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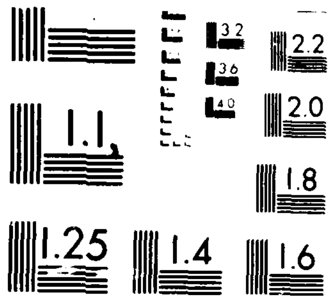

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THE GENERALIZED HESSENBERG REPRESENTATION, NEAR AGGREGATION, AND NEAR UNOBSERVABILITY

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THE GENERALIZED HESSENBERG REPRESENTATION, NEAR AGGREGATION, AND NEAR UNOBSERVABILITY*

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Abstract

Using the Generalized Hessenberg Representation (GHR), the concept of aggregation is extended to systems which nearly aggregate. Near aggregation is given a geometric interpretation. Then near unobservability (defined as an invariant subspace near the null space of C) is introduced and is shown to be equivalent to near aggregation if there exists an appropriately dimensioned invariant subspace. These results depend on the introduction of a topology into the state space, a novel feature of our approach. Finally, near aggregation is shown to correspond to almost pole-zero cancellation for a certain class of systems.

Keywords: *Linear systems, observability, aggregation.*

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1. INTRODUCTION

Model reduction has long been a topic of interest, and it remains relevant to control problems of today. For example, large flexible space structures are modeled by finite element approximations of large order. Unfortunately, the computational constraints of on-board digital controllers make complex control algorithms based on large models infeasible. Thus, the control design requirements call for a simple model for closed-loop control.

Aggregation was proposed by Aoki (1968) as one method of model reduction. This concept was extended by the Generalized Hessenberg Representation (GHR) by Tse, et al (1978). In this report, we investigate further properties of the GHR. In particular, we discuss systems which **nearly aggregate**. This idea has intuitive appeal; however, it turns out to be dependent on the scaling in the system. The usual algebraic formulation of state space models does not directly incorporate scaling in the model. Since the scaling issue appears directly in near aggregation, we incorporate it in the model by attaching an inner product to the state space. This inner product leads to a topological structure in the model which allows us to formalize **near** aggregation. Bart, et al (1980) used a similar idea in the factorization of transfer matrices.

Near aggregation is related to the observability structure of the system through the use of geometrical concepts (Wonham, 1979). To establish this connection, the concept of near unobservability is introduced. Roughly, a system is nearly unobservable if there exists an invariant subspace near the null space of C. It is shown that if an appropriate dimensioned invariant subspace exists, then near aggregation and near unobservability are equivalent concepts from a geometrical point of view. This generalizes the results of Aoki

(1968) and Tse, et al (1978).

It is well known that unobservable systems exhibit a pole-zero cancellation in the transfer function. It is shown below that, under certain conditions, it is possible to generalize this result. That is, under these conditions, nearly unobservable systems exhibit an almost pole-zero cancellation. Hence, near aggregation corresponds, roughly, to removing an almost pole-zero cancellation, a well-known procedure in classical control.

Aggregation has been related to several other model reduction methods (Lindner and Perkins, 1984). The fact that the results here are used as a measure of observability suggests that this approach may be related to balancing (Moore, 1981). However, there are fundamental differences in the two approaches. Balancing requires the computation of the controllability and observability grammians. Hence, this technique is most useful for stable systems. Near unobservability is determined from the algebraic and topological properties of the vector space and operators associated with the system. Stability of the differential equation is not an issue. Secondly, balancing has not been related to zeros. Nonetheless, it would be interesting to compare these two concepts. Unfortunately, a complete theory is not available to date. Some preliminary results may be found in Lindner and Perkins (1984).

The report is organized as follows. Section 2 introduces the GHR and relates it to near aggregation. Section 3 discusses the geometry of the GHR and near aggregation. Section 4 introduces near unobservability and establishes the connection between near aggregation and near unobservability. Section 5 shows how near unobservability is related to almost pole-zero cancellation. Section 6 has the conclusions.

2. THE GHR

Consider the system

$$\dot{x} = Ax + Bu ,$$
$$y = Cx , \tag{2.1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, and (A,B,C) are appropriately dimensioned constant matrices. After i-steps of **chained aggregation** (Tse, et al, 1978), (2.1) is transformed into

$$\dot{x}^i = \begin{bmatrix} \dot{y}^1 \\ \dot{y}_2 \\ \\ \dot{y}^i \\ \dot{x}_r^i \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & O & O \\ F_{22} & F_{22} & F_{23} & O \\ . & & & \\ F_{i,1} & \cdots & \cdots & F_{i,i+1} \\ A_{i+1,1} & & & A_{i+1,i+1} \end{bmatrix} x^i + \begin{bmatrix} G_1 \\ \\ \\ G_i \\ B_{i+1} \end{bmatrix} u , \tag{2.2}$$

$$y = [\, H_1 \;\; 0 \ldots. \;\; 0 \,] \, x^i ,$$

or, more concisely,

$$\begin{bmatrix} \dot{y}_a^i \\ \dot{x}_r^i \end{bmatrix} = \begin{bmatrix} F^i & E^i \\ K^i & A^i \end{bmatrix} \begin{bmatrix} y_a^i \\ x_r^i \end{bmatrix} + \begin{bmatrix} G^i \\ B^i \end{bmatrix} u ,$$
$$y = [H^i \;\; O] \begin{bmatrix} y_a^i \\ x_r^i \end{bmatrix} . \tag{2.3}$$

The representation (2.2) has the property that $\mathcal{N}[H_1] = \mathcal{N}[F_{j,j+1}] = 0$ for $j = 1, \ldots, i-1$. Furthermore, if $\mathcal{N}[F_{i,i+1}] = O$ or $F_{i,i+1} = O$, then the representation (2.2) is the **Generalized Hessenberg Representation** (GHR) of (2.1). If in (2.2) $F_{i,i+1} = O$, then an obvious reduced order model is given by

$$\dot{y}_a^{\,i} = F^i y_a^{\,i} + G^i u\ ,$$
$$y = H^i y_a^{\,i}\ . \tag{2.4}$$

If i=1 in (2.4), then the reduction process is **aggregation** (Aoki, 1968). Thus, the GHR extends the notion of aggregation (Tse, et al, 1978). When $F_{i,i+1} = 0$, we say that (2.1) **exactly** aggregates.

Another direction of extending the notion of aggregation is to consider systems in which none of the super diagonal blocks in (2.2) is zero but one of them is **small**. We formalize this idea with:

Definition 2.1. Given $\mu_o > 0$, we say that (2.2) is μ_o-**aggregable**, if there exists a block $F_{j,j+1}$ such that $\| F_{j,j+1} \| < \mu_o$.

□

Remark 2.2. From the engineering point of view, we are interested in the case when μ_o is small. We describe those systems which are μ_o-aggregable when μ_o is small as **nearly aggregable**. This terminology will be applied again in Def. 4.2 and Def. 5.3 below.

□

Remark 2.3. By considering the dual system, the GHR and Def. 2.1 can be applied to the pair (A,B) in (2.1).

□

It is obvious that $\| F_{j,j+1} \|$ is basis dependent. Thus, for Def. 2.1 to be useful, we must impose further structure on (2.1). In addition to the usual algebraic structure of (2.1), we will attach the natural inner product to the state space which is defined with respect to the given basis of (2.1). This inner product leads to a 2-norm on the underlying vector space given by

$$\| x \| = (x^T x)^{1/2} . \tag{2.5}$$

In this paper we will assume all norms are the 2-norm in (2.5) or the corresponding induced operator norm.

Once (2.1) is given with an inner product, we require that all operations on (2.1) preserve this structure. This implies that all state transformations on (2.1) must be orthogonal. In particular, the GHR (2.2) is constructed using orthogonal transformations (which is also numerically stable (Van Dooren, 1981)).

Lemma 2.4. Suppose that the GHR (2.2) is constructed from (2.1) using orthogonal transformations. Then $\| F_{j,j+1} \|$ is unique.

□

Proof: Consider (2.3) with i=1. The basis associated with this representation is not unique. All allowable transformations which preserve the structure of (2.3) (i.e., $\mathcal{N}[H^1] = 0$) are given by

$$\begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix} \tag{2.6}$$

where T_1 is p×p and T_1 and T_2 are orthogonal. It follows that

$$\| E^1 \| = \| T_1^T E^1 T_2 \| . \tag{2.7}$$

An induction argument completes the proof.

□

Remark 2.5. The choice of the inner product for (2.1) is essentially a scaling issue. It may be implied by the physical variables or it may be a design parameter in the selection of a reduced order model.

□

The rest of this report is devoted to interpreting near aggregability in system theoretic terms. We will start with a geometric interpretation of the GHR in the next section.

3. GEOMETRY OF THE GHR

The following subspaces play a fundamental role below.

Definition 3.1 (Lindner, et al, 1982): The vector $\xi \in X$ is an element of L_j, the **j-th unobservability subspace** if $\xi = x(0)$ implies $y(0) = \dot{y}(0) = \cdots = y^{(j-1)}(0) = 0$. By definition, $L_0 = X$.

□

The unobservability subspaces L_j are intimately related to the GHR. Let e_i be the ith natural basis vector and r_i the dimension of the F_{ii} in (2,2) block in (2.2). Define $\rho_i = \sum_{j=1}^{i} r_j$.

Theorem 3.2 (Lindner, et al. 1982): Assume that (2.2) is a GHR. With respect to this basis:

$$L = \sum_{k=\rho_j+1}^{n} e_k \,, \qquad 1 \leqslant j \leqslant i$$

$$L_{i+1} = 0 \qquad \text{if} \quad F_{i,i+1} \neq 0$$

$$L_{i+1} = L_i \,, \qquad F_{i,i+1} = 0 \,.$$

□

The proof follows immediately from Def. 3.1 and (2.2).

Corollary 3.3. The system (2.1) is exactly aggregable iff L_i coincides with an A-invariant subspace for some i.

□

Proof: Obvious by comparing Theorem 3.2 and (2.2).

Corollary 3.3 gives a geometric interpretation of exact aggregation. If (2.2) is not exactly aggregable, but is nearly aggregable, is there an A-invariant subspace V "near" L? We will obtain an answer to this question below and so obtain a geometric interpretation of near aggregability.

First, to interpret "near" we give a norm on the subspaces of X. Note that this norm is derived from the inner product on X.

Definition 3.4 (Kato, 1966): Let U and V be subspaces of R^n. The gap between U and V is the number

$$\tau(U,V) = \max\{ \sup_{\|u\|=1,\, u\in U} \inf_{v\in V} \| v-u \| , \sup_{\|v\|=1,\, v\in V} \inf_{u\in U} \| v-u \| \} .$$

□

Since we are using the two-norm, $\tau(U,V) = 1$ if U and V have different dimensions.

A useful geometrical interpretation of the gap enters through the use of canonical angles.

Definition 3.5: Let U and V be subspaces of R^n with orthonormal bases U and V, respectively. Let σ_i be the singular values of U^TV. Then the canonical angles between U and V are the numbers

$$\theta_i = \cos^{-1} \sigma_i .$$

□

The gap function is related to canonical angles as follows.

Proposition 3.6 (Stewart, 1973): $\tau(L,V) = |\sin\theta_{max}|$.

□

Hence, if all the canonical angles between two subspaces are small, they are close in the gap topology.

Next, we show how to compute the gap. In what follows, we assume that the system is given in the basis (2.3) and we are trying to find the gap between L_i and an A-invariant subspace V (although the method developed by Stewart (1973), can be used on an arbitrary pair of subspaces).

The angles between two subspaces can be computed in the following way. Suppose that the natural orthonormal basis of R^n yields a basis for $L_i^{\perp}$ and L_i, respectively. In matrix form

$$L = \begin{bmatrix} I_{\rho_i} & O \\ O & I_{n-\rho_i} \end{bmatrix} = [L_c \quad L_i] \tag{3.1}$$

where the first r columns span $L_i^{\perp}$ and the last n-r columns span L_i. Let a second $(n-\rho_i)$ dimensional subspace V and its complement be spanned by the orthogonal basis

$$\bar{P} = \begin{bmatrix} I_r & P \\ -P^T & I_{n-\rho_i} \end{bmatrix} \begin{bmatrix} (I+PP^T)^{-1/2} & O \\ O & (I+P^TP)^{-1/2} \end{bmatrix} \tag{3.2}$$

$$= [V_c V] .$$

To compute the canonical angles between L_i and V, note that

$$L_i^T V = (I + P^T P)^{-1/2} . \tag{3.3}$$

Let P have singular values σ_i. Then the canonical angles between L_i and V are given by

$$\theta_i = \cos^{-1} (1 + \sigma_i^2)^{-1/2} . \tag{3.4}$$

It follows that

$$\sigma_i = \tan \theta_i , \tag{3.5}$$

and

$$\tau(L_j, V) = | \sin \theta_{max} | \leqslant \| P \| = | \tan \theta_{max} | . \tag{3.6}$$

Thus, the matrix P can be used to compute the gap between subspaces.

Let us now return to the problem of interpreting near aggregability in geometric terms. Corollary 3.3 suggests that near aggregation might be characterized geometrically as an invariant subspace near L_j. The GHR suggests that this situation occurs when $\| F_{j,j+1} \|$ is small. To check for this geometry, we look for $(n-\rho_i)$-dimensional invariant subspaces of the form

$$V = \text{sp} \begin{bmatrix} P \\ I \end{bmatrix} . \tag{3.7}$$

If such a subspace exists, then (3.6) holds, from which $\tau(L_j, V)$ can be computed. The following example shows that there exists systems which have this geometry.

Example 3.7. The linearized model of a biomethanization process (Ioannou and Opdenacker, 1983) is given as

$$\begin{bmatrix} \dot{\bar{y}} \\ \\ \dot{z} \end{bmatrix} = \begin{bmatrix} .07 & 1 & 0 & 0 \\ -.0049 & 0 & 1 & 0 \\ -30.164 & -78.631 & -21.654 & .0955 \\ 1 & 0 & 0 & -.07 \end{bmatrix} \begin{bmatrix} \bar{y} \\ \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u ,$$

$$y = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} \bar{y} \\ \\ z \end{bmatrix} . \tag{3.8}$$

For simplicity, we assume that the inner product is the natural one with respect to the basis in (3.8).

The model (3.8) is a GHR. Note that $F_{3,4} = .0955$ is clearly small. Looking over all one-dimensional eigenvectors, we find that the eigenvector corresponding to the pole[1] at -0.75 is computed as

$$v = [P^T \quad 1] = [\,-.49E-2, +.725E-3, -.787E-4, 1\,] . \tag{3.9}$$

Therefore, the angle between $V = sp(v)$ and L_3 is

$$\theta = \tan^{-1} \| P \| = 0.28^\circ , \tag{3.10}$$

and

$$\tau(V, L_3) = \sin\theta = 0.005 . \tag{3.11}$$

□

Below, we will show that under certain conditions a nearly aggregable system does have an invariant subspace near L_j for some j. But first, it is obvious from Theorem 3.2 that exact aggregation and near aggregation are intimately related to observability. In the next section we will restate the geometry above in terms of observability and then show how the observability property is related to near aggregability.

---

[1] See also Example 5.10 below.

4. NEAR AGGREGATION AND NEAR UNOBSERVABILITY

It is clear from Definition 3.1 that $L_1 = N[C]$ and $L_i \subset L_{i+1}$. Therefore, if L_i coincides with an A-invariant subspace, (2.1) is unobservable.

Lemma 4.1 (Aoki, 1968, Tse, et al, 1978): The system (2.1) is exactly aggregable iff it is unobservable.

□

The analysis of Section 3 suggested that nearly aggregable systems might have an A-invariant subspace close to L_i. The observability interpretation above suggests the following definition for this geometrical condition.

Definition 4.2 (Lindner and Perkins, 1984): Given $0 < \epsilon_0 < 1$, we say the system (2.1) is ϵ_0**-unobservable** if there exists an A-invariant subspace V such that

$$\tau(V, L_i) < \epsilon_0$$

for some i.

□

(See Remark 2.2.)

We might guess that nearly aggregable systems are nearly unobservable. However, this is not true in general as the following example shows.

Example 4.3 Consider

$$\dot{x} = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u , \qquad (4.1)$$

$$y = [\,1 \quad 0\,]\, x .$$

For $\mu>0$, there does not exist a 1-dimensional invariant subspace so (4.1) cannot be nearly unobservable.

□

Roughly speaking, nearly aggregable systems are nearly unobservable if there exists an appropriately dimensioned invariant subspace. To develop this idea, we first give conditions under which these invariant subspaces of a given dimension exist. The following result by Stewart (1973) is useful.

Suppose that $\bar{P}$ in (3.2) is used to define a change of basis in R^n. (Note that $\bar{P}$ can be considered a kind of generalized rotation.) Since V is A-invariant, the (1,2) block in the new representation should be zero. Using the representation of A in (2.3), we have

$$V_c^T A V = (I + PP^T)^{-1/2} (F^i P - PA^i + E^i - PK^i P)(I + PP^T)^{-1/2} = 0 . \qquad (4.2)$$

Each solution P of the Riccati equation in (4.2) corresponds to a particular $n-\rho_i$ dimensional invariant subspace V of the form (3.7). Thus, (4.2) will have a solution P if and only if subspaces of the form (3.7) exist. If P satisfies (4.2) we can rewrite that equation as

$$F^i P - PA^i = PK^i P - E^i . \qquad (4.3)$$

We can bound the roots of (4.3) as follows. First note that

$$T(P) = F^i P - PA^i \qquad (4.4)$$

is a linear operator in P. Hence, if K_i is small enough the quadratic term in (4.3) can be neglected and we obtain the linear equation

$$T(P) = -E^{i} . \tag{4.5}$$

This equation has a solution if T(P) is invertible. It can be shown (Stewart, 1973) that if (4.5) has a solution then (4.3) has a solution (under appropriate conditions). Thus, existence of solutions of (4.3) is related to the invertibility of T(P). So motivated we define

Definition 4.4 (Stewart, 1973): The **separation** of F^{i} and A^{i}, denoted by δ, is defined as

$$\delta = \begin{cases} \| T^{-1} \|^{-1} & 0 \notin \lambda(T) \\ 0 & 0 \in \lambda(T) \end{cases} .$$

where $\lambda(T)$ denotes the eigenvalues of T.

□

Remark 4.5. It is well known that T has nonzero eigenvalues iff F^{i} and A^{i} have no common eigenvalues.

□

Remark 4.6. Varah (1979) discusses the separation function.

□

Also note that from the approximate linear equations we get

$$\| P \| \leqslant \| T^{-1} \| \cdot \| E^{i} \| . \tag{4.6}$$

The ideas sketched above are stated precisely in the following theorem.

Theorem 4.7 (Stewart, 1973): Let $\delta = \| T^{-1} \|^{-1}$, $\gamma = \| E^{i} \|$, $\eta = \| K^{i} \|$. Then if

$$\frac{\gamma\eta}{\delta^2} < \frac{1}{4} \qquad (T.1)$$

there exists a matrix P which satisfies (4.9) such that

$$\| P \| \leqslant \frac{2\gamma}{\delta}. \qquad (T.2)$$

□

Remark 4.8. Statement (T.2) of Theorem 4.7 provides an estimate of the distance from L_i to V which is computed in terms of the known data in (2.3).

□

Remark 4.9 As $\| E^i \| \to 0$ then $V \to L_i$ (in the gap topology) as suggested by the GHR. However, we see that the "smallness" of $\| E^i \|$ is measured against the separation of F^i and A^i.

□

Definition 4.10. The decomposition (2.3) is called **separable** if the separation of F^i and A^i is nonzero.

□

In the following we will relate separable decompositions to near unobservability and near aggregability. The idea is to study the system structure as $\| F_{i,i+1} \| \to 0$ in one case, and as $V \to L_i$ in the other. To accomplish this analysis we will parameterize the system matrix. Note that these parameterizations only change the operator on X but leave the underlying structure of X (specifically the inner product) invariant.

Consider (2.3) and parameterize the system in μ by replacing $F_{i,i+1}$ with $\mu F_{i,i+1}$.

Corollary 4.11. Suppose that (2.3) is separable. Then there exists a $\bar{\mu}$ such that for all $0 < \mu \leqslant \bar{\mu}$ there exists an $(n-\rho_i)$-dimensional A-invariant subspace.

□

Proof: Immediate from Theorem 4.7.

□

The connection to near unobservability is now obvious. Again parameterize (2.3) on μ as above. Let ϵ_0 be given.

Proposition 4.12. Suppose that (2.3) is separable. Then there exists a μ_0 such that for all $0 < \mu \leqslant \mu_0$, the system is ϵ_0-unobservable.

□

Proof: By Corollary 4.11 there exists $\bar{\mu}$ which guarantees the existence of an $(n-\rho_i)$-dimensional invariant subspace. Select $\mu_0 < \bar{\mu}$ such that

$$\mu_0 < \frac{\delta \epsilon_0}{2 \| E^i \|}$$

Theorem 4.7 together with (4.7) implies that $\| P \| < \epsilon_0$. Now the proposition follows from (3.6).

□

We can establish a converse to Proposition 4.12 as follows. In (2.3), suppose there exists an invariant subspace V which has a basis of the form (3.7). From (2.3) define a parameterized set of systems as follows:

$$\dot{x} = \bar{F}^i(\epsilon)\, x + \bar{G}^i u\,, \qquad y = \bar{H}^i x\,. \tag{4.8}$$

where $\bar{G}^i$ and $\bar{H}^i$ are identical to (2.3) and

$$F^{1}(\epsilon) = \begin{bmatrix} I & -\epsilon P \\ O & I \end{bmatrix} \begin{bmatrix} \Lambda_1 & O \\ \Lambda_{12} & \Lambda_2 \end{bmatrix} \begin{bmatrix} I & \epsilon P \\ O & I \end{bmatrix}. \tag{4.9}$$

Note that at $\epsilon = 1$, (4.8) and (2.3) are identical. Also note that $\tau(L_1, V) \to 0$ as $\epsilon \to 0$.

Let μ_o be given.

Proposition 4.13. Consider the system defined by (4.8). There exists an $\epsilon_o > 0$ such that for all $0 < \epsilon < \epsilon_o$, (4.8) is μ_o-aggregable.

□

Proof: From (4.9) we have

$$E^1(\epsilon) = \epsilon\Lambda_1 P - P\Lambda_2 \epsilon - \epsilon^2 P\Lambda_{21} P \tag{4.10}$$

so that if

$$\begin{aligned} \| E^1(\epsilon) \| &\leqslant \| \Lambda_1 P - P\Lambda_2 \| \, \epsilon + \| P\Lambda_{21} P \| \, \epsilon^2 \\ &= a\epsilon + b\epsilon^2 < \mu_o \end{aligned} \tag{4.11}$$

then (4.8) is μ_o-aggregable. The last inequality in (4.11) is satisfied for all $0 < \epsilon < \epsilon_o$ if ϵ_o is chosen less than the positive root of

$$b\epsilon^2 + a\epsilon - \mu_o = 0 . \tag{4.12}$$

□

5. POLE-ZERO CANCELLATION

In the previous sections we have related (near) aggregation to (near un-) observability. We can extend these concepts to transfer function matrices by generalizing the following well known result: If the system is unobservable then the transfer function matrix exhibits a pole-zero cancellation. Thus, exact aggregation corresponds to pole-zero cancellation. Below, we will relate near aggregation and near unobservability to almost pole-zero cancellation.

Consider (2.3). In this section we assume that

$$G^i = \begin{bmatrix} O \\ \cdot \\ \cdot \\ \cdot \\ O \\ G_i \end{bmatrix} \tag{5.1}$$

and that G_i is nonsingular. This assumption is equivalent to assuming the number of inputs equals the number of outputs and the first nonzero Markov parameter is nonsingular.

Define the following state transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} I & O \\ X & I \end{bmatrix} \begin{bmatrix} y_a^i \\ x_r^i \end{bmatrix} . \tag{5.2}$$

$$X = [O \quad -B_{i+1}G_i^{-1}] .$$

Substituting (5.2) into (2.3) we obtain

$$\begin{bmatrix} \dot{z}_1 \\ \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix} + \begin{bmatrix} G^i \\ \\ O \end{bmatrix} u \, .$$

$$y = [H^i \quad O] \begin{bmatrix} z_2 \\ \\ z_2 \end{bmatrix} . \tag{5.3}$$

Remark 5.1. It is shown in Lindner (1982) that (5.3) has the following properties:

1) $V^* = L_i = \mathrm{sp} \begin{bmatrix} O \\ \\ z_2 \end{bmatrix}$, where V^* is the largest (A,B)-invariant subspace in N[C] (Wonham, 1979).

2) $R^* = 0$, where R^* is the largest controllability subspace in N[C] (Wonham, 1979), and,

3) the invariant zeros of (5.3) are the eigenvalues of S_4.

It is interesting that the invariant zeros should be explicitly displayed in (5.3). We will use this structure below.

□

Remark 5.2. The natural inner product implied by the basis in (5.3) is not the same as the natural inner product implied by the basis in (2.1) (see (5.2)). In this section we assume that the underlying system leads to no natural inner product for (2.1). In this case the choice of inner product is arbitrary and we choose the natural inner product associated with the basis in (5.3). See also, Remark 5.9 below.

□

To see where we are going, suppose that the system in (5.3) has an $(n-\rho_i)$-dimensional invariant subspace V of the form

$$V = \mathrm{sp} \begin{bmatrix} P \\ I \end{bmatrix}. \qquad (5.4)$$

Furthermore, suppose that (5.3) is ϵ_0-unobservable so that

$$\| P \| \leqslant \frac{\epsilon_0}{\sqrt{1-\epsilon_0}} . \qquad (5.5)$$

(see (3.6)). Using (3.2) to define a change of basis $\omega = \bar{P}z$, it can be shown (Stewart, 1973) that

$$\lambda(S) = \lambda(S_1 - PS_3) \cup \lambda(S_4 + S_3P) . \qquad (5.6)$$

Recall that $p_i \in \lambda(S_4+S_3P)$ are the poles of (5.3) (see (5.6)) and $z_i \in \lambda(S_4)$ are the invariant zeros of (5.3) (Remark 5.1). Therefore, if S_3P is a "small" perturbation of the eigenvalues of S_4, then (5.3) should exhibit an almost pole-zero cancellation.

Note that how close the poles and zeros are depends on the block S_3 in (5.3). This block, which is associated with the controllability structure of (5.3), has not played a direct role in our results so far. Intuitively, its appearance in our analysis says that the controllability structure as well as the observability structure plays a role in determining almost pole zero cancellations. (This is in contrast to cancellations in which only one concept is involved per cancellation.) We can conclude both concepts in our analysis here by extending Definition 2.1. Let $\mu_0 > 0$ be given.

Definition 5.3. The system (5.3) is μ_0-**input/output aggregable** if $\| S_2 \| < \mu_0$ and $\| S_3 \| < \mu_0$.

□

Remark 5.4. It is easily seen by duality that all of the previous result on observability have corresponding results on controllability. By the dual of the results of Section 4, near input aggregability can be related to near uncontrollability.

□

We can characterize the deviation of the eigenvalues of S_4, say, from the eigenvalues of S by applying an eigenvalue perturbation theorem.

Theorem 5.5 (Bauer and Fike, 1960): Let $X^{-1}S_4X = \text{diag}(\lambda_i)$ and define $K(X) = \|X\| \cdot \|X^{-1}\|$. Let μ be an eigenvalue of S_4+S_3P. Then

$$\min_i |\lambda_i - \mu| \leqslant K(X) \|S_3P\| .$$

□

Theorem 5.6. Assume that (5.3) is μ_0-input/output aggregable and ϵ_0-unobservable. Also, assume S_4 in (5.3) has simple structure. Then

$$\min_j |p_j - z_i| \leqslant K(X) \frac{\mu_0 \epsilon_0}{\sqrt{1-\epsilon_0}} .$$

□

Proof: Combine Theorem 5.3 and (5.5).

□

Remark 5.7. Roughly speaking, Theorem 5.4 says that a system which is nearly input/output aggregable and nearly unobservable exhibits an almost pole zero cancellation.

□

Example 5.8. Consider the transfer function

$$G(s) = \frac{s}{(s+\epsilon)(s+1)} . \tag{5.7}$$

Realizing (5.7) in controllable canonic form we obtain

$$\begin{bmatrix} \dot{y}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} -(1+\epsilon) & -\epsilon \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \, ,$$

$$y = [\,1 \quad 0\,] \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} \tag{5.8}$$

Then *we can check the near unobservability of (5.8) by computing a basis of the form* (5.4) for the eigenvector associated with the eigenvalue ϵ. In fact, P is given by

$$P = \tan\theta = -\epsilon \, . \tag{5.9}$$

This equation clearly shows the relationship between near aggregation and near unobservability. These two concepts are related to pole-zero cancellation by applying Theorem 5.6:

$$|\epsilon - 0| \leqslant 1 \cdot |\epsilon| \, . \tag{5.10}$$

Equation (5.10) is a state space version of the pole-zero cancellation evident in (5.7).

Note that these results do not depend on eigenvalue separation.

□

Remark 5.9. Note that (5.8) is a μ_0-aggregable for $\epsilon < \mu_0$ but μ_0-input/output aggregable for $1 < \mu_0$. Suppose we scale z_1 by

$$\bar{z}_1 = \sqrt{\epsilon}\, z_1 \, . \tag{5.11}$$

After substituting (5.11) into (5.8), the new representation is μ_0-input/output aggregable for any $\mu_0 \geqslant \sqrt{\epsilon} \cdot 1$ but not μ_0-input/output aggregable for any $\mu_0 < \sqrt{\epsilon}$. On the other hand a scaling similar to (5.11) could make (5.8) μ_o-aggregable for any μ_o. Apparently, there is a natural μ_o for input/output aggregation associated with (5.8). This idea is completely worked out for single-input single-output systems by Lindner (1984). It is shown that the geometric structure leads to a natural basis in the state space. The metric associated with that basis is then a natural setting for near unobservability.

□

Example 5.10. Consider again the system in Example 3.7. This system is nearly unobservable. The transfer function of this model is

$$\frac{Y(s)}{U(s)} = P(s) = \frac{(s + .07)}{(s+.0749)(s+4.04984)(s+17.1785)(s+.35055)} \qquad (5.12)$$

which shows an almost pole-zero cancellation.

□

Example 5.11. This example shows that when the assumption at the beginning of the section is relaxed, the results fail. Consider the transfer function matrix (Rosenbrock, 1970)

$$P(s) = \begin{bmatrix} \frac{1}{(s+1)} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+2)} \end{bmatrix}. \qquad (5.13)$$

This matrix has Smith-McMillan form

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)^2(s+2)^2} & 0 \\ 0 & s+2 \end{bmatrix} \qquad (5.14)$$

which shows a pole and zero at -2 but does not exhibit a pole-zero cancellation. A state space representation of this system is given by

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ \hline 3 & -7 & -4 & 1 \\ 1 & -4 & -1 & -2 \end{array}\right] \begin{bmatrix} y \\ z \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \hline 1 & -3 \\ 0 & 0 \end{array}\right] u \, . \tag{5.15}$$

$$y = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} y \\ z \end{bmatrix} .$$

Note that G_1 is singular in (5.15). Also note that $V = L_2$.

It can be checked that the Jordan form of A in (5.15) has a block of order 2 corresponding to each eigenvalue $\lambda=-1$ and $\lambda=-2$. Therefore, there are two one-dimensional invariant subspaces which are possibly near L_2. The eigenvector corresponding to $\lambda=-2$ is

$$v_{-2}^T = [\,.889, -.111, -.444, 1\,] = [\,P^T \quad 1\,] \, . \tag{5.16}$$

The angle between V_{-2} and L_2 is

$$\Theta = \tan^{-1} \| P \| = 45^\circ \tag{5.17}$$

so that

$$\tau(V_{-2}, L_2) = \sin \Theta = 0.707 \, . \tag{5.18}$$

The eigenvector corresponding to $\lambda = -1$ is

$$V_{-1}^T = [\,-.250, -.50, -1.75, 1\,] \, . \tag{5.19}$$

We have

$$\tau(V_{-1}, L_2) = \sin 61.4^\circ = 0.878 \, . \tag{5.20}$$

Since $\pi(V, L_2) = 1$ for any larger dimensional invariant subspace, we conclude that the system is not nearly unobservable. Therefore, we would not predict any almost pole-zero cancellation. Also note that (5.15) is not nearly aggregable.

□

6. CONCLUSIONS

We have generalized the notion of aggregation to near aggregation by using the GHR. This generalization required the introduction of a metric into the state space model, a novel feature of our approach. We then showed that if there exists an appropriately dimensioned invariant subspace, then near aggregation was equivalent to near unobservability from a geometric viewpoint.

Under certain restrictions, it is possible to interpret the previous results as pole-zero cancellations in the transfer functions. In order to interpret near unobservability in terms of pole-zero cancellations, we had to assume the first non-zero Markov parameter was nonsingular. This assumption can be relaxed for single-input-single-output systems (Lindner, 1984). The multivariable case is under investigation.

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