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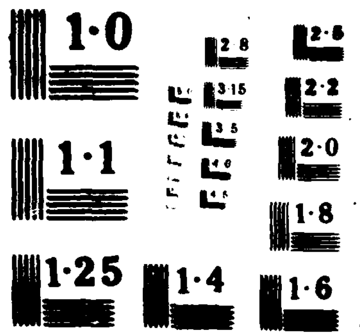
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ON THE CYCLABILITY OF

k-CONNECTED (k+1)-REGULAR GRAPHS

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ON THE CYCLABILITY OF

k-CONNECTED (k+l)-REGULAR GRAPHS

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1. Introduction

In the past fifteen years or so, there have been quite a number of papers dealing with variations on the following general theme. Given a graph G and a positive integer m, $m \leq |V(G)|$, find non-trivial conditions on G which will guarantee that given a set $S = \{v_1, \ldots, v_m\} \subseteq V(G)$, there exists a cycle C_S containing S. In the special case $m = |V(G)|$, we are dealing with conditions for the existence of Hamiltonian cycles, in itself a subject studied extensively by many graph theorists.

For the most recent survey of the subject for general m, the reader is directed to Holton [1983] and Plummer (1983). In particular, some interesting questions remain unsettled in the special case of regular graphs. Let $C(m)$ denote the class of all graphs which have the property that every set of m points lie on some cycle. The largest m for which $G \in C(m)$ is called the cyclability of G. Now suppose $k \geq 3$ and let $f(k)$ denote the largest integer j such that in every k-connected k-regular graph every j points lie on some cycle. It was proved by Holton (1982) and independently by Kelmans and Lomonosov (1982a) that $f(k) \geq k + 4$. This lower bound for $f(k)$ is not believed to be best possible. For example, Holton, McKay, Plummer and Thomassen (1984) proved that $f(3) = 9$. This result was also obtained by Kelmans and Lomonosov independently and announced without proof in (1982a). Meredith (1973) constructed an infinite family of graphs which show, among other things, that $F(k) \geq 10k - 11$. Thus a rather large gap in possible values for $f(k)$ remains at this writing. Recently, McQuaig and Rosenfeld (1984) have shown that for all even $k \geq 4$, there are infinite families of k-connected k-regular graphs with cyclabilities $8k - 4$ when $k \equiv 0 \pmod 4$ and $8k - 5$ when $k \equiv 2 \pmod 4$.

More recently, interest has been generated in the related question of cyclability of k-connected r-regular graphs for $r \geq k + 1$. First of all, Dirac (1960) proved that for any k-connected graph, regular or not, the cyclability is at least k. It is interesting to note that in the case of k-connected $(k + 1)$-regular graphs having k even, the Dirac bound cannot be improved. To see this, consider the complete bipartite graph $K_{k,k+1}$ where the bipartition sets U and W have $|U| = k$ and $|W| = k + 1$ respectively. The cyclability of $K_{k,k+1}$ is clearly k. We can easily modify $K_{k,k+1}$ to yield a graph H_k which is k-connected and $(k+1)$-regular by replacing each point of W by a copy of the graph obtained from K_{k+2} by deleting a matching of cardinality $\frac{k}{2}$. (Figure 1.1 shows how this is done for $k = 4$.)

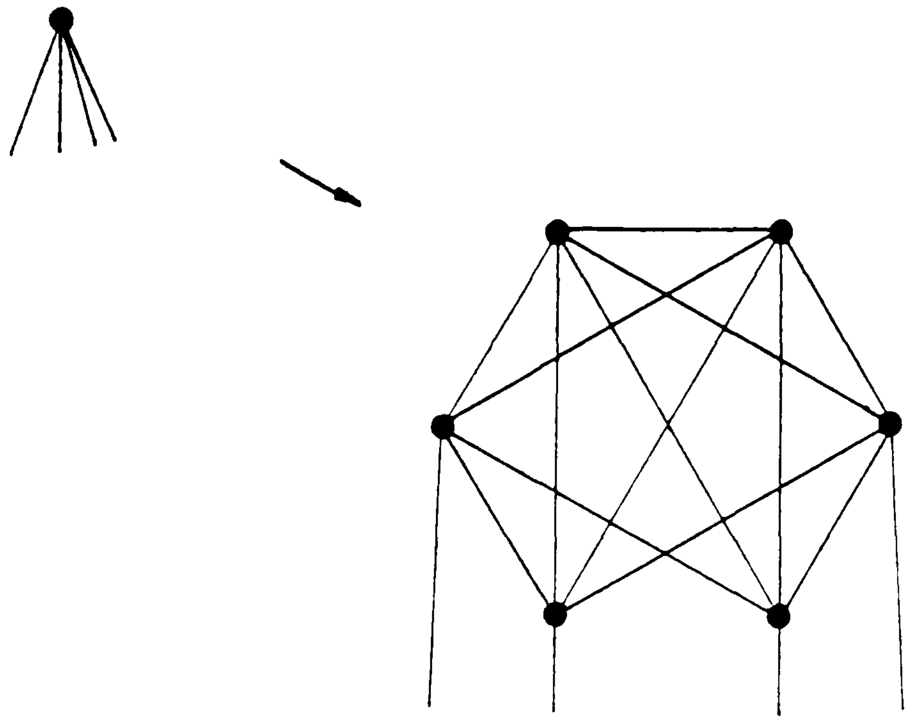


Figure 1.1

More generally, with k still even, but $r \geq k + 1$, Holton (1982) has constructed other graphs which are k-connected and r-regular, but which do not lie in $C(k + 1)$ and hence have cyclability precisely k.

Now suppose k is odd. If $r \geq k + 2$, Holton (1982) has constructed k-connected r-regular graphs which do not lie in $C(k + 1)$. This, again together with Dirac's bound, shows that any k-connected r-regular graph has cyclability = k, as long as k is odd and $r \geq k + 2$.

So in a sense, the only case left unsettled here is that of k-connected $(k + 1)$-regular graphs for $k \geq 3$ and k odd.

One can do a bit better than the Dirac bound here as was shown by Holton (1982), and independently by Kelmans and Lomonosov (1982b), via the following result.

<u>Theorem 1.1</u>. In any k-connected $(k + 1)$-regular graph with $k \geq 3$ and odd, any $k + 2$ points lie on a cycle.

Thus the cycability of such graphs is bounded below by $k + 2$.

In fact, Kelmans and Lomonosov (1982b) claimed that the conclusion of Theorem 11 can be improved to $k + 3$, but this claim is false, at least for $k = 3$. For a counterexample due to the present authors, see Holton, (1983). Since Kelmans and Lomonosov did not publish the proof of the $k + 3$ bound, the situation for k odd and $k \geq 5$ is presently unknown, at least to the present authors. In his 1982 paper, Holton goes on to show that if k is odd and $k \geq 3$ and if $h(k)$ is the largest positive integer m for which all k-connected $(k+1)$-regular graphs lie in $C(m)$, then $h(k) \leq 9k$.

In the present paper, we will prove that, in fact, $h(k) \leq 2k - 1$. (This result was announced without proof by Holton (1983).) To accomplish this, we shall construct, given $k \geq 3$ and odd, a graph G_k which is k-connected and $(k+1)$-regular, but which has a set of $2k$ points which do not lie on a common cycle. The procedure will be as follows. First we construct a graph G'_k which is k-connected and which has all points with degree either k or $k+1$, but which has a set of $2k$ points not lying on any cycle. Then we modify G'_k first to obtain an intermediate graph G''_k and then, in turn, modify G''_k to obtain a k-connected $(k+1)$-regular graph G_k having a set of $2k$ points which lie on no common cycle.

The construction is done in two slightly different ways depending upon whether $k \equiv 1 \pmod 4$ or $k \equiv 3 \pmod 4$. The reader is encouraged to refer to graphs G'_5 and G'_7 to help understand the constructions in general. (See Figure 2.1.)

2. The Construction of G'_k.

Let $k \geq 3$ be an odd integer. In all cases G'_k will be a bipartite graph with bipartition $(X \cup Y \cup Y', Z \cup Z')$ where

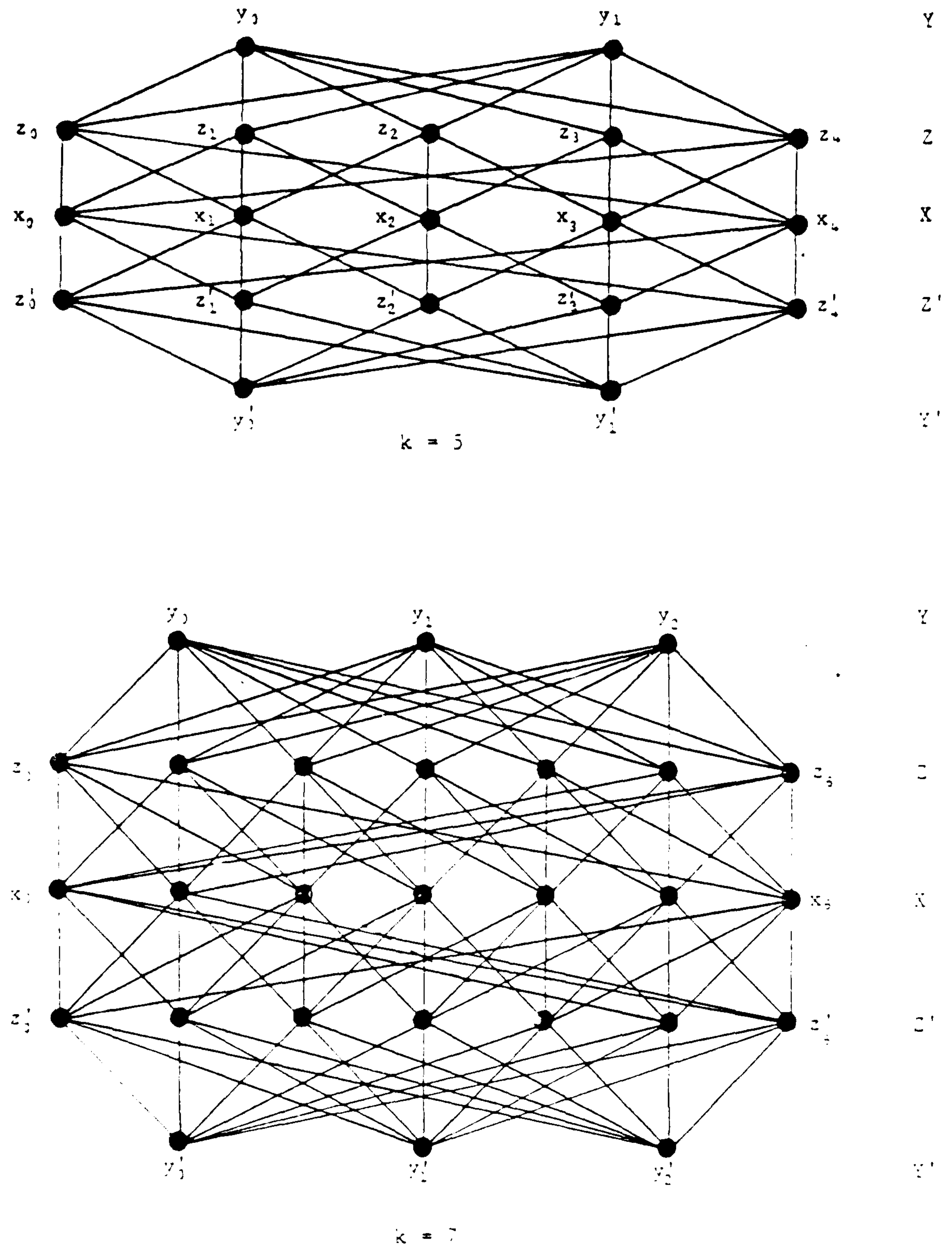
$$X = \{x_0, x_1, \ldots, x_{k-1}\},$$

$$Y = \{y_0, \ldots, y_{\frac{k-3}{2}}\}, \qquad Y' = \{y'_0, \ldots, y'_{\frac{k-3}{2}}\},$$

$$Z = \{z_0, \ldots, z_{k-1}\}, \qquad Z' = \{z'_0, \ldots, z'_{k-1}\}.$$

The lines in G'_k are defined as follows. Every point of Y (respectively Y') is adjacent to every point in Z (respectively Z'). For the remaining adjacencies we split the description into two cases. Suppose $k \equiv 1 \pmod 4$. For each $i = 0, \ldots, k-1$, both z_i and z'_i are adjacent to $x_i, x_{i-1}, x_{i+1}, \ldots, x_{i-\frac{k-1}{4}}, x_{i+\frac{k-1}{4}}$ where subscripts are taken modulo k. In the case in which $k \equiv 3 \pmod 4$, for $i = 0, \ldots, k-1$, both z_i and z'_i are adjacent to $x_i, x_{i-1}, x_{i+1}, \ldots, x_{i-\frac{k-3}{4}}, x_{i+\frac{k+1}{4}}$, where again the subscripts are taken modulo k.

The modulo k "circular symmetry" for adjacencies among the x_i's, z_j's and z'_k's is important to bear in mind and will prove to drastically reduce the number of cases we will have to treat in order to prove that G'_k is k-connected.



y_0
y_1
Y
z_0
z_1
z_2
z_3
z_4
Z
x_0
x_1
x_2
x_3
x_4
X
z'_0
z'_1
z'_2
z'_3
z'_4
Z'
y'_0
y'_1
Y'
k = 5
y_0
y_1
y_2
Y
z_0
z_6
Z
x_0
x_6
X
z'_0
z'_6
Z'
y'_0
y'_1
y'_2
Y'
k = 7

Figure 2.1

Finally, we note that G'_3 is just the well-known Hershel graph.

3. The connectivity of G'_k.

Note that in G'_k we have $\deg u = k$ for $u \in \cup Z \cup Z'$ and $\deg u = k+1$ for $u \in X$.

We now proceed to prove that G'_k is k-connected. To this end, let u and v be two distinct points in $V(G)$. We must find k openly disjoint paths joining u and v. We shall refer to such a family of paths as openly disjoint u - v paths. Here openly disjoint (hereafter abbreviated as o.d.) means that the paths joining u and v are otherwise pairwise point disjoint. We shall often refer to a set of k openly disjoint u - v paths as a k-skein joining u and v (or as a u - v k-skein).

1. First suppose $\{u,v\} \subseteq Y$. Say $u = y_0$ and $v = y_1$. Then $y_0z_0y_1, y_0z_1y_1, \ldots, y_0z_{k-1}y_1\}$ suffices as the u - v k-skein. The k-connection between two points of Y' follows by symmetry.

2. Suppose $u \in Y$ and $v \in Y'$. Without loss of generality, assume $u = y_0$ and $v = y'_0$. Then $\{y_0z_0x_0z'_0y'_0, \ldots, y_0z_{k-1}x_{k-1}z'_{k-1}y'_0\}$ suffices.

For the rest of the cases, we will treat the congruence classes $k \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$ separately.

First suppose $k \equiv 1 \pmod 4$. (Thus $k \geq 5$.)

3a. Suppose $u \in Y$ and $v \in Z$, say $u = y_0$ and $v = z_{\frac{k-1}{2}}$. Note that for $i = 0,1,\ldots,\frac{k-3}{2}$, we have z_i adjacent to $x_{\frac{k-1}{4}+i}$. So let

$$P_i = y_0\, z_i\, x_{\frac{k-1}{4}+i}\, z_{\frac{k-1}{2}}, \qquad \text{for } i = 0, \ldots, \tfrac{k-3}{2},$$

$$Q_i = y_0\, z_{\frac{k-1}{2}+i}\, y_i\, z_{\frac{k-1}{2}}, \qquad \text{for } i = 1, \ldots, \tfrac{k-3}{2} \qquad \text{and let}$$

$$R_1 = y_0\, z_{\frac{k-1}{2}} \quad \text{and} \quad S_1 = y_0\, z_{k-1}\, x_{\frac{3k-3}{4}}\, z_{\frac{k-1}{2}}.$$

Then $\{P_0, \ldots, P_{\frac{k-3}{2}}, Q_1, \ldots, Q_{\frac{k-1}{2}}, R_1, S_1\}$ is a u - v k-skein.

4a. Suppose $u \in Y$ and $v \in X$. Without loss of generality, suppose $u = y_0$ and $v = x_{\frac{k-1}{2}}$.

Let $P_i = y_0 \, z_{\frac{k-1}{4}+i} \, x_{\frac{k-1}{2}}$, for $i = 0, \ldots, \frac{k-1}{2}$.

Now let $Q_i = y_0 \, z_i \, x_{\frac{k-1}{4}+i} \, z'_{\frac{k-1}{4}+i} \, x_{\frac{k-1}{2}}$, for $i = 0, \ldots, \frac{k-5}{4}$

and let the "mirror images" of the Q_i's about the axis $z_{\frac{k-1}{2}} \, x_{\frac{k-1}{2}} \, z'_{\frac{k-1}{2}}$ be

$$R_i = x_{\frac{k-1}{2}} \, z'_i \, x_i \, z_{\frac{k-1}{4}+i} \, y_0, \quad \text{for } i = \frac{k+1}{2}, \ldots, \frac{3k-3}{4}.$$

We then have a total of $\frac{k+1}{2} + \frac{k-1}{4} + \frac{3k-3}{2} - \frac{k-1}{2} = k$ o.d. u - v paths as desired.

5a. Suppose $u \in Y$ and $u \in Z'$. Without loss of generality, let $u = y_0$ and $v = z'_{\frac{k-1}{2}}$. Then let

$$P_i = y_0 \, z_{\frac{k-1}{4}+i} \, x_{\frac{k-1}{4}+i} \, z'_{\frac{k-1}{2}}, \quad i = 0, \ldots, \frac{k-1}{2},$$

$$Q_i = y_0 \, z_i \, x_i \, z'_i \, y_i \, z'_{\frac{k-1}{2}}, \qquad i = 0, \ldots, \frac{k-5}{4}, \quad \text{and let}$$

$$R_i = y_0 \, z_{\frac{3k+1}{4}+i} \, x_{\frac{3k+1}{4}+i} \, z'_{\frac{3k+1}{4}+i} \, y'_{\frac{k-1}{4}+i} \, z'_{\frac{k-1}{2}} \quad \text{for } i = 0, \ldots, \frac{k-5}{4}.$$

We then have a total of $\frac{k+1}{2} + \frac{k-1}{4} + \frac{k-1}{4} = k$ o.d. u - v paths as sought.

6a. Suppose u and v are both in Z.

First note that any pair of z_i's have at least one common neighbour in X (and in fact, there are pairs of z_i's which have exactly one common neighbour). For example, (and for the sake of symmetry when working with the drawing in this case) let $u = z_{\frac{k-1}{4}}$ and let $v = z_{\frac{3k-3}{4}}$. Now let

$$P_i = z_{\frac{k-1}{4}} \, y_i \, z_{\frac{3k-3}{4}}, \qquad i = 0, \ldots, \frac{k-3}{2} \quad \text{and let}$$

$$P_{\frac{k-1}{2}} = z_{\frac{k-1}{4}} \, x_{\frac{k-1}{4}} \, z_{\frac{3k-3}{4}}. \quad \text{Next let}$$

$$Q_i = z_{\frac{k-1}{4}} \, x_i \, z'_i \, y_i \, z'_{\frac{k+1}{2}+i} \, x_{\frac{k+1}{2}+i} \, z_{\frac{3k-3}{4}} \quad \text{for } i = 0, \ldots, \frac{k-3}{2}.$$

Then we have a total of $\frac{k-1}{2}+1 + \frac{k-1}{2} = k$ o.d. $u-v$ paths as desired.

Now if the two z_i's chosen for u and v have $r \geq 2$ common neighbours in X, then in addition to the $\frac{k+1}{2}$ paths of type P_i above we get $r-1$ more of the form ux_jv. Taking these together with $\frac{k-1}{2}-(r-1)$ of type Q_i above, we get a total of $\frac{k+1}{2}+r-1+\frac{k-1}{2}-(r-l) = k$ o.d. $u-v$ paths as desired.

7a. Suppose $u \in Z$ and $v \in X$. Without loss of generality, assume $u = z_{\frac{k-1}{2}}$. There are now two cases to consider.

First suppose that $v \notin \Gamma(u) \cap X = \Gamma(z_{\frac{k-1}{2}}) \cap X$. (Here and throughout the rest of this paper $\Gamma(u)$ denotes the neighbourhood of u.) Let M denote the "vertical" matching of $\Gamma(z_{\frac{k-1}{2}})$ into all lines of which are of the form $x_i z'_i$. Then $|M| = \frac{k+1}{2}$ and we can find $\frac{k+1}{2}$ o.d. $u-v$ paths using M, r of them of length 3 where r is the number of neighbours of v in Z' which are covered by M and $\frac{k+1}{2}-r$ of length 5 which are of the form $v z'_j y'_k z'_m x_m z_{\frac{k-1}{2}}$, where $z'_m x_m \in M$, but $z'_m \notin \Gamma(v) \cap Z'$. On the other hand, v always has at least $\frac{k+1}{2}-1 = \frac{k-1}{2}$ neighbours in Z which are not equal to $z_{\frac{k-1}{2}}$ and these can be used to form an additional $\frac{k-1}{2}$ o.d. $u-v$ paths. Again, then, we get $\frac{k+1}{2}+\frac{k-1}{2} = k$ o.d. $u-v$ paths as required.

So we may suppose that $v \in \Gamma(u) \cap X = \Gamma(z_{\frac{k-1}{2}}) \cap X$. This time we have $\frac{k-1}{2}$ o.d. $u-v$ paths of length 3 of form $z_{\frac{k-1}{2}} y_i z_j v$, the line $z_{\frac{k-1}{2}} v$ and $\frac{k-1}{2}$ additional paths of length 3 or 5 obtained as follows. Consider the matching M' of $\Gamma(u) \cap X$ "vertically" into Z'; that is, all lines of M' are of the form $x_i z'_i$. Delete from M' the line covering v and denote by M'' the resulting matching of size $\frac{k-1}{2}$. Now if M'' covers a neighbour of v we get a path of length 3, while if a line e of M'' does not cover a point of $\Gamma(v) \cap Z'$, we can find a $u-v$ path of length 5 using e by detouring through Y'.

Now if u and v have $r \geq 2$ common neighbours, then it is easy to see that there is still a set of $\frac{k-1}{2}$ $u-v$ paths of length 2 or 4 where the paths of length 4 are of the form $z_{\frac{k-1}{2}} y_j z_m x_m z'_i$, where $z_{\frac{k-1}{2}} = u$ and $z'_i = v$. Then one can find an additional $\frac{k+1}{2}-r$ $u-v$ paths of length 4 of the form $z_{\frac{k-1}{2}} x_m z'_m y_j z'_i$ and the remaining $k-\left(r+\frac{k+1}{2}-r+\frac{k+1}{2}-r\right) = r-1$ paths of length 6 having the form

$z_{\frac{k-1}{2}}\, y_j\, z_m\, x_m\, z'_m\, y'_n\, z_j$.

8a. Suppose $u \in Z$ and $v \in Z'$. Without loss of generality, let $u = z_{\frac{k-1}{2}}$. Now note that regardless of where v is in set Z', $r = |(\Gamma(u) \cap X) \cap (\Gamma(v) \cap X)| > 0$. So let $s_1 = |(\Gamma(u) \cap X) - (\Gamma(v) \cap X)|$, let $s_2 = |(\Gamma(u) \cap X) - (\Gamma(v) \cap X)|$ and $s_3 = |X - (\Gamma(u) \cup \Gamma(v))|$. Then clearly $r + s_1 + s_2 + s_3 = k$. Those members of X counted by r give rise to r o.d. $u - v$ paths of length 2. For each point x_i of X counted by s_1 take line $x_i z'_i$, for each counted by s_2 take line $x_i z_i$ and for each counted by s_3 take the path $z_i\, x_i\, z'_i$. The lines $x_i\, z'_i$ counted by s_1 give rise to $u - v$ paths of length 4 all of form $u\, x_i\, z'_i\, y'_j\, v$, those counted by s_2 yield $u - v$ paths of length 4 of form $u\, y_j\, z_i\, x_i\, v$ and those counted by s_3 yield $u - v$ paths of length 6 all of the form $u\, y_j\, z_i\, x_i\, z'_i\, y_j\, v$. Altogether, these form a collection of k o.d. $u - v$ paths.

9a. Finally, suppose both u and $v \in X$. Let $u = x_i$ and $v = x_j$. Consider $\Gamma(x_i) \cap Z = N_Z(x_i)$. If $z_m \in N_Z(x_i)$ then if it is also in $\Gamma(x_j) \cap Z = N_Z(x_j)$ we have a path of length 2 - namely $x_i\, z_m\, x_j$ - joining x_i and x_j. On the other hand, if $Z_m \in N_Z(x_i - N_Z(x_j)$ then we have a path of length 4 - namely $x_i\, z_m\, y_n\, z_s\, x_j$ - joining x_i and x_j. This yields a total of $\frac{k+1}{2}$ o.d. $u - v$ paths and they all lie within $G'_k[X \cup Z \cup Y]$. But clearly there is a second set of $\frac{k+1}{2}$ o.d. $u - v$ paths (the reflections of the first set of paths in the X axis) which, together with the first set yields a total of $k + 1$ o.d. $u - v$ paths as sought.

Now let us suppose $k \equiv 3 \pmod 4$.

3b. Suppose $u \in Y$ and $v \in Z$. Without loss of generality, suppose $u = y_0$ and $v = z_{\frac{k-1}{2}}$.

Let $P_i = y_0\, z_i\, x_{\frac{k+1}{4}+i}\, z_{\frac{k-1}{2}}$, for $i = 0, \ldots, \frac{k-3}{2}$,

$Q_i = y_0\, z_{\frac{k-1}{2}+i}\, y_i\, z_{\frac{k-1}{2}}$, for $i = 1, \ldots, \frac{k-3}{2}$,

$R_1 = y_0\, z_{\frac{k-1}{2}}$ and $S_1 = y_0\, z_{k-1}\, x_{\frac{3k-1}{4}}\, z_{\frac{k-1}{2}}$.

Then $\{P_0, \ldots, P_{\frac{k-3}{2}}, Q_1, \ldots, Q_{\frac{k-3}{2}}, R_1, S_1\}$ is a k-skein joining u and v.

4b. Suppose $u \in Y$ and $v \in X$. Without loss of generality, suppose $u = y_0$ and $v = x_{\frac{k-1}{2}}$.

Let $P_i = y_0 z_{\frac{k-3}{4}+i} x_{\frac{k-1}{2}}$, $i = 0, \ldots, \frac{k-1}{2}$. Now if $k \neq 3$, let

$Q_i = y_0 z_i x_{\frac{k+1}{4}+i} z'_{\frac{k+1}{4}+i} x_{\frac{k-1}{2}}$, $i = 0, \ldots, \frac{k-7}{4}$ and if $k = 3$, let $Q_0 = \phi$.

Also let $R_i = x_{\frac{k-1}{2}} z'_i x_{i+1} z_{\frac{k+1}{4}+i} y_0$, for $i = \frac{k-1}{2}, \ldots, \frac{3k-5}{4}$. We then have

a total of $\frac{k+1}{2} + \frac{k-3}{4} + \frac{3k-5}{4} - \frac{k-3}{2} = k$ o.d. u - v lines in all cases.

5b. Suppose $u \in Y$ and $v \in Z'$. Without loss of generality, let $u = y_0$ and $v = z'_{\frac{k-1}{2}}$.

Then let $P_i = y_0 z_{\frac{k+1}{4}+i} x_{\frac{k+1}{4}+i} z'_{\frac{k-1}{2}}$ for $i = 0, \ldots, \frac{k-1}{2}$.

$Q_i = y_0 z_i x_i z'_i y'_i z'_{\frac{k-1}{2}}$ for $i = 0, \ldots, \frac{k-3}{4}$, and

and let $R_i = y_0 z_{\frac{3k+3}{4}+i} x_{\frac{3k+3}{4}+i} z'_{\frac{3k+3}{4}+i} y'_{\frac{k+1}{4}+i} z'_{\frac{k-1}{2}}$ for $i = 0, \ldots, \frac{k-7}{4}$, when $k \geq 7$.

(For $k = 3$, let $R_3 = \phi$.)

We then have a total of $\frac{k+1}{2} + \frac{k+1}{4} + \frac{k-3}{4} = k$ o.d. u - v paths as desired.

6b. Suppose u and v are both in Z. Again, as in 6a, note that every pair of x_i's have at least one common neighbour in X and in fact there are pairs with exactly common neighbour. Let $u = z_{\frac{k-3}{4}}$ and $v = z_{\frac{3k-5}{4}}$, for example.

First suppose $k \geq 7$.

Now let

$P_i = z_{\frac{k-3}{4}} y_i z_{\frac{3k-5}{4}}$, $i = 0, \ldots, \frac{k-3}{2}$ and let

$P_{\frac{k-1}{2}} = z_{\frac{k-1}{4}} x_{\frac{k-1}{2}} z_{\frac{3k-3}{4}}$. Next let

$Q_i = z_{\frac{k-3}{4}} x_i z'_i y_i z'_{\frac{k+1}{2}+i} x_{\frac{k+1}{2}+i} z_{\frac{3k-5}{4}}$ for $i = 0, \ldots, \frac{k-3}{2}$.

Thus we obtain a total of $\frac{k-1}{2}+1 + \frac{k-1}{2} = k$ o.d. $u - v$ paths as desired. If $k = 3$, then 3 o.d. $u - v$ paths are obvious.

Now if $k \geq 7$ and the 2 z_i's chosen for u and v have $r \geq 2$ common neighbours in X, then in addition to the $\frac{k+1}{2}$ paths of type P_i above, we get $r - 1$ more of the form $u\,x_j\,v$. Taking these together with $\frac{k-1}{2} - (r-1)$ of type Q_i above, we get a total of $\frac{k+1}{2} + r - 1 + \frac{k-1}{2} - (r-1) = k$ o.d. $u - v$ paths as desired.

The proofs of Cases 7b $(u \in Z,\ v \in X)$, 8b $(u \in Z,\ v \in Z')$ and 9b $(u, v \in X)$ are identical to those for Cases 7a, 8a and 9a respectively.

This completes the proof that G'_k is k-connected.

4. The Construction of G_k.

Recall that in graph G'_k each point in $Y \cup Y' \cup Z \cup Z'$ has degree k, while each point in X has degree $k+1$. We now proceed to construct a $(k+1)$-regular graph G_k from G'_k as follows.

First consider each line joining some $y_i \in Y$ to a $z_j \in Z$. Insert a new "midpoint" on this line and call it α_{ij}. Similarly, insert a midpoint β_{ij} on each line joining a z_i to an x_j. Midpoints are similarly inserted on lines joining a y'_i to a z'_j and on lines joining a z'_i to an x_j. They are called α'_{ij} and β'_{ij} respectively.

Now we replace each point of $Y \cup Y' \cup Z \cup Z'$ with a set of points as follows.

First suppose $k \equiv 1 \pmod 4$. For each $i \in \{0, \ldots, \frac{k-5}{2}\}$, replace y_i by a set A_i of $2k$ new points joined two by two to midpoints $\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{i(k-1)}$ respectively. Now replace $y_{\frac{k-3}{2}}$ by a set $B_{\frac{k-3}{2}}$ consisting of $2k$ points, k of them joined one at a time to each $\alpha_{\frac{k-3}{2}j}$ for $j = 0, \ldots, k-1$ and the remaining k joined to yet another new point b. Replace the y'_i's with sets A'_i and $B'_{\frac{k-3}{2}}$ in a symmetric manner.

Next, replace each $z_j \in Z$ with a set C_j of points as follows. For each line of the form $\alpha_{ij} z_j$ for $i \neq \frac{k-3}{2}$ insert $k-1$ new points into C_j and join each to α_{ij}. Also replace $\alpha_{\frac{k-3}{2}j} z_j$ with an additional k new points. Furthermore, for each line of the form $\beta_{jr} x_r$, insert k new points into C_j and join each to β_{jr}. (See Figure 4.1a.) Thus altogether, C_j contains $\frac{(k-3)}{2}(k-1) + k + \frac{(k+1)}{2}k = \frac{2k^2-k+3}{2}$ points, which since $k \equiv 1 \pmod 4$ is an even number.

Thus when $k \equiv 1 \pmod 4$, all of the sets A_i, C_j and $B_{\frac{k-3}{2}}$ contain an even number of points.

"Mirror image" sets A'_i, $B'_{\frac{k-3}{2}}$, C'_j and point b' are constructed analogously.

Now since each of the sets A_i, A'_i, $B_{\frac{k-3}{2}}$, $B'_{\frac{k-3}{2}}$, C_j, C'_j have more than k points and each is even, we may invoke Lemma 4a of Wang and Kleitman (1973) to conclude that there exists a k-connected k-regular graph on each of these sets of points. Insert such a k-regular graph on each such point set. Finally, join points b and b'. Clearly, the resulting graph G_k is $(k+1)$-regular.

Now suppose $k \equiv 3 \pmod 4$. In this case, we can construct a $(k+1)$-regular G_k which is even simpler than that built for the case $k \equiv 1 \pmod 4$ in that no "special" replacement for $y_{\frac{k-3}{2}}$ is necessary.

In G'_k, insert midpoints α_{ij} and β_{ij} as before. For each $i = 1, \ldots, \frac{k-3}{2}$ replace y_i by a set A_i of $2k$ points joined two by two to each α_{ij}. Replace each z_j by a set C_j consisting of $\frac{k-1}{2}(k-1)$ joined $k-1$ at a time to each of the $\frac{k-1}{2}$ different α_{ij}'s. Also add $k\frac{k-1}{2}$ additional points to C_j joined k at a time to each of the different β_{ij}'s. (See Figure 4.1b.)

Once again construct the "mirror image" sets A'_i and C'_j analogously.

Now each A_i and A'_i contains $2k$ points while each C_j and C'_j contains $\frac{(k-1)^2}{2} + \frac{k(k-1)}{2} = \frac{1}{2}(2k^2 - 3k+1)$ points which is also an even number since $k \equiv 3 \pmod 4$. Thus again by the Wang and Kleitman result we can construct k-connected k-regular graphs on each of these sets and hence obtain our $(k+1)$-regular graph G_k.

$k \equiv 1 \pmod 4$:

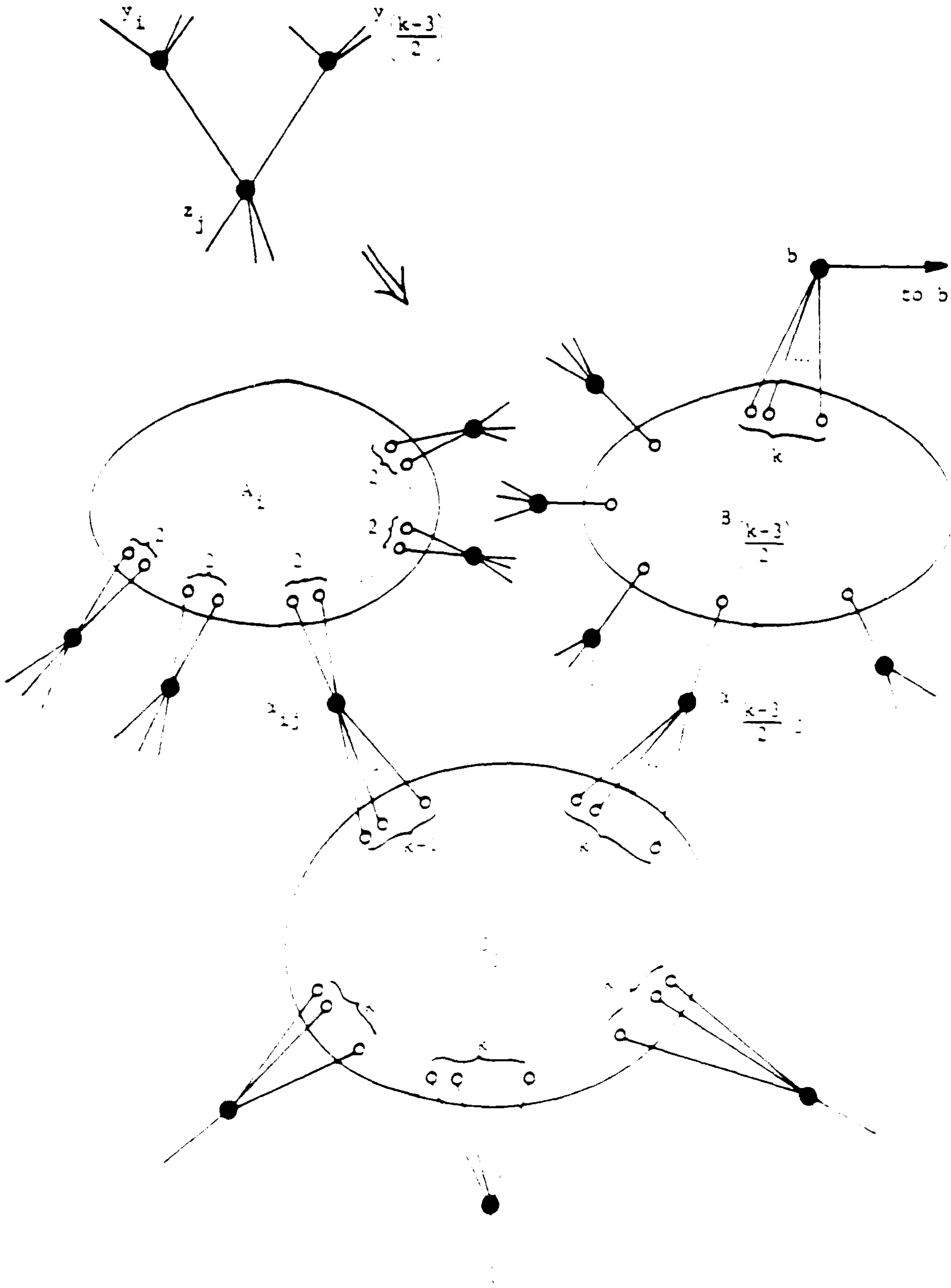


Figure 4 (a)

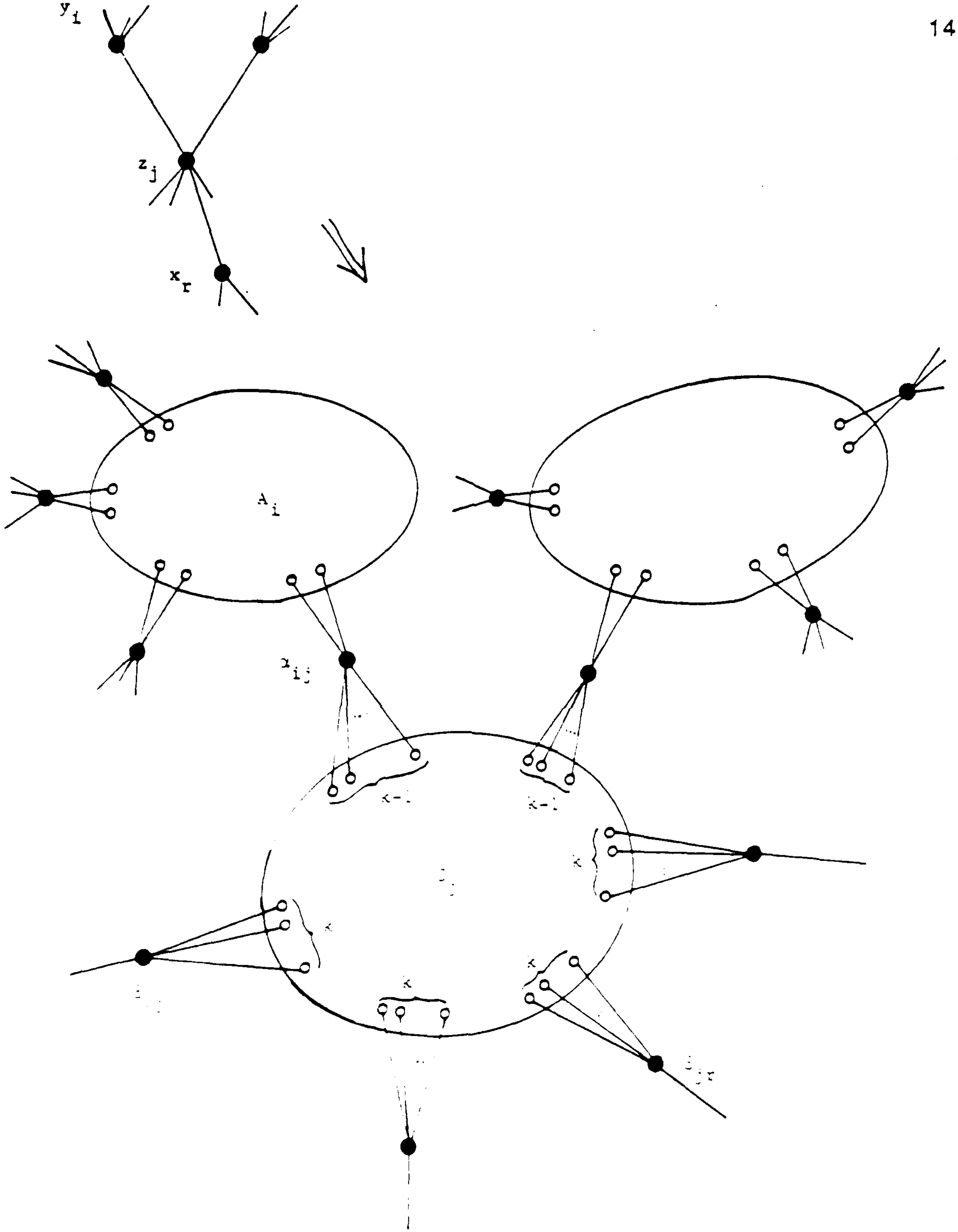


Figure 4.1 (b)

5. <u>The Connectivity of G_k</u>.

To prove that G_k is k-connected we proceed in two steps. First we consider an intermediate graph G''_k obtained from G'_k by inserting only the C_j's. (From this point on we shall denote the subgraphs guaranteed by the Wang and Kleitman result on A_i by $<A_i>$, on $B_{\frac{k-3}{2}}$ by $<B_{\frac{k-3}{2}}>$, etc.)

We now proceed to show G''_k to be k-connected. Let u and v be two distinct points in G''_k.

Suppose first that neither u nor v is a midpoint.

1. If u and v lie in the same C_j then there exist k o.d. u - v paths <u>in $<C_j>$</u>, since $<C_j>$ is k-connected. The analogous result holds when u and v lie in the same C'_j.

2. If u and v lie in two different C_j's, C'_j's or one in a C_j and the other in a C'_j, then there exist k o.d. u - v paths since such a set of paths exists in G'_k. More precisely, suppose $u \in C_u$ and $v \in C_v$. In C_u for each midpoint adjacent to C_u choose a point in C_u different from u. (Henceforth we shall refer to such a point as a <u>foot</u> of this midpoint in C_u.) This is possible because each midpoint has at least $k - 1 > 2$ such feet in C_u. So the feet selected in this way form a set of k distinct points in C_u different from u. Now since $<C_u>$ is k-connected by a well-known variation of Menger's Theorem, there exists a fan of paths in $<C_u>$ from u to each of the k feet chosen.

Repeat this procedure in $<C_v>$ and use these two k-fans, together with suitable pieces of the k o.d. paths in G'_k joining $<C_u>$ contracted to a point to $<C_v>$ contracted to a point.

This argument is also valid if u and v are in the same C_j, the same C'_j or one is in a C_j and the other in a C'_j.

3. Suppose $u \in Y \cup Y' \cup X$ and $v \in C_j$ or C'_j. Without loss of generality, suppose $v \in C_j$. Since G'_k is k-connected, there exist k o.d. u - v paths in G'_k and using the argument of Case 2, we can find k o.d. u - v paths in G''_k.

4. If $\{u,v\} \subseteq Y \cup Y' \cup X$, then k o.d. u - v paths are found using the k-connectedness of G'_k and the fact that all $<C_j>$'s are themselves k-connected.

So it remains to treat the cases when at least one of u and v is a midpoint. Note that in G''_k, the midpoints have degree k if they lie between Y and Z or between Y' and Z', and they have degree $k+1$ if they lie between X and Z or between X and Z'.

First suppose both u and v are midpoints in G''_k.

Let us now first consider the case when u and v are adjacent to the same $<C_i>$ (or $<C'_j>$). Then u and v are adjacent to at least k-1 different points of C_i respectively. By Menger's Theorem there are at least k-1 o.d. u - v paths in the subgraph $<C_i \cup \{u,v\}>$ of G''_k. Call them $P_1, \ldots, P_{k-1}$. Also since G'_k is 2- connected, there is a cycle N in G'_k containing the lines L_u and L_v (whose midpoints are u and v) and hence $N - L_u - L_v$ is a path which may be used to construct a path Q joining u and v which is openly disjoint from all the P_i's. Thus $\{P_1, \ldots, P_{k-1}, Q\}$ is the desired u - v k-skein.

Now suppose u and v are adjacent (as midpoints) to different $<C_j>$'s, say C_u and C_v respectively. Now in G'_k the 2 points corresponding to the contractions of $< C_u >$ and $< C_v >$ are joined by k o.d. paths. Call them $P_1, \ldots, P_k$. One of these - say P_1 - uses line L_u. Choose k-1 distinct feet of u in C_u. Call this set U_1. Also for each path P_i, $i \neq 1$, choose exactly one foot in C_u. Call this set U_2. We then have $U_1 \cup U_2 \subseteq C_u$, $U_1 \cap U_2 = \phi$ and $|U_1| = |U_2| = k-1$. Since $<C_u>$ is (k-1)-connected, by Menger's Theorem there exist k-1 totally disjoint paths in $<C_u>$ joining the points of U_1 to those of U_2. A similar argument applies to $<C_v>$. Using these paths within $<C_u>$ and $<C_v>$ as well as paths $P_1, \ldots, P_k$, we can construct k o.d. u - v paths in G''_k.

Finally, suppose u is a midpoint in G''_k, but v is not. Suppose u is adjacent to $<C_u>$. But this is even simpler than the proceding case. In G'_k, let $P_1, \ldots, P_k$ be k o.d. paths joining the contraction to a point of $<C_u>$ with point v. As before, let U_1 be a set of k-1 feet of u in C_u and choose U_2 so that it contains precisely one foot of each of the rest of the midpoints adjacent to $< C_u >$. Then $|U_1| = |U_2| =$ k-1, $U_1 \cap U_2 = \phi$ and since $<C_u>$ is (k-1)-connected we can proceed as before to get k o.d. u - v paths.

This completes the proof that G''_k is k-connected.

Now insert the A_i's, A'_i's, (and $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$ if $k \equiv 1 \pmod 4$)) into G''_k. Also insert points b and b' together with their respective k-fans to $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$. But do not join b and b' yet.

Actually, we will now show that $G_k - b - b'$ is k-connected. So suppose $\{u,v\} \subseteq V(G_k) - \{b,b'\}$.

I. Suppose $\{u,v\} \cap (A_i \cup A'_j) = \phi$, for all i,j and $\{u,v\} \cap (B_{\frac{k-3}{2}} \cup B'_{\frac{k-3}{2}}) = \phi$ when $k \equiv 1 \pmod 4$. Then since G''_k is k-connected there are k o.d. $u-v$ paths $P_1, \ldots, P_k$ in G''_k. Since all $\langle A_i \rangle$'s, $\langle A'_j \rangle$'s, $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$ are connected, paths $P_1, \ldots, P_k$ give rise to k o.d. paths $Q_1, \ldots, Q_k$ joining u and v in $\underline{G_k}$.

Before proceeding to the next case, we state and prove the following statement.

<u>Claim</u> (a) If $y_i \in Y$ corresponds to inserted subgraph A_i (respectively $B_{\frac{k-3}{2}}$) in G_k and if $L_1, \ldots, L_k$ are the k lines incident with y_i in G'_k, then given any point $u \in A_i$ (respectively $B_{\frac{k-3}{2}}$) there exists a k-fan in A_i (respectively $B_{\frac{k-3}{2}}$) which can be extended to a k-fan joining u to the midpoints of $L_1, \ldots, L_k$.

(b) Analogous statements hold for $y'_i \in Y'$ with respect to A'_i (respectively $B'_{\frac{k-3}{2}}$).

<u>Proof of Claim</u>. We prove only part (a) as (b) is proved in just the same way. Suppose y_i corresponds to A_i. Choose any point $u \in A_i$. Then u is one of exactly two feet in A_i of some midpoint x_i. Suppose w is the other of these two feet. Form a set U of k points by including w and exactly one of the two feet of all the other $k-1$ midpoints adjacent to A_i. Then since $u \notin U$ and $\langle A_i \rangle$ is k-connected, there exists a fan of paths from u to each of the k points in U which in turn leads to the k-fan sought.

Now suppose y_i corresponds to $B_{\frac{k-3}{2}}$. That is, $y_i = y_{\frac{k-3}{2}}$. Let $u \in B_{\frac{k-3}{2}}$. There are two cases to consider.

First suppose u is the foot of new point b. Then since u is not a foot of any of the midpoints of L_1, ..., L_k and since $B_{\frac{k-3}{2}}$ is k-connected, there is a k-fan of paths from u to the (unique) foot of each L_i in $B_{\frac{k-3}{2}}$. There are k such feet and this fan clearly extends to one from u to each of the k midpoints of $L_1, ..., L_k$.

So suppose u is the foot of some L_j in $B_{\frac{k-3}{2}}$. Without loss of generality, suppose u is the foot of L_1. Then since $B_{\frac{k-3}{2}}$ is k-connected, there is a fan at u to the feet in $B_{\frac{k-3}{2}}$ of each of the $k - 1$ lines $L_2, ..., L_k$. These $k - 1$ paths, together with the line from the foot of L_1 to the midpoint of L_1, clearly extends to a fan from u to the midpoint of each of $L_1, ..., L_k$ as desired. This completes the proof of the Claim.

2. Now suppose at least one of u,v lies in an A_i, A'_i, $B_{\frac{k-3}{2}}$, or $B'_{\frac{k-3}{2}}$, but that u and v do not both lie in the same one of these sets. Since G^*_k is k-connected, there are k o.d. $u - v$ paths in G^*_k which together with the fans guaranteed by the above Claim, where necessary, yield k o.d. $u - v$ paths in G_k.

3. If both u and v lie in the same $\langle A_i \rangle$, $\langle A'_i \rangle$, $\langle B_{\frac{k-3}{2}} \rangle$ or $\langle B'_{\frac{k-3}{2}} \rangle$, then since all of these subgraphs are k-connected, there exist k o.d. $u - v$ paths as desired.

Thus $G_k - b - b'$ is k-connected. It remains now to add points b and b', join them to k points each in $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$ as described earlier. But if we join b to its k points, the resulting graph $G_k - b'$ is k-connected by Menger's Theorem and then joining b' to its k neighbours, the resulting graph $G_k - bb'$ is k-connected by the same reasoning. But then adding line bb' we obtain G_k which must be k-connected. Clearly G_k is $(k + 1)$-regular.

Finally, we note that trivially the $2k$ points of $Z \cup Z'$ lie on no common cycle in G_k since $Z \cup Z'$ is an independent set and $|V(G_k) - (Z \cup Z')| = |Y \cup Y' \cup X| = k - 1$.

References

G.A. Dirac

1960. In abstrakten Graphen vorhandene vollstandige 4-Graphen und ihre Unterteilungen, Math. Nachr., 22, 1960, 61-85.

D.A. Holton

1982. Cycles through specified vertices in k-connected regular graphs, Ars Combinatoria, 13, 1982, 129-143.

1983. Cycles in graphs, Combinatorial Mathematics X (Proceedings, Adelaide 1982), Lecture Notes in Math. Vol 1036, Springer-Verlag, Berlin, 1983, 24-48.

D.A. Holton, B.D. McKay, M.D. Plummer and C. Thomassen

1982. A nine point theorem for 3-connected graphs, Combinatorica, 2, 1982, 53-62.

A.K. Kelmans and M.V. Lomonosov

1982a. When m vertices in a k-connected graph cannot be walked round along a simple cycle, Discrete Math., 38, 1982, 317-322.

1982b. On cycles through given vertices of a graph, Abstracts Amer. Math. Soc. No. 82T - 05 - 245, 3, 1982, 255.

W.D. MCCuaig and M. Rosenfeld

1984. Cyclability of r-regular r-connected graphs, Bull. Austral. Math. Soc., 29, 1984, 1-11.

G.H.J. Meredith

1973. Regular n-valent n-connected non-Hamiltonian non-n-edge colorable graphs, J. Combin. Theory Ser. B., 14, 1973, 55-60.

M.D. Plummer

1983 Some recent results on cycle traversability in graphs, Ann. Discrete Math., 20, 1983, 255 - 262.

D.L. Wang and D.J. Kleitman

1973. On the existence of n-connected graphs with prescribed degrees ($n \geq 2$), Networks, 3, 1973, 225-239.

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