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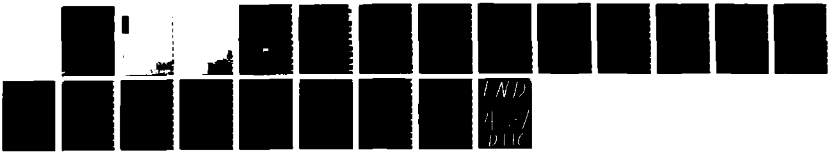

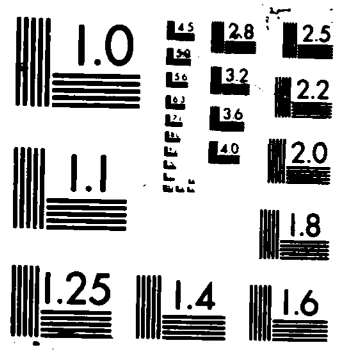


ROCOPY RESOLUTION TEST CHART

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# CONFIDENCE INTERVAL FOR PARAMETER *n* IN A BINOMIAL DISTRIBUTION


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## ABSTRACT

This research memorandum presents a simple procedure to approximate a confidence interval for the parameter $n$ in a binomial distribution. A simulation procedure to verify the coverage of confidence intervals is presented in appendix A. An interactive computer program is included in appendix B. The program is written in the FORTRAN language, which is readily available in most computing environments. Tables with 90-percent and 95-percent confidence coefficients are included in appendix C.

## TABLE OF CONTENTS

| | Page |
|---|---|
| Introduction | 1 |
| Background | 2 |
| Procedure | 3 |
| Remarks | 4 |
| References | 5 |
| Appendix A: Simulation Procedure | A-1–A-2 |
| Appendix B: An Interactive Computer Program | B-1–B-3 |
| Appendix C: Selected Tables | C-1–C-2 |

# INTRODUCTION

Consider the binomial distribution $b(n, p)$, where the parameter $n$ represents the number of trials and the parameter $p$ represents the probability of success. When $n$ is fixed in advance after observing $k$ successes, the usual problem is to estimate the probability of success $p$ in the experiment. Situations may also arise when $n$ becomes the unknown parameter of interest. If $p$ is assumed to be known and $k$ successes have been observed, the experimenter would be interested in estimating $n$ instead. Examples include the estimation of the total number of herds in a small area of Kruger Park, South Africa, with the number of herds being observed on several occasions. A detailed discussion of this was described by Carroll and Lombard [1]. This kind of problem has a direct implication to some naval operations, more specifically, search problems. When some threats have been detected in a certain region, the total number of threats in that region becomes a major concern. An estimate of the unknown quantity would be an important consideration in the decision-making process.

The next section presents the background and recent development of the problem of estimating $n$ in a binomial distribution. The third section derives the procedure of estimating $n$ in the form of a confidence interval. The last section consists of some concluding remarks. A simulation procedure, an interactive computer program, and selected tables are included in the appendixes.

## BACKGROUND

The usual problem to which a binomial distribution is applied is the estimation of $p$ given $k$ successes among $n$ trials. Although the problem of estimating $n$ has a long history, it has not been investigated intensively until recently. Olkin, Petkau, and Zidek [2] noted that both the method of moments estimator and the maximum likelihood estimator of $n$ are "highly unstable." They proposed some stabilized versions of these estimators. Blumenthal and Dahiya [3] offered an alternative stabilized maximum likelihood estimator by modifying the likelihood function. Carroll and Lombard [1] examined these estimators by applying a beta prior distribution to the probability of success $p$. They claimed that their estimators compared favorably with those introduced by Olkin et al [2]. Most recently, Casella [4] proposed to assess the stability of the estimation problem and to choose an appropriate point estimator based on the assessment. Each of the methods reported above require complicated iterations based on repeated counts of the number of successes. In most of our applications, it is often not feasible to recreate the same situations and to repeat the experiment over again. Therefore, the proposed procedure in this study is derived with the assumption that there is only one sample available. Unlike the works cited above, which provide point estimators of the parameter $n$, the computationally simple procedure developed here provides a confidence interval for the parameter.

This procedure is particularly useful for those users who need a quick and easy estimate of $n$ but have only limited computer capability. Confidence intervals are approximated by applying the central limit theorem. Because the normal approximation of the binomial distribution is involved, the confidence levels may not be obtained exactly. However, a simulation showed that they can, on the average, indeed achieve the specified confidence levels indicated by confidence coefficients. The procedure will be detailed in the next section.

## PROCEDURE

Let $Y$ be a random variable having a binomial distribution, $b(n,p)$. Then,

$$E(Y) = np \ ,$$

and

$$Var(Y) = np(1-p) \ .$$

By the central limit theorem, the fraction

$$\frac{Y-np}{\sqrt{np(1-p)}}$$

converges in distribution to $Z$, where $Z$ is a random variable having a standard normal distribution, $N(0,1)$. Therefore,

$$\text{Prob}\left\{-z_{\alpha/2} < \frac{Y-np}{\sqrt{np(1-p)}} < z_{\alpha/2}\right\} \doteq 1-\alpha \ ,$$

where $z_{\alpha/2}$ is the upper $100\alpha/2-$percentage value from a standard normal distribution table (e.g., $z_{0.05} = 1.645$). For a given confidence coefficient $1-\alpha$, a confidence interval for $n$ can be obtained by solving the inequality:

$$-z_{\alpha/2} < \frac{Y-np}{\sqrt{np(1-p)}} < z_{\alpha/2} \ .$$

And the double inequality can be reduced to the following quadratic inequality:

$$(Y-np)^2 < z_{\alpha/2}^2 np(1-p) \ .$$

When the number of successes has been observed (i.e., $Y = y$), the above expression is a quadratic inequality in $n$ and can be solved by the familiar quadratic formula. The solutions would give confidence limits for $n$ with a given confidence coefficient $1-\alpha$.

## REMARKS

The procedure described in the preceding section provides an easy, quick way to approximate a confidence interval for $n$ at a given confidence level when $p$ is known. In most naval operations, timing can be crucial. Scenarios may be changed in a short period of time, and it is therefore necessary to produce estimates quickly. The interactive computer program would help the users obtain an interval estimate of the parameter $n$ in a binomial distribution. The program can be adapted to almost any computer. Tables with confidence coefficients 90 percent and 95 percent are included in tables C-1 and C-2, respectively. One of the merits of the procedure is its simplicity in computation. When computers and tables are not available, the problem can still be solved by using a pocket calculator.

Because the derivation of the confidence intervals involves the normal approximation to a binomial distribution, the resulting confidence level might not be obtained due to the approximation. But this was not observed in a simulation study in which 5,000 replications were simulated at each combination of probability of success and confidence coefficient in tables C-1 and C-2. The results showed that the percent of the trials that the confidence limits were calculated using the methodology reported here did include the known values of the parameter $n$.

To apply this procedure effectively, the probability of success is assumed to be known. This assumption may not be fulfilled in practice. Yet prior information and subjective assessment could be used in determining a reasonable and acceptable probability of success.

## REFERENCES

[1] R. U. Carroll and F. Lombard. "A Note on $N$ Estimators for the Binomial Distribution," Journal of the American Statististical Association, 80 (1985): 423-426

[2] I. Olkin, A. J. Petkau and J. V. Zidek. "A Comparison of $n$ Estimators for the Binomial Distribution," Journal of the American Statistical Association, 76 (1981): 637-642

[3] S. Blumenthal and R. C. Dahiya. "Estimating Binomial Parameter $N$," Journal of the American Statistical Association, 76 (1981): 903-909

[4] G. Casella. "Stabilizing Binomial $n$ Estimators," Journal of the American Statistical Association, 81 (1986): 172-175

# APPENDIX A

# SIMULATION PROCEDURE

In this simulation, it is assumed that the confidence coefficient, $1 - \alpha$, and the probability of success $p$ are specified. For the given $p$, let $J$ be a large positive integer. $J$ may be chosen to be the right endpoint of the confidence interval with the largest possible number of successes under consideration. For example, when $1 - \alpha = 90$ percent and $p = 0.1$, $J$ may be chosen to be 341. (See table C-1.)

After $J$ has been determined, select a random integer $n_i$ between 1 and $J$, for $i = 1,2, \ldots, B$, where $B$ is the number of replicates intended for this simulation. Then select $n_i$ random numbers between 0 and 1, say

$$\rho_i^{(1)}, \rho_i^{(2)}, \ldots, \rho_i^{(n_i)} .$$

Define

$$a_s = \begin{cases} 1 & \text{if } \rho_i^{(s)} \leq p \\ 0 & \text{otherwise} . \end{cases}$$

Let

$$k_i = \sum_{s=1}^{n_i} a_s .$$

$k_i$ represents then the number of successes in the ith simulation. According to this $k_i$, there exists a confidence interval $[l_i, u_i]$. This confidence interval can be either computed by using the interactive computer program in appendix B or obtained from the tables included in appendix C. Define

$$b_i = \begin{cases} 1 & \text{if } l_i \leq n_i \leq u_i \\ 0 & \text{otherwise} . \end{cases}$$

Repeat the process $B$ times in the similar manner. And then,

$$\widehat{1 - \alpha} = \sum_{i=1}^{B} b_i / B$$

will give the percentage that $n$ is included in the intervals for a given $p$. If $\widehat{1 - \alpha}$ is close to $1 - \alpha$, the proposed method would be satisfactory for approximating confidence intervals for $n$.

In this study, with $1 - \alpha = 90$ percent and 95 percent and $B = 5{,}000$ replications, the results show that $\widehat{1 - \alpha}$ is approximately equal to 90 percent and 95 percent, respectively. For other values of $1 - \alpha$, simulation also shows similar results. It indicates that the proposed method provides a quick and easy way to construct confidence intervals for the parameter $n$ in a binomial distribution.

# APPENDIX B

# AN INTERACTIVE COMPUTER PROGRAM

```
C     CONFIDENCE INTERVAL FOR PARAMETER n IN A BINOMIAL DISTRIBUTION
C
C
C     THE PURPOSE OF THIS CODE IS TO ACCEPT INPUT PARAMETERS
C     (CONFIDENCE LEVEL, AND PROBABILITY OF SUCCESS),
C     AND INVOKE A ROUTINE TO COMPUTE CONFIDENCE INTERVALS
C     USING THE CENTRAL LIMIT THEOREM METHODOLOGY. A REPORT
C     DEPICTING CONFIDENCE INTERVALS IS PRODUCED AND WRITTEN
C     TO THE CRT.
C
C     PROGRAM INPUT  1). CONLEV
C                    2). PRSUCC
C     PROGRAM OUTPUT 1). CONFIDENCE INTERVALS
C
C     VARIABLE DEFINITIONS
C
C     PRSUCC    - PROBABILITY OF SUCCESS
C     CONLEV    - CONFIDENCE LEVEL
C     VT_MATRIX - COMPUTED TABLE OF LOWER AND UPPER LIMITS OF CONFIDENCE
C                 INTERVALS FOR K = 1 THROUGH 25 SUCCESSES, AND FOR
C                 SELECTED CONFIDENCE LEVEL AND PROBABILITY OF SUCCESS
C     K         - NUMBER OF SUCCESSES
C
C     DATA DECLARATION/INITIALIZATION
C
      INTEGER  VT_MATRIX(25,2), CONLEV
      REAL K
C
C     ACCEPT CONFIDENCE LEVEL FROM INPUT DEVICE, EDIT VALUE,
C     AND ASSIGN APPROPRIATE Z VALUE
C
   25 PRINT *, ' '
      PRINT *, 'ENTER CONFIDENCE LEVEL
      PRINT *, ' '
      PRINT *, 'CONFIDENCE LEVEL MUST BE 60, 70, 80, 90, 95, 98, OR 99'
      PRINT *, ' '
      ACCEPT 30, FLOAT_CONLEV
   30 FORMAT (F80.0)
      CONLEV = FLOAT_CONLEV
      IF (CONLEV .EQ. 99) THEN
         ZCONST = 2.576**2.0
      ELSEIF (CONLEV .EQ. 98) THEN
         ZCONST = 2.326**2.0
      ELSEIF (CONLEV .EQ. 95) THEN
         ZCONST = 1.960**2.0
      ELSEIF (CONLEV .EQ. 90) THEN
         ZCONST = 1.645**2.0
      ELSEIF (CONLEV .EQ. 80) THEN
         ZCONST = 1.282**2.0
      ELSEIF (CONLEV .EQ. 70) THEN
         ZCONST = 1.040**2.0
      ELSEIF (CONLEV .EQ. 60) THEN
         ZCONST = .8400**2.0
      ELSE
         GOTO 25
      ENDIF
C
C     ACCEPT PROBABILITY OF DETECTION FROM INPUT DEVICE AND EDIT VALUE
C
   35 PRINT *, ' '
```

```
      PRINT *, 'ENTER PROBABILITY OF DETECTION (MUST BE BETWEEN .01 AND
     &.50)'
      PRINT *, ' '
      ACCEPT 40, PRSUCC
   40 FORMAT (F80.2)
      IF (PRSUCC .GE. .01 .AND. PRSUCC .LE. .50) THEN
         GOTO 45
      ELSE
         PRINT *, ' '
         PRINT *, 'PROBABILITY OF DETECTION MUST BETWEEN .01 AND .50'
      GOTO 35
      ENDIF
C
C     INVOKE ROUTINE TO COMPUTE CONFIDENCE INTERVALS
C
C     COMPUTE FUNCTION CONSTANTS
C
   45 A = PRSUCC**2.0
      DO 48 I=1,25
         K = I
         B = -PRSUCC * (  (2.0*K) +  (ZCONST*(1.0-PRSUCC))  )
         C = K**2.0
C
C        INVOKE QUADRIATIC FORMULA USING FUNCTION CONSTANTS
C
         DISCRI = (B**2.0) - (4.0*A*C)
         SQRTDI = SQRT(DISCRI)
C
         XL = (-B - SQRTDI) / (2.0*A)
         XU = (-B + SQRTDI) / (2.0*A)
C
C        STORE LOWER AND UPPER ENDPOINTS
C
         IXL = XL
         IXU = XU
         VT_MATRIX(I,1) = IXL+1
         VT_MATRIX(I,2) = IXU
C
   48 CONTINUE
C
C     GENERATE REPORT
C
      WRITE (6,50)
   50 FORMAT (1H1,25X,'CONFIDENCE INTERVALS')
      WRITE (6,55)
   55 FORMAT (1H0,15X,'"NUMBER OF OBJECTS DETECTED IN A REGION"')
      WRITE (6,60) CONLEV
   60 FORMAT (1H0,1X,'CONFIDENCE LEVEL = ',I3,'%')
      WRITE (6,65) PRSUCC
   65 FORMAT (1H ,1X,'PROBABILITY OF DETECTION = ',F3.2)
      WRITE (6,70)
   70 FORMAT (1H0,15X,'NO OF SUCCESSES',11X,'LOWER',8X,
     &         'UPPER')
      WRITE (6,75)
   75 FORMAT (1H ,15X,' OBSERVED (K)',13X,'LIMIT',8X,
     &         'LIMIT')
      WRITE (6,80)
   80 FORMAT (1H ,15X,' ------------',13X,'-----',8X,
     &        '-----')
      DO 110 I=1,25
```

```
         WRITE (6,100) I, VT_MATRIX(I,1), VT_MATRIX(I,2)
  100    FORMAT (1H ,21X,I2,18X,I4,9X,I4)
  110 CONTINUE
C
      STOP
      END
```

# APPENDIX C

# SELECTED TABLES

**TABLE C-1**

90-PERCENT CONFIDENCE INTERVAL FOR $n$

| Number of | Probability of success | | | | | |
|---|---|---|---|---|---|---|
| successes | .01 | .1 | .2 | .3 | .4 | .5 |
| 1 | 23,445 | 3,41 | 2,19 | 1,12 | 1,8 | 1,6 |
| 2 | 67,601 | 7,57 | 4,27 | 3,17 | 3,11 | 2,8 |
| 3 | 121,747 | 13,71 | 7,34 | 5,21 | 4,15 | 4,11 |
| 4 | 181,887 | 19,85 | 10,41 | 7,26 | 6,18 | 5,14 |
| 5 | 245,1023 | 26,99 | 14,47 | 10,30 | 8,21 | 6,16 |
| 6 | 312,1156 | 33,112 | 17,54 | 12,34 | 9,25 | 8,19 |
| 7 | 381,1287 | 40,125 | 21,60 | 14,39 | 11,28 | 10,21 |
| 8 | 453,1415 | 47,137 | 24,66 | 17,43 | 13,31 | 11,24 |
| 9 | 525,1542 | 54,150 | 28,73 | 20,47 | 15,34 | 13,26 |
| 10 | 600,1668 | 62,163 | 32,79 | 22,51 | 17,37 | 14,28 |
| 11 | 675,1793 | 70,175 | 36,85 | 25,55 | 19,40 | 16,31 |
| 12 | 752,1916 | 77,187 | 40,91 | 27,59 | 21,43 | 18,33 |
| 13 | 829,2039 | 85,199 | 44,97 | 30,63 | 23,46 | 19,35 |
| 14 | 908,2160 | 93,211 | 48,103 | 33,67 | 25,49 | 21,38 |
| 15 | 987,2281 | 101,223 | 52,109 | 36,71 | 28,52 | 23,40 |
| 16 | 1066,2402 | 109,235 | 56,115 | 38,75 | 30,54 | 24,42 |
| 17 | 1146,2521 | 117,247 | 60,121 | 41,79 | 32,57 | 26,45 |
| 18 | 1227,2641 | 125,259 | 64,127 | 44,82 | 34,60 | 28,47 |
| 19 | 1309,2759 | 134,271 | 68,132 | 47,86 | 36,63 | 30,49 |
| 20 | 1390,2878 | 142,283 | 73,138 | 50,90 | 38,66 | 31,51 |
| 21 | 1473,2995 | 150,294 | 77,144 | 52,94 | 40,69 | 33,54 |
| 22 | 1555,3113 | 158,306 | 81,150 | 55,98 | 42,72 | 35,56 |
| 23 | 1638,3230 | 167,318 | 85,156 | 58,102 | 45,74 | 37,58 |
| 24 | 1721,3346 | 175,329 | 89,161 | 61,105 | 47,77 | 38,60 |
| 25 | 1805,3463 | 184,341 | 94,167 | 64,109 | 49,80 | 40,63 |

## TABLE C-2

95-PERCENT CONFIDENCE INTERVAL FOR $n$

| Number of successes | Probability of success .01 | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|---|
| 1 | 18,562 | 2,52 | 2,24 | 1.14 | 1,10 | 1,7 |
| 2 | 56,725 | 6,68 | 4,32 | 3,20 | 2,13 | 2,10 |
| 3 | 103,877 | 11,83 | 6,39 | 5,24 | 4,17 | 3,13 |
| 4 | 157,1024 | 17,98 | 9,46 | 6,29 | 5,21 | 5,15 |
| 5 | 215,1165 | 23,112 | 12,53 | 9,34 | 7,24 | 6,18 |
| 6 | 277,1304 | 29,126 | 15,60 | 11,38 | 9,27 | 7,20 |
| 7 | 341,1440 | 36,139 | 19,67 | 13,42 | 10,30 | 9,23 |
| 8 | 407,1573 | 42,152 | 22,73 | 16,47 | 12,33 | 10,25 |
| 9 | 475,1705 | 49,165 | 26,80 | 18,51 | 14,37 | 12,28 |
| 10 | 545,1835 | 56,178 | 29,86 | 20,55 | 16,40 | 13,30 |
| 11 | 616,1964 | 64,191 | 33,92 | 23,59 | 18,43 | 15,33 |
| 12 | 689,2091 | 71,203 | 37,99 | 26,63 | 20,46 | 17,35 |
| 13 | 762,2218 | 79,216 | 41,105 | 28,68 | 22,49 | 18,38 |
| 14 | 837,2344 | 86,228 | 44,111 | 31,72 | 24,52 | 20,40 |
| 15 | 912,2469 | 94,241 | 48,117 | 33,76 | 26,55 | 22,42 |
| 16 | 988,2593 | 101,253 | 52,123 | 36,80 | 28,58 | 23,45 |
| 17 | 1064,2716 | 109,265 | 56,129 | 39,84 | 30,61 | 25,47 |
| 18 | 1142,2839 | 117,278 | 60,135 | 41,88 | 32,64 | 27,49 |
| 19 | 1220,2961 | 125,290 | 64,141 | 44,92 | 34,67 | 28,52 |
| 20 | 1298,3082 | 133,302 | 68,147 | 47,96 | 36,70 | 30,54 |
| 21 | 1377,3203 | 141,314 | 72,153 | 50,99 | 38,73 | 32,56 |
| 22 | 1456,3324 | 149,326 | 76,159 | 52,103 | 40,75 | 33,59 |
| 23 | 1536,3444 | 157,338 | 80,165 | 55,107 | 42,78 | 35,61 |
| 24 | 1617,3564 | 165,350 | 85,171 | 58,111 | 45,81 | 37,63 |
| 25 | 1697,3683 | 173,361 | 89,177 | 61,115 | 47,84 | 38,65 |

END
4-87
DTIC