure 19. Operating Characteristics for $N=4$, $K=8$, $M=64$

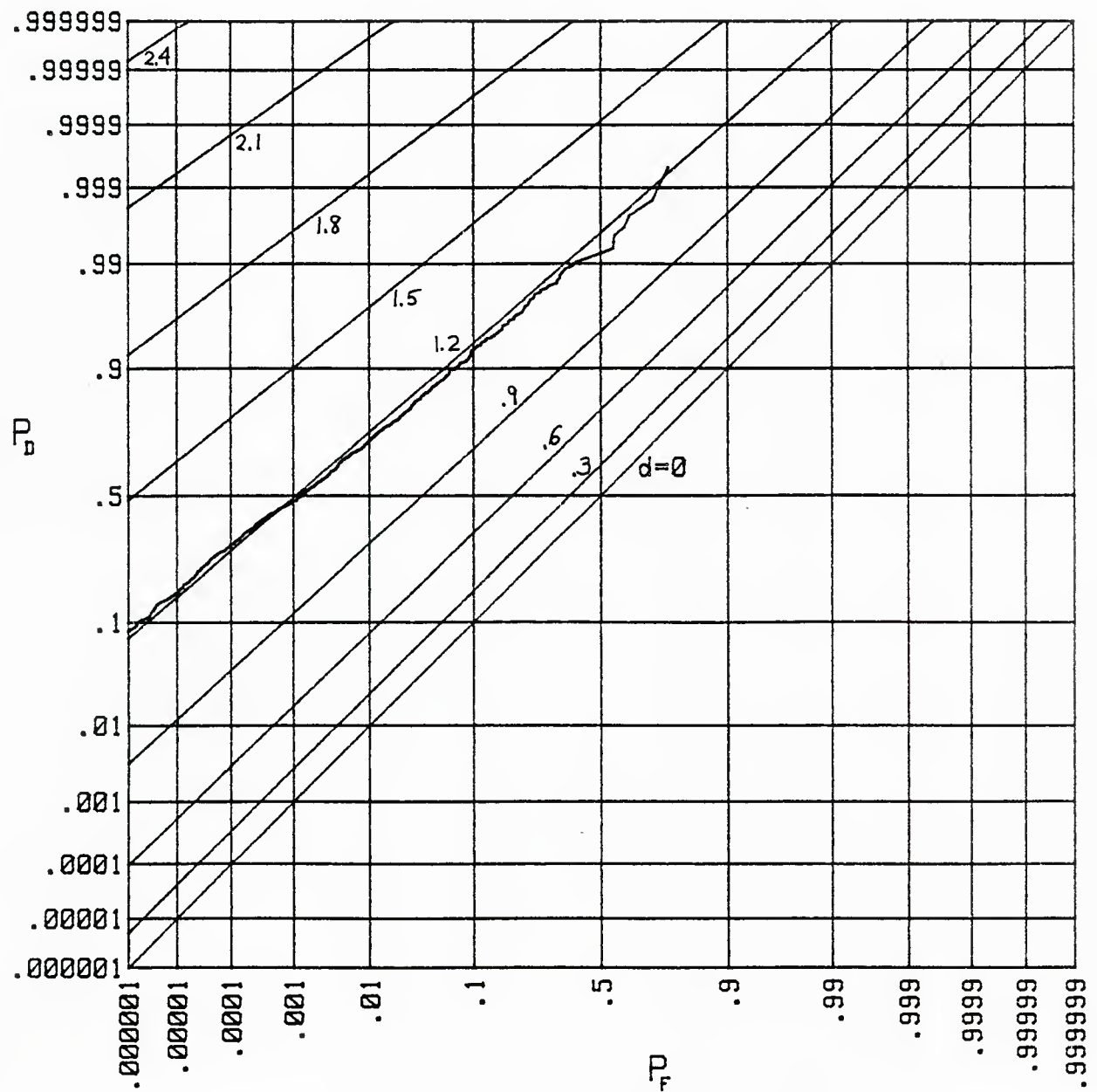


Figure 20. Operating Characteristics for N=8, K=1, M=32

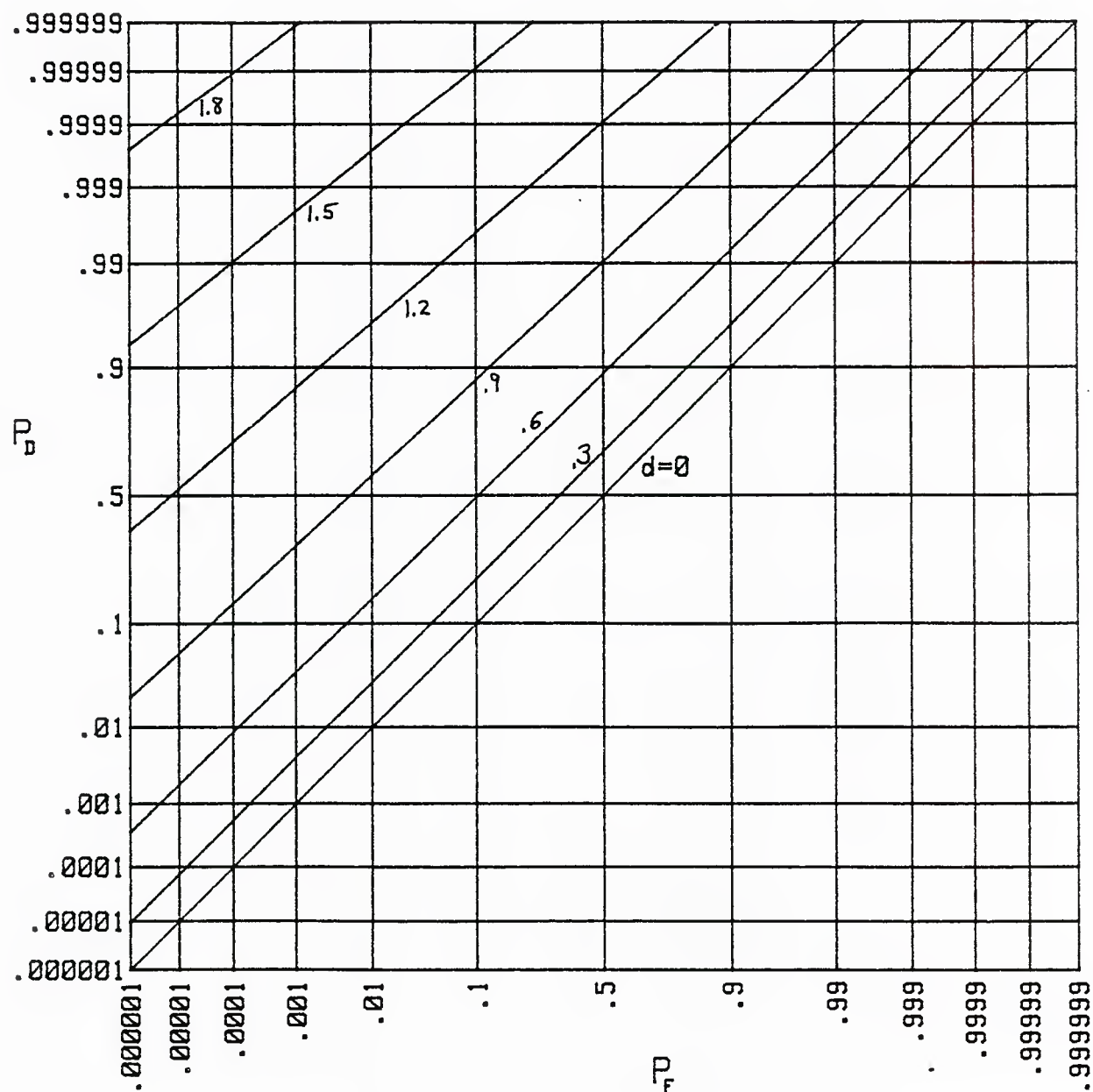


Figure 21. Operating Characteristics for $N=8$, $K=1$, $M=64$

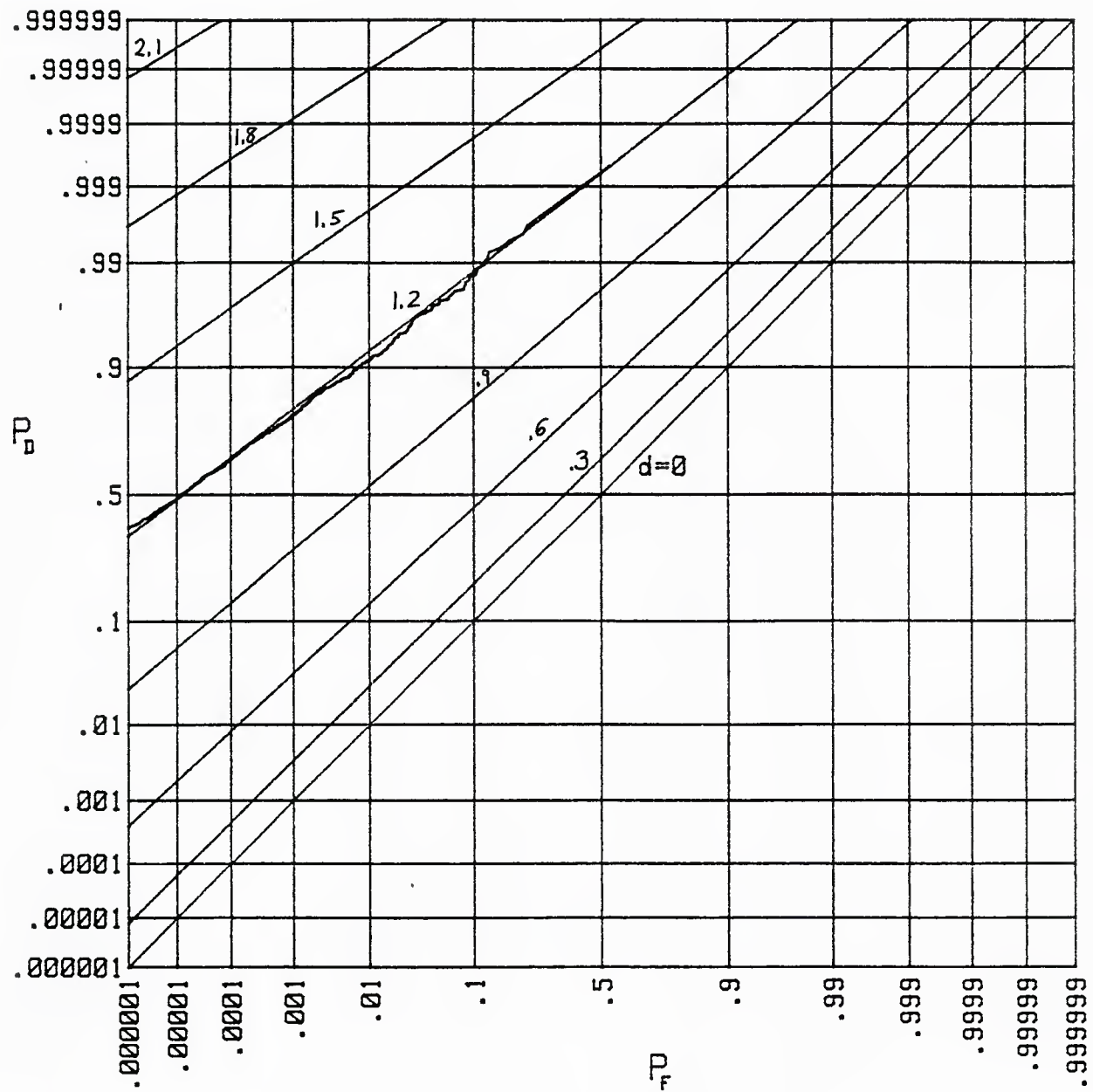


Figure 22. Operating Characteristics for $N=8$, $K=2$, $M=32$

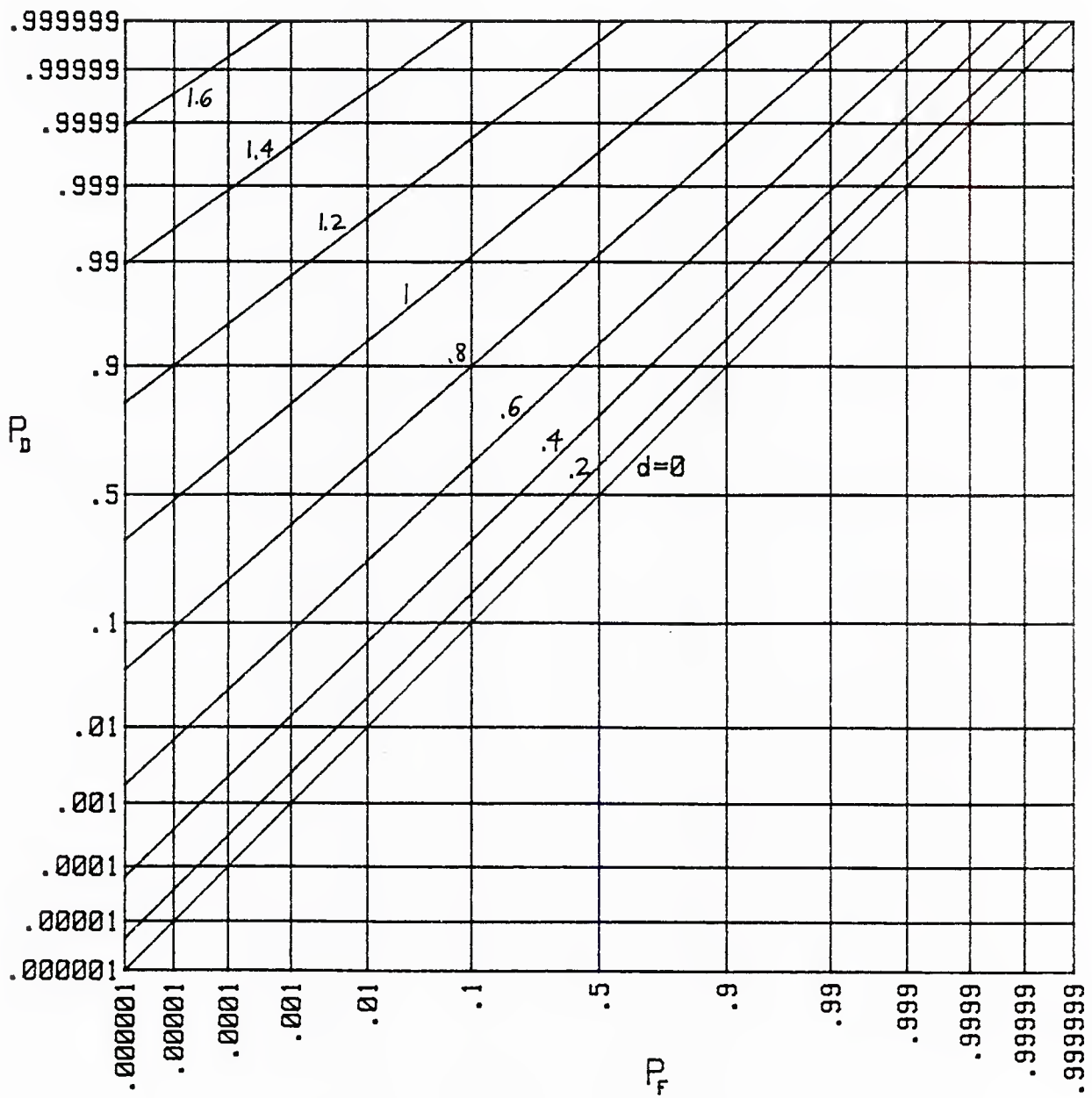


Figure 23. Operating Characteristics for $N=8$, $K=2$, $M=64$

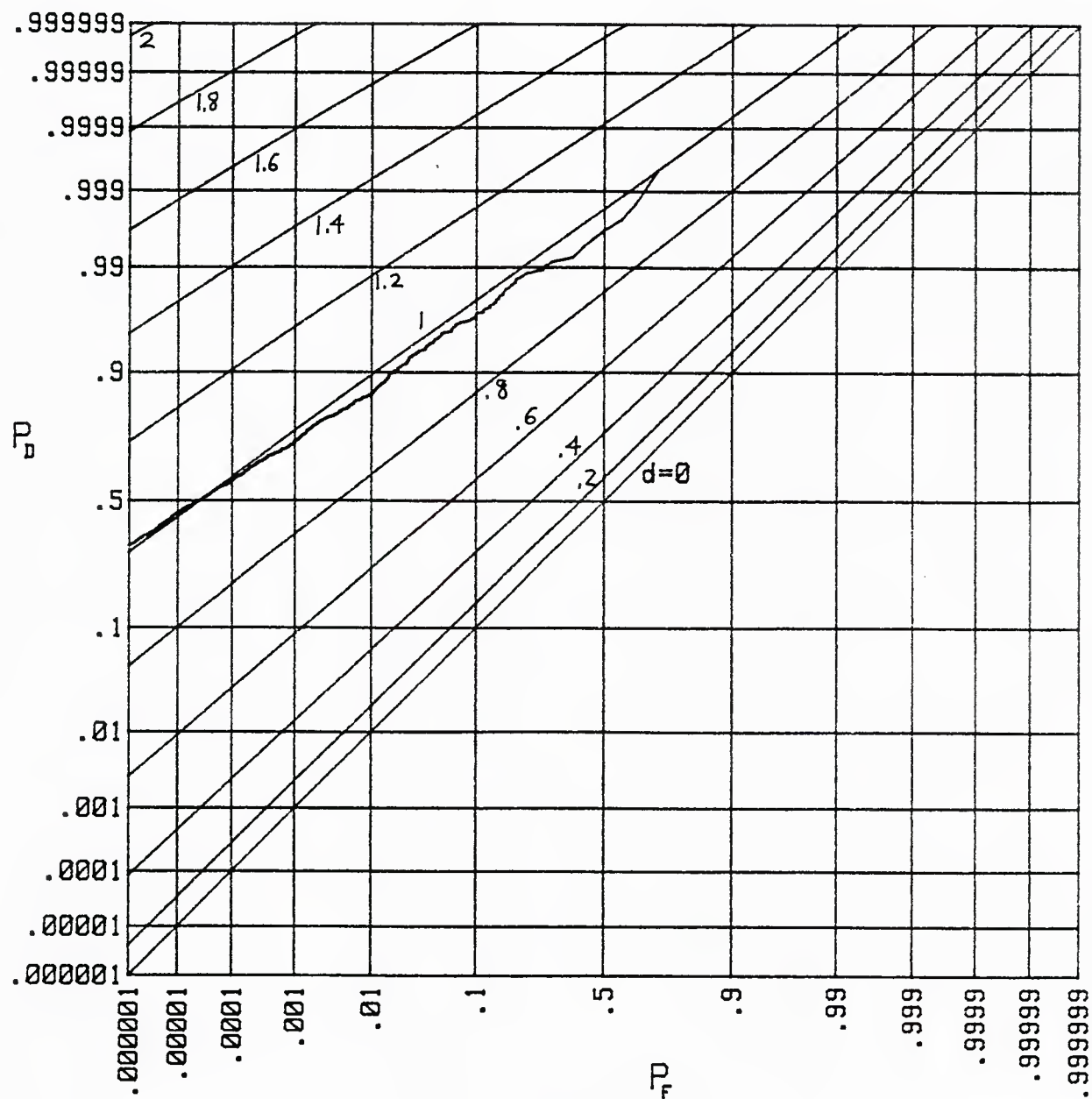


Figure 24. Operating Characteristics for $N=8$, $K=4$, $M=32$

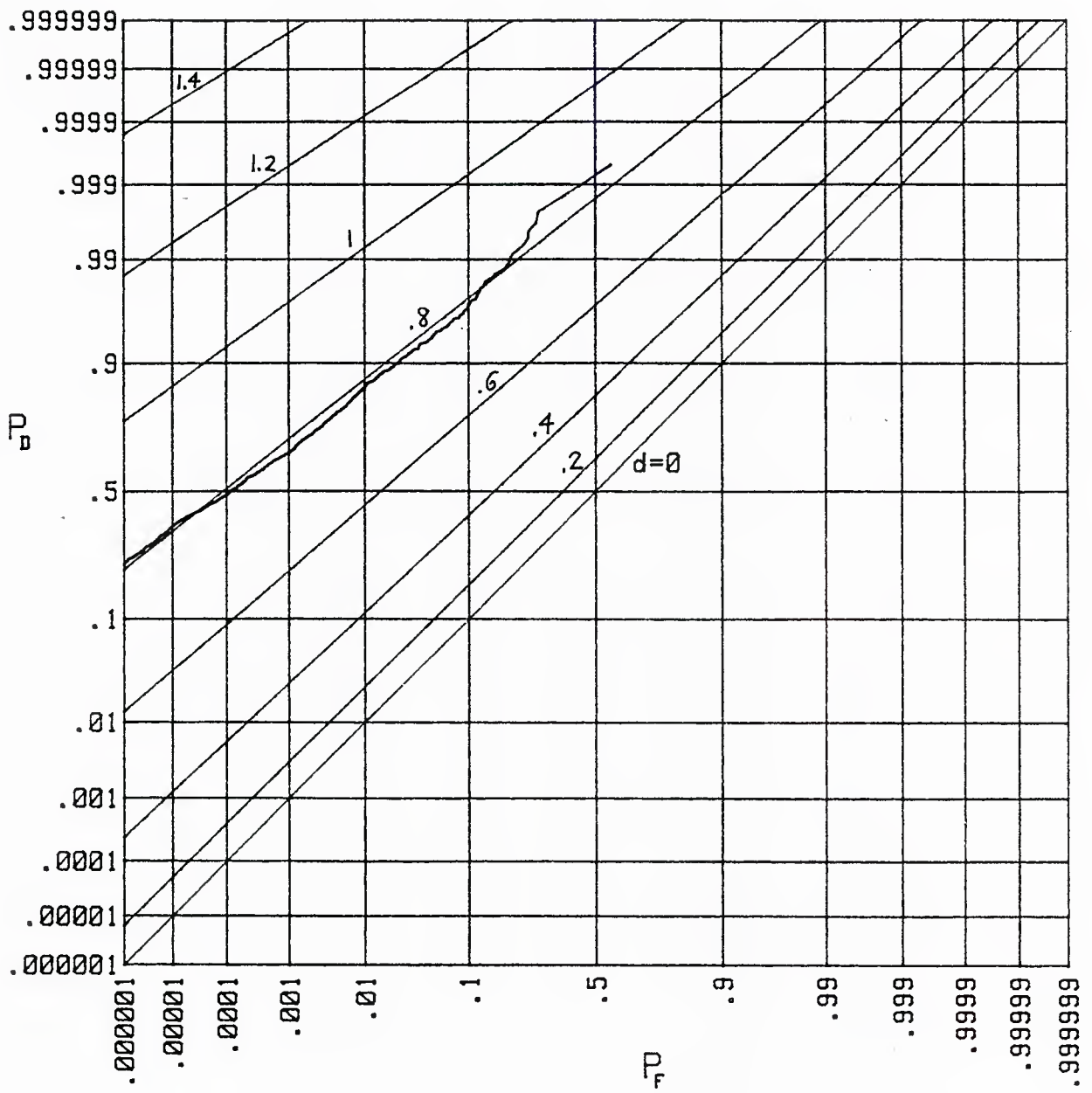


Figure 25. Operating Characteristics for $N=8$, $K=4$, $M=64$

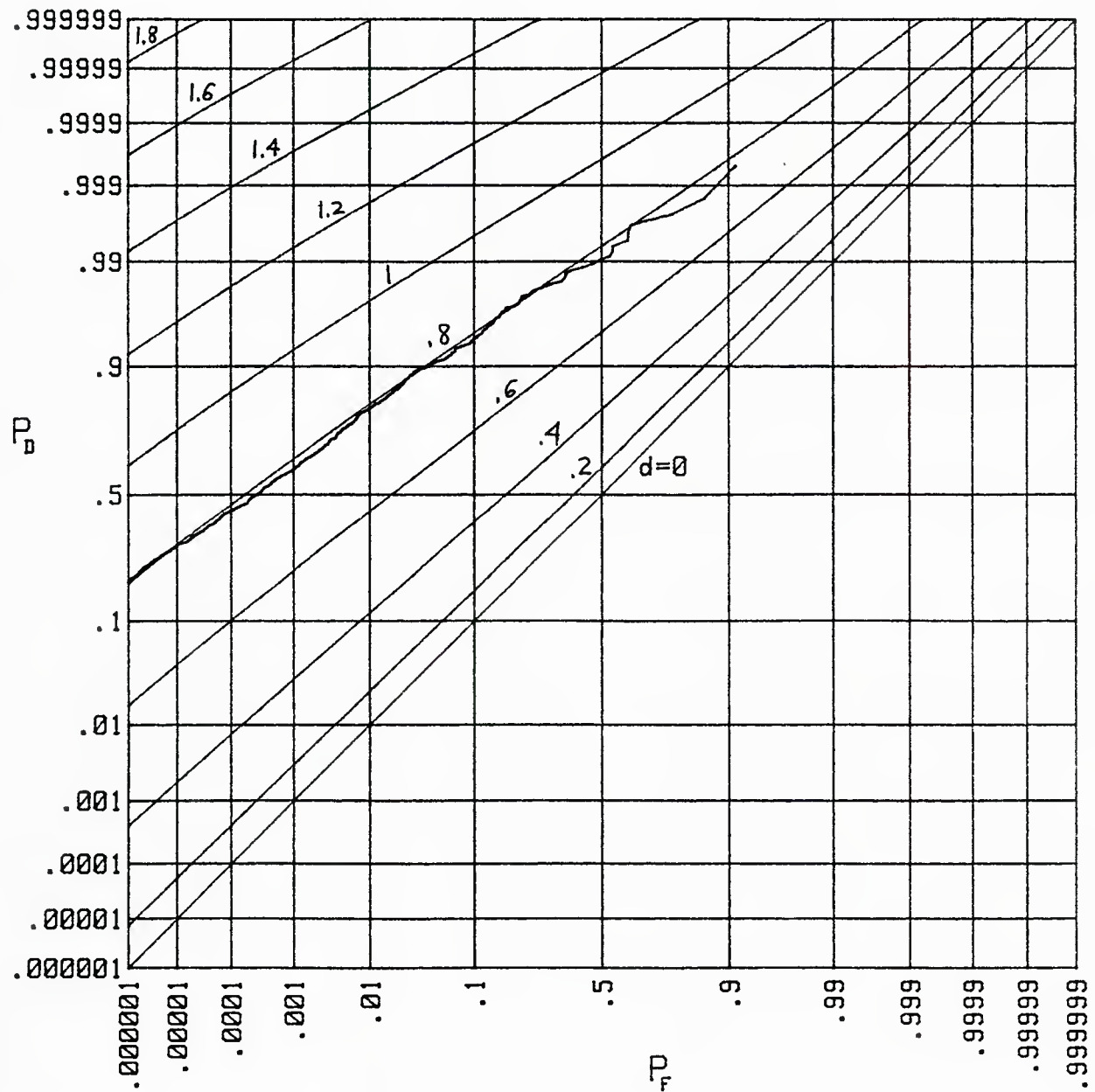


Figure 26. Operating Characteristics for $N=8$, $K=8$, $M=32$

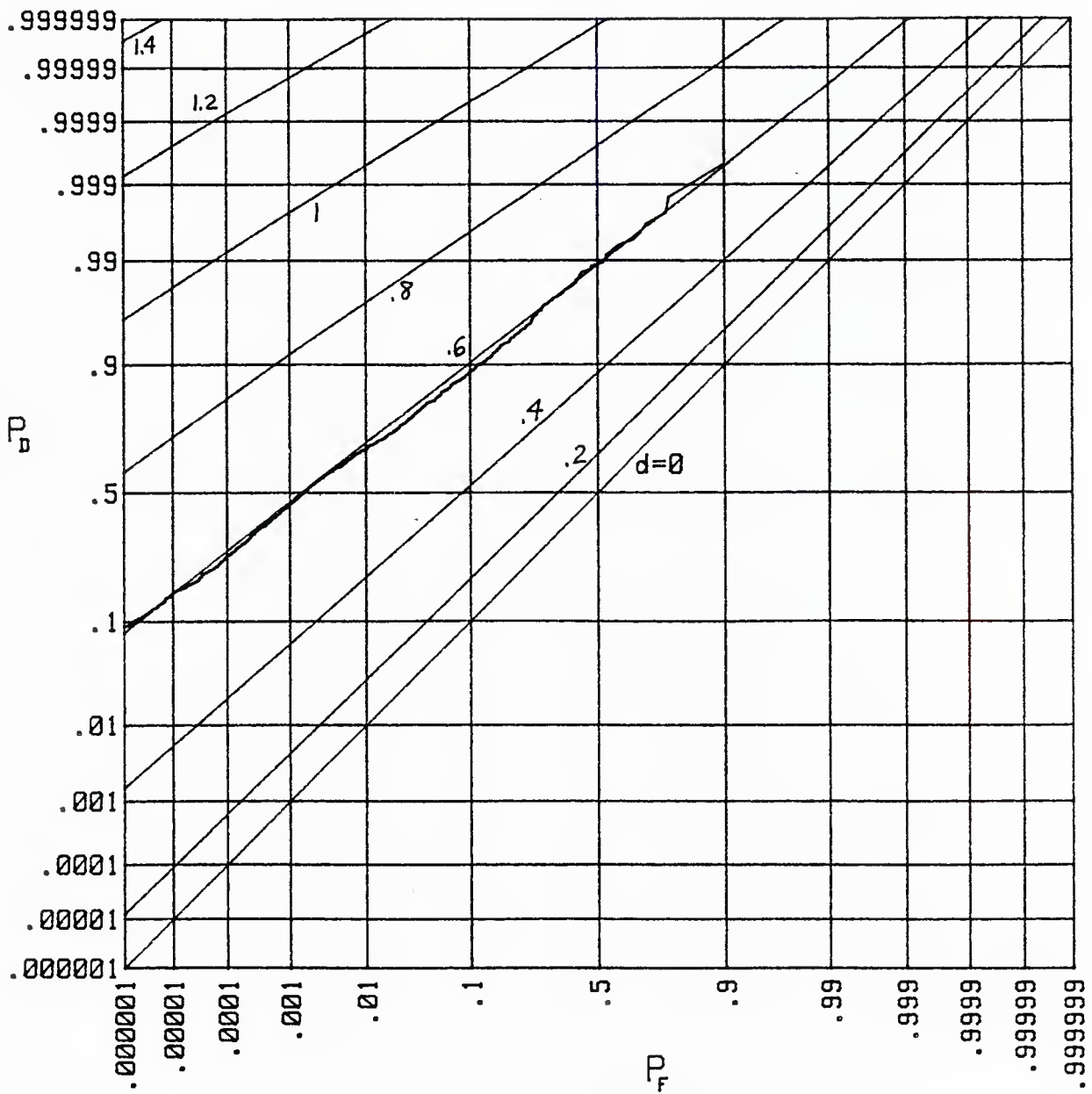


Figure 27. Operating Characteristics for $N=8$, $K=8$, $M=64$

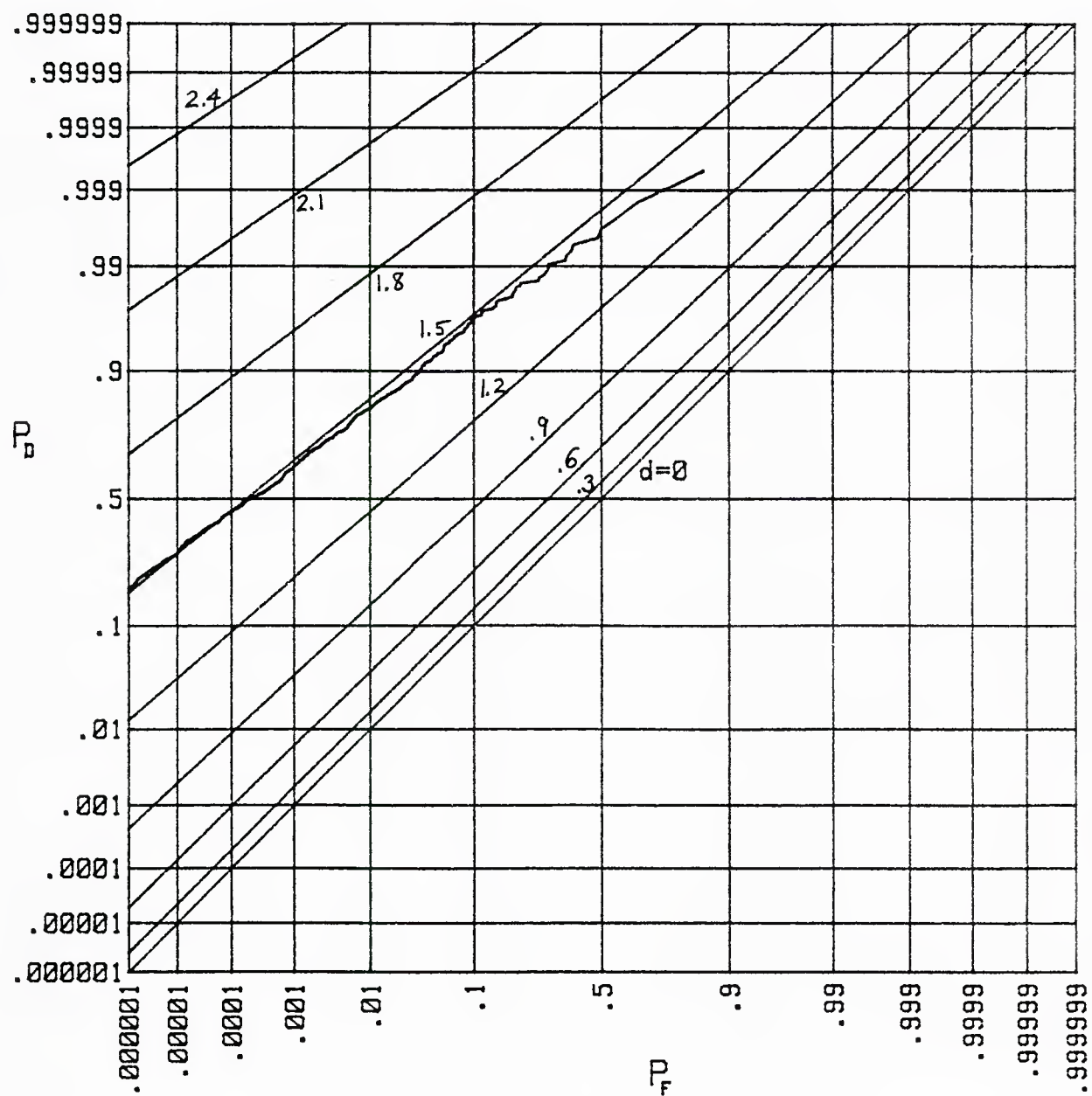


Figure 28. Operating Characteristics for $N=16$, $K=1$, $M=32$

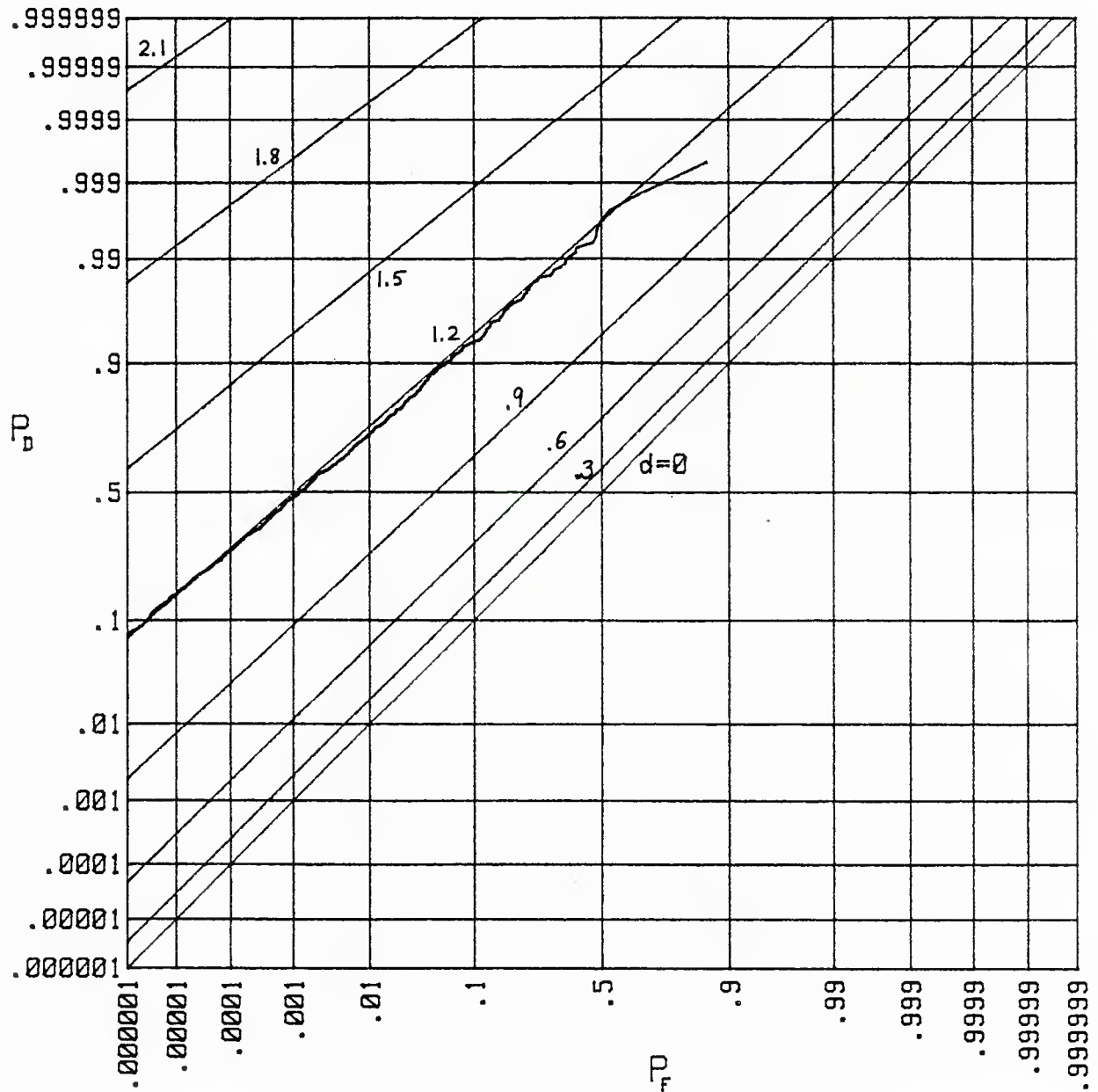


Figure 29. Operating Characteristics for $N=16$, $K=1$, $M=64$

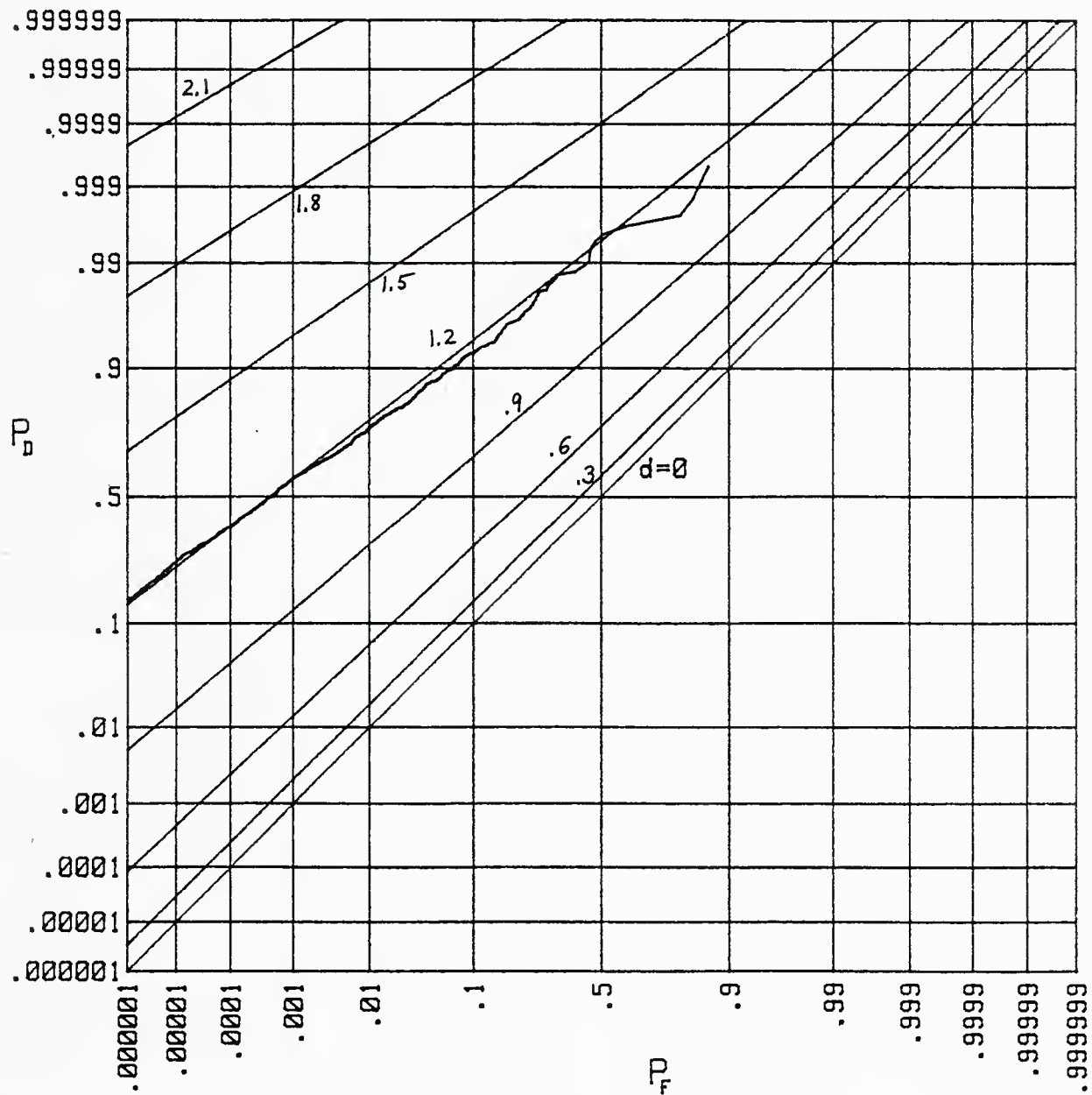


Figure 30. Operating Characteristics for $N=16$, $K=2$, $M=32$

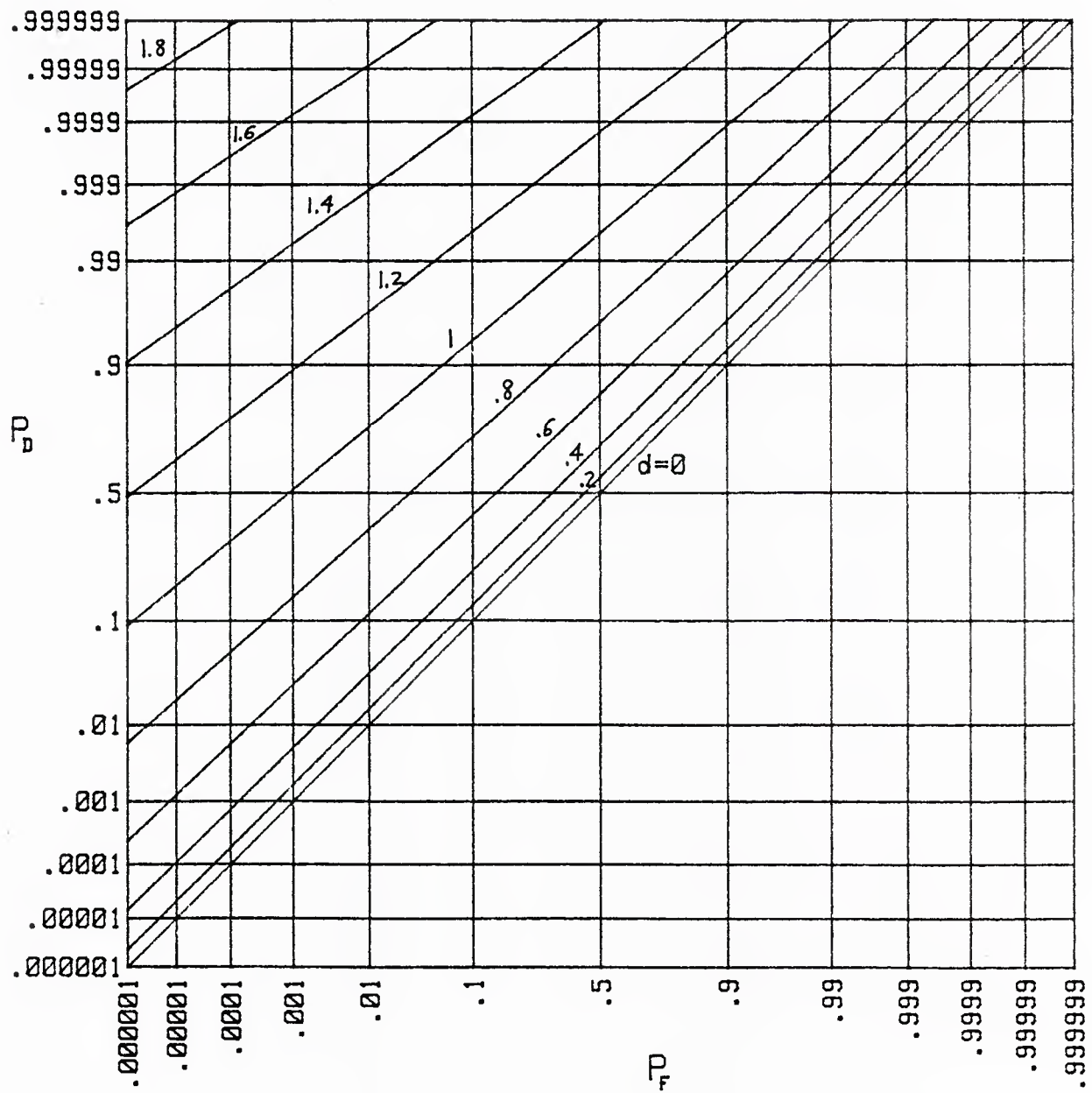


Figure 31. Operating Characteristics for $N=16$, $K=2$, $M=64$

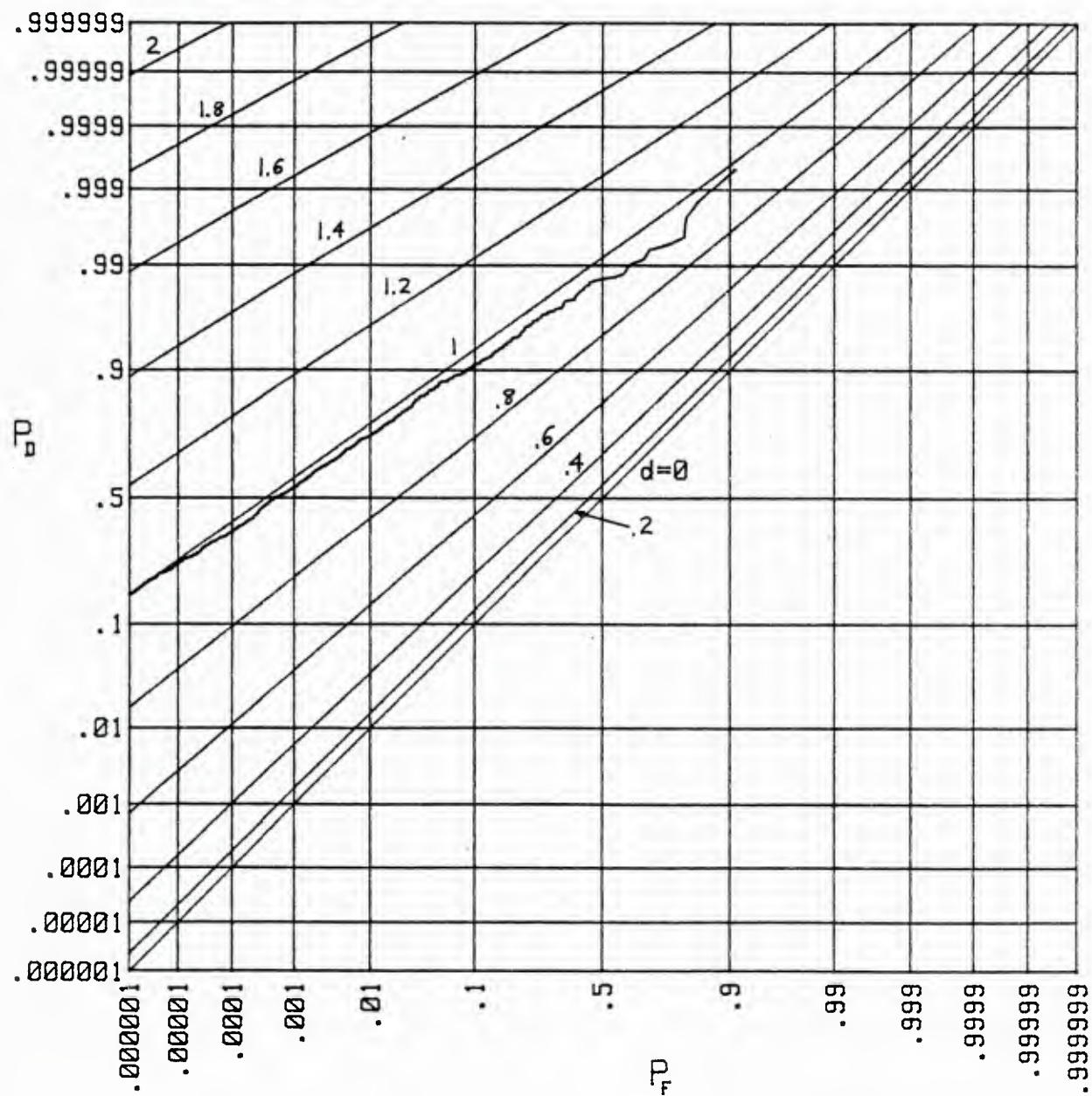


Figure 32. Operating Characteristics for $N=16$, $K=4$, $M=32$

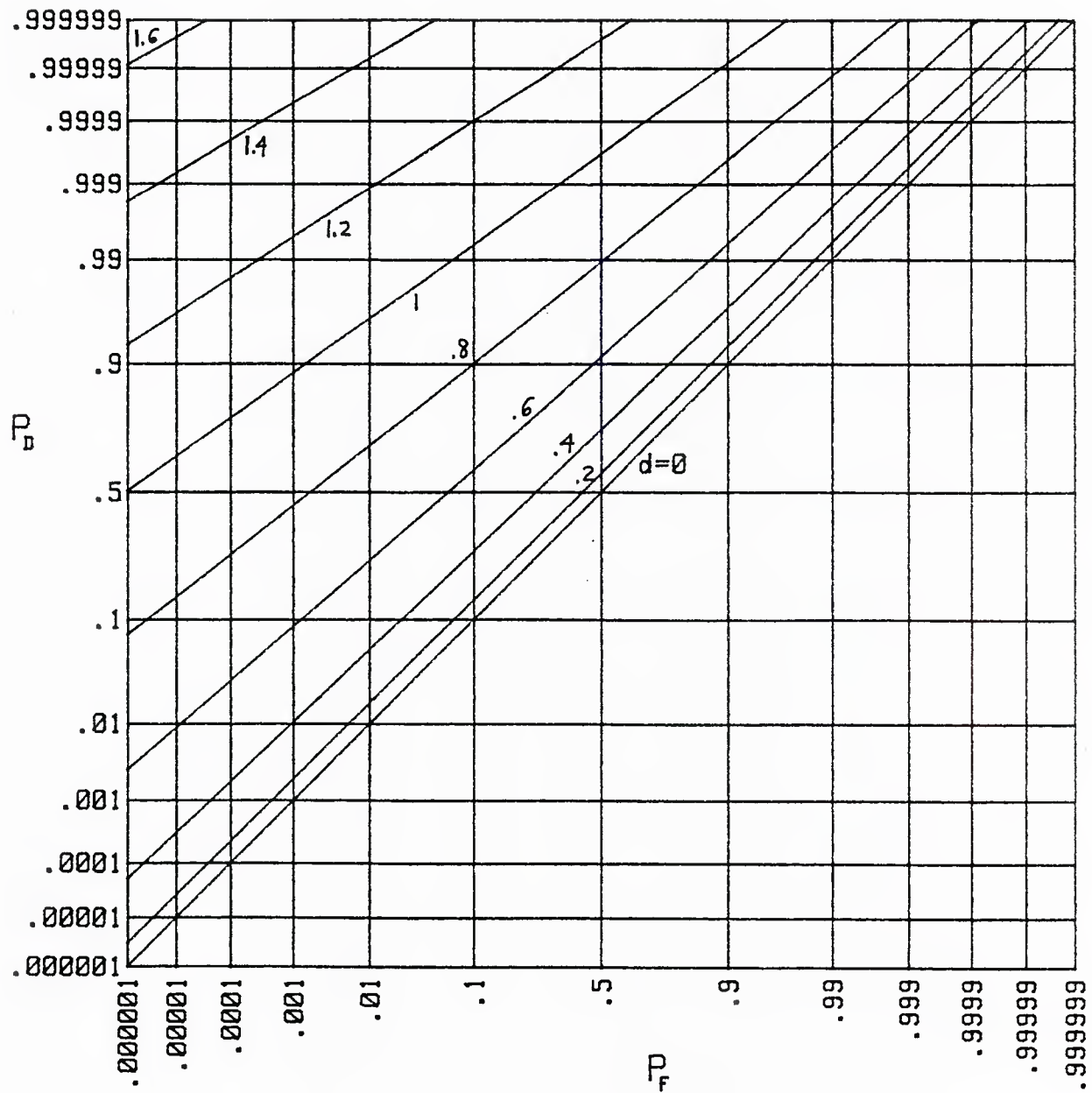


Figure 33. Operating Characteristics for $N=16$, $K=4$, $M=64$

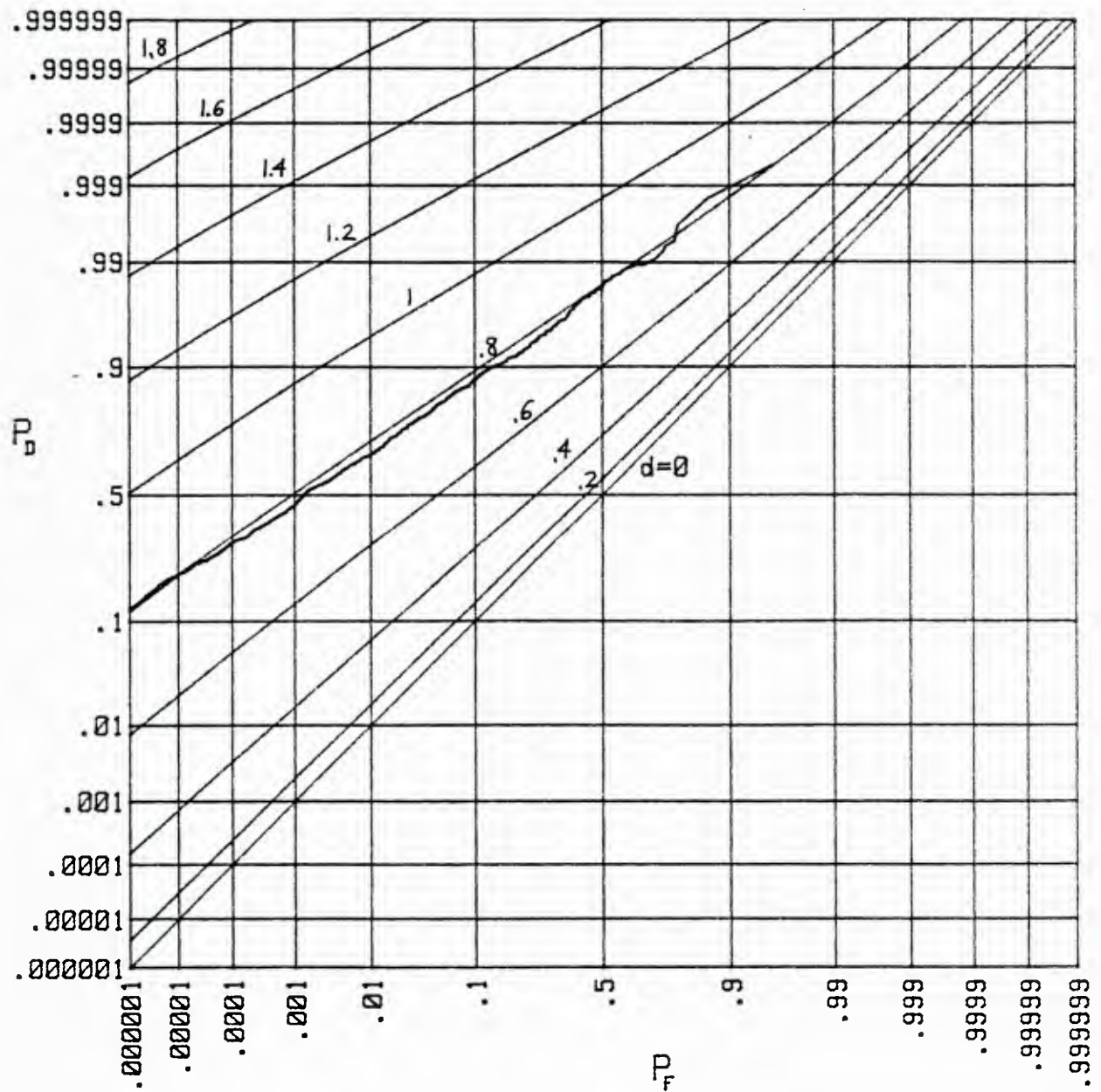


Figure 34. Operating Characteristics for $N=16$, $K=8$, $M=32$

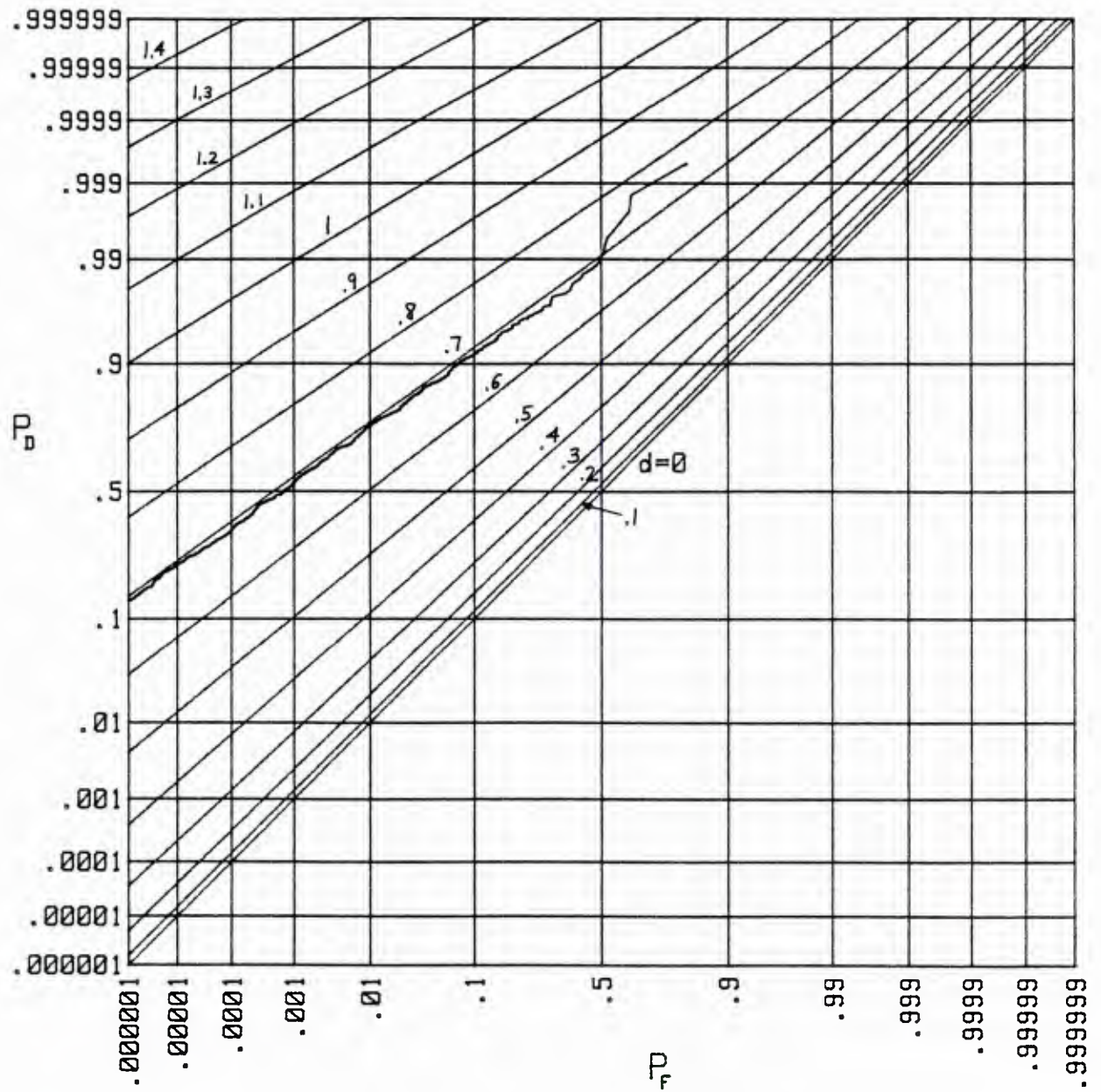


Figure 35. Operating Characteristics for $N=16$, $K=8$, $M=64$

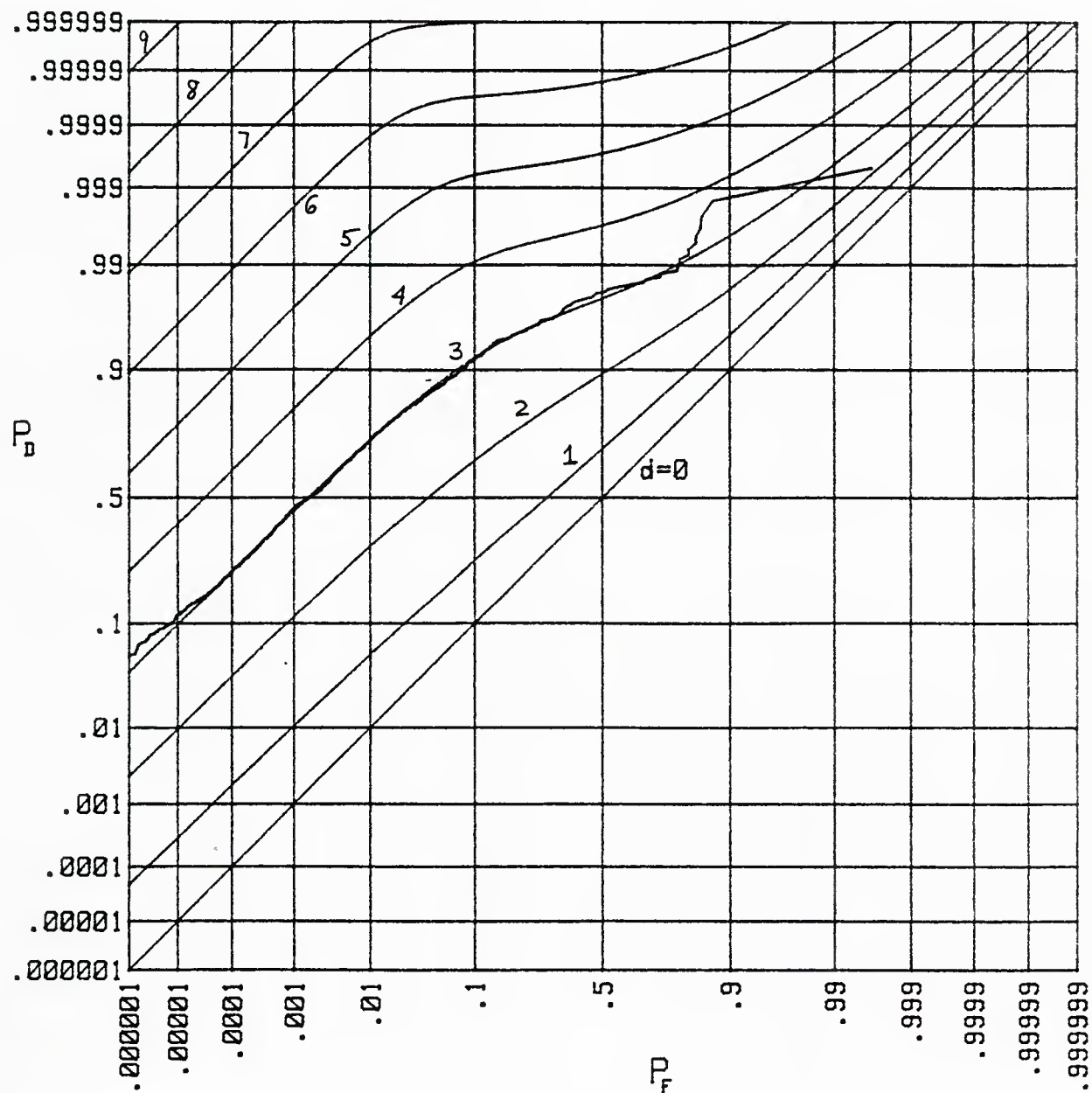


Figure 36. Operating Characteristics for $N=4$, $K=1$, $M=1$

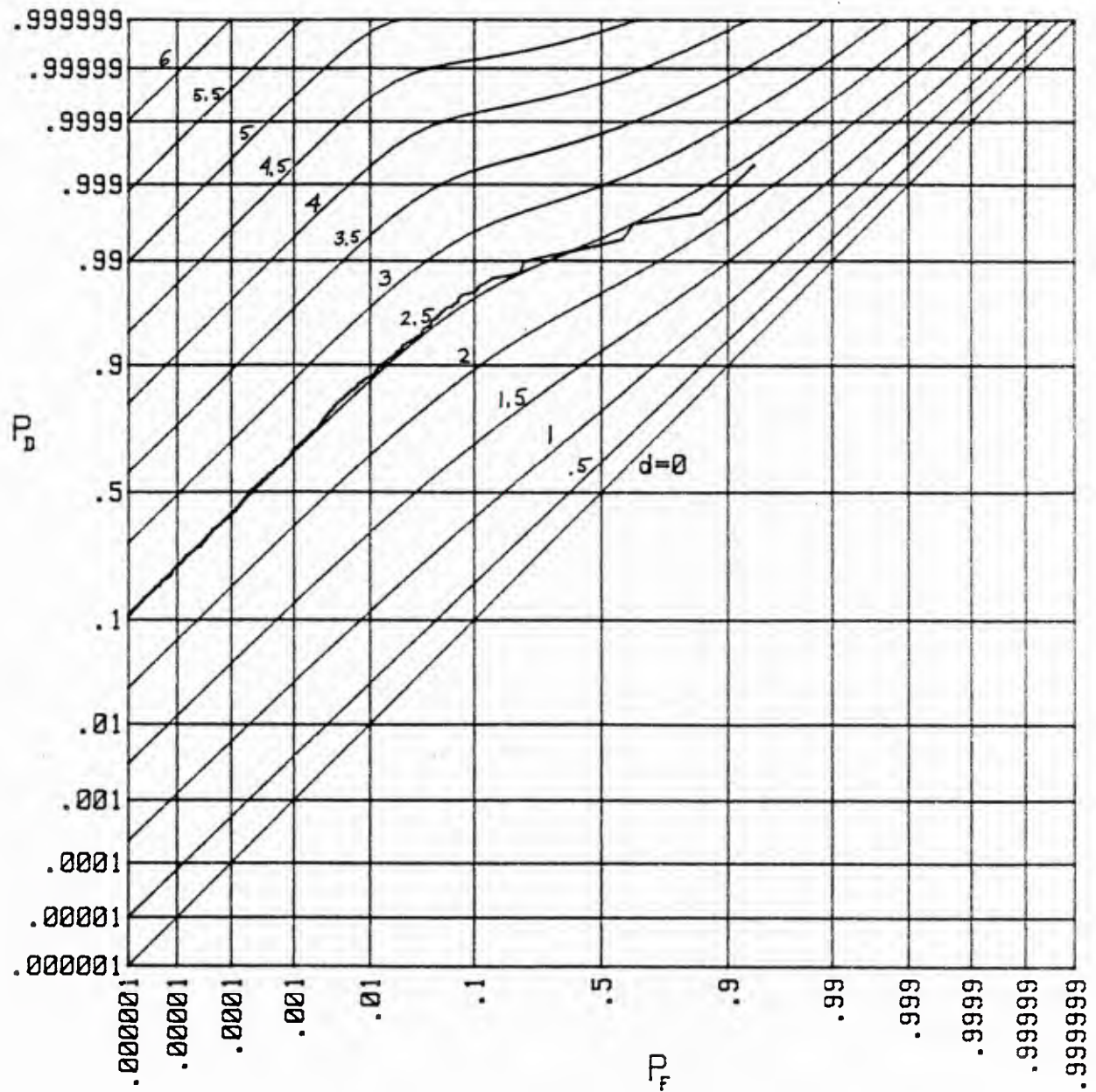


Figure 37. Operating Characteristics for $N=4$, $K=2$, $M=2$

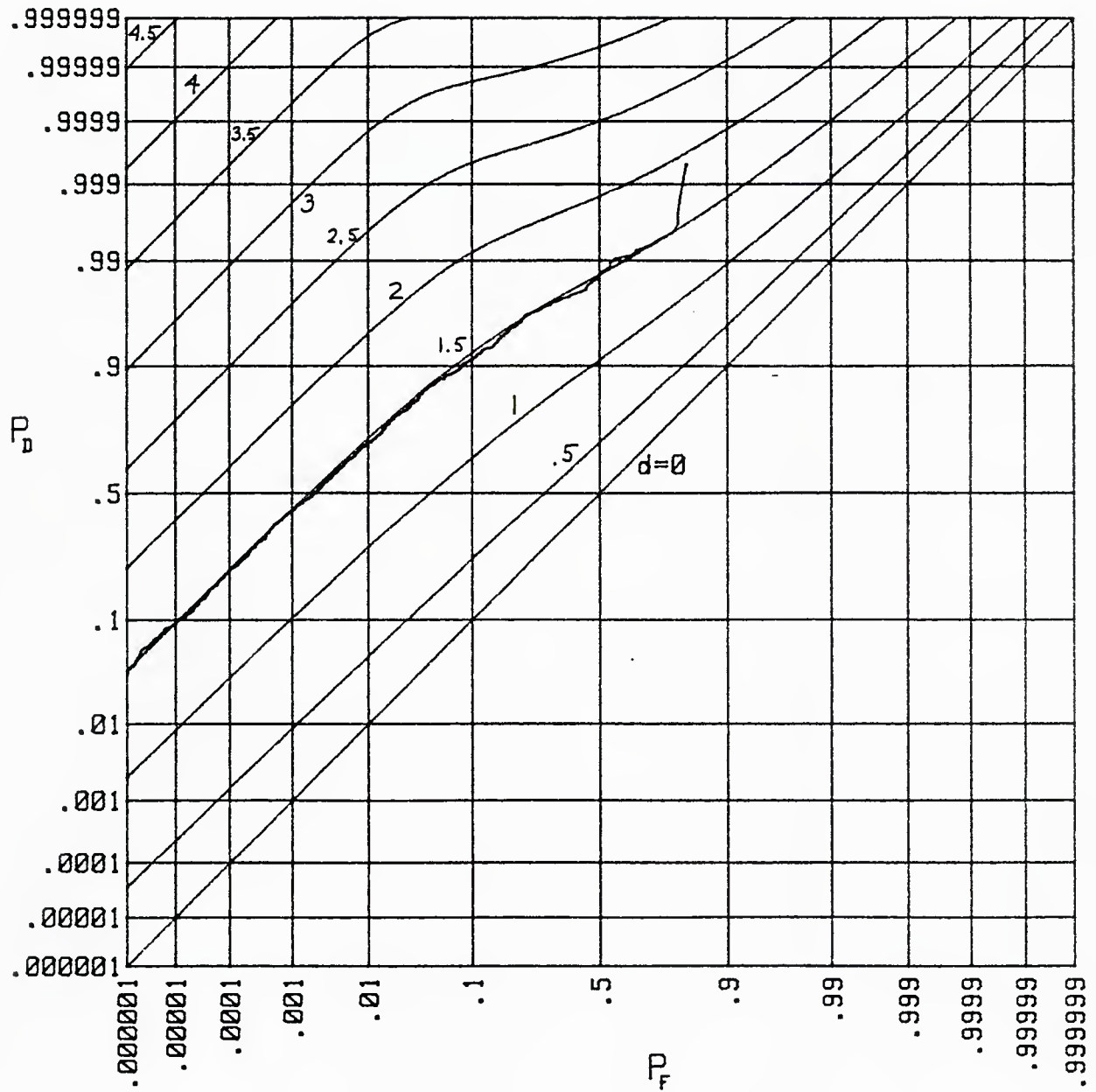
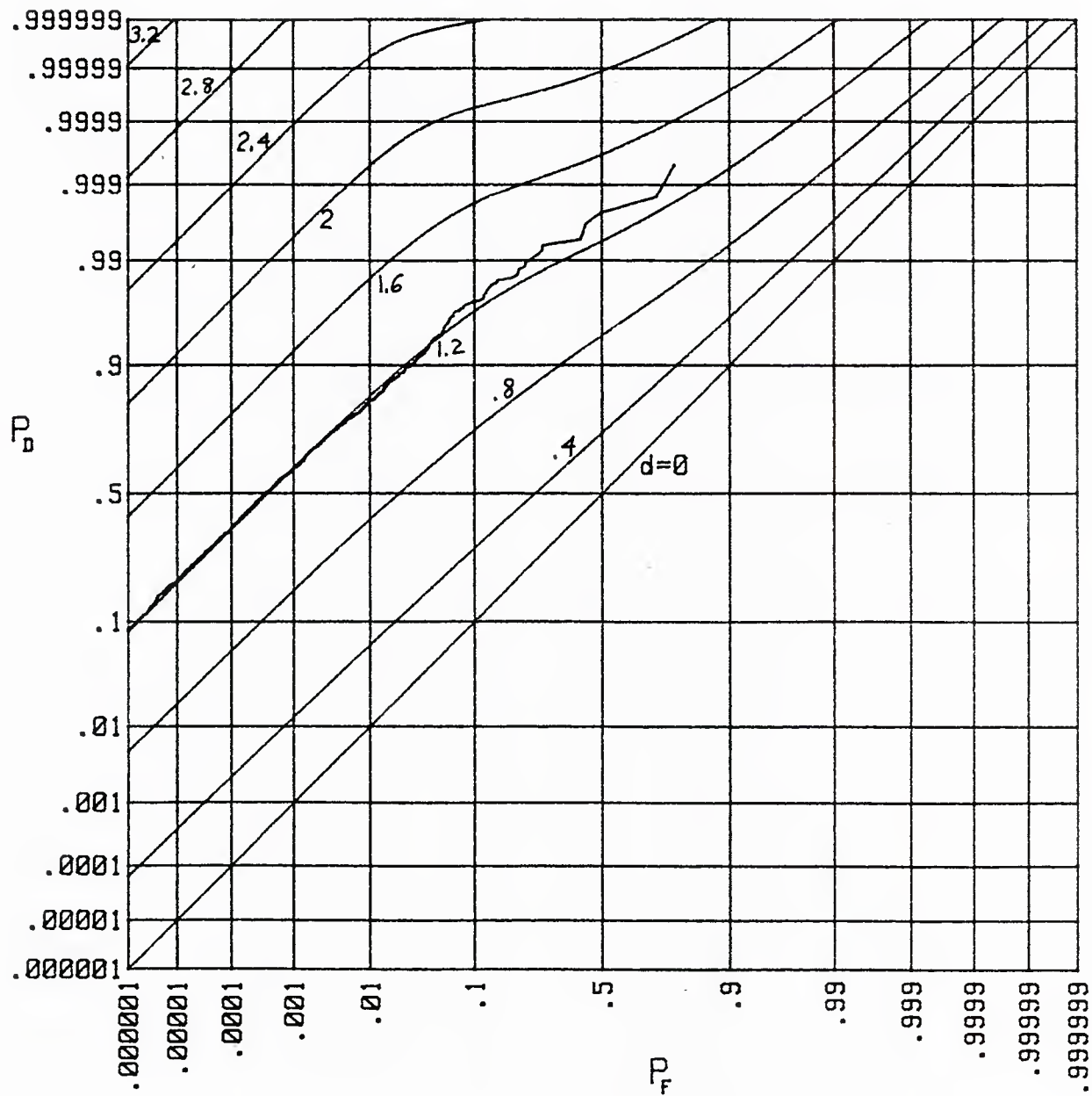


Figure 38. Operating Characteristics for $N=4$, $K=4$, $M=4$

Figure 39. Operating Characteristics for $N=4$, $K=8$, $M=8$

APPENDIX A. CHARACTERISTIC FUNCTION OF DECISION VARIABLE

With reference to figure 1, let $\underline{l} = (l_0, l_1, \dots, l_{M-1})$ be the time sequence of values of $l(t)$ at the output of the Greatest-Of device leading to a particular value of the decision variable $w(t)$ at some arbitrary time instant. Then the conditional characteristic function of $w(t)$ is, since the inputs $\{x_n\}$ to the system are all statistically independent in time,

$$f_w(\xi|\underline{l}) = \prod_{j=0}^{M-1} f_{l_j}(\xi) \quad , \quad (A-1)$$

where the individual characteristic functions in (A-1) are

$$f_{l_j}(\xi) = \left\{ \begin{array}{ll} f_x^{(1)}(\xi) & \text{if signal is present in channel no. } l_j \\ f_x^{(0)}(\xi) & \text{if signal is absent from channel no. } l_j \end{array} \right\} \quad . \quad (A-2)$$

Recall that $f_x^{(h)}(\xi)$ is the characteristic function of a single channel input under hypothesis h ; $h = 0$ or 1 .

H_0 : No signal present in any input channel

The time sequence of selector outputs $\{v(t)\}$ is one of statistically independent random variables, regardless of whether sequence $\{l(t)\}$ remains at the same value (channel number) for several successive time instants, as shown in figures 2 and 3. So the conditional characteristic function of $w(t)$ is, under H_0 ,

$$f_w^{(0)}(\xi|\underline{l}) = \left[f_x^{(0)}(\xi) \right]^M \quad , \quad (A-3)$$

independent of the particular sequence $\underline{l}$. Therefore the unconditional characteristic function of $w(t)$ is

$$f_w^{(0)}(\xi) = \left[f_x^{(0)}(\xi) \right]^M \quad . \quad (A-4)$$

Thus, the cumulative distribution function of decision variable $w(t)$ under H_0 can be easily evaluated [10,11] and the false alarm probability can be accurately calculated. The false alarm probability is independent of the number of input channels, N , and the amount of pre-averaging, K . The false alarm probability depends solely on the noise-only statistics at a single channel input, the amount of post-averaging, and the threshold. The threshold need not be increased as N is increased, in order to maintain a fixed false alarm probability. Arbitrary input statistics of $\{x_n(t)\}$ are allowed, except for the independent identically-distributed assumption.

H_1 : Signal present in one unknown input channel

The conditional characteristic function of $w(t)$, using the independence in time of $\{x_n(t)\}$, is now given by

$$f_w^{(1)}(\xi|\underline{l}) = \left[f_x^{(1)}(\xi) \right]^T \left[f_x^{(0)}(\xi) \right]^{M-T}, \quad (\text{A-5})$$

where T is the number of times that the values of the sequence $\underline{l}$ correspond with the correct (unknown) signal channel. Thus the unconditional characteristic function of the decision variable $w(t)$ is

$$\begin{aligned} f_w^{(1)}(\xi) &= \sum_{T=0}^M \text{Prob}(T) f_w^{(1)}(\xi|\underline{l}) \\ &= \sum_{T=0}^M \text{Prob}(T) \left[f_x^{(1)}(\xi) \right]^T \left[f_x^{(0)}(\xi) \right]^{M-T}, \end{aligned} \quad (\text{A-6})$$

where $\text{Prob}(T)$ is the probability of exactly T correspondences of $\underline{l}$ with the correct signal channel.

Unfortunately, evaluation of $\text{Prob}(T)$ is very difficult. For example, using the independence of the channel inputs, the simplest case of $T=M$ yields

$$\text{Prob}(T=M) = \int_{-\infty}^{+\infty} du_1 \dots \int_{-\infty}^{+\infty} du_M p_a^{(1)}(u_1, \dots, u_M) \\ * \left[\int_{-\infty}^{u_1} dv_1 \dots \int_{-\infty}^{u_M} dv_M p_a^{(0)}(v_1, \dots, v_M) \right]^{N-1}, \quad (\text{A-7})$$

where $p_a^{(h)}(\dots)$ is the joint probability density function of M adjacent time samples of one channel input to the Greatest-Of. However, probability density functions $p_a^{(0)}(v_1, \dots, v_M)$ and $p_a^{(1)}(u_1, \dots, u_M)$ cannot be factored, because neighboring time samples of $a_n(t)$, at a particular pre-averager output, are highly dependent. (If $\{a_n(t)\}$ samples were independent in time, as for $K=1$, this would simplify to

$$\text{Prob}(T=M) = \left\{ \int_{-\infty}^{+\infty} du p_a^{(1)}(u) \left[p_a^{(0)}(u) \right]^{N-1} \right\}^M. \quad (\text{A-8})$$

So we must resort to an approximate analysis under H_1 in order to determine $\text{Prob}(T)$.

Approximate Analysis under H_1

The plots in figures 2 and 3, of $\{a_n(t)\}$ vs. time sample t , reveal that a particular pre-averager output can dominate for many adjacent time samples (even if there is no signal present). Thus a typical plot of $\ell(t)$ appears as in figure A-1.

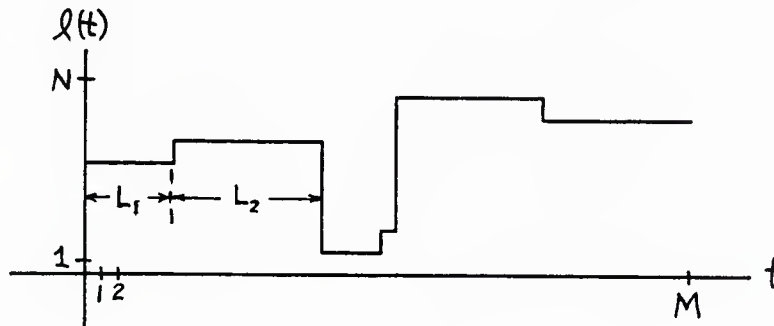


Figure A-1. Sample Plot of $\ell(t)$

The effective number of "independent samples", I_ℓ , of $\ell(t)$ in an interval of length M can be considerably less than M ; it depends critically on K , the amount of pre-averaging. And since $\ell(t)$ depends directly on the pre-averager outputs $\{a_n(t)\}_1^N$, it appears that the effective number of independent

samples, I_a , of one of the pre-averager outputs in an interval M should be approximately the same as I_x . This assumption precludes the dependence of I_x on N , the number of channels; how good this presumption is must be borne out by comparison of analytical results with simulation.

The effective number, for a time sequence $a_n(t)$ observed for an interval M , is given by [12, eq. B-12]

$$I_a = \frac{M}{\sum_{m=-M}^M \left(1 - \frac{|m|}{M}\right) \rho_a(m)}, \quad (A-9)$$

where $\rho_a(m)$ is the normalized covariance of stationary process $a_n(t)$; i.e., for any n ,

$$\rho_a(m) = \frac{E\{\tilde{a}_n(t) \tilde{a}_n(t-m)\}}{E\{\tilde{a}_n^2(t)\}}, \quad (A-10)$$

where $\tilde{a}_n(t)$ is the ac component of $a_n(t)$. But since $a_n(t)$ is the sum of K independent time samples of $x_n(t)$, there follows

$$\rho_a(m) = \begin{cases} 1 - \frac{|m|}{K} & \text{for } |m| < K \\ 0 & \text{otherwise} \end{cases}, \quad (A-11)$$

regardless of whether signal is present or not in the channel under consideration. Substitution of (A-11) in (A-9) yields

$$I_x \cong I_a = \begin{cases} \frac{3M^2}{3KM+1-K^2} & \text{for } K \leq M \\ \frac{3KM}{3KM+1-M^2} & \text{for } M \leq K \end{cases}. \quad (A-12)$$

A reasonable approximation to (A-12) is $I_x \cong M/K$ for $M \geq K$; however, we use the more accurate result (A-12) in the following.

Now let probability

$$Q_a \equiv \text{Prob} \{ \text{signal channel dominates Greatest-Of on one trial} \}. \quad (\text{A-13})$$

Using the statistical independence between input channels, this quantity is given by

$$Q_a = \int du p_a^{(1)}(u) \left[p_a^{(0)}(u) \right]^{N-1}; \quad (\text{A-14})$$

this is the special case of (A-7) or (A-8) for $T=1$. Then in an interval of length M , where there are I_x effectively-independent samples (as given by (A-12)), the probability that the signal channel dominates the Greatest-Of exactly n times is

$$\text{Prob}\{n; I_x\} = \binom{I_x}{n} Q_a^n (1-Q_a)^{I_x-n} \quad \text{for } 0 \leq n \leq I_x. \quad (\text{A-15})$$

We are presuming I_x to be an integer for now; we shall alleviate this restriction shortly.

Now define

$$L = \frac{M}{I_x}. \quad (\text{A-16})$$

This is the average number of time samples, in interval length M , for which a particular channel dominates the Greatest-Of. In terms of figure A-1, L is the average of lengths $L_1, L_2, \dots$. We then approximate the actual $\ell(t)$ plot in figure A-1 by that in figure A-2; there are I_x independent steps in interval M , each of length L .

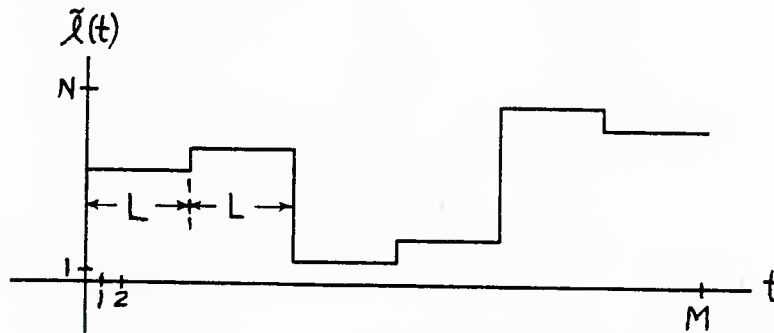


Figure A-2. Approximation $\tilde{\ell}(t)$ to $\ell(t)$

Under this approximation, the random variable T defined in (A-5)-(A-6) can only take on the values

$$T = 0 \text{ or } L \text{ or } 2L \text{ or } \dots \text{ or } M, \quad (\text{A-17})$$

that is

$$T = nL \quad \text{for} \quad 0 \leq n \leq I_L. \quad (\text{A-18})$$

Therefore the conditional characteristic function (A-5) becomes

$$f_w^{(1)}(\xi|L) = [f_x^{(1)}(\xi)]^{nL} [f_x^{(0)}(\xi)]^{M-nL} \quad \text{for} \quad 0 \leq n \leq I_L. \quad (\text{A-19})$$

Employment of (A-15) now yields the unconditional characteristic function of decision variable $w(t)$ under H_1 as

$$\begin{aligned} f_w^{(1)}(\xi) &= \sum_{n=0}^{I_L} \text{Prob}\{n; I_L\} f_w^{(1)}(\xi|L) \\ &= \sum_{n=0}^{I_L} \binom{I_L}{n} Q_a^n (1-Q_a)^{I_L-n} [f_x^{(1)}(\xi)]^{nL} [f_x^{(0)}(\xi)]^{M-nL} \\ &= \left\{ Q_a [f_x^{(1)}(\xi)]^L + (1-Q_a) [f_x^{(0)}(\xi)]^L \right\}^{I_L}. \end{aligned} \quad (\text{A-20})$$

This is the main result of this appendix.

To review: the characteristic functions $f_x^{(h)}(\xi)$ are those for the system inputs $\{x_n(t)\}$ under hypotheses $h = 0$ and 1 ; I_L is given by (A-12) for various amounts of pre-averaging K and post-averaging M ; L is given by (A-16); and Q_a is given by (A-14), where $p_a^{(h)}(u)$ is the first-order probability density function of a pre-averager output under hypothesis h . Since $a_n(t)$ is the sum of K independent random variables, we have characteristic functions

$$f_a^{(h)}(\xi) = \left[f_x^{(h)}(\xi) \right]^K \quad \text{for } h = 0 \text{ and } 1, \quad (\text{A-21})$$

which can be used to determine probability density functions $p_a^{(h)}(u)$. The fundamental inputs required are N , K , M , $f_x^{(1)}(\xi)$, and $f_x^{(0)}(\xi)$.

For Q_a near 1, it is necessary to numerically evaluate $1-Q_a$ separately from (A-14); via physical reasoning or an integration by parts on (A-14), there follows

$$1-Q_a = (N-1) \int du p_a^{(1)}(u) p_a^{(0)}(u) \left[p_a^{(0)}(u) \right]^{N-2}. \quad (\text{A-22})$$

We employ both (A-14) and (A-22) in the program that follows.

Special Cases of Characteristic Function $f_w^{(1)}(\xi)$

By taking the derivative of (A-20) with respect to ξ and then setting $\xi=0$, we find the mean value of $w(t)$ under H_1 as

$$\lambda_w^{(1)}(1) = M \left[Q_a \lambda_x^{(1)}(1) + (1-Q_a) \lambda_x^{(0)}(1) \right], \quad (\text{A-23})$$

where $\lambda_r^{(h)}(n)$ is the n -th cumulant of random variable r under hypothesis h . Although (A-23) is deduced from the approximate characteristic function $f_w^{(1)}(\xi)$ given in (A-20), this result is in fact exact, as may be easily verified by an independent analysis.

From (A-20), we also can extract the variance of decision variable $w(t)$; there follows

$$\lambda_w^{(1)}(2) = M \left[Q_a \lambda_x^{(1)}(2) + (1-Q_a) \lambda_x^{(0)}(2) + L Q_a (1-Q_a) \left\{ \lambda_x^{(1)}(1) - \lambda_x^{(0)}(1) \right\}^2 \right]. \quad (\text{A-24})$$

Since the exact calculation of the variance of $w(t)$ is very difficult, we have no check on this result.

As the input signal strength approaches zero, $f_x^{(1)}(\xi) \rightarrow f_x^{(0)}(\xi)$, and

(A-20) yields $f_w^{(1)}(\xi) \rightarrow [f_x^{(0)}(\xi)]^M$, which checks the H_0 result in (A-4).

The following are a series of checks, that are obviously correct. If $N=1$, then $Q_a=1$ from (A-14), and*

$$f_w^{(1)}(\xi) = [f_x^{(1)}(\xi)]^M \quad . \quad (A-25)$$

On the other hand, if $M=1$, then $I_d=1$ from (A-21), and

$$f_w^{(1)}(\xi) = Q_a f_x^{(1)}(\xi) + (1-Q_a) f_x^{(0)}(\xi) \quad ; \quad (A-26)$$

then probability density function

$$p_w^{(1)}(u) = Q_a p_x^{(1)}(u) + (1-Q_a) p_x^{(0)}(u) \quad (A-27)$$

giving detection probability

$$\begin{aligned} P_D &= \text{Prob} \{w > \Lambda \mid H_1\} = \int_{\Lambda}^{\infty} du p_w^{(1)}(u) \\ &= Q_a [1 - p_x^{(1)}(\Lambda)] + (1-Q_a) [1 - p_x^{(0)}(\Lambda)] \\ &= 1 - Q_a p_x^{(1)}(\Lambda) - (1-Q_a) p_x^{(0)}(\Lambda) \quad , \end{aligned} \quad (A-28)$$

where Λ is the value of the threshold in figure 1. Finally, if $K=1$, then $I_d=M$ from (A-12), and

$$f_w^{(1)}(\xi) = [Q_a f_x^{(1)}(\xi) + (1-Q_a) f_x^{(0)}(\xi)]^M \quad ; \quad (A-29)$$

this result and Q_a in (A-14) are exact in this case, as noted in (A-8).

* For the Gaussian example in (C-1), we have the exact result for $N=1$,

$$\text{Prob} \{w > \Lambda \mid H_h\} = \Phi \left(\frac{M \mu_X^{(h)} - \Lambda}{\sqrt{M} \sigma_X} \right) \quad .$$

APPENDIX B. NON-INTEGGER POWER OF A CHARACTERISTIC FUNCTION

The quantities L and I_k in (A-20) will generally not be integer. So we may encounter the problem that even if $f(\xi)$ is a legal characteristic function, $[f(\xi)]^\nu$ may not be, for non-integer ν . (If it is a legal characteristic function, we ignore the following procedure.) We circumvent the illegal characteristic function problem by approximating

$$[f(\xi)]^\nu \quad \text{by legal characteristic function } [f(\alpha\xi)]^I f(\beta\xi) \quad , \quad (B-1)$$

where I is the integer part of ν , and α, β are determined by equating the first two cumulants of these two expressions in (B-1). There follows

$$\nu = I\alpha + \beta, \quad \nu = I\alpha^2 + \beta^2 \quad . \quad (B-2)$$

The solution of these two equations is

$$\alpha = \frac{\nu I + (\nu I(1+I-\nu))^{1/2}}{I(I+1)} \quad , \quad \beta = \nu - I\alpha \quad . \quad (B-3)$$

Observe that I, α, β are functions solely of ν , and are independent of the statistics $f(\xi)$. We find generally that $\alpha \geq 1$ and $0 \leq \beta < 1$; observe in figure B-1 that α approaches 1 as ν increases.

For ν an integer, there follows $I = \nu, \alpha = 1, \beta = 0$, as expected. Also, for a Gaussian random variable, replacement (B-1) is in fact exact, as both functions yield

$$\exp(i\xi\mu\nu - \frac{1}{2}\xi^2\sigma^2\nu) \quad \text{for any } \nu \quad , \quad (B-4)$$

where μ and σ^2 are the mean and variance of the Gaussian random variable.

We now apply this procedure to (A-20). Define, as in (B-3),

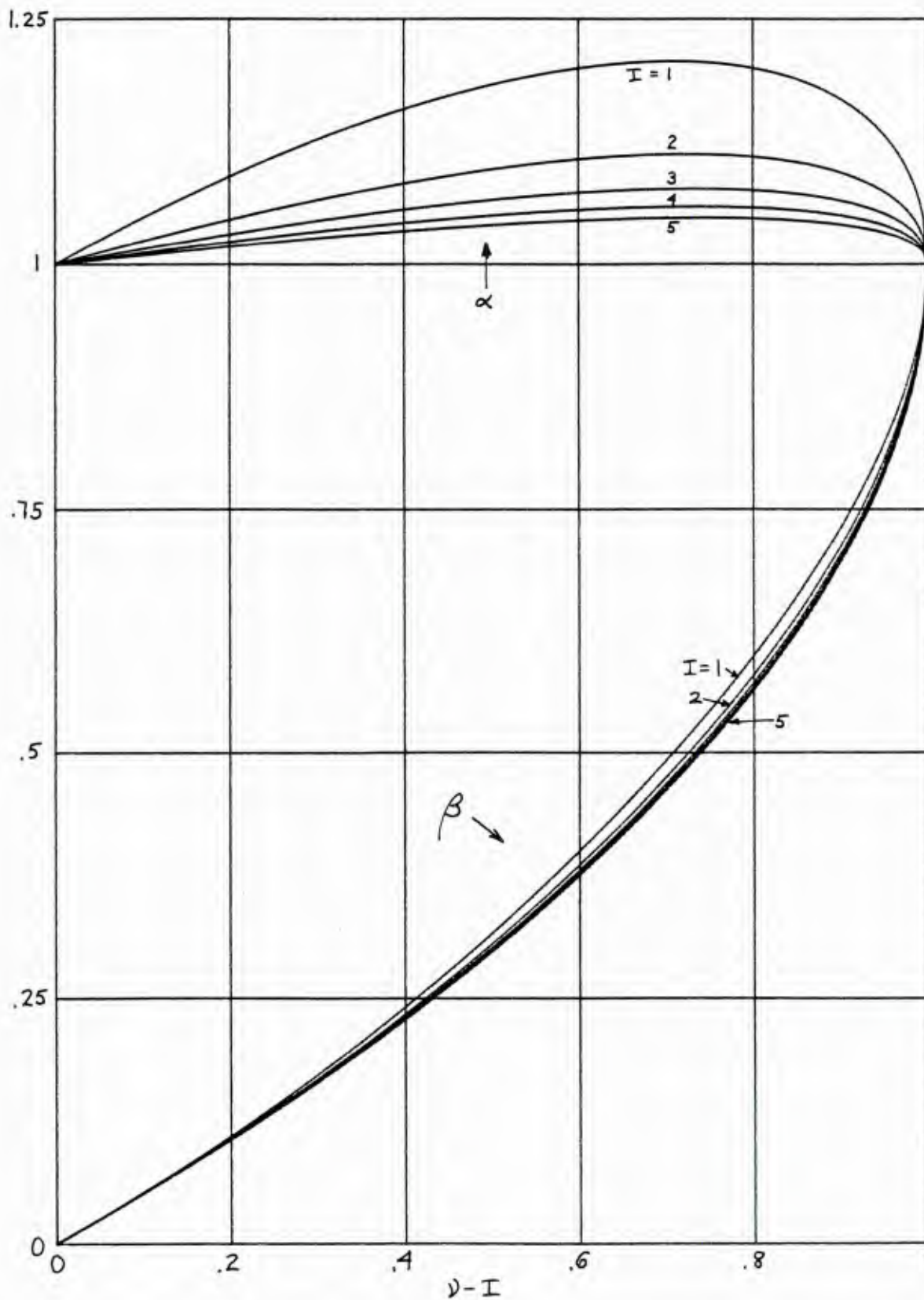


Figure B-1. Parameters of Approximation (B-1)

$I_1 = \text{INT}(L)$ with attendant α_1, β_1 ;

$I_2 = \text{INT}(I_1)$ with attendant α_2, β_2 . (B-5)

Then the interior terms of (A-20) are replaced by

$$\left[f_x^{(h)}(\xi) \right]^L \cong \left[f_x^{(h)}(\alpha_1 \xi) \right]^{I_1} f_x^{(h)}(\beta_1 \xi) \cong f_1^{(h)}(\xi) \quad . \quad (\text{B-6})$$

The bracketed quantity in (A-20) then becomes

$$Q_a f_1^{(1)}(\xi) + (1-Q_a) f_1^{(0)}(\xi) \cong f_1(\xi) \quad . \quad (\text{B-7})$$

Finally, (A-20) becomes

$$f_w^{(1)}(\xi) = \left[f_1(\xi) \right]^{I_2} = \left[f_1(\alpha_2 \xi) \right]^{I_2} f_1(\beta_2 \xi)$$

$$= \left[Q_a f_1^{(1)}(\alpha_2 \xi) + (1-Q_a) f_1^{(0)}(\alpha_2 \xi) \right]^{I_2} \left[Q_a f_1^{(1)}(\beta_2 \xi) + (1-Q_a) f_1^{(0)}(\beta_2 \xi) \right] \quad (\text{B-8})$$

$$= \left\{ Q_a \left[f_x^{(1)}(\alpha_1 \alpha_2 \xi) \right]^{I_1} f_x^{(1)}(\beta_1 \alpha_2 \xi) + (1-Q_a) \left[f_x^{(0)}(\alpha_1 \alpha_2 \xi) \right]^{I_1} f_x^{(0)}(\beta_1 \alpha_2 \xi) \right\}^{I_2} \\ * \left\{ Q_a \left[f_x^{(1)}(\alpha_1 \beta_2 \xi) \right]^{I_1} f_x^{(1)}(\beta_1 \beta_2 \xi) + (1-Q_a) \left[f_x^{(0)}(\alpha_1 \beta_2 \xi) \right]^{I_1} f_x^{(0)}(\beta_1 \beta_2 \xi) \right\} \quad (\text{B-9})$$

in terms of the input characteristic functions $f_x^{(h)}$. This form is useful when $f_x^{(h)}$ are available numerically. An alternative form is available from (B-8) as

$$f_w^{(1)}(\xi) = \left[Q_a f_1^{(1)}(\beta_2 \xi) + (1-Q_a) f_1^{(0)}(\beta_2 \xi) \right] \\ * \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n} \left[f_1^{(1)}(\alpha_2 \xi) \right]^n \left[f_1^{(0)}(\alpha_2 \xi) \right]^{I_2-n} \quad . \quad (\text{B-10})$$

This is useful if the products of characteristic functions can be handled easily or have easily-recognized probability density functions.

APPENDIX C. GAUSSIAN INPUT EXAMPLE

The inputs $\{x_n(t)\}$ to the system of figure 1 are Gaussian with characteristic function

$$f_x^{(h)}(\xi) = \exp(i\xi u_x^{(h)} - \frac{1}{2} \xi^2 \sigma_x^2) \quad \text{for } h = 0 \text{ and } 1 \quad (C-1)$$

The input means are $u_x^{(0)}$ and $u_x^{(1)}$ under hypothesis H_0 and H_1 respectively, while the input variance is common value σ_x^2 under both hypotheses.

Since $a_n(t)$ is the sum of K statistically independent random variables, the characteristic function of $a_n(t)$ is

$$f_a^{(h)}(\xi) = \exp(i\xi u_x^{(h)} K - \frac{1}{2} \xi^2 \sigma_x^2 K) \quad (C-2)$$

The corresponding probability density function and distribution function are

$$p_a^{(h)}(u) = \frac{1}{\sqrt{K} \sigma_x} \phi\left(\frac{u - K u_x^{(h)}}{\sqrt{K} \sigma_x}\right) \quad (C-3)$$

and

$$P_a^{(h)}(u) = \Phi\left(\frac{u - K u_x^{(h)}}{\sqrt{K} \sigma_x}\right), \quad (C-4)$$

where

$$\phi(u) \equiv (2\pi)^{-1/2} \exp(-u^2/2), \quad \Phi(u) \equiv \int_{-\infty}^u dv \phi(v) \quad (C-5)$$

Substitution of (C-3) and (C-4) in (A-14) and (A-22) yields

$$Q_a = q(\sqrt{K} d, N-1),$$

$$1 - Q_a = q_1(\sqrt{K} d, N-1), \quad (C-6)$$

where

$$q(c,m) \equiv \int dx \phi(x) [\Phi(x+c)]^m ,$$

$$q_1(c,m) \equiv m \int dx \phi(x) [\Phi(x)]^{m-1} \Phi(x-c) , \quad (C-7)$$

and

$$d \equiv \frac{\mu_X^{(1)} - \mu_X^{(0)}}{\sigma_X} . \quad (C-8)$$

The quantity d is called the input deflection. Special cases of (C-7) are $q(c,0) = 1$ and $q(c,1) = \Phi(c/\sqrt{2})$.

The characteristic function of decision variable $w(t)$ under H_0 is immediately available by substitution of (C-1) in (A-4):

$$f_w^{(0)}(\xi) = \exp(i\xi \mu_X^{(0)} M - \frac{1}{2} \xi^2 \sigma_X^2 M) . \quad (C-9)$$

Then the probability density function of w under H_0 is

$$p_w^{(0)}(u) = \frac{1}{\sqrt{M} \sigma_X} \phi\left(\frac{u - M\mu_X^{(0)}}{\sqrt{M} \sigma_X}\right) \quad (C-10)$$

and the false alarm probability is

$$P_F = \text{Prob} \{w > \lambda | H_0\} = \int_{\lambda}^{\infty} du p_w^{(0)}(u) = \Phi(\gamma) , \quad (C-11)$$

where λ is the threshold in figure 1, and

$$\gamma \equiv \frac{M \mu_X^{(0)} - \lambda}{\sqrt{M} \sigma_X} . \quad (C-12)$$

Under H_1 , we employ (C-1) in (B-6) and get

$$f_1^{(h)}(\xi) = [f_X^{(h)}(\xi)]^L = \exp(i\xi \mu_X^{(h)} L - \frac{1}{2} \xi^2 \sigma_X^2 L) , \quad (C-13)$$

which is exact for any L (as noted in (B-4)); there is no need here to take the integer part of L , as was done in (B-5) and (B-9) for the general case.

We now use (C-13) in (B-10) and find characteristic function

$$f_w^{(1)}(\xi) = \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n} \cdot \left[Q_a \exp \left(i \xi m_1(n) - \frac{1}{2} \xi^2 \sigma^2 \right) + (1-Q_a) \exp \left(i \xi m_2(n) - \frac{1}{2} \xi^2 \sigma^2 \right) \right] \quad , \quad (C-14)$$

where

$$\begin{aligned} m_1(n) &\equiv u_x^{(0)} M + \left(u_x^{(1)} - u_x^{(0)} \right) \left(\beta_2 + \alpha_2 n \right) L, \\ m_2(n) &\equiv u_x^{(0)} M + \left(u_x^{(1)} - u_x^{(0)} \right) \alpha_2 n L, \\ \sigma^2 &\equiv \sigma_x^2 M. \end{aligned} \quad (C-15)$$

Here we have used (B-2) and (B-5) in the form

$$\begin{aligned} I_{\mathcal{L}} &= I_2 \alpha_2 + \beta_2, \\ I_{\mathcal{L}} &= I_2 \alpha_2^2 + \beta_2^2, \end{aligned} \quad (C-16)$$

and (A-16) in the form $M = L I_{\mathcal{L}}$.

The probability density function of decision variable $w(t)$ under H_1 is then

$$p_w^{(1)}(u) = \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n} \cdot \left[Q_a \frac{1}{\sigma} \phi \left(\frac{u - m_1(n)}{\sigma} \right) + (1-Q_a) \frac{1}{\sigma} \phi \left(\frac{u - m_2(n)}{\sigma} \right) \right] \quad . \quad (C-17)$$

Finally the detection probability is

$$P_D = \text{Prob} \{w > \mathcal{L} | H_1\} = \int_{\mathcal{L}}^{\infty} du p_w^{(1)}(u)$$

$$= \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n} \left[Q_a \Phi\left(\frac{m_1(n) - \mathcal{L}}{\sigma}\right) + (1-Q_a) \Phi\left(\frac{m_2(n) - \mathcal{L}}{\sigma}\right) \right] \quad (C-18)$$

In order to put this result in a form consistent with that for the false alarm probability in (C-11), we use (C-12), (C-15), (C-8), and (A-16), to get

$$P_D = \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n}$$

$$* \left[Q_a \Phi\left(\gamma + d\sqrt{M} \frac{\beta_2 + \alpha_2}{I_{\mathcal{L}}} n\right) + (1-Q_a) \Phi\left(\gamma + d\sqrt{M} \frac{\alpha_2}{I_{\mathcal{L}}} n\right) \right] \quad (C-19)$$

To review: d is the input deflection as given by (C-8); M is the amount of post-averaging; $I_{\mathcal{L}}$ is the effective number of independent samples of $\mathcal{L}$, as given by (A-12) in terms of K and M ; I_2 is the integer part of $I_{\mathcal{L}}$; α_2 and β_2 are the solutions of (C-16) and can be found via (B-3); Q_a and $1-Q_a$ are probabilities given by (C-6) and (C-7); and γ is a modified threshold given by (C-12) such that the false alarm probability is simply

$$P_F = \Phi(\gamma) \quad (C-20)$$

As $\mu_X^{(1)} \rightarrow \mu_X^{(0)}$, then $d \rightarrow 0$, and (C-19) tends to (C-20) as it should. For $N=1$, then $Q_a=1$ and (C-19) reduces to $P_D = \Phi(\gamma + d\sqrt{M})$. Equations (C-19) and (C-20) for the detection probability and false alarm probability are the end results of this appendix. The fundamental inputs are N , K , M , d .

Simulation

Since the absolute scales of $\mu_X^{(1)}$, $\mu_X^{(0)}$, and σ_X do not enter separately into the detection probability and false alarm probability, but only through the combination

$$d = \frac{\mu_X^{(1)} - \mu_X^{(0)}}{\sigma_X} \quad (C-21)$$

we let $\mu_x^{(0)} = 0$ and $\sigma_x = 1$ in a simulation. Then from (C-11) and (C-12),

$$P_F = \Phi(-\mathcal{L}/\sqrt{M}); \quad (C-22)$$

this result is exact, so there is no need to simulate the false alarm case.

Under H_1 , suppose we make a total of T_t trials, getting decision variables

$\{w_k\}_1^{T_t}$, ordered such that $w_1 \leq w_2 \leq \dots \leq w_{T_t}$. Then the detection probability

and false alarm probability are given by

$$\left. \begin{aligned} P_D &= \text{Prob} \{w > \mathcal{L} | H_1\} \equiv 1 - \frac{k - \frac{1}{2}}{T_t} \\ P_F &= \text{Prob} \{w > \mathcal{L} | H_0\} = \Phi(-w_k/\sqrt{M}) \end{aligned} \right\} \text{ at } \mathcal{L} = w_k \text{ for } 1 \leq k \leq T_t. \quad (C-23)$$

Sample simulation results for various values of d ($= \mu_x^{(1)}$) are superposed on the analytical results, given by (C-19) and (C-20), in figures 4 through 35.

APPENDIX D. PROGRAM FOR OPERATING CHARACTERISTICS

Since Q_a and $1-Q_a$ are evaluated separately according to (C-6) and (C-7), a partial check on accuracy is furnished by printing out their sum. Also, the array F(*) holds the binomial terms in the upper line of (C-19); its sum should be 1 and is also printed out.

```

10   T$="Figure 39.  Operating Characteristics"
20   N=4                      ! NUMBER OF INPUT CHANNELS
30   K=8                      ! LENGTH OF PRE-AVERAGERS
40   M=8                      ! LENGTH OF POST-AVERAGER
50   Dmax=4                   ! MAXIMUM VALUE FOR DEFLECTION d
60   Dinc=.4                  ! INCREMENT IN d
70   Dsim=1.2                 ! VALUE OF d IN SIMULATION
80   Tt=1000                  ! TOTAL NUMBER OF TRIALS
90   OUTPUT 0;"NUMBER OF INPUT CHANNELS      N =";N
100  OUTPUT 0;"LENGTH OF PRE-AVERAGERS      K =";K
110  OUTPUT 0;"LENGTH OF POST-AVERAGER      M =";M
120  OUTPUT 0;"MAXIMUM VALUE FOR d          Dmax =";Dmax
130  OUTPUT 0;"INCREMENT IN d              Dinc =";Dinc
140  OUTPUT 0;"VALUE OF d IN SIMULATION     Dsim =";Dsim
150  OUTPUT 0;"TOTAL NUMBER OF TRIALS      Tt =";Tt
160  OUTPUT 0;"
170  INTEGER I,J,Ns,It
180  T=3*K*M
190  IF K<M THEN 220
200  I1=T/(T+1-M*M)
210  GOTO 230
220  I1=3*M*M/(T+1-K*K)
230  OUTPUT 0;"EFFECTIVE NUMBER OF INDEPENDENT SAMPLES I1 =";I1
240  OUTPUT 0;"
250  I2=INT(I1)
260  T=I2*I1
270  A2=(T+SQR(T*(1+I2-I1)))/(I2*(I2+1))
280  B2=I1-I2*A2
290  IF N=1 THEN 310
300  OUTPUT 0;"          Qa","          1-Qa","          Qa+(1-Qa)","          SUM F(*)"
310  REDIM F(0:I2)
320  DIM F(0:64)
330  PLOTTER IS "9872A"
340  OUTPUT 705;"VS5"
350  LIMIT 10,100,0,235                      ! LABELLING
360  MOVE 51,25
370  DIM T$(37)
380  N$=" for N="
390  K$=", K="
400  M$=", M="
410  B$=CHR$(8)
420  CSIZE 3,.5
430  LONG 5
440  LABEL T$;N$;B$;N;B$;K$;B$;K;B$;M$;B$;M
450  LIMIT 10,100,65,235                      ! GRID
460  LOCATE 12,98,12,98
470  DATA 1E-6,1E-5,1E-4,.001,.01,.1,.5
480  DATA .9,.99,.999,.9999,.99999,.999999
490  DIM P(1:13),Y(1:13)
500  READ P(*)

```

```

510   FOR I=1 TO 13
520   Y(I)=FNInvphi(P(I))   ! NORMAL PROBABILITY SCALES
530   NEXT I
540   S=Y(1)
550   B=Y(13)
560   SCALE S,B,S,B
570   FOR I=1 TO 13
580   MOVE S,Y(I)
590   DRAW B,Y(I)
600   NEXT I
610   FOR I=1 TO 13
620   MOVE Y(I),S
630   DRAW Y(I),B
640   NEXT I
650   MOVE S,S
660   DRAW B,B
670   CSIZE 3,.5
680   LORG 1
690   MOVE .4,.2
700   LABEL "d=0"
710   LORG 8
720   FOR I=1 TO 13
730   MOVE S+.1,Y(I)
740   LABEL P(I)
750   NEXT I
760   R=.5*(Y(7)+Y(8))
770   MOVE S-1,R
780   CSIZE 4,.5
790   LORG 8
800   LABEL "P"
810   MOVE S-1,R-.1
820   CSIZE 2,.5
830   LORG 6
840   LABEL "D"
850   CSIZE 3,.5
860   LDIR PI/2
870   LORG 8
880   FOR I=1 TO 13
890   MOVE Y(I),S+.05
900   LABEL P(I)
910   NEXT I
920   MOVE R,S-1
930   CSIZE 4,.5
940   LDIR 0
950   LORG 8
960   LABEL "P"
970   MOVE R,S-1.1
980   CSIZE 2,.5
990   LORG 6
1000  LABEL "F"

```

```

1010 Sk=SQR(K)                                ! ANALYTICAL RESULTS
1020 Sm=SQR(M)
1030 P13=P(13)
1040 REDIM Xp(1:Tt),Yp(1:Tt)
1050 DIM Xp(1:1000),Yp(1:1000)
1060 FOR Ds=Dinc TO Dmax STEP Dinc            ! DEFLECTION d IN (6)
1070 Dk=Ds*Sk
1080 Dm=Ds*Sm
1090 IF N>1 THEN 1130
1100 MOVE S,S+Dm
1110 DRAW B,B+Sm
1120 GOTO 1450
1130 Qa=FNQs(Dk,N-1)                          ! (C-6)
1140 Qa1=FNQs1(Dk,N-1)                       ! (C-6)
1150 F(0)=Qa1^I2
1160 FOR Ns=1 TO I2
1170 F(Ns)=F(Ns-1)*(I2+1-Ns)*Qa/(Ns*Qa1)      ! BINOMIAL TERMS IN (C-19)
1180 NEXT Ns
1190 OUTPUT 0;Qa,Qa1,Qa+Qa1,SUM(F)
1200 T=Dm/I1
1210 Ta2=T*A2
1220 Tb2=T*B2
1230 J=0
1240 FOR Gam=S TO B STEP (B-S)/50
1250 J=J+1
1260 Xp(J)=Gam
1270 G2=Gam+Tb2
1280 Pd=0
1290 FOR Ns=0 TO I2
1300 V=FNPhi(G2+Ta2*Ns)
1310 V1=FNPhi(Gam+Ta2*Ns)
1320 Pd=Pd+F(Ns)*(Qa*V+Qa1*V1)              ! (C-19)
1330 NEXT Ns
1340 D=Pd-.5
1350 IF ABS(D)<.49999999999 THEN 1380
1360 Yp(J)=100*SGN(D)
1370 GOTO 1390
1380 Yp(J)=FNInuphi(Pd)
1390 IF Pd>P13 THEN 1410
1400 NEXT Gam
1410 FOR I=1 TO J
1420 PLOT Xp(I),Yp(I)
1430 NEXT I
1440 PENUP
1450 NEXT Ds
1460 PENUP

```



```

1470 L4=-LOG(4)
1480 R=PI/180
1490 Km=K+M
1500 REDIM X(1:Km,1:N),W(1:Tt)
1510 DIM X(1:72,1:16),W(1:1000)
1520 FOR It=1 TO Tt
1530 FOR I=1 TO Km
1540 FOR Ns=1 TO N STEP 2
1550 H=1E-5*INT(1E5*R)
1560 L=R-H
1570 U=FRAC(31622000000*L)
1580 V=FRAC(777021*H)
1590 W=FRAC(777021*L)
1600 R=FRAC(U+V+W+1E-11)
1610 V1=R-.5
1620 H=1E-5*INT(1E5*R)
1630 L=R-H
1640 U=FRAC(31622000000*L)
1650 V=FRAC(777021*H)
1660 W=FRAC(777021*L)
1670 R=FRAC(U+V+W+1E-11)
1680 V2=R-.5
1690 S2=V1*V1+V2*V2
1700 IF S2>.25 THEN 1550
1710 Q2=(L4-LOG(S2))/S2
1720 Q2=SQR(Q2+Q2)
1730 X(I,Ns)=V1*Q2
1740 IF Ns+1>N THEN 1760
1750 X(I,Ns+1)=V2*Q2
1760 NEXT Ns
1770 X(I,1)=X(I,1)+Dsim
1780 NEXT I
1790 Wt=0
1800 FOR I=K+1 TO Km
1810 Yt=-9E99
1820 FOR Ns=1 TO N
1830 A=0
1840 FOR J=1 TO K
1850 A=A+X(I-J,Ns)
1860 NEXT J
1870 IF Yt>A THEN 1900
1880 Yt=A
1890 Lt=Ns
1900 NEXT Ns
1910 Wt=Wt+X(I,Lt)
1920 NEXT I
1930 W(It)=Wt
1940 NEXT It
1950 MAT SORT W
1960 F=1/Sm
1970 FOR It=1 TO Tt
1980 Pd=1-(It-.5)/Tt
1990 Yp(It)=FNINUPHI(Pd)
2000 Xp(It)=T=-W(It)*F
2010 IF T<S THEN 2030
2020 NEXT It
2030 FOR I=1 TO MIN(It,Tt)
2040 PLOT Xp(I),Yp(I)
2050 NEXT I
2060 PENUP
2070 END
2080 !

```

! SIMULATION RESULTS
! SEED FOR RANDOM NUMBERS

! GENERATE TWO
! INDEPENDENT
! NORMALIZED
! GAUSSIAN
! RANDOM VARIABLES
! VIA
! ACCEPTANCE-
! REJECTION
! METHOD

! SIGNAL IN CHANNEL 1

! PRE-AVERAGER OUTPUT

! OR-ING OPERATION
! LARGEST PRE-AVERAGER OUTPUT
! CHANNEL NUMBER OF LARGEST PRE-AVERAGER OUTPUT

! POST-AVERAGER OUTPUT

```

2090 DEF FNQs(Cs,Ms)                                ! (C-7), LINE 1
2100 INTEGER K
2110 IF Ms=0 THEN RETURN 1
2120 IF Ms>1 THEN 2150
2130 V=FNPhi(Cs*.707106781187)
2140 RETURN V
2150 A=-7
2160 B=7
2170 S=(FNS(A,Cs,Ms)+FNS(B,Cs,Ms))* .5
2180 N=2
2190 H=(B-A)*.5
2200 F=(B-A)/3*.398942280401
2210 T=0
2220 FOR K=1 TO N-1 STEP 2
2230 T=T+FNS(A+H*K,Cs,Ms)
2240 NEXT K
2250 S=S+T
2260 Vo=V
2270 V=(S+T)*F
2280 PRINT V
2290 IF ABS(1-Vo/V)<=1E-9 THEN RETURN V
2300 N=N+N
2310 H=H*.5
2320 F=F*.5
2330 GOTO 2210
2340 FNEND
2350 !
2360 DEF FNQs1(Cs,Ms)                                ! (C-7), LINE 2
2370 INTEGER K
2380 IF Ms=0 THEN RETURN 0
2390 IF Ms>1 THEN 2420
2400 V=FNPhi(-Cs*.707106781187)
2410 RETURN V
2420 A=-7
2430 B=7
2440 S=(FNS1(A,Cs,Ms)+FNS1(B,Cs,Ms))* .5
2450 N=2
2460 H=(B-A)*.5
2470 F=(B-A)/3*Ms*.398942280401
2480 T=0
2490 FOR K=1 TO N-1 STEP 2
2500 T=T+FNS1(A+H*K,Cs,Ms)
2510 NEXT K
2520 S=S+T
2530 Vo=V
2540 V=(S+T)*F
2550 PRINT V
2560 Du=ABS(V-Vo)
2570 IF (Du<=1E-10) AND (Du/V<=1E-6) THEN RETURN V
2580 N=N+N
2590 H=H*.5
2600 F=F*.5
2610 GOTO 2480
2620 FNEND
2630 !

```

```

2640 DEF FNS(X,Cs,Ms) ! INTEGRAND OF (C-7), LINE 1
2650 T=FNPhi(X+Cs)
2660 RETURN EXP(-.5*X*X)*T^Ms
2670 FNEND
2680 !
2690 DEF FNS1(X,Cs,Ms) ! INTEGRAND OF (C-7), LINE 2
2700 T=FNPhi(X)
2710 T1=FNPhi(X-Cs)
2720 RETURN EXP(-.5*X*X)*T^(Ms-1)*T1
2730 FNEND
2740 !
2750 DEF FNPhi(X) ! (C-5)
2760 INTEGER J
2770 IF ABS(X)>5.14 THEN 3030
2780 A=.282842712475*X
2790 C=COS(A)
2800 S=SIN(A)
2810 B=C+C
2820 A=B*C-1
2830 C=A*(1.2536751E-18+B*7.10005E-20+A*7.4517E-21)
2840 C=A*(1.533423425E-16+B*1.01649277E-17+C)
2850 C=A*(1.36760444757E-14+B*1.0601364636E-15+C)
2860 C=A*(8.89786526722E-13+B*8.06068838945E-14+C)
2870 C=A*(4.22616144318E-11+B*4.46968229249E-12+C)
2880 C=A*(1.46660614234E-9+B*1.80848587810E-10+C)
2890 C=A*(3.72252349369E-8+B*5.34275027603E-9+C)
2900 C=A*(6.91927520325E-7+B*1.15330990944E-7+C)
2910 C=A*(9.43281169838E-6+B*1.82066316364E-6+C)
2920 C=A*(9.44909268810E-5+B*2.10404583073E-5+C)
2930 C=A*(6.97183792408E-4+B*1.78228016255E-4+C)
2940 C=A*(3.80150767985E-3+B*1.10860645342E-3+C)
2950 C=A*(.0153985726157+B*.00507906961220+C)
2960 C=A*(.0467755234325+B*.0172439625887+C)
2970 C=A*(.108630245023+B*.0439773381941+C)
2980 C=A*(.201339747265+B*.0869894549959+C)
2990 C=A*(.330501521917+B*.144227226362+C)
3000 C=.703225002744+B*.247255168140+C
3010 Phi=.5+.0450158158079*X+.5*S*C
3020 RETURN Phi
3030 IF X>7 THEN RETURN 1
3040 N=MAX(6,INT(69/ABS(X)),INT(525/(X*X)))+1
3050 S=A=1
3060 B=1/X
3070 C=B*B
3080 FOR J=1 TO N
3090 A=(1-J-J)*A*C
3100 S=S+A
3110 NEXT J
3120 Phi=.398942280401*EXP(-.5*X*X)*ABS(B)*S
3130 IF X>0 THEN Phi=1-Phi
3140 RETURN Phi
3150 FNEND
3160 !

```

```

3170 DEF FNInvphi(Z)      ! INVERSE FUNCTION TO PHI IN (C-5)
3180 INTEGER N
3190 DIM T(0:20),A(0:20)
3200 DATA .992885376619,.120467516143,.0160781993421,.00268670443716
3210 DATA .49963473024E-3,.988982186E-4,.2039181276E-4,.432727162E-5
3220 DATA .93808141E-6,.20673472E-6,.461597E-7,.1041668E-7,.23715E-8
3230 DATA .54393E-9,.12555E-9,.2914E-10,.679E-11,.159E-11,.37E-12
3240 DATA .912158803418,-.0162662818677,.43355647295E-3,.21443857007E-3
3250 DATA .262575108E-5,-.302109105E-5,-.1240606E-7,.6240661E-7,-.54012E-9
3260 DATA -.142321E-8,.3438E-10,.3358E-10,-.146E-11,-.81E-12,.5E-13,.2E-13
3270 DATA .956679709020,-.0231070043091,-.00437423609751,-.57650342265E-3
3280 DATA -.1096102231E-4,.2510854702E-4,.1056233607E-4,.275441233E-5
3290 DATA .43248450E-6,-.2053034E-7,-.4389154E-7,-.1768401E-7,-.399129E-8
3300 DATA -.18693E-9,.27292E-9,.13282E-9,.3183E-10,.167E-11,-.204E-11
3310 DATA -.965E-12,-.22E-12
3320 X=Z+Z-1
3330 A=B=ABS(X)
3340 IF A>=.8 THEN B=SQR(-LOG(4*Z*(1-Z)))
3350 IF A<.8 THEN 3450
3360 IF A<.9975 THEN 3410
3370 Nmax=20
3380 RESTORE 3270
3390 Y=-.559457631330*B+2.28791571626
3400 GOTO 3480
3410 Nmax=15
3420 RESTORE 3240
3430 Y=-1.54881304237*B+2.56549012315
3440 GOTO 3480
3450 Nmax=18
3460 RESTORE 3200
3470 Y=12.5*Z*(Z-1)+2.125
3480 REDIM A(0:Nmax)
3490 READ A(*)
3500 Y2=Y+Y.
3510 T(0)=1
3520 T(1)=Y
3530 FOR N=2 TO Nmax
3540 T(N)=Y2*T(N-1)-T(N-2)
3550 NEXT N
3560 R=0
3570 FOR N=Nmax TO 0 STEP -1
3580 R=R+A(N)*T(N)
3590 NEXT N
3600 Invphi=SGN(X)*B*R*1.41421356237
3610 RETURN Invphi
3620 FNEND

```

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