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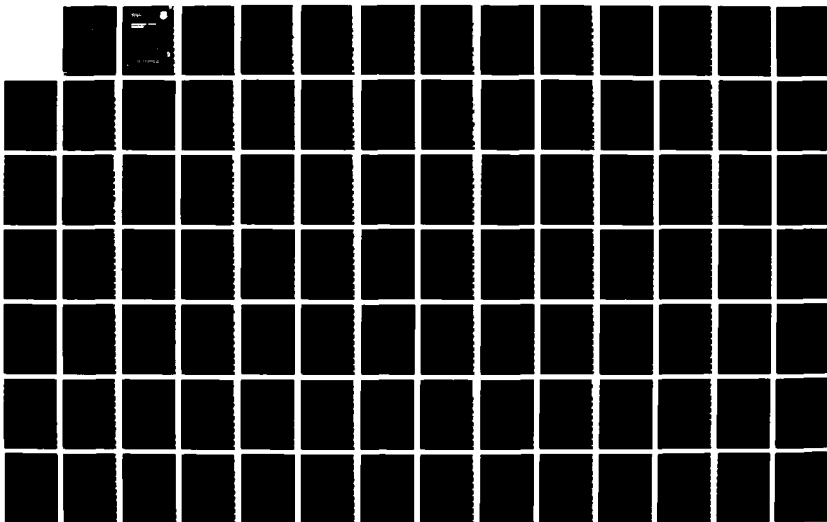
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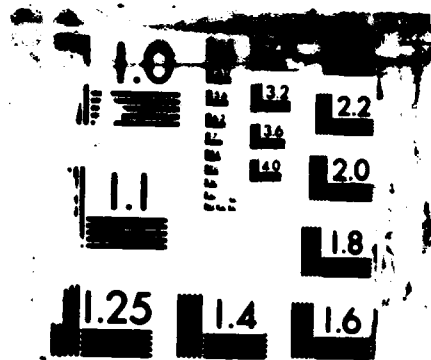
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MULTIDIMENSIONAL FILTERING INVESTIGATIONS

Stevens Institute of Technology

Dr. Sankar Basu

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE				Form Approved OMB No. 0704-0188	
1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS N/A		
2a. SECURITY CLASSIFICATION AUTHORITY N/A			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A			5. MONITORING ORGANIZATION REPORT NUMBER(S) RADC-TR-86-231		
4. PERFORMING ORGANIZATION REPORT NUMBER(S) N/A			7a. NAME OF MONITORING ORGANIZATION Rome Air Development Center (DCCD)		
6a. NAME OF PERFORMING ORGANIZATION Stevens Institute of Technology		6b. OFFICE SYMBOL (If applicable)		7b. ADDRESS (City, State, and ZIP Code) Griffiss AFB NY 13441-5700	
6c. ADDRESS (City, State, and ZIP Code) Department of Electrical Engineering & Computer Science Hoboken NJ 07030			9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F30602-81-C-0185		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable)		10. SOURCE OF FUNDING NUMBERS	
8c. ADDRESS (City, State, and ZIP Code) Washington DC 20332		PROGRAM ELEMENT NO. 61102F		PROJECT NO. 2305	TASK NO. J8
				WORK UNIT ACCESSION NO. PI	
11. TITLE (Include Security Classification) MULTIDIMENSIONAL FILTERING INVESTIGATIONS					
12. PERSONAL AUTHOR(S) Dr. Sankar Basu					
13a. TYPE OF REPORT Final		13b. TIME COVERED FROM Jan 86 TO Sep 86		14. DATE OF REPORT (Year, Month, Day) February 1987	
15. PAGE COUNT 142					
16. SUPPLEMENTARY NOTATION N/A					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB-GROUP	Filters) Multi-Dimensional Filtering, Signal Processing)		
17	02	01	Passive Filters) Lossless Filters)		
17	04				
19. ABSTRACT (Continue on reverse if necessary and identify by block number) New classes of multivariable polynomials arising in studies of passive multidimensional systems have been identified and their properties have been studied. The problem of structurally passive synthesis of multidimensional digital filters as a cascade interconnection of more elementary building blocks has been addressed via the factorization matrix. The problem of sampling rate alteration of deterministic multidimensional signals is addressed on the basis of frequency domain description of the signal. Experiments with digitized images are also reported to demonstrate the performance of the designed interpolation and decimation schemes. <i>2 reports included</i>					
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL Richard N. Smith			22b. TELEPHONE (Include Area Code) (315) 330-3224		22c. OFFICE SYMBOL RADC (DCCD)

DD Form 1473, JUN 86

Previous editions are obsolete.

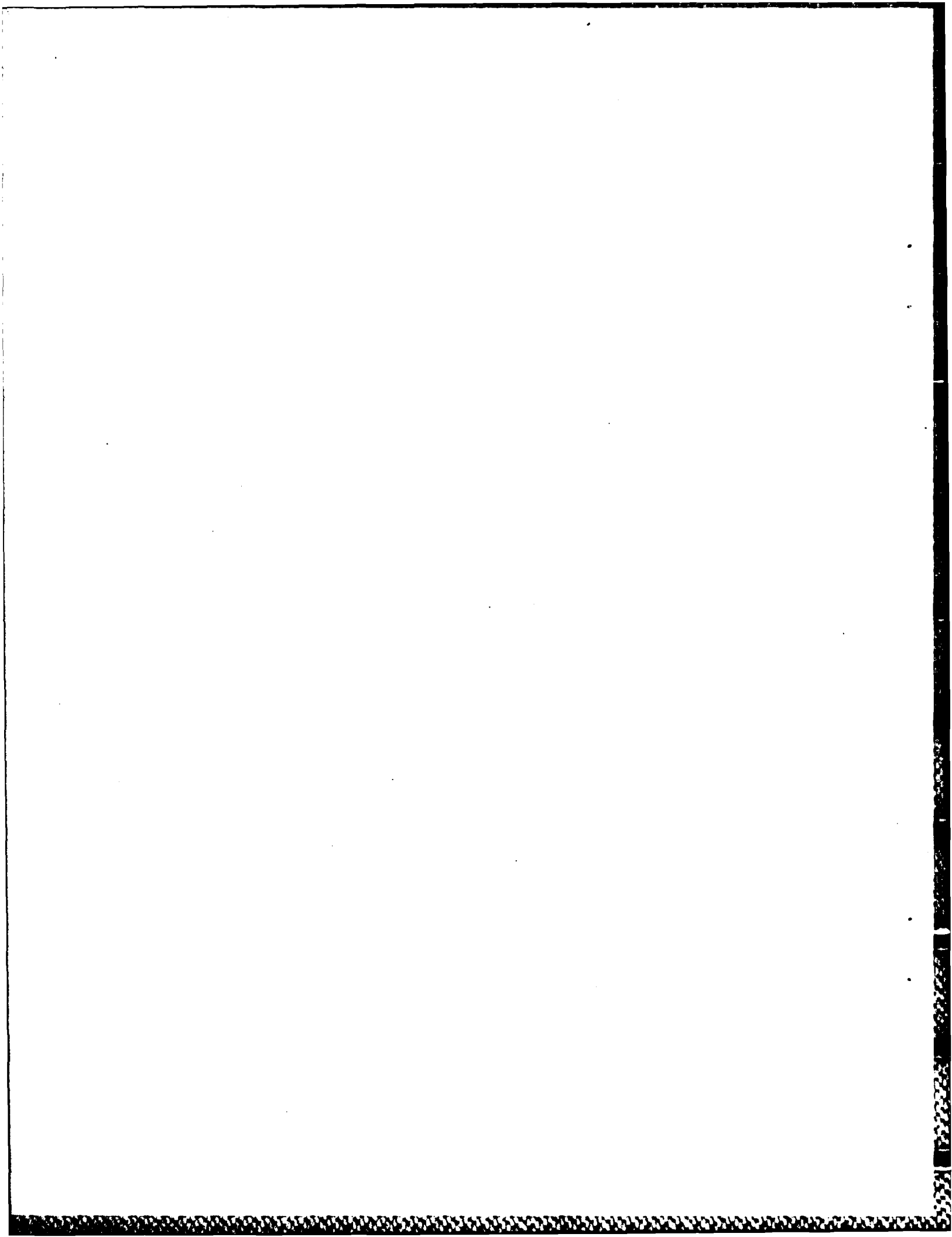
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INDEX

	page
1. Introduction	1
2. Simple proofs of stability results on multidimensional polynomials	6
3. New results on stable multidimensional polynomials	38
4. Realization of structurally passive multidimensional digital filters	81
5. Multidimensional interpolation and decimation schemes for fast implementation	100
6. Conclusions.....	121
Figures.....	130

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CHAPTER 1

INTRODUCTION

The necessity of dealing with signals and systems having two or more independent parameters is now well recognized. Specific areas where multidimensional signal processing find applications include image processing, remote sensing, target tracking, robotics, geophysical and astronomical data processing. A wide variety of problems belonging to these categories can be dealt with either by the methods of multidimensional frequency selective filtering or by employing multidimensional modelling schemes. For most practical applications two specific types of multidimensional filters are of interest. To exemplify the situation it may be recalled that in image processing type applications the processing scheme is mandated to be rotation invariant which requires that pass/stop regions of the frequency response of the filter be spherically symmetric in the multidimensional frequency hyperspace. On the other hand, in the problem of discriminating image regions moving with different velocities (e.g., a target against a background) the dynamic scene can be modelled as a time dependent intensity distribution $s(x,y,t)$ or its Fourier transform $S(u,v,f)$ where u, v are the spatial frequencies and f is the temporal frequency. We then have:

$$S(x,y,t) = m(x-x_0-c_x t, y-y_0-c_y t) + \\ [1-\lambda(x-x_0-c_x t, y-y_0-c_y t)] \cdot b(x-d_x t, y-d_y t) \\ (1.1)$$

where $m(x,y)$ and $b(x,y)$ are textural functions of the object and the background; $\lambda(x,y)$ is a mask function whose value is one in regions where the object is present and zero in

regions where no object is present; (x_0, y_0) is the coordinate of the sensor. Considering the fourier transform of (1.1) we then have:

$$S(u, v, f) = M(u, v, f) + B(u, v, f) - \Lambda(u, v, f) * B(u, v, f) \quad (1.2)$$

where $*$ denotes 3-D convolution. Analysis of this equation reveals that the spectral energies of image regions moving with different velocities are concentrated in different planes in the 3-D (u, v, f) frequency space. Therefore, they can be discriminated by linear spatio-temporal filters the frequency responses of which are fitted to different velocity planes.

Furthermore, due to possible interrelations and redundancies in the large amount of image data one needs a spatio-temporal 3-D stochastic model of the data field, where again two of the dimensions indicate the space coordinates and the third dimension represents the time coordinate. The need for estimation, prediction, noise filtering etc. associated with problems such as displacement estimation, movement compensated prediction of time varying images thus become apparent.

Considerations other than those mentioned above are also highly important in the successful operation of a multidimensional recursive digital filter when realization in hardware is sought. These are: (i) insensitivity to coefficient perturbation i.e., numerical stability to counter the effects of inevitable rounding and truncation of signal values (ii) fault tolerance i.e., insensitivity of the overall filter performance to sudden and unexpected faults in digital circuitry (iii) feasibility of hardware implementation in currently emerging high speed architectures e.g., the systolic architecture and finally, (iv) the property that it can be conveniently made adaptive when the

signal characteristics vary in the spatial or in the temporal domain.

The class of one-dimensional digital filters which has been found to satisfy all of the desirable characteristics mentioned above can, from a fundamental point of view, be broadly categorized as the structurally passive filters. Structurally passive digital filters are those which are not only passive (i.e., nonenergy generating in a discrete sense) from the input output point of view, but the most elementary building blocks which constitute the filter structure are also. The fact that structural passivity, in addition to being responsible for the properties of numerical stability, fault tolerant and adaptivity, can also be exploited towards the goal of implementing the filter in highspeed VLSI structures is now known in the one-dimensional context. The well known lattice filters, wave digital filters and the orthogonal filters belong to this class of filters.

Steps towards the analysis and design of multidimensional structurally passive digital filters have been taken in the investigation reported here. Since an appropriate transform domain description of multidimensional passive (or lossless) linear shift invariant quarter plane filters requires a proper notion of stable multidimensional polynomials previously not considered in the literature our investigation starts from a reexamination of multidimensional stability concepts from a fundamental standpoint. In chapter 2 it is shown that all of the known results on stable multidimensional polynomials can be derived via the elementary artifice of a continuity property of zeros of a polynomial as a function of its coefficients. Several previously unrecognized classes of multidimensional stable polynomials crucial to the transform domain description of multidimensional passive continuous systems are identified for the first time in chapter 3 by utilizing analytical

techniques which are largely similar to those used in chapter 2. Certain theoretical results on the analysis of passive multidimensional continuous systems are also derived here. Synthesis of multidimensional structurally passive recursive digital filters directly in the discrete domain form the contents of chapter 4. Motivated by one-dimensional examples such as the digital lattice filters and its potential desirability in pipelined implementation, the further topological constraint that the filter consist of cascade interconnection of elementary passive building blocks is imposed in the problem formulation. Necessary and sufficient conditions for the feasibility of such synthesis are derived. Since the design problems of many of the known one-dimensional structurally passive filters can be viewed as special cases of the results developed, new algorithms for one-dimensional design fall out as a byproduct of this discussion.

More specific practical questions of image interpolation and decimation are examined in chapter 5 by formulating the problems in terms of frequency domain digital filtering. Certain computational FIR-type structures are derived which exploit full advantages of combined pipelineability and parallelism when high speed hardware implementation of the resulting filters are sought. Experiments with real as well as synthetic two-dimensional data are reported.

Conclusions and recommendations for further research are chalked out in chapter 6.

Each of the following chapters are self contained and can be read independently. For similar discussions in the open literature we refer to the publications in [1], [2], [3] and [4] in the following.

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- [1] S. Basu and A. Fettweis, Simple proofs of stability related results on multidimensional polynomials, to be published in International Journal on Circuit Theory and Applications.
- [2] A. Fettweis and S. Basu, New results on stable multidimensional polynomials: part I - continuous case, to be published in IEEE Trans. on Circuits and Systems.
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- [4] S. Basu and M. Omrani, Multidimensional digital interpolation and decimation schemes for fast implementation, 20th Asilomar Conference on Signals, Systems and Computers, Pacific grove, California.

CHAPTER 2

SIMPLE PROOFS OF STABILITY RELATED PROPERTIES OF MULTIDIMENSIONAL POLYNOMIALS

2.1. Introduction

The criterion for bounded-input-bounded-output property of multidimensional (k-D) linear shift invariant systems, when the rational transfer function associated with the system does not have non-essential singularities of the second kind on the distinguished boundary of the polydomain under consideration, is well established [1-5]. More specifically, if the transfer function of a k-dimensional discrete time system is given by $H(\underline{z})$ as in (2.1), where

$$H(\underline{z}) = A(\underline{z})/B(\underline{z}) \quad (2.1)$$

$A(\underline{z}) = A(z_1, z_2, \dots, z_k)$ and $B(\underline{z}) = B(z_1, z_2, \dots, z_k)$ are relatively prime polynomials in the k variables $\underline{z} = (z_1, z_2, \dots, z_k)$ then under the restrictive hypothesis that $A(\underline{z})$ and $B(\underline{z})$ do not have any common zero on the distinguished boundary $|z_i| = 1$, $i = 1$ to k (also to be denoted as $|\underline{z}| = 1$ in the forthcoming discussion) of the polydomain $|z_i| \leq 1$, $i = 1, 2, \dots, k$ (also to be expressed as $|\underline{z}| \leq 1$), the system produces a bounded output in response to a bounded input if and only if $B(\underline{z})$ is a strict sense Hurwitz polynomial, i.e., (2.2) is satisfied.

$$B(\underline{z}) \neq 0 \text{ for } |\underline{z}| \leq 1 \quad (2.2)$$

Since for a given polynomial $B(\underline{z})$ it is not possible to test for condition (2.2) directly, a number of alternative but equivalent conditions, which are easier to test for, have been derived by Strintzis, DeCarlo, et. al. [7] and others. A

summary of all these results are available in the works of Jury [8] and Bose [1]. A variety of different methods of proofs of these results have appeared so far in the literature. Strintzis in [6] uses analytic function theory, DeCarlo et. al. uses homotopy theory expounded in Rudin's book [9]. It has been pointed out [1,10] that the results just mentioned can be derived as special cases of Rudin's theorem (Theorem 4.7.2, pp.87 in [9]). Delsarte, Genin and Kamp [10] have shown that all these results including Rudin's theorem can be proved via a number of elementary one-dimensional (1-D) steps. However, the proofs given in [10] still require the use of some function theoretic results, which may be inaccessible to an engineering reader. The present report deals with proofs of the above results, which are very simple and highly intuitive as well. The technique dwells on the fact that the zeros of a polynomial can be viewed as continuous functions of its coefficients. A correct and complete statement of this continuity property, which includes all possible degenerate cases (although restricted to polynomials in one variable), is available e.g. in [11] (Theorem 4, p.19, including footnote), [14] (§44), and [14a] (p.200). The technique has already proven to be a very powerful tool in recent studies on passive multivariable network theory [12]. A similar effort in this direction is noted in [13]. However, [13] deals with the two-variable case ($k = 2$) only, and a fully appropriate discussion of results including all possible degenerate cases are not given.

A complete statement of the continuity property of zeros of a multivariable polynomial as a function of its coefficients, which includes all the degenerate cases, is given in Section 2.2. Section 2.3 contains proofs of the main theorems of Anderson, Jury [16] and those due to Strintzis [6], DeCarlo et al. [7]. Another result originally proved by DeCarlo, Murray and Saeks [7] forms the main topic of discussion in Section 2.4, where it is again shown that all related results

can be derived from the continuity property of zeros of a polynomial as a function of its coefficients. In section 2.5 it is shown how the results of section 2.3 and 2.4, when used with further continuity type arguments lead up to Rudin's theorem mentioned earlier. At this point a few generalizations of Rudin's theorem are also proven as consequences of discussions of earlier sections of the report. Finally, the report is summarized and conclusions are drawn in Section 2.6.

2.2. Continuity property of the zeros of a polynomial as a function of its coefficients.

Due to the fact that the utilization of the aforementioned continuity property in stability related problems in the context of multi-dimensional polynomials is the main contribution of this report and that a proper formulation of the property needed for our purposes has not until recently been known to be available, at least not in the engineering literature, we undertake to give a brief exposition of the result without giving any details of proofs.

The one variable version of the following result is well known [11] in mathematical literature. For a proof see, for example, [14]. Let $g(\underline{z}) = \sum_{\underline{v} \in N} A_{\underline{v}} \underline{z}^{\underline{v}}$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$, where N denotes the set of k -tuples $\underline{v}$ such that $A_{\underline{v}}$ is not zero. Also let $||\underline{z}||^2 = |z_1|^2 + |z_2|^2 + \dots + |z_k|^2$. Then, if the coefficient $A_{\underline{v}}$ are assumed to be variable quantities with certain initial values $A_{\underline{v}_0}$, and $g_0(\underline{z})$ be the corresponding expression for $g(\underline{z})$, the following two mutually exclusive cases can arise:

- (1) $g_0(\underline{z})$ is identically zero, i.e., $A_{\underline{v}_0} = 0$ for all $\underline{v}_0 \in N$.
- (2) $g_0(\underline{z}) \neq 0$, i.e., there exists at least one $\underline{v} \in N$ such that $A_{\underline{v}_0} \neq 0$.

If in this latter case there exists a $\underline{z}_0$ with finite $||\underline{z}_0||$ such that $g_0(\underline{z}_0) = 0$, then to any $\epsilon > 0$ we can make correspond a $\delta > 0$ such that for $|A_{\underline{v}} - A_{\underline{v}_0}| < \delta$, for all $\underline{v} \in N$, there exists a value of $\underline{z}$ for which we have $||\underline{z} - \underline{z}_0|| < \epsilon$ as well as $g(\underline{z}) = 0$. The proof of the above result follows from its one-variable counterpart, and is available in [15].

For the present purpose, however, a different formulation of the above principle proves to be more useful and is elaborated upon next. Assume the A_v for each $v \in N$ to be continuous functions of some vector $\underline{t}$, say, $A_v = A_v(\underline{t})$. Consequently, we can write $g = g(\underline{z}, \underline{t})$. Consider first a fixed value $\underline{t} = \underline{t}_0$ of $\underline{t}$ such that $g(\underline{z}_0, \underline{t}_0) = 0$ for some $\underline{z} = \underline{z}_0$. Then one of the following two mutually exclusive cases must hold true:

- (1) $A_v(\underline{t}_0) = 0$ for all $v \in N$, i.e., $g(\underline{z}, \underline{t}_0) = 0$ for all $\underline{z} = (z_1, z_2, \dots, z_k)$.
- (2) $A(\underline{t}_0) \neq 0$ for at least one $v \in N$.

In the latter case, if we move $\underline{t}$ along a continuous curve from its initial value $\underline{t}_0$, it is possible to move also each z_i , $i = 1$ to k along certain continuous curves in their respective z_i - planes in such a way that $g(\underline{z}, \underline{t}) = 0$ continues to hold true, until a value $\underline{t} = \underline{t}_f$ of $\underline{t}$ is reached such that either $A_v(\underline{t}_f) = 0$ for all $v \in N$ or, for $\underline{t}$ approaching $\underline{t}_f$, $|z_i| \rightarrow \infty$ for at least one i in $1 \leq i \leq k$. For $k=1$, this result amounts simply to a reformulation of a corresponding result in [14a] (§39), and it can be easily extended to the case $k>1$ by using the same simple approach as in Appendix 1 of [15].

2.3. Simplified proofs of results of Strintzis [6] and DeCarlo et. al. [7] and Anderson and Jury [16]

We will first need the following lemma.

Lemma 2.1: Let $f(\underline{z}) = f(z_1, z_2, \dots, z_k)$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$. Assume that there exists a $\underline{z}_0 = (z_{10}, z_{20}, \dots, z_{k0})$ in $|\underline{z}_0| \leq 1$ such that $f(\underline{z}_0) = 0$, and $|z_{i0}| = 1$ holds for v of the k variables z_i , where v is any integer $0 \leq v < k-1$. Then there exists a $\underline{z}'_0 = (z'_{10}, z'_{20}, \dots, z'_{k0})$ in $|\underline{z}'_0| \leq 1$ such that $f(\underline{z}'_0) = 0$ and $|z'_{i0}| = 1$ holds for at least $(v+1)$ of the z'_{i0} . In particular, such a zero then exists with $z'_{i0} = z_{i0}$ for all those i for which $|z_{i0}| = 1$.

Proof: No loss of generality occurs in assuming that $|z_{i0}| = 1$ for $i = 1$ to v . Let us freeze the variables z_i at $z_i = z_{i0}$ for $i = 1$ to v . If $f(\underline{z})$ is independent of at least one of the variables z_{v+1} to z_k , say z_k , then the proof is immediate by choosing $z'_{i0} = z_{i0}$ for $i = 1$ to $k-1$ and any z'_{k0} with $|z'_{k0}| = 1$.

Next, assume that $f(\underline{z})$ involves (i.e., actually depends on) z_{k-1} and z_k . Let us now freeze the variables z_i at $z_i = z_{i0}$ for $i = 1$ to $(k-2)$, where thus $|z_{i0}| = 1$ for $i = 1$ to v and $|z_{i0}| < 1$ for $i = (v+1)$ to $(k-2)$, and consider a continuous path Γ_{k-1} from $z_{k-1,0}$ leading up to the unit circle in the z_k plane. As z_{k-1} is moved continuously along Γ_{k-1} the variable z_k can be moved along continuous path Γ_k starting from z_{k0} such that $f(\underline{z}) = 0$ remains satisfied. The only exception that could arise is that for some $z_{k-1} = \alpha_{k-1}$ on Γ_{k-1} , with $|\alpha_{k-1}| < 1$, the polynomial f_1 defined by $f_1(z_{k-1}, z_k) = f(z_{10}, \dots, z_{k-2,0}, z_{k-1}, z_k)$ is zero for all z_k ; the proof is then completed by choosing $z'_{i0} = z_{i0}$ for $1 \leq i \leq k-2$, $z'_{k-1,0} = \alpha_{k-1}$, and any z'_{k0} such that $|z'_{k0}| = 1$. If however no such α_{k-1} exists, then by invoking the continuity property of zeros of a polynomial, mentioned in Section 2.2, it follows that either z_{k-1} or z_k (or both) will reach the unit circle

in the corresponding z_i plane. We would then have constructed a zero $\underline{z}'_0$ of $f(\underline{z})$ such that $|z'_{i0}| = 1$ for at least $(v+1)$ of the k variables z_i .

We are now in a position to prove the theorem due to Strintzis[6] and DeCarlo et. al.[7] mentioned earlier.

Theorem 2.2: If $f(\underline{z})$ is a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$ then $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$ if and only if the following conditions simultaneously hold true:

- (a) $f(\underline{z}) \neq 0$ for $|\underline{z}| = 1$, i.e., $|z_i| = 1$ for $i = 1$ to k .
- (b) if j is any integer in $1 \leq j \leq k$, then $f(\underline{z}) \neq 0$ for $z_i = r_{ij}$, $i = 1, 2, \dots, (j-1), (j+1), \dots, k$, and $|z_j| \leq 1$, where the r_{ij} 's are some complex numbers with $|r_{ij}| = 1$.

Proof: Necessity of the theorem is obvious. To prove sufficiency let us assume that there exists a $\underline{z}_0 = (z_{10}, z_{20}, \dots, z_{k0})$ with $f(\underline{z}_0) = 0$, $|z_{i0}| \leq 1$ for $i=1$ to k . In view of lemma 2.1 we may assume that $(k-1)$ of the k variables $(z_1, z_2, \dots, z_k)$ are located on the unit circles in the corresponding z_i -planes. A renumbering of the variables, if necessary, will show that no loss of generality occurs in assuming $|z_{i0}| = 1$ for $i = 1$ to $(k-1)$. Due to condition (a) of the theorem, we then have $|z_{k0}| < 1$. This latter conclusion, however, implies in particular that $f(\underline{z})$ is not independent of z_k . Next we successively move the variables z_i , $i=1$ to $(k-1)$ from z_{i0} to r_{ik} continuously along the arcs Γ_i of the corresponding unit circles. The variable z_k may then be moved along a path Γ_k starting from z_{k0} such that $f(\underline{z}) = 0$ remains satisfied. In this process it is impossible to have $f(\underline{z}) = 0$ for some $z_i = r_{i0}$ on Γ_i , $1 \leq i \leq k-1$, and all z_k because otherwise a contradiction with condition (a) of the theorem is easily arrived at by choosing z_k at any point on the unit circle $|z_k| = 1$. Invoking the continuity property of the zeros of a polynomial, it then follows that

Γ_k can be chosen to be a continuous path in the z_k -plane and the process described above can only have one of the following two outcomes. Either we reach $z_i = \gamma_{ik}$, $i = 1$ to $(k-1)$ with z_k remaining such that $|z_k| < 1$ or the path Γ_k described by z_k has at least one point in common with the unit circle $|z_k| = 1$. In the former case, the condition (b) of the theorem is violated, whereas in the latter case the condition (a) of the theorem is violated. The proof of the theorem is thus complete.

Remark: The proof does not assume that $f(\underline{z})$ is a polynomial or that it is a holomorphic function. All that is required is that the continuity property presented in Section 2.2 holds. A similar remark holds for other results in this paper.

We next need to prove a result due to Anderson and Jury [16] via the continuity argument. We need, however, the following lemma, which will also prove to be useful in other developments to follow.

Lemma 2.3: Let $f(\underline{z})$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$ and let the set of indices $i = 1$ to k be the union of two disjoint subsets I_1 and I_2 . Then $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$ if and only if the following conditions hold true simultaneously:

(a) $f(\underline{z}) \neq 0$ if $z_i = a_i$ for $i \in I_1$; $|z_i| \leq 1$ for $i \in I_2$ where the a_i , for all $i \in I_1$, are some complex numbers such that $|a_i| \leq 1$.

(b) $f(\underline{z}) \neq 0$ if $|z_i| \leq 1$ for $i \in I_1$; $|z_i| = 1$ for $i \in I_2$.

Proof: Necessity is obvious. To prove sufficiency, let it be assumed for contradiction that $f(\underline{z})$ has a zero at $\underline{z} = \underline{z}_0 = (z_{10}, z_{20}, \dots, z_{k0})$ with $|\underline{z}_0| \leq 1$. By virtue of Lemma 2.1, no loss of generality occurs in assuming that there exists a

fixed integer μ such that $|z_{\mu 0}| \leq 1$ and that $|z_{i 0}| = 1$ for all other $i \in (I_1 \cup I_2)$. In view of condition (b) of the present lemma, we must have in fact $|z_{\mu 0}| < 1$ and $\mu \in I_2$. We move the variables z_i for $i \in I_1$ from their initial locations $z_i = z_{i 0}$ on $|z_{i 0}| = 1$ along continuous curves Γ_i lying in the respective unit disc $|z_i| \leq 1$ and leading up to the terminal points $z_i = a_i$, while the variables z_i for $i \in I_2$, $i \neq \mu$ are held frozen at their corresponding values $z_i = z_{i 0}$. The variable z_μ can then trace out a contour Γ_μ in the z_μ -plane so that $f(\underline{z}) = 0$ is satisfied. Note first that as long as z_i is on Γ_i for all $i \in I_1$ and $z_i = z_{i 0}$ for $i \in I_2 \setminus \{\mu\}$, the polynomial $f(\underline{z})$ cannot be zero for all z_μ because otherwise an arbitrary choice of z_μ on $|z_\mu| = 1$ yields a value of the k -tuple $\underline{z}$ with $|z_i| \leq 1$ for $i \in I_1$ and $|z_i| = 1$ for $i \in I_2$ such that $f(\underline{z}) = 0$, which contradicts condition (b) of the present lemma. However, with this alternative excluded, it would be possible, due to condition (a) and the continuity argument, to arrive at the same unpermitted situation in a way in which z_μ reaches a point with $|z_\mu| = 1$ by moving continuously along Γ_μ while the z_i , for $i \in I_1$, remain on their respective Γ_i .

Remark: If $k=2$ and $I_1=\{1\}$ and $I_2=\{2\}$ then lemma 2.3 coincides with a result well known (see e.g., [2] and [13b]) in the literature.

Corollary 2.3.1: Let $f(\underline{z})$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$. Then $f(\underline{z}) \neq 0$ for $|\underline{z}| \leq 1$ if and only if the following two conditions hold true simultaneously:

- (a) $f(a, z_2, z_3, \dots, z_k) \neq 0$ for $|z_i| \leq 1$, $i = 2$ to k , where a is some complex number with $|a| \leq 1$.
- (b) $f(z_1, z_2, \dots, z_k) \neq 0$ for $|z_1| \leq 1$, and $|z_i| = 1$, $i=2$ to k .

Proof: Follows from lemma 2.3 via the choice of $I_1 = \{1\}$ and

$$I_2 = \{2, 3, \dots, k\}.$$

The proof of the theorem due to Anderson and Jury [16] mentioned earlier is given next.

Theorem 2.4: If $f(\underline{z})$ is a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$ then $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$ if and only if for some complex number a with $|a| \leq 1$ the following conditions simultaneously hold true:

- (1) $f(\underline{z}) \neq 0$ for $|z_1| \leq 1$, and $|z_j| = 1$, $j = 2$ to k .
- (2) $f(\underline{z}) \neq 0$ for $z_1 = a$, $|z_2| \leq 1$, and $|z_j| = 1$, $j = 3$ to k .
- (3) $f(\underline{z}) \neq 0$ for $z_i = a$, $i = 1, 2$, $|z_3| \leq 1$, and $|z_j| = 1$, $j = 4$ to k .
- .
- .
- .
- (k) $f(\underline{z}) \neq 0$ for $z_i = a$, $i = 1$ to $(k-1)$, $|z_k| \leq 1$.

Proof: Necessity is obvious. To prove sufficiency, let us define the n -variable ($1 \leq n \leq k$) polynomial B_n from $f(\underline{z})$ by freezing each of the first $(k-n)$ variables equal to a , i.e., $z_i = a$ for $i = 1$ to $(k-n)$. Note that $B_k = f(\underline{z})$ and $B_1 = B_1(z_k) = f(a, \dots, a, z_k)$. We first claim that $B_n \neq 0$ in $|z_i| \leq 1$, for $i = (k-n+1)$ to k for each n in $1 \leq n \leq k$. The proof of this assertion is via induction on n .

Obviously, due to condition (k) of the theorem, $B_1 = B_1(z_k) \neq 0$ in $|z_k| \leq 1$. The assertion is, therefore, true for $n = 1$. Assume now that our assertion is correct for B_n i.e., $B_n = B_n(z_{k-n+1}, \dots, z_k) \neq 0$ for $|z_i| \leq 1$, $i = (k-n+1)$ to k . Note that this implies $B_{n+1}(a, z_{k-n+1}, \dots, z_k) = B_n(z_{k-n+1}, \dots, z_k) \neq 0$ for $|z_i| \leq 1$, $i = (k-n+1)$ to k . However, condition $(k-n)$ of the theorem states that $B_{n+1}(z_{k-n}, \dots, z_k) \neq 0$ for $|z_{k-n}| \leq 1$ and $|z_i| = 1$, $i = (k-n+1)$ to k . The last two conditions, in

view of corollary 2.3.1, imply that $B_{n+1} \neq 0$ for $|z_i| \leq 1$, $i = (k-n)$ to k . The proof of the theorem by induction is thus complete.

2.4. Proof of a theorem of DeCarlo, Murray, and Saeks [7] and its extension

In this section we undertake the proof of the theorem of DeCarlo, et.al. [7] based on continuity property of zeros of polynomials. We note that this theorem is not mentioned by Srintzis [6], and its proof in [7] makes use of homotopy theoretic arguments. Subsequently it was pointed out [1,10] that the result can be considered to be a special case of Rudin's theorem [9]. We will need to have the following lemma as a preperation for our proof of the theorem.

Lemma 2.5: Let $f(\underline{z})$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$ such that $f(\underline{z}) \neq 0$ for $|\underline{z}| = 1$, and the set of indices $i = 1$ to k be the union of disjoint subsets of indices I_1, I_2 and I_3 . If $f(\underline{z}) \neq 0$ in $|z_i| \leq 1$ for $i \in I_1$, $|z_i| = 1$ for $i \in I_2$, and $z_i = \gamma_i$, for $i \in I_3$, where the γ_i 's are some complex numbers such that $|\gamma_i| = 1$, then we also have $f(\underline{z}) \neq 0$ in $|z_i| \leq 1$ for $i \in I_1$; $|z_i| = 1$ for $i \in I'_2$, and $z_i = \gamma_i$ for $i \in I'_3$, where j is any integer belonging to I_3 , $I'_2 = I_2 \cup \{j\}$, and $I'_3 = I_3 \setminus \{j\}$.

Note that the lemma is meaningful only if I_3 is nonempty and that we may also assume I_1 to be nonempty since otherwise the result is trivial. However, I_2 may be empty.

Proof: Let it be assumed for the purpose of contradiction, that $f(\underline{z}_0) = 0$, where $\underline{z}_0 = (z_{10}, z_{20}, \dots, z_{k0})$ is such that $|z_{i0}| \leq 1$ for $i \in I_1$, $|z_{i0}| = 1$ for $i \in I'_2$, and $z_{i0} = \gamma_i$ for $i \in I'_3$. In view of Lemma 2.1 and the fact that $f(\underline{z}) \neq 0$ for $|\underline{z}| = 1$ we may assume that for one of the $i \in I_1$, say for $i = \mu$, we have $|z_{\mu 0}| < 1$ and that $|z_{i0}| = 1$ for all $i \in I'_1$, where $I'_1 = I_1 \setminus \{\mu\}$.

Consider the polynomial $f_1(z_\mu, z_j)$ obtained by freezing in $f(\underline{z})$ the rest of the variables as follows: $z_i = z_{i0}$ for $i \in I'_1 \cup I_2$ and $z_i = \gamma_i$ for $i \in I'_3$. Clearly, $f_1(z_{\mu 0}, z_{j0}) = 0$.

but by the hypothesis of the lemma we have $f_1(z_\mu, z_j) \neq 0$ for $|z_\mu| \leq 1$, $z_j = \gamma_j$ as well as for $|z_\mu| = |z_j| = 1$. Consider then a directed arc Γ_j of the circle $|z_j| = 1$ originating from $z_j = z_{j0}$ and terminating at $z_j = \gamma_j$. Obviously, for any fixed $z_j \in \Gamma_j$ the polynomial $f_1(z_\mu, z_j)$ cannot be zero for all z_μ because otherwise a contradiction with the fact that $f(z) \neq 0$ in $|z| = 1$ would immediately be arrived at by choosing z_μ to be arbitrarily located on $|z_\mu| = 1$. By invoking the continuity argument we can thus move z_j continuously on Γ_j from z_{j0} towards γ_j and simultaneously move z_μ continuously from $z_{\mu 0}$ such that $f_1(z_\mu, z_j) = 0$ remains satisfied. We will then reach a situation either with $|z_\mu| = |z_j| = 1$ or in which $|z_\mu| \leq 1$, $z_j = \gamma_j$, which both have been seen to be impossible.

The following result is immediately obtained by repeated application of Lemma 2.5.

Lemma 2.6: Let $f(\underline{z})$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$ such that $f(\underline{z}) \neq 0$ for $|\underline{z}| = 1$, and the set of indices $i = 1$ to k be the disjoint union of two subsets I_1 and I_2 . Then if $f(\underline{z}) \neq 0$ in $|z_i| \leq 1$ for $i \in I_1$ and $z_i = \gamma_i$ for $i \in I_2$ where the γ_i 's are some complex numbers such that $|\gamma_i| = 1$, then $f(\underline{z}) \neq 0$ in $|z_i| \leq 1$ for $i \in I_1$, and $|z_i| = 1$ for $i \in I_2$. Note again that the set result is trivial if I_1 is empty.

Lemma 2.7: The transformation

$$z_1 = (u-v)/(1-v^*u), \quad z_2 = (u+v)/(1+v^*u) \quad (2.3a,b)$$

has the following properties:

Property 2.1: If $v = 0$ then $z_1 = z_2 = u$.

Property 2.2: If $|u| = 1$ and v is such that $v^* \neq \pm 1/u$, then $|z_1| = |z_2| = 1$.

Property 2.3: If $|u| < 1$ and $|v| < 1$, then $|z_1| < 1$ and $|z_2| < 1$.

Property 2.4: For any z_1 and z_2 with $|z_1| < 1$ and $|z_2| < 1$ there exist u and v with $|u| < 1$, $|v| < 1$ such that (2.3) is satisfied. (Note: (2.3) does not represent a simple one-to-one transformation, but Property 2.4 is in a sense the converse of Property 2.3.)

Proof: Properties 2.1 and 2.2 are easily verified. For proving Property 2.3, consider any fixed v in $|v| < 1$. Then z_1 is an analytic function of u in the domain $|u| \leq 1$. Furthermore, from property 2.2, $|z_1| = 1$ for all u on the boundary $|u| = 1$ of the domain $|u| \leq 1$. Thus, maximum modulus theorem implies that $|z_1| < 1$ for $|u| < 1$. Similar arguments hold for z_2 .

For proving Property 2.4, note first that if $z_1 = z_2$ then the proof immediately follows by choosing $u = z_1 = z_2$ and $v = 0$. Henceforth $z_1 \neq z_2$ will be assumed. By eliminating u from (2.3a) and (2.3b) it follows that:

$$(1 - |v|)^2 / 2|v| = (z_1 + \exp(j\alpha))(z_2 \exp(-j\alpha) - 1) / (z_1 - z_2) \quad (2.4)$$

where $v = |v| \exp(j\alpha)$.

We next claim that by choosing α properly the right hand side of (2.4) can be made to be equal to a finite real and positive number. To substantiate this claim consider the angle γ defined by $(z_1 + \exp(j\alpha))(z_2 \exp(-j\alpha) - 1) = |c| \exp(j\gamma)$. Since $|z_1| < 1$ and $|z_2| < 1$, it is obvious that $|c| \neq 0$, and γ is thus well defined. Furthermore, γ satisfies (2.5).

$$\exp(2j\gamma) = \{(\xi + z_1)/(1 + z_1^* \xi)\} \{(\xi - z_2)/(1 - z_2^* \xi)\} \quad (2.5)$$

where $\xi = \exp(j\alpha)$.

Since the right hand side of (2.5) is the product of two

allpass functions of the variable ξ it is well known that as α increases continuously by 2π , the angle associated with each of the two factors also increases continuously by an amount 2π . Consequently, γ then also increases continuously by an amount 2π , i.e., by proper choice of α , γ can be given any arbitrary value. The proof of our claim then follows from the fact that γ is the angle associated with the numerator of the right hand side of (2.4), and that the corresponding denominator depends only on the given quantities z_1 and z_2 . By solving (2.4) for $|v|$ it then easily follows that one of the two solutions satisfies $0 < |v| < 1$.

Finally, since (2.6) follows from (2.3), by using arguments similar to

$$u = (z_1 + v)/(1 + v^* z_1) = (z_2 - v)/(1 - v^* z_2) \quad (2.6)$$

those used in the proof of property 2.3, it follows that $|v| < 1$ together with either $|z_1| < 1$ or $|z_2| < 1$, implies that $|u| < 1$.

We prove the theorem due to DeCarlo et. al. [7] next.

Theorem 2.7: Let $f(\underline{z})$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$. Then $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$ if and only if the following conditions hold true simultaneously:

- (a) $f(\underline{z}) \neq 0$ in $|\underline{z}| = 1$
- (b) $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$, where $z_i = z$ for each $i=1$ to k

First proof of Theorem 2.7:

We first provide a particularly simple proof of theorem 2.7 for the two-variable case i.e., when $k=2$. Subsequently, it will be shown via induction on k that if the theorem is true for $k=2$, then it is true for any k . For all this, no use will be made of the results obtained in Section 2.3.

Only sufficiency needs to be proved, the necessity being obvious.

(i) $k=2$.

Consider the function $h(u,v)$ in (2.7) obtained from $f(z_1, z_2)$ via the transformation (2.3), where n_1 , and n_2 are degrees of $f(z_1, z_2)$ in z_1 and z_2 respectively.

$$h(u,v) = (1-v^*u)^{n_1}(1+v^*u)^{n_2} f(z_1, z_2) \quad (2.7)$$

The function $h(u,v)$ is not a polynomial in u and v , but may be considered as a polynomial in u whose coefficients are polynomials in v and v^* . The coefficients just mentioned are, therefore, continuous functions of v .

Property 2.1 of Lemma 2.7 along with condition (b) of theorem 2.7 then yields:

$$h(u,0) = f(u,u) \neq 0 \text{ for } |u| < 1 \quad (2.8)$$

Furthermore, it follows from property 2.2 and condition (a) of theorem 2.7 that:

$$h(u,v) \neq 0 \text{ for } |u| = 1, |v| < 1 \quad (2.9)$$

We next claim that condition (2.8) and (2.9) imply that $h(u,v) \neq 0$ for $|u| < 1$ and $|v| < 1$. To show this we assume for contradiction that $h(u_0, v_0) = 0$ with $|u_0| < 1$, $|v_0| < 1$. Consider a continuous directed arc Γ_v in the complex v -plane originating from $v=v_0$ and terminating in $v=0$. For any fixed v on Γ_v , $h(u,v)$ cannot be zero for all u because otherwise a contradiction with (2.9) would be arrived at by choosing u to be arbitrarily located on $|u|=1$. Furthermore, since as v describes Γ_v , u describes a continuous path in the u -plane, it follows from (2.8), by invoking the continuity argument, that there must exist a $v=v'_0$ on Γ_v and a corresponding $u=u'_0$

with $|u'_0|=1$ such that $h(u'_0, v'_0)=0$, which again contradicts (2.9), thus proving the claim that $h(u, v) \neq 0$ in $|u| < 1, |v| < 1$.

If $f(z_1, z_2)=0$ for some $z_1=z_{10}$ and $z_2=z_{20}$ in $|z_1| < 1$ and $|z_2| < 1$, then it follows that from (2.7) and property 2.4 of Lemma 2.7 that there exists u_0 and v_0 in $|u| < 1$ and $|v| < 1$ such that $h(u_0, v_0)=0$, which however contradicts the conclusions of the last paragraph. Thus $f(z_1, z_2) \neq 0$ in $|z_1| < 1, |z_2| < 1$.

It only remains to show that $f(z_1, z_2) \neq 0$ if one of the two variables z_1 and z_2 is strictly inside and the other on the corresponding unit circle. Assume e.g., $f(z_{10}, z_{20})=0$ for $|z_{10}|=1, |z_{20}| < 1$. Then $f(z_{10}, z_2)$ cannot be zero for all z_2 because otherwise an arbitrary choice of z_2 on $|z_2|=1$ would violate condition (a) of theorem (2.8). The continuity property of zeros then implies that by moving the variable z_1 from $z_1=z_{10}$ to inside the unit circle by an arbitrarily small amount it would be possible to construct a zero of $f(z_1, z_2)$ in $|z_1| < 1, |z_2| < 1$, the impossibility of which has been demonstrated in the previous paragraph.

(ii) $k > 2$.

Assume for the purpose of induction that the theorem is true for $(k-1)$ variables, with $k > 2$, and consider the polynomial $f_1(z, z_k)$ of two variables z_1, z_2 defined as $f_1(z, z_k) = f(z, \dots, z, z_k)$. Then it obviously follows from condition (a) of Theorem 2.7 that $f_1(z, z_k) \neq 0$ for $|z| = 1$ and $|z_k| = 1$, whereas condition (b) implies that $f_1(z, z) \neq 0$ for $|z| \leq 1$. The last two conditions, due to the proof already given for case (i), imply that $f_1(z, z_k) \neq 0$ for $|z| \leq 1$ and $|z_k| \leq 1$. In particular, $f_1(z, z_k) = f(z, \dots, z, z_k) \neq 0$ for $|z|=1$ and $|z_k| \leq 1$. The latter conclusion along with condition (a) and Lemma 2.6 imply that $f(z_1, z_2, \dots, z_k) \neq 0$ for $|z_i|=1, i=i$ to $(k-1)$ and $|z_k| \leq 1$.

Consider next the $(k-1)$ variable polynomial defined as $f_2(z_1, z_2, \dots, z_{k-1}) = f(z_1, z_2, \dots, z_{k-1}, z_{k0})$, where z_{k0} is any fixed complex number in $|z_k| \leq 1$. It then follows that $f_2(z, z, \dots, z) \neq 0$ for $|z| \leq 1$, because otherwise $f_1(z, z_{k0}) = f(z, \dots, z, z_{k0})$ would have a zero in $|z| \leq 1$, the impossibility of which has already been demonstrated. Furthermore, from the concluding sentence of the last paragraph it follows that $f_2(z_1, z_2, \dots, z_{k-1}) \neq 0$ for $|z_i| = 1$, $i=1$ to $(k-1)$. Therefore, by invoking the induction hypothesis we can assert that $f_2(z_1, z_2, \dots, z_{k-1}) \neq 0$ in $|z_i| \leq 1$, $i=1$ to $(k-1)$, which in view of the fact that z_{k0} is arbitrarily located in $|z_k| \leq 1$, in turn implies that $f(\underline{z}) \neq 0$ for $|\underline{z}| \leq 1$.

Corollary 2.7.1: The polynomial $f(\underline{z})$ in $\underline{z} = (z_1, z_2, \dots, z_k)$ is devoid of zeros in $|\underline{z}| \leq 1$ if and only if the following conditions simultaneously hold true:

- (a) $f(\underline{z}) \neq 0$ for $|\underline{z}| = 1$
- (b) $f(\underline{g}(z)) \neq 0$ in $|z| \leq 1$, where $\underline{g}(z) = (g_1(z), g_2(z), \dots, g_k(z))$, and each $g_i(z)$, for $i = 1$ to k are functions which are analytic in $|z| \leq 1$ with the further property that they map unit circles into unit circles and unit discs into unit discs.

Proof: Let us note that any function $g_i(z)$ which maps unit disc into unit disc, unit circle into unit circle, and is also analytic in $|z| \leq 1$ is a rational function which can be written [19, p.12] as in (2.10), where the constants γ_i and $\alpha_{i\nu}$ satisfy $|\gamma_i| = 1$, $|\alpha_{i\nu}| < 1$.

$$z_i = g_i(u_i) = \gamma_i \prod_{\nu=1}^{\nu_i} (u_i - \alpha_{i\nu}) / (1 - \alpha_{i\nu}^* u_i), \nu_i \geq 1 \quad (2.10)$$

Furthermore, we also have that:

$$|g(u_i)| \leq 1 \text{ for } |u_i| \leq 1 \quad (2.11)$$

Consider the polynomial $g(\underline{u})$ in $\underline{u}=(u_1, u_2, \dots, u_k)$, defined as in (2.12), where n_i is the partial degree of $f(\underline{z})$ in z_i .

$$h(\underline{u}) = f(\underline{z}) \prod_{i=1}^k \prod_{v=1}^{v_i} (1 - \alpha_{i,v}^* u_i)^{n_i} \quad (2.12)$$

Due to condition (b) of the present theorem it then follows that $h(\underline{u}) \neq 0$ in $|\underline{u}| \leq 1$, where each $u_i = u$ for $i=1$ to k . Furthermore, it follows from equation (2.11) and condition (a) of the present theorem that $h(\underline{u}) \neq 0$ for $|\underline{u}|=1$. Invoking theorem 2.7 it then follows that $h(\underline{u}) \neq 0$ in $|\underline{u}| \leq 1$.

Next, consider any $\underline{z}=(z_1, z_2, \dots, z_k)$ with $|\underline{z}| \leq 1$. For each corresponding z_i we can compute a u_i by means of equation (2.10) i.e., by solving an algebraic equation of degree $v_i > 0$. Hence, we can find a $\underline{u}=(u_1, u_2, \dots, u_k)$ such that (2.10) is satisfied, and due to (2.11), we then have $|\underline{u}| \leq 1$. Hence, it follows from (2.12) that $f(\underline{z}) \neq 0$ for $|\underline{z}| \leq 1$.

Remark: We note that Theorem 2.7 is a special case of Corollary 2.7.1 when the choice $g_i(z) = z$ for all $i = 1$ to k is made.

The following multidimensional ($k > 2$) versions of results stated in [10] and [1], respectively, follow immediately from Corollary 2.7.1.

Corollary 2.7.2: If $f(\underline{z})$ is a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$ then $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$ if and only if the following conditions simultaneously hold true:

- (a) $f(\underline{z}) \neq 0$ in $|\underline{z}| = 1$
 - (b) $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$ with $z_i = c_i z^{p_i}$, $i=1$ to k
- where the c_i 's are some unimodular constants (i.e., $|c_i| = 1$), and the p_i 's are some positive integers (i.e., $p_i > 0$).

Corollary 2.7.3: Let $f(\underline{z})$ be a polynomial in $\underline{z} =$

$(z_1, z_2, \dots, z_k)$. Then $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$ if and only if $f(\underline{z}) \neq 0$ for $|\underline{z}| \leq 1$ with $z_i = z \exp(j\beta_i)$, $i = 1$ to k , where the β_i 's are arbitrary real numbers.

Remark: An alternative proof of theorem 2.7, which heavily makes use of the results developed earlier in this report, but does not necessitate the technique of induction on the number of variables, can be formulated along lines explained later. We first consider the following theorem.

Theorem 2.8: Let $f(\underline{z})$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$. Then $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$ if and only if the following conditions simultaneously hold true:

- (a) $f(\underline{z}) \neq 0$ in $|\underline{z}|=1$
- (b) $f(\underline{z}) \neq 0$ for $z_j = a$, and $|z_i| \leq 1$ for $i \neq j$ where a is some complex constant such that $|a| \leq 1$ and where j is some integer $1 \leq j \leq k$.
- (c) $f(\underline{z}) \neq 0$ if $|z_j| \leq 1$ and $z_i = \gamma_i$ for the indices $i \neq j$, where the γ_i 's are some complex numbers such that $|\gamma_i| = 1$.

Proof: Necessity of the theorem is obvious. To prove sufficiency we first note that no loss of generality occurs if we assume $j = 1$. In view of Lemma 2.6, with $I_1 = \{1\}$, conditions (a) and (c) of the theorem together imply that $f(\underline{z}) \neq 0$ if $|z_1| \leq 1$ and $|z_i| = 1$ for $i = 2$ to k . However, this latter conclusion along with condition (b) of the theorem, due to Corollary 2.3.1, implies that $f(\underline{z}) \neq 0$ in $|\underline{z}| \leq 1$.

Second proof of Theorem 2.7:

Consider a polynomial $g(\underline{u})$ in $\underline{u} = (u_1, u_2, \dots, u_k)$ obtained by making in the polynomial $f(\underline{z})$ the substitutions: $z_k = u_k$ and $z_i = u_i u_k$ for each $i = 1$ to $(k-1)$. Conditions (2.13), (2.14) then immediately follow from condition (b) of Theorem 2.7,

whereas (2.15) follows from condition (a) of Theorem 2.7 and the fact that $|\underline{u}|=1$ implies $|\underline{z}|=1$.

$$\begin{aligned} g(u_1, \dots, u_{k-1}, 0) &= f(0, 0, \dots, 0) \neq 0 \\ \text{for all finite } u_i, i=1 \text{ to } k-1 \end{aligned} \quad (2.13)$$

$$g(1, \dots, 1, u_k) = f(u_k, \dots, u_k) \neq 0 \text{ for } |u_k| \leq 1 \quad (2.14)$$

$$g(u_1, \dots, u_k) \neq 0 \text{ for } |\underline{u}| = 1 \quad (2.15)$$

Invoking Theorem 2.8 on $g(\underline{u})$, with $j=k$, $a=0$, and $\gamma_i=1$, $i=1$ to $k-1$, it then follows from (2.13), (2.14), and (2.15) that $g(\underline{u}) \neq 0$ for $|\underline{u}| \leq 1$.

We next note that $|\underline{u}| \leq 1$, thus $f(\underline{z})=g(\underline{u}) \neq 0$, if any of the following conditions hold: (a) $|z_1| \leq 1$ and $|z_i|=1$ for $i=2$ to k , (b) $z_i=0$ for $i=1$ to $j-1$, $|z_j| \leq 1$, and $|z_i|=1$ for $i=j+1$ to k , where j is any integer such that $2 \leq j \leq k-1$, (c) $z_i=0$ for $i=1$ to $k-1$ and $|z_k| \leq 1$. Hence, due to Theorem 2.4, $f(\underline{z}) \neq 0$ for $|\underline{z}| \leq 1$, thus providing an alternative proof of Theorem 2.7.

2.5. Proof of Rudin's theorem and an Extension of Rudin's Theorem

In this section we undertake the proof of Rudin's theorem using continuity argument. A generalization of Rudin's theorem is also reported. Our proof of Rudin's theorem is based entirely on continuity arguments and an application of Corollary 2.7.2. Proofs previously reported in the literature have been obtained by appealing to homotopy theory [9]. Our extension of Rudin's theorem is a further generalization of the results discussed in [17] and involves a simple application of our lemma 2.3 and lemma 2.6, along with the Rudin's theorem itself. We first state and prove the conventional form of Rudin's theorem for convenience of exposition.

Theorem 2.9 (Rudin [1],[9]): The polynomial $f(\underline{z})$ in $\underline{z} = (z_1, z_2, \dots, z_k)$ is devoid of zeros in $|\underline{z}| \leq 1$ if and only if the following conditions simultaneously hold true:

- (a) $f(\underline{z}) \neq 0$ for $|\underline{z}| = 1$
- (b) $f(\underline{g}(z)) \neq 0$ in $|z| \leq 1$, where $\underline{g}(z) = (g_1(z), g_2(z), \dots, g_k(z))$, and each $g_i(z)$, for $i = 1$ to k are continuous functions of z with nonnegative winding numbers with respect to the origin with the further property that they map unit circles into unit circles, and unit discs into unit discs.

Note that this theorem reduces to corollary 2.7.1 if the $g_i(z)$'s are assumed to be analytic in $|z| < 1$; indeed this added assumption ensures that $g_i(z)$'s are all-pass functions (i.e., unit functions [19]). A proof such as in [10], however, is not fully correct because of the fact that if the functions $g_i(z)$ are merely continuous the principle of argument may not be invoked. An approach modified wherever required will therefore be used.

Proof: Necessity is obvious. To prove sufficiency, Let $c_i = g_i(1)$, and $p_i \geq 0$ be the winding number of $g_i(z)$ with respect to the origin for all $i = 1$ to k . Also, let Γ_0 and Γ_1 be the contours described in the complex plane defined respectively by the functions $f(c_1 z^{p_1}, c_2 z^{p_2}, \dots, c_k z^{p_k})$ and $f(g_1(z), g_2(z), \dots, g_k(z))$, as the variable z describes the unit circle $z = \exp(j\theta)$ beginning from $\theta = 0$ and ending in $\theta = 2\pi$ in the anticlockwise direction. Clearly, Γ_0 and Γ_1 are closed contours lying in the finite complex plane, each with the same initial and terminal points at $f(c_1, c_2, \dots, c_k)$.

We first claim that the number of encirclements of the origin of the complex plane by the contours Γ_0 and Γ_1 are the same. To prove this assertion we define the functions $\psi_i(\theta)$ for $0 \leq \theta \leq 2\pi$ as: $\psi_i = \arg(g_i(\exp(j\theta)))$ for each $i = 1$ to k . Note that since each $g_i(z)$ is a continuous mapping of the unit circle into the unit circle, $\psi_i(\theta)$'s are continuous functions of θ , and we can also write:

$$g_i(\exp(j\theta)) = \exp(j\psi_i(\theta)) \text{ for } i = 1 \text{ to } k \quad (2.16)$$

In particular, we have (2.17a), whereas (2.17b) follows from the fact that the winding number of $g_i(z)$ with respect to the origin is p_i for each $i = 1$ to k . Consider next a complex valued function $h(t, \theta)$ defined for $0 \leq t \leq 1$, $0 \leq \theta \leq 2\pi$ as in (2.18), where the $\phi_i(\theta)$'s in (2.18) are as defined in (2.19).

$$c_i = g_i(1) = \exp(j\psi_i(0)), \quad \psi_i(2\pi) = 2\pi p_i + \psi_i(0) \quad (2.17a, b)$$

$$h(t, \theta) = f(\phi_1(\theta), \phi_2(\theta), \dots, \phi_k(\theta)) \quad (2.18)$$

$$\phi_i(\theta) = \exp(jt\psi_i(\theta) + j(1-t)(\psi_i(0) + p_i\theta)) \quad (2.19)$$

for each $i = 1$ to k .

Substituting $t=1$ in (2.19) and subsequently making use of (2.18) and (2.19) we have $h(1,\theta) = f(g_1(z), g_2(z), \dots, g_k(z))$ with $z = \exp(j\theta)$. Similarly, by substituting $t = 0$ in (2.19) and subsequently making use of (2.18), (2.19) and (2.17a) we obtain $h(0,\theta) = f(c_1 z^{p_1}, c_2 z^{p_2}, \dots, c_k z^{p_k})$ with $z = \exp(j\theta)$. Also, it follows from (2.17), (2.18) and (2.19) that $h(t,0) = h(t,2\pi) = f(c_1, c_2, \dots, c_k)$ for all t in $0 \leq t \leq 1$. Furthermore, since the $\psi_i(\theta)$'s are continuous functions of θ , and $f(\underline{z})$ is a polynomial and hence a continuous function of $\underline{z} = (z_1, z_2, \dots, z_k)$, it follows in view of (2.18) and (2.19) that $h(t,\theta)$ is a continuous function of t and θ . The function $h(t,\theta)$, when viewed as a function of θ only, can therefore be thought of as representing a family of closed contours with their initial and terminal points fixed at $f(c_1, c_2, \dots, c_k)$, which are continuously parametrized by the variable t in such a way that we obtain Γ_0 when $t = 0$ and Γ_1 when $t = 1$. In addition, since it follows from (2.19) that $|\phi_i(\theta)| = 1$ for each i , we have from condition (a) of the present theorem that $h(t,\theta) \neq 0$ for all $0 \leq t \leq 1$, $0 \leq \theta \leq 2\pi$. Therefore, the function $h(t,\theta)$ can be taken to represent a continuous deformation of the contour Γ_0 into Γ_1 with the initial and terminal points fixed at $f(c_1, c_2, \dots, c_k)$ such that at any intermediate stage of the continuous deformation process the contour may never pass through the origin $(0, j0)$ of the complex plane. It, therefore, follows that the number of encirclements of the origin $(0, j0)$ by Γ_0 and Γ_1 are the same.¹

Next, by using arguments similar to the one used above we show that the number of encirclements of the origin by the

¹This intuitive notion is more formally expressed by saying that if Γ_0 and Γ_1 are Ω -homotopy of each other then they are Ω -homologous (theorem 13.15 in [18]). Here, Ω is the set of all complex numbers except the origin $(0, j0)$.

contour Γ_1 is zero. Consider the family of circles described by the complex function $k(r,s)$ of real variables $r,s \in [0,1]$ defined as $k(r,s) = \exp(2\pi js)$. Also, consider the function $K(r,s) = f(g(k(r,s)))$. Then $K(1,s)$ describes Γ_1 and $K(0,s) = f(g(0))$ for all s . Thus, if r changes from 1 to 0, the contour Γ described by $K(r,s)$ for $s \in [0,1]$ contracts continuously from Γ_1 into the point $f(g(0)) \neq 0$. Hence the number of encirclements of the origin must be zero for Γ sufficiently close to 0, and it must therefore be equal to zero also for $r=1$, because due to condition (b) of the present theorem Γ never goes through the origin for $r \in (0,1)$.²

Thus the number of encirclement of the origin by the contour Γ_0 is zero. Since Γ_0 is the image of the unit circle $|z|=1$ due to the mapping defined by the function $f(c_1 z^{p_1}, c_2 z^{p_2}, \dots, c_k z^{p_k})$, it follows by using the principle of argument that $f(c_1 z^{p_1}, c_2 z^{p_2}, \dots, c_k z^{p_k}) \neq 0$ in $|z| \leq 1$. Since $p_i \geq 0$, and from (2.17a) we have that $|c_i| = |g_i(1)| = 1$ this latter statement along with condition (a), due to Corollary 2.7.2, completes the proof the present theorem.

Before we can undertake the proof of the extended version of Rudin's theorem reported in [17], we need the following lemma.

Lemma 2.8: Let $f(\underline{z})$ be a polynomial in $\underline{z} = (z_1, z_2, \dots, z_k)$ such that $f(\underline{z}) \neq 0$ in $|\underline{z}| = 1$. Also, let the indices $i = 1$ to k be the union of disjoint sets of indices J_1, J_2 and J_3 . If $f(\underline{z})$ satisfies: (a) $f(\underline{z}) \neq 0$ for $|z_i| \leq 1, i \in J_1; z_i = \alpha_i, i \in (J_2 \cup J_3)$, and (b) $f(\underline{z}) \neq 0$ for all $\underline{z}$ with $z_i = \beta_i$ for $i \in J_1, z_i = g_i(z)$ for $i \in J_2$ and $|z| \leq 1$, and $z_i = \gamma_i$ for $i \in J_3$, where the $g_i(z)$'s are functions satisfying the same

²This step does not indeed follow from the principle of argument.

conditions as those in Rudin's theorem, and the $\alpha_i, \beta_i, \gamma_i$'s are some complex numbers such that $|\alpha_i| = |\gamma_i| = 1$ and $|\beta_i| \leq 1$, then $f(\underline{z}) \neq 0$ for $|z_i| \leq 1, i \in (J_1 \cup J_2); z_i = \gamma_i, i \in J_3$.

Proof: Since $f(\underline{z}) \neq 0$ in $|\underline{z}| = 1$, invoking Lemma 2.6 along with condition (a) of the present lemma yields that:

$$f(\underline{z}) \neq 0 \text{ for } |z_i| \leq 1, i \in J_1; |z_i| = 1, i \in (J_2 \cup J_3) \quad (2.20)$$

Condition (2.20) implies, in particular, that $f(\underline{z}) \neq 0$ for $z_i = \beta_i, i \in J_1; |z_i| = 1, i \in J_2$ and $z_i = \gamma_i, i \in J_3$, which along with condition (b), due to Rudin's theorem, implies that $f(\underline{z}) \neq 0$ for $z_i = \beta_i, i \in J_1, |z_i| \leq 1, i \in J_2$ and $z_i = \gamma_i, i \in J_3$. This latter conclusion, along with a particularization of (2.20) via the choice of $z_i = \gamma_i, i \in J_3$, due to Lemma 2.3, yields the desired result.

We can now prove the generalized version of Rudin's theorem stated as follows.

Theorem 2.11: Let the set of indices $i = 1$ to k be the union of n disjoint subsets of indices $I_j, j = 1, 2, \dots, n$. Then the polynomial $f(\underline{z})$ in $\underline{z} = (z_1, z_2, \dots, z_k)$ is devoid of zeros in $|\underline{z}| \leq 1$ if and only if the following hold true simultaneously:

(a) $f(\underline{z}) \neq 0$ in $|\underline{z}| = 1$

(b) for each $\mu, \mu = 1, 2, \dots, n$, the polynomials obtained by setting $z_i = \beta_{i\mu}, i \in I_j, j = 1$ to $(\mu-1); z_i = g_{i\mu}(z), i \in I_\mu$ and $z_i = \gamma_{i\mu}, i \in I_j, j = (\mu+1)$ to n in $f(\underline{z})$ are devoid of zeros in $|\underline{z}| \leq 1$, where the $g_{i\mu}(z)$'s are functions satisfying the same constraints as those satisfied by the $g_i(z)$'s in Rudin's theorem, and the $\beta_{i\mu}$'s and $\gamma_{i\mu}$'s are some complex numbers such that $|\beta_{i\mu}| \leq 1$ and $|\gamma_{i\mu}| = 1$.

Proof: Only sufficiency needs to be proved. We claim that for all m in $1 \leq m \leq n$ the polynomial B_m obtained by freezing z_i at $z_i = \gamma_{im}$, $i \in \bigcup_{j=m+1}^n I_j$ satisfies $B_m \neq 0$ in $|z_i| \leq 1$, $i \in \bigcup_{j=1}^m I_j$. The proof of this assertion is via induction on m . Since B_1 is obtained from $f(\underline{z})$ by setting $z_i = \gamma_{i1}$, $i \in \bigcup_{j=2}^n I_j$ and $|\gamma_{i1}| = 1$, it follows from condition (a) that $B_1 \neq 0$ for $|z_i| = 1$, $i \in I_1$. Furthermore, condition (b) with $\mu = 1$ implies that $B_1 \neq 0$ in $|z| \leq 1$ with $z_i = g_{i1}(z)$, $i \in I_1$. The last two conditions, in view of Rudin's theorem, prove that $B_1 \neq 0$ in $|z_i| \leq 1$, $i \in I_1$. Therefore, our assertion is true for $m = 1$. We next assume that the assertion is true for m , with $1 \leq m < n-1$ i.e., $f(\underline{z}) \neq 0$ for $|z_i| \leq 1$, $i \in \bigcup_{j=1}^m I_j$ and $z_i = \gamma_{im}$, $i \in \bigcup_{j=m+1}^n I_j$. Applying Lemma 2.8 on $f(\underline{z})$ with

$J_1 = \bigcup_{j=1}^m I_j$, $J_2 = I_{m+1}$, $J_3 = \bigcup_{j=m+2}^n I_j$ along with condition (a) and condition (b) of the present lemma, with $\mu = m+1$, it directly follows that $f(\underline{z}) \neq 0$ in $|z_i| \leq 1$ for $i \in \bigcup_{j=1}^m I_j$; $z_i = \gamma_{i,m+1}$; $i \in \bigcup_{j=m+2}^n I_j$. Therefore, $B_{m+1} \neq 0$ in $|z_i| \leq 1$,

$i \in \bigcup_{j=1}^{m+1} I_j$. The proof of our assertion via the induction is thus complete. The theorem then follows by noting that B_n and $f(\underline{z})$ are identical in all the variables z_i , $i = 1$ to k .

Remark: We note that Rudin's theorem is a special case of Theorem 2.11 when $n=1$. Also if $\beta_{i\mu} = \beta_i$ for all μ then we obtain the extension of Rudin's theorem reported in [17]. In this sense Theorem 2.11 can be considered as a slightly generalized version than that reported in [17].

As a corollary to the above theorem, we have the following result reported in [20].

Corollary 2.11.1: The polynomial $f(\underline{z})$ in $\underline{z} = (z_1, z_2, \dots, z_k)$ is devoid of zeros in $|\underline{z}| \leq 1$ if and only if the following hold true simultaneously.

(a) $f(\underline{z}) \neq 0$ in $|\underline{z}| = 1$

(b) for each μ , $\mu = 1, 2, \dots, k$, the polynomials obtained by setting $z_i = \beta_i$, $i = 1$ to $(\mu-1)$, $z_i = z$, $i = \mu$, and $z_i = \gamma_{i\mu}$, $i = (\mu+1)$ to k in $f(z)$ are devoid of zeros in $|z| \leq 1$, where β_i 's and γ_i 's are some constants such that $|\beta_i| \leq 1$ and $|\gamma_{i\mu}| = 1$.

Proof: The proof of the above corollary clearly follows from Theorem 2.11 by choosing $n = k$, $I_j = \{j\}$ for $j = 1, 2, \dots, k$, $\beta_{i\mu} = \beta_i$ and $g_{i\mu}(z) = z$ for each i and μ .

Remark: It is possible to formulate a more direct proof of Corollary 2.11.1 without making use of the Rudin's theorem via the use of Lemma 2.3, Lemma 2.6 and a strategy similar to the one adopted for the proof of Theorem 2.11. However, the details of such a proof is omitted from the present discussion for the sake of brevity.

Also note that if $\beta_i = 0$ for all i , and $\gamma_{i\mu} = 1$ for all i and μ then Corollary 2.11.1 coincides with a result stated and proved in Theorem 2.4 in [7].

2.6. Conclusion

A correct formulation of the continuity property of zeros of a multivariable polynomial as a function of its coefficients has been discussed. This property has been used to derive all the conditions for a polynomial $f(\underline{z})$ in $\underline{z} = (z_1, z_2, \dots, z_k)$ to be devoid of zeros in $|\underline{z}| \leq 1$ previously known in the literature. The proofs presented here are believed to be simple, rigorous and more intuitive than those published earlier. The present report deals with results arising from studies of bounded-input-bounded-output property of multidimensional discrete time systems only. It is well known [1,10] that unlike in one-dimension, the obvious continuous domain analogs of certain discrete time domain results do not hold true unless proper modifications necessitated by the non-compactness of the right half polydomain are duly made. The proofs of these results in the multidimensional ($k > 2$) context based on the continuity property of the zeros of a polynomial as a function of its coefficients can also be worked out along lines similar to those discussed here. This discussion will be the content of a forthcoming report [21].

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CHAPTER 3

NEW RESULTS ON STABLE MULTIDIMENSIONAL POLYNOMIALS

3.1. Introduction

Recent studies on the scattering parameter description of passive multidimensional systems have given rise to a new class of multidimensional Hurwitz polynomials, called scattering Hurwitz polynomials, by allowing zeros of restricted nature to occur on the boundary of the domain under consideration [1]. Originally, two different definitions of these polynomials were introduced, the equivalence of which has been demonstrated via an extension of the maximum modulus theorem for analytic functions of several complex variables [2], and some of their properties have been discussed in [3]. Nontrivial properties of scattering Hurwitz polynomials occurring in discrete time domain applications are considered in [4] and their testing procedures are elaborated in the two-dimensional context in [5]. An alternative approach to deriving the basic properties of scattering Hurwitz polynomials has also been offered [6]. The present report, however, will be organized such that all proofs are complete without requiring any knowledge of [6].

The aforementioned investigation is carried further in the present report by classifying a wider variety of multidimensional polynomials occurring in transfer function descriptions of passive systems. Whereas the scattering Hurwitz polynomials occur as the denominators of bounded functions, reactance Hurwitz and immittance Hurwitz polynomials, as defined later, are characterized as the denominators (and hence the numerators) of reactance functions and immittance functions respectively. Related other results on properties of multidimensional polynomials

and rational functions are discussed in this context. Some related investigations in [7] have been brought to the attention of the authors.

Notations, terminologies and definitions and some general properties of multivariable polynomials to be used in the rest of the report are introduced in Section 3.2. Properties of widest-sense Hurwitz polynomials and self-paraconjugate Hurwitz polynomials are discussed in Section 3.3. Properties of scattering Hurwitz polynomials previously unpublished in the literature are discussed in Section 3.4. Section 3.5 deals with properties of positive functions, whereas the reactance Hurwitz and immittance Hurwitz polynomials are defined and their properties studied in Sections 3.6 and 3.7 respectively. In Section 3.8 a few results potentially useful for testing the positivity property of multidimensional rational functions are derived. A nontrivial result concerning the property of nonnegativity of the real part of a rational function, analytic in the right half-polyplane, in terms of the behavior of the function on the distinguished boundary of the domain of holomorphy is to be noted in this connection. Finally results obtained are summarized, and conclusions are drawn in Section 3.9.

3.2. Notation, terminology, definitions and general properties of multivariable polynomials.

The following is a partial glossary of notations and terminologies to be used in succeeding discussions.

A polynomial g in k variables $p_1, p_2, \dots, p_k$ will be denoted simply by g or $g(p_1, p_2, \dots, p_k)$ or by $g(p)$, where p denotes the k -tuple of variables $p = (p_1, p_2, \dots, p_k)$. We will also write $g = g(p)$ or $g = g(p_1, p_2, \dots, p_k)$ to indicate that g is a polynomial in k -variables. We take for granted that a polynomial g in k variables may be independent of one or more of the variables $p_1, p_2, \dots, p_k$. The k -variable polynomial g will be said to involve a variable p_i if the indeterminate p_i actually exists in at least one of the monomials composing the polynomial g . If $g = g(p)$ involves p_i then the $(k-1)$ variable polynomials obtained by assigning arbitrary values to p_i cannot all be identical. A polynomial $g = g(p)$ will be called nonconstant if g involves at least one of the variables $p_1, p_2, \dots, p_k$. The set of integers 1 to k will be designated by I i.e., $I = \{1, 2, \dots, k\}$. By $i_1, i_2, \dots, i_k$ we designate a permutation of the integers $1, 2, \dots, k$. A nonconstant factor of the form d^v where v is an integer ≥ 2 is said to be a multiple factor, and a nonconstant factor that is irreducible is called an irreducible factor.

If $g = g(p)$ is written as a polynomial in p_i , $i \in I$, as

$$g = \sum_{v=0}^{n_i} A_v p_i^v, \quad (3.1)$$

where the coefficients A_v are polynomials in the remaining variables, with $A_{n_i} \neq 0$, then n_i is called the partial degree of g in the variable p_i and is to be denoted by $\deg_i g$. Two polynomials will be said to be relatively prime if they do not have a nonconstant common factor. The terms factor coprime and proper factor are also to be used respectively

for the terms relatively prime and nonconstant factor.

The asterisk $*$, when used as a superscript along with any scalar expression (or a constant), will indicate complex conjugation. The paraconjugate of a polynomial (also of a rational function) g is defined as: $g_* = g_*(p) = g^*(-p_1^*, -p_2^*, \dots, -p_k^*)$. A polynomial g is said to be self-paraconjugate if $g_* = Cg$, where C is a constant (necessarily unimodular, i.e., $|C|=1$). g is said to be paraeven or paraodd if $C = 1$ or $C = -1$ respectively. Sometimes it will be appropriate to write the k -tuples p as: $p = (p_1, p')$, where p' indicates the $(k-1)$ -tuple $p' = (p_2, p_3, \dots, p_k)$. Correspondingly we will also write: $g = g(p_1, p')$. A polynomial f in the variables $p_2, p_3, \dots, p_k$ will be expressed as $f = f(p_2, p_3, \dots, p_k)$ or equivalently as $f = f(p')$. For any specific $i \in I$, we will use the phrase " $(k-1)$ -variable polynomial obtained by freezing the variable p_i in $g(p)$ " to mean the $(k-1)$ variable polynomial obtained from $g(p)$ by assigning a fixed value to p_i . A second subscript to the variables p_i , $1 \leq i \leq k$ such as p_{i0} , will usually mean a fixed value of p_i . Correspondingly the notations $p_0 = (p_{10}, p_{20}, \dots, p_{k0})$ and $p'_0 = (p_{20}, p_{30}, \dots, p_{k0})$ etc. will be used. The notation $\text{Rep } p > \underline{0}$ (or $\text{Rep } p' > \underline{0}$) will be taken to mean $\text{Rep } p_i > 0$ for all $i \in I$ (or, $i \in \{2, 3, \dots, k\}$ correspondingly), etc. (all p_i involved being obviously assumed finite). The symbol ω will be used exclusively for designating real numbers. Thus, notations such as $p_i = j\omega_i$ and $p_{i0} = j\omega_{i0}$ imply that $\text{Rep } p_i = 0$, $\text{Rep } p_{i0} = 0$ respectively. Similarly $\underline{\omega}$ is used for the k -tuple of real numbers $(\omega_1, \omega_2, \dots, \omega_k)$, i.e., $p = j\underline{\omega}$ implies $\text{Rep } p = \underline{0}$ etc. Wherever appropriate, definitions and notations discussed so far also apply if g is more generally a rational function in p . The notation $|| \cdot ||$ denotes any suitable norm, say, the Euclidean norm,

$$||p||^2 = \sum_{i=1}^k |p_i|^2.$$

Let P be a set of k -tuples $p = (p_1, p_2, \dots, p_k)$, where all p_i belong to the same number field K (hereafter always the field of real numbers or the field of complex numbers). We will say a certain property holds for almost all values if a variable may be equal to any element of the field except, possibly, finitely many of them. The set of all values that the variable may then take is said to be almost complete. The symbol Ω will be reserved to denote the set P when the variables are restricted to be real.

Definition 3.2.1a: We say that P is a sequentially almost complete set of order $m \geq 1$, with $m \leq k$, if there exists a permutation $i_1, i_2, \dots, i_k$ of the integers $1, 2, \dots, k$ such that all $p \in P$ can be generated in the following way: There exists an almost complete set $K_1 \subset K$ such that any $p_{i_1} \in K_1$ may be chosen. For any choice thus made, assuming $m \geq 2$, there exists an almost complete set $K_2 \subset K$ (possibly depending on the particular $p_{i_1} \in K_1$ selected) such that any $p_{i_2} \in K_2$ may be chosen. Again for any choice thus made, assuming $m \geq 3$, there exists an almost complete set $K_3 \subset K$ (possibly depending on the particular p_{i_1} and p_{i_2} selected) such that any $p_{i_3} \in K_3$ may be chosen, etc. If $m = k$ this process is continued until we have reached p_{i_k} . If $m < k$, once we have reached i_m there exists at least one $(k-m)$ -tuple $(p_{i_{m+1}}, \dots, p_{i_k})$ (possibly depending on the particular p_{i_1} to p_{i_m} selected) that may be chosen. Finally, we may extend the above definition to the situation $m = 0$ by saying that in this case the set P is not empty.

Definition 3.2.1b: P is sequentially infinite of order m , $1 \leq m \leq k$, if it can be generated as in Definition 3.2.1a except for replacing everywhere the term "almost complete set" by the term "infinite set".

Note: In Definitions 3.2.1a and 3.2.1b, the permutation

$(i_1, i_2, \dots, i_k)$ will be called the ordering of P . If $i_\mu = \mu$, $\mu = 1$ to k , then P is said to be ordered naturally.

Definition 3.2.1c: P is sequentially exceeding $n = (n_1, n_2, \dots, n_k)$ with order m , where $m \leq k$, if it can be generated as in Definition 3.2.1a except that the terms "an almost complete set $K_i \subset K$ " etc. are replaced by "a set $K_i \subset K$ comprising at least $n_i + 1$ elements" etc., the n_i , $i = 1$ to k , being finite, nonnegative integers.

An obvious interrelationship between the sets defined above is also summarized in the following theorem.

Theorem 3.2.1: A sequentially almost complete set of order m is also sequentially infinite of order m , and a set of the latter type is sequentially exceeding $\underline{n} = (n_1, n_2, \dots, n_k)$ with order m , and this for any choice of the $\underline{n}$.

Definition 3.2.1d: P is almost complete of order m , infinite of order m , or exceeding $n = (n_1, n_2, \dots, n_k)$ with order m if $P = P_1 \times P_2 \times \dots \times P_k$, the sets P_i , $i = 1$ to k , being non-empty and such that at least m of them are almost complete, infinite, or contain a number of elements larger than the corresponding integer n_i , respectively, where $m \leq k$.

Some properties of the type of sets just defined, as they relate to the sets defined earlier in Definitions 3.2.1a, 3.2.1b and 3.2.1c are mentioned in the following.

Theorem 3.2.2: A set, P , of k -tuples, p , that is

1. almost complete of order m is sequentially almost complete of order m ,
2. infinite of order m is sequentially infinite of order m ,

3. exceeding $\underline{n} = (n_1, n_2, \dots, n_k)$ with order m is sequentially exceeding $\underline{n} = (n_1, n_2, \dots, n_k)$ with order m .

Proofs of these results clearly follow from a close examination of the definitions of the sets involved.

For the purpose of the present report we will also adopt the following definitions. More definitions and terminologies will be introduced as they occur in the main body of the text.

Definition 3.2.2: A polynomial $g = g(p)$ is widest-sense Hurwitz if $g(p) \neq 0$ for $\text{Re } p > 0$.

Definition 3.2.3: A polynomial $g = g(p)$ is strict-sense Hurwitz if $g(p) \neq 0$ for $\text{Re } p \geq 0$.

Definition 3.2.4: A polynomial $g = g(p)$ is scattering Hurwitz if the following conditions simultaneously hold true:

- (i) $g(p) \neq 0$ for $\text{Re } p > 0$, i.e., g is widest-sense Hurwitz

(3.2)

- (ii) g and g_* are relatively prime polynomials.

(3.3)

Definition 3.2.5: A polynomial $g = g(p)$ is a self-paraconjugate Hurwitz polynomial if it is a widest-sense Hurwitz polynomial and is self-paraconjugate.

Definition 3.2.6: A polynomial $g = g(p)$ is reactance Hurwitz if it can be written as a constant (possibly complex) multiplied by the paraeven or paraodd part of a scattering Hurwitz polynomial.

Definition 3.2.7: An immittance Hurwitz polynomial is the product of a scattering Hurwitz and a reactance Hurwitz polynomial.

Definition 3.2.8: A function $F(p)$ is called a positive function if $\operatorname{Re} F(p) \geq 0$ everywhere in $\operatorname{Re} p > 0$, where F is holomorphic.

The positive function $F = jC$, where C is a real constant is said to be trivial. All other positive functions are non-trivial.

Definition 3.2.9: A paraodd rational positive function is called a reactance function.

Note that the Definitions 3.2.8 and 3.2.9 do not assume the function under consideration to be a real function.

Theorem 3.2.3: If $g(p)$ is a polynomial in k -variables such that the set of zeros of g comprises a sequentially infinite set of order k , then g is identically equal to zero. More generally, let $n_i = \deg_i g$, $i = 1$ to k . If $g = 0$ for all $p \in P$, where P is sequentially exceeding $\underline{n} = (n_1, n_2, \dots, n_k)$ with order k , then g is identically equal to zero.

Proof: Assume that a set P of the type mentioned exists, but that $g \neq 0$. We may assume P to be ordered naturally. Consider g as a polynomial in p' , the coefficients $A_v(p_1)$ of which are polynomials in p_1 only; the A_v are of degree $\leq n_1$. Obviously, there exists a v' such that $A_{v'}(p_1) \neq 0$. Hence, among the values of p_1 to be considered for forming P , there must exist at least one, say p_{10} , with $A_{v'}(p_{10}) \neq 0$. The polynomial g_1 , defined by $g_1(p') = g(p_{10}, p')$ is then not identically zero.

Next, we proceed with g_1 as before with g i.e., consider g_1 as a polynomial in $(p_3, p_4, \dots, p_k)$ the coefficients $B_v(p_2)$ of

which are polynomials in p_2 etc.... Finally, we arrive at $g(p_0) \neq 0$ where $p_0 \in P$. This, however, is a contradiction.

Lemma 3.2.4: For any non-constant polynomial g , there exists a sequentially almost complete set, P , of order $(k-1)$ such that $g(p) = 0$ for all $p \in P$.

Proof: We may assume that g involves p_k . Freeze p_1 at p_{10} and consider the polynomial g_1 defined by $g_1(p') = g(p_{10}, p')$. Since the coefficients of g_1 are polynomials in p_{10} , there are at most finitely many values of p_{10} for which $g_1(p')$ is independent of p_k . For any other choice of p_{10} we may apply the same argument to g_1 as formerly to g , with p_2 taking the role of p_1 etc. Finally, we find a polynomial $g_{k-1}(p_k)$ that still involves p_k and thus has at least one zero.

Theorem 3.2.5: If f and g are polynomials in k variables then f and g have a proper common factor if and only if the set of zeros that are common to f and g is sequentially infinite of order $(k-1)$.

Proof: Necessity is obvious in view of Lemma 3.2.4. To prove sufficiency, let P be the set mentioned in the statement of the theorem and assume that p_1 is the last variable selected in forming P , i.e., in the terminology of Definition 3.2.1a, that we have $i_k = 1$. Let P' be the set of $(k-1)$ -tuples p' involved in forming P . There exist polynomials u, v and w such that $uf + vg = w$, where $w = w(p')$ is a polynomial independent of p_1 , $\deg_1 u < \deg_1 g$ and $\deg_1 v < \deg_1 f$; a polynomial w thus defined has the property that $w = 0$ holds if and only if f and g have a common proper factor involving p_1 [8]. Furthermore, since P' is sequentially infinite of order $(k-1)$ it is clear from Theorem 3.2.3 that $w(p')$ is actually identically zero, i.e., that f and g have indeed a common factor.

Lemma 3.2.6: Let g be a polynomial in p . Let us select one of the p_i , say $p_{i'}$, and let us freeze $p_{i'}$ at say, $p_{i',0}$. There exist at most finitely many choices of $p_{i',0}$ such that any partial degree of g is lowered.

Proof: We may assume $i' = 1$. Write g as a polynomial in p' whose coefficients are polynomials in p_1 . Clearly, there are at most finitely many values of $p_{1,0}$ such that the leading one of these coefficients becomes zero.

Theorem 3.2.7: Let f and g be two relatively prime polynomials. For any m such that $1 \leq m < k$ let us freeze m of the variables p_i , say for $i = i_1$ to i_m , at corresponding values $p_{i,0}$. Let f_1 and g_1 be the resulting polynomials in the remaining variables. Then there exists a sequentially almost complete set P_m , of m -tuples of order m such that for $(p_{i_1,0}, p_{i_2,0}, \dots, p_{i_m,0}) \in P_m$, the polynomials f_1 and g_1 are still relatively prime.^{m0} Furthermore, for any ordering chosen there exists a set P_m with the property given.

Proof: It is enough to prove the theorem for $m = 1$ and to assume that the variable to be frozen is p_k . Define u , v and w as for the proof of Theorem 3.2.5, and let u_1 , v_1 and w_1 be the polynomials resulting from the former set of polynomials by freezing p_k at $p_{k,0}$. Clearly, w is not identically zero, and there are at most finitely many values of $p_{k,0}$ for which at least one of the relations $\deg_1 f_1 < \deg_1 f$, $\deg_1 g_1 < \deg_1 g$ (cf. Lemma 3.2.6) or $w_1(p_2, \dots, p_{k-1}) = 0$ could hold. For all other choices of $p_{k,0}$ the conditions for ensuring that f_1 and g_1 have no common factor involving p_1 are fulfilled. In a similar way one can show that there are at most finitely many $p_{k,0}$ for which f_1 and g_1 have a common factor involving any of the variables p_2 to p_{k-1} .

Theorem 3.2.8: For any polynomial g that is not identically

zero there exists a set, P , of k -tuples p , that is sequentially almost complete of order k and such that $g(p) \neq 0$ for $p \in P$. Furthermore, a set P with the given property exists for any choice of the ordering of this set.

Proof: It is sufficient to assume natural ordering. Let g_1 be the polynomial in p' obtained by freezing p_1 at p_{10} , i.e., $g_1(p') = g(p_{10}, p')$. The coefficients of g_1 are polynomials in p_{10} . Hence, $g_1(p') \neq 0$ for almost all choices for p_{10} . For any of these, we may apply the same argument to g_1 as before to g except that p_2 now plays the same role as p_1 had etc. Finally, we arrive at a polynomial g_{k-1} that depends on p_k alone and is not identically zero, which is thus different from zero for almost all p_k .

Theorem 3.2.9: If a polynomial g is devoid of zeros in the region $\text{Re } p \geq 0$ as well as in the region $\text{Re } p \leq 0$ then g is a nonzero constant.

Proof: If g is not a constant let us assume that it involves, say, the variable p_1 . Let A be the leading coefficient of g when writing it as a polynomial in p_1 ; clearly, A is a polynomial in p' and is not identically zero. According to Theorem 3.2.8 there exists a sequentially almost complete set, P' , of $(k-1)$ -tuples such that $A(p') \neq 0$ for $p' \in P'$. Hence, we may consider a fixed value $p'_0 = j\omega'_0$ such that $g_0(p_1) = g(p_1, j\omega'_0)$ still involves p_1 . Then there exists a $p_1 = p_{10}$ such that $g(p_{10}, \omega'_0) = 0$. This, however, leads to a contradiction, because both $\text{Re } p_{10} \geq 0$ and $\text{Re } p_{10} \leq 0$ are impossible due to our hypothesis.

Theorem 3.2.10: Let g be a polynomial in p and assume that there exists an $i \in I$ and a fixed value p_0 such that $g(p) = 0$ if $p_i = p_0$ and the remaining p_i take any arbitrary value belonging to a sequentially infinite set of order $k-1$. Then g contains $(p - p_0)$ as factor.

Proof: We may assume $i'=1$. Write g as a polynomial in p' whose coefficients are polynomials in p_1 . In view of Theorem 3.2.3, with k replaced by $k-1$, all these coefficients are zero for $p_1=p_0$.

Theorem 3.2.11: If $g(p)$ is a polynomial and $g(p_0) \neq 0$ for some p_0 then there exists an $\eta > 0$ such that $g(p) \neq 0$ for all p in the neighbourhood $||p-p_0|| < \eta$. More generally, for any $\epsilon > 0$ there exists an $\eta > 0$ such that $|g(p)| > |g(p_0)| - \epsilon$ for $|p-p_0| < \eta$.

Proof: Since g is a polynomial and hence a continuous function of p , for any given $\epsilon > 0$ there exists an $\eta > 0$ such that for all p satisfying $||p-p_0|| < \eta$ we have that $|g(p)-g(p_0)| < \epsilon$, i.e., $|g(p)| > |g(p_0)| - \epsilon$, thus $|g(p)| > 0$ if we choose $\epsilon < |g(p_0)|$.

Theorem 3.2.12: Let g be a polynomial in p having a zero at p_0 . Let U be any neighborhood of p_0 . There exists a sequentially infinite set $P \subset U$ of k -tuples p that is of order $k-1$ and such that $g(p)=0$ for $p \in P$.

Proof: Since any polynomial can be decomposed into a product of irreducible factors, it is sufficient to assume that g is irreducible. On the other hand, the result is true for $k=1$. Assume thus that it holds for $k-1$, we will show that it remains valid for k . For this, consider g as a polynomial g_1 in p' whose coefficients are polynomials in p_1 and write $p_0=(p_{10}, p_0')$. In view of Theorem 3.2.10, g_1 cannot be identically zero for $p_1=p_{10}$ since otherwise g would not be irreducible. Hence, we may move p_1 from p_{10} to a position p_{11} close to p_{10} , apply to g_1 the continuity property of the zeros of a polynomial in several variables [2], and conclude that there exists a p_1' such that $g_1(p_1')=0$ for $p_1=p_{11}$. By assumption, there exists thus a sequentially infinite set P'

of $(k-1)$ -tuples p' that is of order $k-2$ and such that $g_1(p')=0$, thus $g(p_1, p')=0$, for $p' \in P'$ and $p_1 = p_{11}$. Furthermore, for any p_{11} sufficiently close to p_{10} any corresponding value p' then resulting in the way just explained is such that $(p_{11}, p') \in U$. Hence, a set P as required does indeed exist.

This result together with Theorem 3.2.5 yields immediately the following corollary:

Corollary 3.2.12.1: Let g and h be two relatively prime polynomials in p and let p_0 be a common zero of g and h . Then in any neighborhood of p_0 there exist points p for which, say, $g(p)=0$ and $h(p) \neq 0$.

The following theorem is known in a more general form from the theory of functions in several complex variables [10], but is included for the sake of completeness and in order to point out a simple proof based on the above results.

Theorem 3.2.13: Consider the function $F=h/g$ where g and h are relatively prime polynomials in p . If F is known to be bounded for all those p in a domain D where $g(p) \neq 0$, then $g(p) \neq 0$ for all $p \in D$.

Proof: Without loss of generality, we may assume $|F(p)| \leq 1$ for $p \in D$. Assume that there exists a point $p_0 \in D$ such that $g(p_0)=0$. In view of Corollary 3.2.12 it is sufficient to assume $h(p_0) \neq 0$. Due to continuity of the polynomial g we may state that for any $\epsilon > 0$ there exists an $\eta_1 > 0$ such that $|g(p_1)| < \epsilon$ for all p_1 satisfying $\|p_1 - p_0\| < \eta_1$. Similarly, due to Theorem 3.2.11, for any $\epsilon > 0$ there exists an η_2 such that $|h(p_1)| < |h(p_0)| - \epsilon$ if $\|p_1 - p_0\| < \eta_2$. Clearly, we may choose $\epsilon < |h(p_0)|/2$, in which case the expression for $h(p_1)$ becomes $|h(p_1)| > \epsilon$. Choose then $\eta > 0$ such that $\eta \leq \eta_1$ and $\eta \leq \eta_2$. Due to Theorem 3.2.8, we can choose p_1 such that $\|p_1 - p_0\| < \eta$ and $g(p_1) \neq 0$, i.e., by assumption, that $|h(p_1)/g(p_1)| \leq 1$. This is

in contradiction with $|g(p_1)| < \epsilon$ and $|h(p_1)| > \epsilon$.

3.3. Properties of Widest Sense Hurwitz Polynomials and Self-paraconjugate Hurwitz Polynomials

The proof of the following Theorem is rather trivial:

Theorem 3.3.1: Let g be a widest-sense Hurwitz polynomial in p . The following holds:

(i) Freeze one of the p_i , say $p_{i'}$, at $p_{i',0}$ with $\operatorname{Re} p_{i',0} > 0$. The resulting polynomial in the remaining $k-1$ variables p_i is also widest-sense Hurwitz.

(ii) Factors and products of widest-sense Hurwitz polynomials are also widest-sense Hurwitz.

Lemma 3.3.2: Let g be a widest-sense Hurwitz polynomial. Let g_1 be the polynomial in $k-1$ variables obtained by freezing any one of the variables p_i , say $p_{i'}$, at $p_{i',0} = j\omega_{i',0}$. Then,

(i) there exists an almost complete set, Ω' , of real numbers such that g_1 is widest-sense Hurwitz if $\omega_{i',0} \in \Omega'$ and that g_1 is identically zero if $\omega_{i',0} \notin \Omega'$;

(ii) There exists an almost complete real set $\Omega'' \subset \Omega'$ such that for $\omega_{i',0} \in \Omega''$, g_1 is widest-sense Hurwitz and has, in its $k-1$ variables, the same partial degrees as g .

Proof: We may assume $i' = 1$. Write g as a polynomial in p' whose coefficients are polynomials in p_1 .

(i) There are at most finitely many values of p_1 , thus a fortiori finitely many such values with $\operatorname{Re} p_1 = 0$, for which all these coefficients can become zero. Choosing $j\omega_{1,0}$ different from any of these values, moving p_1 slightly from $j\omega_{1,0}$ into $\operatorname{Re} p_1 > 0$, and applying the continuity property of the zeros of a polynomial, we see that a zero of $g_1(p')$ in

$\text{Re } p' > 0$ would imply a zero of g in $\text{Re } p > 0$. The proof of (ii) follows from (i) and Lemma 3.2.6.

The following result follows by applying Lemma 3.3.2 m times:

Theorem 3.3.3: Let g be a widest-sense Hurwitz polynomial in p , and let p_m denote an m -tuple obtained by selecting any $m < k$ of the variables p_1 to p_k . Let us freeze p_m at $p_{m0} = j\omega_{m0}$ and let g_1 be the resulting polynomial in the $k-m$ remaining p_i . Then

- (i) g_1 is either widest-sense Hurwitz or identically zero;
- (ii) more precisely, for any ordering selected, there exists a sequentially almost complete set, Ω'_m , of real m -tuples such that g_1 is widest-sense Hurwitz if $\omega_{m0} \in \Omega'_m$ and that it is identically zero if $\omega_{m0} \notin \Omega'_m$;
- (iii) there exists a sequentially almost complete set, $\Omega''_m \subset \Omega'_m$, of real m -tuples such that, for $\omega_{m0} \in \Omega''_m$, g_1 is widest-sense Hurwitz and has, in its $k-m$ variables, the same partial degrees as g .

Theorem 3.3.4: If g is a proper self-paraconjugate Hurwitz polynomial then there exists a sequentially almost complete set, Ω , of order $(k-1)$, composed of real k -tuples $\underline{\omega}$ such that $g(j\underline{\omega}) = 0$ for any $\underline{\omega} \in \Omega$.

Proof: We may assume that g depends on p_1 . Consider the one-variable polynomial $g_1(p_1) = g(p_1, j\underline{\omega}'_0)$, obtained by freezing p' at $p' = j\underline{\omega}'_0$, where $\underline{\omega}'_0$ is a real $(k-1)$ -tuple. Due to Theorem 3.3.3, there exists a sequentially almost complete set, Ω' , of order $(k-1)$ such that, for any $\underline{\omega}'_0 \in \Omega'$, g_1 is of degree ≥ 1 and is widest-sense Hurwitz. Hence, there exists at least one p_{10} , necessarily with $\text{Re } p_{10} \leq 0$, such that $g_1(p_{10}) = 0$. However, since $g_* = Cg$, $p_{10}' = -p_{10}^*$ is also a

zero of g_1 , where $\operatorname{Re} p'_{10} \leq 0$, i.e., $\operatorname{Re} p_{10} \geq 0$. Hence, $\operatorname{Re} p_{10} = 0$.

Theorem 3.3.5: Let g be a widest-sense Hurwitz polynomial and let d be the greatest common divisor of g and g_* . Then d is self-paraconjugate Hurwitz.

Proof: In view of Lemma 1 in [3], we have $d_* = Cd$ where C is a constant. The rest follows from Theorem 3.3.1.

Theorem 3.3.6: If $g(p)$ is a widest-sense Hurwitz polynomial in p , then $g(p)$ and $g_*(p)$ have a proper common factor if and only if $g(j\omega) = 0$ where the real k -tuple ω can assume any value belonging to a certain sequentially infinite set of order $(k-1)$.

Proof: Necessity follows Theorem 3.3.4 and 3.3.5, sufficiency from Theorem 3.2.5.

Theorem 3.3.7: A widest-sense Hurwitz polynomial, g , can be written as a product of a scattering Hurwitz polynomial and a self-paraconjugate Hurwitz polynomial.

Proof: Write $g = ad$ where d is the greatest common divisor of g and g_* . Then, a is relatively prime with a_* . The rest follows from Theorem 3.3.1 (item (ii)) and 3.3.5.

The following corollary follows immediately:

Corollary 3.3.7.1: An irreducible widest-sense Hurwitz polynomial is either scattering Hurwitz or self-paraconjugate Hurwitz.

Theorem 3.3.8: A polynomial g is self-paraconjugate Hurwitz if and only if $g \neq 0$ for $\operatorname{Re} p > 0$ and for $\operatorname{Re} p < 0$.

Proof: Necessity: Since $g_* = Cg$, a zero for $\text{Re } p < 0$ would imply a zero for $\text{Re } p > 0$ and is hence excluded.

Sufficiency: If $g(p) \neq 0$ for $\text{Re } p > 0$ then due to Theorem 3.3.3 g is the product of a self-paraconjugate Hurwitz factor and a scattering Hurwitz factor. However, factors of the latter type are excluded because they are known to have zeros in $\text{Re } p < 0$ (Theorem 3 in [3]).

Theorem 3.3.9: A polynomial g is self-paraconjugate Hurwitz if and only if all its irreducible factors are self-paraconjugate Hurwitz.

Proof: Sufficiency is quite obvious. To show necessity, observe that, due to Corollary 3.3.7.1 irreducible factors of g are either self-paraconjugate Hurwitz or scattering Hurwitz. Presence of a scattering Hurwitz factor, in view of Theorem 3 in [3], would imply that g has a zero in $\text{Re } p < 0$, which is ruled out by Theorem 3.3.8.

The following corollary follows immediately:

Corollary 3.3.9.1: Factors and products of self-paraconjugate Hurwitz polynomials are self-paraconjugate Hurwitz.

Corollary 3.3.9.2: Let g be a scattering Hurwitz polynomial and h be a self-paraconjugate polynomial. Then, g and h are relatively prime.

Proof: Otherwise, g would contain a proper self-paraconjugate Hurwitz factor and thus would not be relatively prime with g_* . Alternatively, apply Theorem 3.3.8 as well as Theorem 3 in [3].

Lemma 3.3.10: Let g be a polynomial, and let us select one

of the p_1 , say p_1 , and choose $m \geq n_1 = \deg_1 g$. Then the polynomial $g'(n, p') = n^m \cdot g(n^{-1}, p')$ is widest-sense Hurwitz if and only if g is widest-sense Hurwitz.

Proof: Obviously follows from the fact that $g'(n_0, p'_0) = 0$ for $\text{Re } n_0 > 0$, $\text{Re } p'_0 > 0$ if and only if $p_1 = n_0^{-1}$, $p' = p'_0$ is a zero of g in $\text{Re } p > 0$.

Theorem 3.3.11: Let g be a widest-sense Hurwitz polynomial in p and let i be any of the integers 1 to k such that $n_i = \deg_i g \geq 1$. Then $\partial^v g / \partial p_i^v$, for $v = 1, 2, \dots, n_i$ is also widest-sense Hurwitz.

Proof: It is enough to prove the theorem for $v = 1$. Assume $i = 1$. Since the polynomial $g_1(p_1) = g(p_1, p'_0)$ obtained by freezing p' at p'_0 in $\text{Re } p'_0 > 0$ is widest-sense Hurwitz, by invoking a classical result it follows that dg_1/dp_1 is also widest-sense Hurwitz. The result then follows by noting that $\partial g / \partial p_1 = dg_1/dp_1$, when $p = p'_0$.

Note, however, that $\partial g / \partial p_1$ is not even scattering Hurwitz when g is strict sense Hurwitz. As an example, the polynomial $g = p_1 p_2 + p_1 + 1$ is strict sense Hurwitz, but $\partial g / \partial p_2 = p_1$ is only a widest-sense Hurwitz polynomial.

Let us write the polynomial $g(p)$ with $\deg_1 g = n_1$ as

$$g = \sum_{v=0}^{n_1} A_v(p') p_1^v \quad (3.4)$$

where the $A_v(p')$ are polynomials in p' .

Lemma 3.3.12: Let g be a widest-sense Hurwitz polynomial involving one of the variables, say p_1 . If g is written as in (3.4), then for any two integers α, β with $0 \leq \alpha \leq \beta \leq n_1$ and $A_\alpha \neq 0$, there exists a set of positive integers $N_v, v = \alpha, \alpha + 1, \dots, \beta$ such that

$1, \dots, \beta$, such that the polynomial in (3.5) is widest-sense Hurwitz.

$$\sum_{v=\alpha}^{\beta} N_v A_v (p') p_1^{v-\alpha} \quad (3.5)$$

Proof: Due to Theorem 3.3.11, $g_\alpha = \partial^\alpha g / \partial p_1^\alpha$ is widest-sense Hurwitz, while $\deg_1 g_\alpha = (n_1 - \alpha)$. Consequently, due to Lemma 3.3.10, $g'_\alpha = p_1^{n_1-1} g_\alpha(p_1^{-1})$ is also widest sense Hurwitz, with $\deg_1 g'_\alpha = n_1 - \alpha$. Invoking Theorem 3.3.11 again, it

follows that $g_\beta = \partial^{n_1-\beta} g'_\alpha / \partial p_1^{n_1-\beta}$ is widest-sense Hurwitz with $\deg_1 g_\beta = (n_1 - \alpha) - (n_1 - \beta) = (\beta - \alpha)$. Therefore, from Lemma 3.3.10 we have that $g'_\beta = p_1^{n_1-1} g_\beta(p_1^{-1})$ is widest-sense Hurwitz. The proof is then completed by noting that g'_β has the form (3.5).

Theorem 3.3.13: Let g be a widest-sense Hurwitz polynomial, expressed as in (3.4), and μ be any integer $0 \leq \mu \leq n_1$. Then the following hold true: (i) If $A_\mu \neq 0$, then A_μ is widest-sense Hurwitz; in particular, $A_{n_1}(p') \neq 0$ for $\text{Re } p' > 0$. (ii) If $0 \leq \mu, \mu+1, \dots, \mu+\gamma \leq n_1$ and $\gamma > 2$ then it is impossible to have $A_{\mu+1} = A_{\mu+2} = \dots = A_{\mu+\gamma-1} = 0$, $A_\mu \neq 0$, $A_{\mu+\gamma} \neq 0$. (iii) For any μ satisfying $1 \leq \mu \leq n_1 - 1$, if $A_\mu = 0$, $A_{\mu-1} \neq 0$ and $A_{\mu+1} \neq 0$ then $A_{\mu-1}/A_{\mu+1}$ is a positive constant. (iv) If for any μ satisfying $0 \leq \mu < n_1$, $A_\mu \neq 0$, $A_{\mu+1} \neq 0$ then $A_\mu/A_{\mu+1}$ is a positive function.

Proof: (i) Follows immediately by choosing $\alpha = \beta = \mu$ in Lemma 3.3.12.

(ii) We prove this for $\gamma = 3$, the proof being similar for $\gamma > 3$. Lemma 3.3.12, with $\alpha = \mu, \beta = \mu + 3$, implies that $g_3 = (N_{\mu+3} A_{\mu+3} p_1^3 + N_\mu A_\mu)$ is widest sense Hurwitz, where $N_{\mu+3}$ and N_μ are positive integers. However, this latter conclusion is impossible due to the fact that for any fixed p' in $\text{Re } p' > 0$ (where, due to (i), $A_\mu \neq 0$ and $A_{\mu+3} \neq 0$) the cubic equation

$g_3 = 0$ in p_1 has at least one solution in $\text{Rep}_1 > 0$.

(iii) Lemma 3.3.12 with $\alpha = \mu - 1, \beta = \mu + 1$ implies that $g_2 = (N_{\mu+1} A_{\mu+1} p_1^2 + N_{\mu-1} A_{\mu-1})$ is widest sense Hurwitz, where $N_{\mu+1}$ and $N_{\mu-1}$ are positive integers. We may claim that $A_{\mu-1}/A_{\mu+1}$ is real and positive for each β' in $\text{Rep}' > 0$, because otherwise there would exist a $p' = p'_0$ in $\text{Rep}'_0 > 0$ for which the equation $g_2 = 0$ would have a solution in $\text{Rep}_1 > 0$. Since $A_{\mu-1}/A_{\mu+1}$ is thus holomorphic as well as real everywhere in $\text{Rep}' > 0$, invoking a standard result in the theory of functions of complex variables (e.g. the theorem on the minimum of the imaginary part), we have that $A_{\mu-1}/A_{\mu+1}$ is a constant.

(iv) Lemma 3.3.12 with $\alpha = \mu, \beta = \mu + 1$ yields that $g_1 = N_{\mu+1} A_{\mu+1} p_1 + N_{\mu} A_{\mu}$ is widest-sense Hurwitz. Clearly, if $A_{\mu}/A_{\mu+1}$ has negative real part for some $p' = p'_0$ in $\text{Rep}'_0 > 0$ then $g_1(p_{10}, p'_0) = 0$, where $p_{10} = -[N_{\mu} A_{\mu}(\beta'_0)]/[N_{\mu+1} A_{\mu+1}(\beta''_0)]$. Since $\text{Rep}_{10} > 0$, the latter conclusion is impossible due to the widest-sense Hurwitz property of g_1 .

Theorem 3.3.14: Let g be a widest-sense Hurwitz polynomial in p and g_1 be the $(k-1)$ -variable polynomial obtained by freezing any one of the variables, say p_1 , at $p_1 = p_{10}$ in $\text{Rep}_1 \geq 0$. Then $\deg_i g = \deg_i g_1$ for the remaining variables $p_i, i = 2$ to k except possibly for finitely many values of p_{10} on $\text{Rep}_1 = 0$.

Proof: Write g as a polynomial in p' whose coefficients are polynomials in p_1 . Let $A = A(p_1)$ be any one of these coefficients that is not identically zero for all p_1 . By repeated application of Theorem 3.3.13, item (i), it follows that $A(p_1)$ is widest-sense Hurwitz. Hence, $A(p_{10}) \neq 0$ for $\text{Re } p_{10} > 0$ and there are at most finitely many $p_{10} = j\omega_{10}$ for which $A(j\omega_{10}) = 0$ (cf. also Lemma 3.3.2).

3.4. Properties of scattering Hurwitz polynomials.

Theorem 3.4.1: If g is a scattering Hurwitz polynomial then g cannot have a zero for a p_0 with $\text{Rep}_{i0} = 0$ for one of the i and $\text{Rep}_{i0} > 0$ for the remaining i .

Proof: We may assume $\text{Rep}_{i0} = 0$ for $i = 1$ i.e., $p_{10} = j\omega_{10}$. In view of Lemma 3.3.2, a zero of the type just mentioned would require $g(j\omega_{10}, p') = 0$ for all p' . Thus, in view of Theorem 3.2.10, $(p_1 - j\omega_{10})$ would be a factor of g . Since $(p_1 - j\omega_{10})$ is self-paraconjugate, this is excluded due to (3.3).

It is important to note, however, that a scattering Hurwitz polynomial can indeed have zeros e.g. for $p_1 = j\omega_1, p_2 = j\omega_2$ and $\text{Rep}_i > 0, i = 3$ to k . Consider the polynomial $g = p_1 p_2 + p_2 p_3 + p_3 p_1 + p_1 p_2 p_3$, which is scattering Hurwitz [1], but $g=0$, when $p_1 = p_2 = 0$.

Theorem 3.4.2: Let g be a scattering Hurwitz polynomial and let i' be any specific one of the $i = 1$ to k . Then the $(k-1)$ -variable polynomial g_1 obtained by freezing $p_{i'}$ at $p_{i'} = j\omega_{i',0}$ is also scattering Hurwitz and has the same partial degrees in the remaining variables as g , with the possible exception of at most finitely many values of $\omega_{i',0}$.

Proof: We may assume $i' = 1$. Referring to Theorem 3.4.1, $g_1(p') = g(j\omega_{10}, p') \neq 0$ in $\text{Rep}' > 0$. Furthermore, $g(p)$ and $g_*(p)$ do not have any nonconstant common factor. Therefore, due to Theorem 3.2.7 with $m=1$, the polynomials $g_1(p')$ and $g_{1*}(p') = g^*(j\omega_{10}, -p'^*) = g_*(j\omega_{10}, p')$ can have a common nonconstant factor for at most finitely many values of ω_{10} . Furthermore, in view of Lemma 3.2.6, a lowering of a partial degree can also occur at most for finitely many values of ω_{10} .

In view of the above theorem and Definition 3.2.1a it is also possible to state the more general result:

Theorem 3.4.3: Let g be a scattering Hurwitz polynomial. For any m such that $1 \leq m < k$ consider the $(k-m)$ -variable polynomial g' obtained by freezing m of the p_i at $p_i = j\omega_{i0}$ for, say, $1 \leq i \leq m$. Then there exists a sequentially almost complete set, Ω_m , of order m of m -tuples such that for $(\omega_{10}, \omega_{20}, \dots, \omega_{m0}) \in \Omega_m$, g' is still scattering Hurwitz, with the same partial degrees in p_{m+1} to p_k as g . Furthermore, any ordering may be chosen for the set Ω_m .

Proof: Proof of the above theorem follows by sequentially freezing, in any order, the variables p_i to p_m on the imaginary axis and observing Theorem 3.4.2.

Theorem 3.4.4: A polynomial g in k variables is scattering Hurwitz if and only if (i) g is widest-sense Hurwitz, and (ii) the set of real k -tuples $\underline{\omega}$ such that $g(j\underline{\omega}) = 0$ does not form a sequentially infinite set of order $(k-1)$.

Proof: Follows immediately from Definition 3.2.4, and Theorem 3.3.6.

Lemma 3.4.5: Let g be a scattering Hurwitz polynomial. Freeze one of the p_i , say $p_{i'}$, at $p_{i',0}$ with $\text{Re } p_{i',0} > 0$. The resulting polynomial, g_1 , in the remaining p_i is still a scattering Hurwitz polynomial, and the partial degrees of g_1 in these remaining p_i are the same as for g .

Proof: Without loss of generality we may assume that $i' = 1$ i.e., $g_1 = g_1(p')$. Obviously g_1 is widest sense Hurwitz (Theorem 3.3.1). We show in the following that g_1 and g_{1*} are relatively prime. If g_1 and g_{1*} have a nontrivial common factor, then due to Theorem 3.3.6 there exists a sequentially infinite set, Ω' of real $(k-1)$ -tuples $\underline{\omega}'$ of

order $(k-2)$ such that $g(p_{10}, \underline{\omega}') = 0$ for all $\underline{\omega}' \in \Omega'$. Thus, applying Theorem 3.3.3, item (i), for $m = k-1$, we conclude that $g(p_1, j\underline{\omega}') = 0$ for all p_1 and all $\underline{\omega}' \in \Omega'$. We therefore have, in particular, $g(j\omega_1, j\underline{\omega}') = 0$ for all $\underline{\omega}' \in \Omega'$ and any arbitrary ω_1 . Thus g would be zero for real k -tuples $\underline{\omega}$ belonging to a sequentially infinite set of order $(k-1)$, which is impossible in view of Theorem 3.4.4. Furthermore, the preservation of the partial degrees follows from Theorem 3.3.14.

Repeated use of Lemma 3.4.5 yields the following more general result:

Theorem 3.4.6: If g is a scattering Hurwitz polynomial in p then for $1 \leq m < k$ the m -variable polynomial, g' , obtained by freezing $(k-m)$ of the p_i at $p_i = p_{i0}$ in $\text{Re } p_i > 0$ is scattering Hurwitz and the partial degrees of g' are the same as the corresponding partial degrees of g .

Theorem 3.4.7: If g is a scattering Hurwitz polynomial expressed as in (1) with $\deg_i g = n_i$, then $A_v \neq 0$ for each $v = 0, 1, 2, \dots, n_i$.

Proof: Assume $i = 1$. For any $\underline{p}' = \underline{p}'_0$ with $\text{Re } p'_0 > 0$, the polynomial $g_1(p_1) = g(p_1, \underline{p}'_0)$, due to Theorem 3.4.6, is a scattering Hurwitz polynomial in the single variable p_1 (i.e., a Hurwitz polynomial in the classical sense [3]) of degree equal to $n_1 = \deg_1 g$ and thus has nonzero coefficients for $v = 0$ to n_1 .

3.5. Rational Positive Functions and Related Results

Theorem 3.5.1: Let $F=h/g$ be a rational positive function in irreducible form (g and h thus being relatively prime polynomials) with $h(p) \neq 0$ (and, obviously, $g(p) \neq 0$). Then,

(i) the polynomial $g+h$ is scattering Hurwitz,

(ii) both g and h are widest-sense Hurwitz.

Proof: (i) Consider the function

$$\rho = (F-1)/(F+1) \quad (3.6)$$

which we can write in the form $\rho=c/d$, c and d being defined by $c=h-g$, $d=h+g$. Clearly, c and d are relatively prime. Then, for $\operatorname{Re} p > 0$ and $d(p) \neq 0$, we have $\operatorname{Re} F(p) \geq 0$ and thus $|\rho(p)| \leq 1$ if $g(p) \neq 0$, and $\rho(p)=1$ if $g(p)=0$. Hence, by Theorem 1 in [3] (where in view of the proof of Theorem 1 in [1], on which the proof of Theorem 1 in [3] is based, (3.4) should be interpreted to mean that the first inequality is known to hold for all those p in $\operatorname{Re} p > 0$ for which $g(p) \neq 0$), d is scattering Hurwitz. (Note that $d(p) \neq 0$ for $\operatorname{Re} p > 0$ follows from Theorem 3.2.13.)

(ii) Since $d(p) \neq 0$ in $\operatorname{Re} p > 0$, we cannot have $g(p)=h(p)=0$ in $\operatorname{Re} p > 0$. If only one of the polynomials g and h has a zero for a p_0 with $\operatorname{Re} p_0 > 0$ we have $\rho = \pm 1$. This, however, is excluded since due to the maximum-modulus theorem [10] we have in fact $|\rho(p)| < 1$ in $\operatorname{Re} p > 0$, except if ρ is a unimodular constant, i.e., if $F=jC$, C being a real constant (in which case $g=1$, $h=jC$).

Theorem 3.5.2: If F is a rational positive function, then

- (i) $\operatorname{Re} F(p) > 0$ if $\operatorname{Re} p > 0$ and F is nontrivial;
- (ii) $\operatorname{Re} F(p) \geq 0$ for any p in $\operatorname{Re} p \geq 0$ where $F(p)$ is holomorphic;
- (iii) assuming $F(p) \neq 0$, $1/F$ is also a rational positive function.

Proof: (i) Define ρ by (3.6) and consider a closed polydomain $\bar{U}$ in the neighborhood of an arbitrarily selected point p_0 with $\operatorname{Re} p_0 > 0$. In view of Theorem 3.5.1, F and ρ are holomorphic in $\bar{U}$. Thus $|\rho(p)| \leq 1$ for $p \in \bar{U}$, while application of the maximum-modulus theorem in its simplest form [10] yields $|\rho(p_0)| < 1$, i.e., $\operatorname{Re} F(p_0) > 0$. The proof of (ii) follows by simple continuity arguments, and that of (iii) follows in an obvious fashion from (i).

Theorem 3.5.3: If g and h are relatively prime polynomials, then $F=h/g$ is a positive function if and only if (i) the polynomial $d=g+h$ is scattering Hurwitz, (ii) $\operatorname{Re} F(j\omega) \geq 0$ for all $p=j\omega$ where $F(p)$ is holomorphic.

Proof: Necessity of (i) follows from Theorem 3.5.1, necessity of (ii) from Theorem 3.5.2 item (ii). For proving sufficiency, observe that for ρ defined again by (3.6) and $d(j\omega) \neq 0$, we have $\operatorname{Re} F(j\omega) \geq 0$ and thus $|\rho(j\omega)| \leq 1$ if $g(j\omega) \neq 0$, and $\rho(j\omega)=1$ if $g(j\omega)=0$, altogether thus $|\rho(j\omega)| \leq 1$ wherever $\rho(j\omega)$ is holomorphic. Hence applying Theorem 1 in [2] (generalized maximum-modulus theorem), we conclude that in $\operatorname{Re} p > 0$ we have $|\rho(p)| < 1$ and thus also $\operatorname{Re} F(p) > 0$. The only exception to this is if ρ is a unimodular constant, in which case F is an imaginary constant, thus a trivial positive function.

Lemma 3.5.4: Let F be a positive function (reactance function). For any m such that $1 \leq m < k$, consider the $(k-m)$ -variable rational function F' obtained by freezing, in F , m of the variables p_i at $p_i = j\omega_{i0}$, say for $i = 1$ to m . Then there exists a sequentially almost complete set, Ω_m , of

order m of real m -tuples such that for any $(\omega_{10}, \omega_{20}, \dots, \omega_{m0}) \in \Omega_m$, F' is still a positive function (reactance function). If in addition, F is given in irreducible form, $F=h/g$, then Ω_m can be chosen such that $F'=h'/g'$ is also in irreducible form, h' and g' being obtained by applying the corresponding freezing operation to h and g , respectively, and that the partial degrees of h' and g' in the remaining variables are the same as for h and g respectively, with Ω_m remaining otherwise as stated.

Proof: We need to prove the lemma for $m = 1$ only. Due to Theorem 3.5.1, g is widest-sense Hurwitz. Also, let $g'(p') = g(j\omega_{10}, p')$. Due to Lemma 3.3.2, there exists an almost complete real set, Ω'_1 , such that $g'(p') \neq 0$ for $\text{Re } p' > 0$ and $\omega_{10} \in \Omega'_1$ and that the partial degrees of h' and g' in p_2 to p_k are the same as those of h and g , respectively. In particular, for $\omega_{10} \in \Omega'_1$, F is regular for $\text{Re } p' > 0$, and by invoking Theorem 3.5.2, it follows that the function $F'(p') = F(j\omega_{10}, p')$, for $\omega_{10} \in \Omega'_1$, satisfies the property that $\text{Re } F'(p') \geq 0$ for $\text{Re } p' > 0$, i.e., it is a positive function.

Furthermore, if F is a reactance function then $F + F_* = 0$ implies that $F' + F'_* = 0$. Consequently, F' is a reactance function.

Also, due to Theorem 3.2.7, there exists an almost complete real set Ω''_1 such that the polynomials $g'(p')$ and $h'(p') = h(j\omega_{10}, p')$ are relative prime for $\omega_{10} \in \Omega''_1$. Therefore, h'/g' is in irreducible form if ω is required to be in the almost complete set $\Omega_1 = (\Omega'_1 \cap \Omega''_1)$.

Lemma 3.5.5a: The numerator and denominator polynomials of a positive function in irreducible form cannot contain self-paraconjugate factors of multiplicity larger than one.

Proof: Let $F = h/g$ be a positive function in irreducible

form, and g contain a proper self-paraconjugate factor, a , of multiplicity v . Let a involve, say, p_1 . Then it follows by invoking Lemma 3.5.4 with $m = k-1$ that there exists a sequentially almost complete real set, Ω' , of order $(k-1)$ such that for any real k -tuple $\underline{w}'_0 \in \Omega'$, the rational function $F_1 = h_1/g_1$, with $h_1(p_1) = h(p_1, j\underline{w}'_0)$ and $g_1(p_1) = g(p_1, j\underline{w}'_0)$, is a one-variable positive function in irreducible form and that the partial degree of g_1 in p_1 is the same as that of g . Obviously, $a_1(p_1) = a(p_1, j\underline{w}'_0)$ is a nonconstant factor of $g_1 = g_1(p_1)$ of multiplicity at least equal to v . Since a is self-paraconjugate, a_1 is also self-paraconjugate. Furthermore, taking into account Theorem 3.3.3, a_1 is widest-sense Hurwitz and, therefore, its zeros are restricted to be on $p_1 = j\omega_1$. Consequently, g_1 has zeros of multiplicity at least v on the $p_1 = j\omega_1$ axis. Since g_1 is the denominator of a positive function in irreducible form, the latter conclusion dictates that $v = 1$. Similar arguments hold for h and g , in view of Theorem 3.5.2. Item (iii).

Lemma 3.5.5b: If $F = h/g$ is a positive function with h and g polynomials then $\deg h - 1 \leq \deg g \leq \deg h + 1$ for each $i = 1, 2, \dots, k$.

Proof: The proof follows by adopting a strategy similar to that utilized in the proof of Lemma 3.5.4a and the fact that a one-variable positive function cannot have poles or zeros of multiple order at infinity.

Theorem 3.5.6: If g is a widest-sense Hurwitz polynomial, then for any $i = 1$ to k with $n_i = \deg g_i$, g_i is a positive function in p_i .

Proof: Assume $i = 1$. Then g_1 is a polynomial in p_1 . By 3.3.14, the polynomial g_1 is a positive function in p_1 if and only if g_1 is a positive function in p_1 with $\deg g_1 = n_1$ and $\deg g_1 = n_1$ and $\deg g_1 = n_1$.

invoking a well known one-variable result we assert that $(dg_1/dp_1)/g$ is a positive function. However, since this latter conclusion is true for any p'_0 in $\text{Rep}' > \underline{0}$, we have $\text{Re}[(\partial g/\partial p_1)/g] \geq 0$ for $\text{Rep} > \underline{0}$.

Finally, for the sake of completeness, we offer a more complete, but obvious version of Theorem 3.5.2, item (i).

Theorem 3.5.7: A rational function F is a nontrivial positive function positive if and only if $\text{Re}F(p) > 0$ for $\text{Rep} > \underline{0}$.

Proof: Sufficiency follows by a simple continuity argument. Necessity has been shown in Theorem 3.5.2, item (i), but it also follows by simple application of the theorem on the minimum of the real part of a holomorphic function. Alternatively, we may freeze p' at a p'_0 with $\text{Rep}'_0 > \underline{0}$, in which case F reduces to a function F_1 in p_1 alone. Applying Theorems 3.3.1, 3.3.14, and 3.5.1, F_1 is found to be a nontrivial positive function, i.e., the proof is reduced to the known one-variable result $\text{Re}F_1(p_1) > 0$ in $\text{Rep}_1 > 0$.

3.6. Reactance Hurwitz Polynomials

The following lemma follows directly from Definition 3.2.6 and Theorem 8 in [3]:

Lemma 3.6.1: A reactance Hurwitz polynomial is self-paraconjugate Hurwitz.

Definition 3.2.6 is justified by the following theorem:

Theorem 3.6.2: 1. If g is a reactance Hurwitz polynomial, there exists a polynomial, h , relatively prime with g , such that h/g is a reactance function in irreducible form.

2. Vice versa, if h/g is a reactance function in irreducible form, the following holds: (i) g and h are reactance Hurwitz polynomials. (ii) For any constant C , the polynomial d defined by $d=g_0+h_0$, $g_0=Cg$, $h_0=Ch$, is scattering Hurwitz. In particular, it is always possible to choose C in such a way that the paraeven and paraodd parts of d are equal to g_0 and h_0 , respectively, or that these parts are equal to h_0 and g_0 , respectively.

Proof: For proving the first statement, observe that in view of Definition 3.2.6 there exist polynomials g_0 and h_0 such that $g_0 = \pm g$, $h_0 = \mp h$, and $g_0 = Cg$ and that g_0+h_0 is scattering Hurwitz, the two upper and the two lower signs corresponding to one another and C being a nonzero constant. By Theorem 7 of [3], h_0/g_0 is then a reactance function in irreducible form and the same is thus true for h/g where $h=h_0/C$.

For proving the second statement, observe first that in view of Theorem 3.5.1, g and h are widest-sense Hurwitz while g_0+h_0 is scattering Hurwitz where $g_0=Cg$ and $h_0=Ch$, C being an arbitrary nonzero constant, and that in view of Definition

3.2.9 we have $h_*/g_* = -h/g$, thus $g_* = \gamma g$, $h_* = -\gamma h$, γ being a constant (necessarily unimodular). If in particular we choose C such that $C^2 = \pm \gamma$ we have $g_{0*} = \pm g_0$, $h_{0*} = \mp h_0$.

Theorem 3.6.3: If g is a self-paraconjugate Hurwitz polynomial then for any $i = 1$ to k with $n_i = \deg_i g \geq 1$, $F = h/g$ is a reactance function where $h = \partial g / \partial p_i$.

Proof: From Theorem 3.5.6 it follows that h/g is a positive function. Assume $i = 1$. If g is written as in (3.4), then

$$g_* = \sum_{v=0}^{n_1} A_{v*} (p') \cdot (-p_1)^v.$$

Since g is self-paraconjugate, $g = Cg_*$ for some constant C with $|C| = 1$. Therefore, $A_v = (-1)^v C A_{v*}$ whence it can be shown that $\partial g / \partial p_1 = -C(\partial g / \partial p_1)_*$. It thus follows that $F_* = -F$. Consequently, F is a reactance function.

Corollary 3.6.3.1: If g is an irreducible, self-paraconjugate Hurwitz polynomial then for any $i = 1$ to k with $n_i = \deg_i g \geq 1$, $(\partial g / \partial p_i) / g$ is a reactance function in irreducible form.

Proof: Since g is irreducible and the partial degree of $\partial g / \partial p_i$ is smaller than that of g in the variable p_i the polynomials $\partial g / \partial p_i$ and g are relatively prime. The proof is then completed by observing Theorem 3.6.3.

Theorem 3.6.4: A polynomial g is reactance Hurwitz if and only if all its irreducible factors are self-paraconjugate Hurwitz and it contains no multiple factors.

Proof: **Necessity:** If g is a reactance Hurwitz polynomial, invoking Theorem 3.6.2 there exists a reactance function, thus a positive function, $F = h/g$, such that h and g are relatively prime. In view of Lemma 3.6.1 and Theorem 3.3.9,

irreducible factors of g are necessarily self-paraconjugate. It then immediately follows from Lemma 3.5.5a that g cannot contain multiple factors.

Sufficiency: Let g_i , $i = 1$ to v , be the irreducible, distinct, self-paraconjugate factors of g . Let g_i involve the variable $p_i \in \{p_1, p_2, \dots, p_k\}$ and consider the polynomial h

defined by $h/g = \prod_{i=1}^v (h_i/g_i)$, $h_i = \partial g_i / \partial p_i$.

From Corollary 3.6.3.1 we conclude on the one hand that h/g is a reactance function, and on the other that each h_i is relatively prime with the corresponding g_i . Thus, since by assumption the g_i are pairwise mutually prime, h is relatively prime with g . The proof is then completed by observing Theorem 3.6.2, second part, item (i)

Corollary 3.6.4.1: Products of reactance Hurwitz polynomials, that are pairwise relatively prime are reactance Hurwitz polynomials. Conversely, any factor of a reactance Hurwitz polynomial is also a reactance Hurwitz polynomial.

Proof: Follows immediately from Theorem 3.6.4

Corollary 3.6.4.2: A polynomial, g , is reactance Hurwitz if and only if it is self-paraconjugate Hurwitz and contains no multiple factor.

Proof: Follows from Theorems 3.3.9 and 3.6.4.

Corollary 3.6.4.3: Any scattering Hurwitz polynomial is relatively prime with any reactance Hurwitz polynomial.

Proof: Follows from Corollary 3.3.9.2 and Lemma 3.6.1.

Theorem 3.6.5: Let g be a reactance Hurwitz polynomial. For

any integer m such that $1 \leq m < k$, freeze m of the variables p_i at $p_i = j\omega_{i0}$, say for $i = 1$ to m . Consider the polynomial g' in p_{m+1} to p_k defined by $g'(p_{m+1}, \dots, p_k) = g(j\omega_{10}, \dots, j\omega_{m0}, p_{m+1}, \dots, p_k)$. Then there exists a sequentially almost complete set, Ω_m , of order m of real m -tuples such that for $(\omega_{10}, \omega_{20}, \dots, \omega_{m0}) \in \Omega_m$, g' is still reactance Hurwitz and that its partial degrees in p_{m+1} to p_k are the same as for g .

Proof: Due to Theorem 3.6.2 there exists a polynomial h such that $F = h/g$ is a reactance function in irreducible form. Applying Lemma 3.5.4 to the rational function $F = h/g$, we conclude that Ω_m may be chosen in such a way that the resulting $F' = h'/g'$ is a reactance function in irreducible form, with g' having the same partial degrees in p_{m+1} to p_k as g . The rest of the proof follows by applying item (1) of the second part of Theorem 3.6.2 to F' .

3.7. Properties of Immittance Hurwitz Polynomials

The following theorems justify the characterization of immittance Hurwitz polynomials as given in Definition 3.2.7.

Theorem 3.7.1: If $F = h/g$ is a positive function in irreducible form, then both h and g are immittance Hurwitz polynomials.

Proof: g is widest sense Hurwitz in view of Theorem 3.5.1. Invoking Theorem 3.3.7 it follows that g is product of a scattering Hurwitz polynomial, g_1 , and a self-paraconjugate Hurwitz polynomial, g_2 . By virtue of Corollary 3.3.9.1, Lemma 3.5.5a, and Corollary 3.6.4.2, g_2 is reactance Hurwitz. The same argument holds for h .

Corollary 3.7.1.1: Factors of immittance Hurwitz polynomials are immittance Hurwitz. Conversely, products of immittance Hurwitz polynomials that do not have any common self-paraconjugate factors are also immittance Hurwitz.

Proof: Obviously follows from Definition 3.2.7 and Corollary 3.6.4.1.

Theorem 3.7.2: Every immittance Hurwitz polynomial is the numerator or denominator of a positive function in irreducible form.

Proof: Since the reciprocal of a positive function is also a positive function, it is enough to prove the theorem for the denominator polynomial, g . Let $g = ab$, where a and b are, respectively, the scattering Hurwitz and the reactance Hurwitz factors of g . Then, due to Theorem 3.6.2, there exists a polynomial c that is relatively prime with b and such that c/b is a reactance function. Let us define $F = 1 + (a_0/a) + \dots + (a_n/a)$. F can also be written in the form $F = h/g$,

where $h = ab + a_*b + ac$. Since a_*/a is an all-pass function, $1 + (a_*/a)$ is a positive function (cf. Lemma 2 of [3]), and the same is true of F . According to Corollary 3.6.4.3, a and b are also relatively prime. Furthermore, in view of the definition of a scattering Hurwitz polynomial, a is relatively prime with a_* . Hence, since $h = a(b+c) + a_*b = (a+a_*)b + ac$, h can not be divisible by any factor of a or of b . The proof is thus completed by observing that any common irreducible factor of h and g would have to be a factor of either a or b .

3.8. Tests for Positive Functions and Related Results

Results, which are useful in identifying positive functions based on their behavior on the distinguished boundary of the domain of holomorphy have already been proved essential [11] in problems related to multivariable network synthesis and are elaborated in this section.

Theorem 3.8.1: Assume that the rational function $F = h/g$ satisfies the following properties: (i) g is a scattering Hurwitz polynomial (ii) $\deg_i h \leq \deg_i g$, $i = 1$ to k ; (iii) $\operatorname{Re} F(j\omega) \geq 0$, where ω is any real k -tuple such that $F(p)$ is holomorphic at $p=j\omega$. Then F is a positive function. Note that we have not assumed the polynomials h and g to be relatively prime.

Proof: The validity of the theorem for $k = 1$ is classically known. To prove the result in the general case, via induction on the number of variables, we assume that the theorem be true for $k-1$ variables, where $k \geq 2$.

Let us freeze one of the variables, say p_1 , at $j\omega_{10}$ and define the rational function in $k-1$ variables, $F_1 = h_1/g_1$, where $h_1(p') = h(j\omega_{10}, p')$ and $g_1(p') = g(j\omega_{10}, p')$. Due to Theorem 3.4.2, there exists an almost complete set, Ω_1 , of real numbers such that for all $\omega_{10} \in \Omega_1$, $g_1(p')$ is scattering Hurwitz and $\deg_i g = \deg_i g_1$, thus $\deg_i h_1 \leq \deg_i g_1$, for all $i = 2$ to k .

Since $g_1(j\omega') = g(j\omega_{10}, \omega')$, the condition $g_1(j\omega') \neq 0$ implies $g(j\omega_{10}, j\omega') \neq 0$. Thus, $\operatorname{Re} F_1(j\omega') = \operatorname{Re} F(j\omega_{10}, j\omega') \geq 0$ for $g_1(j\omega') \neq 0$.

All the prerequisites for the validity of the present theorem are, therefore, satisfied by the $(k-1)$ -variable rational function F_1 . Hence, by induction hypothesis, $\operatorname{Re} F_1(p') \geq 0$

for $\text{Re } p' > 0$. Therefore, if $\omega_{10} \in \Omega_1$ then (3.7) applies.

$$\text{Re } F(j\omega_{10}, p') \geq 0 \text{ for all } \text{Re } p' > 0. \quad (3.7)$$

Let us now freeze the variables p' at an arbitrary point p'_0 with $\text{Re } p'_0 > 0$ and define the rational function $F_0 = h_0/g_0$ in the variable p_1 only, where $h_0(p_1) = h(p_1, p'_0)$ and $g_0(p_1) = g(p_1, p'_0)$. Due to Theorem 3.4.6 with $m=k-1$ the polynomial g_0 is scattering Hurwitz, thus Hurwitz in the classical sense (cf. Theorem 2 of [3]) and $\deg_1 g_0 = \deg_1 g \geq \deg_1 h \geq \deg_1 h_0$. Furthermore, it follows from (3.7) that $\text{Re } F_0(j\omega_1) \geq 0$ for $\omega_1 \in \Omega_1$. Hence it follows from a classical result that $\text{Re } F_0(p_1) \geq 0$ for all p_1 with $\text{Re } p_1 > 0$. Therefore, we conclude that $\text{Re } F(p) \geq 0$ for $\text{Re } p > 0$.

The following comments on the above result are in order. In the one-variable case the proof of the above result follows by invoking the maximum modulus theorem on the function $\exp(-F)$. In the multidimensional situation, however, a maximum modulus theorem which allows for special types of singularities on the boundary of the domain of holomorphy is not found in the literature. Reference [2] gives a version of maximum-modulus theorem where non-essential singularities of the second kind are allowed to occur on the boundary of the domain of holomorphy, but the proof is restricted to rational functions only. The non-rationality of the function $\exp(-F)$ makes it impossible to use results of [2], in the present context, thereby calling for an independent proof of Theorem 3.8.1.

Note that Theorem 3.8.1 can be generalized to include the possibility that the domain of holomorphy of F be a cartesian product of domains other than half-planes (e.g., discs) in the variables p_i , $i = 1$ to k .

The following partial results are of some interest.

Theorem 3.8.2a: If $F = d/cg$ is a positive function in p written in irreducible form where g is a scattering Hurwitz polynomial, c is a reactance Hurwitz polynomial that is the product of reactance Hurwitz polynomials in one-variable only, and d is a polynomial such that $\deg_i d \leq \deg_i (cg)$, $i = 1$

to k , then F can be decomposed as $F = \sum_{v=1}^n F_v + (d_1/g)$, where each F_v is a one-variable reactance function and (d_1/g) is a positive function in p , with $\deg_i d_1 \leq \deg_i g$ for each $i = 1$ to k .

Proof: Write $c = c_1 c_2 \dots c_n$, where $c_v = (p'_v - j\omega_{v0})$, $v = 1$ to n , each p'_v being one of the p_1 to p_k and two ω_{v0} being necessarily distinct if the corresponding p'_v represents the same p_1 . We claim that the rational functions K_v defined in (3.8a) are constants. We show this for $v=1$, assuming $p'_v = p_1$, in which case K_1 could be a function of p' .

$$K_v(p') = \lim_{p'_v \rightarrow j\omega_{v0}} [(p'_v - j\omega_{v0})d/cg] ; \quad \zeta = d - cg \sum_{v=1}^n K_v/c_v \quad (3.8a, b)$$

For each $p' = p'_0$ in $\text{Re } p'_0 > 0$, $K_1(p'_0)$ is the residue of the positive function $F'(p_1) = F(p_1, p'_0)$ at the pole $p_1 = j\omega_{10}$ and is hence positive and thus, in particular, real and finite. Therefore, $K_1(p')$ is real and holomorphic in $\text{Re } p' > 0$. Consequently, invoking a standard result from the theory of functions of complex variables, it follows that K_1 is independent of p' ; it is thus a positive constant. Consider next the polynomial $\zeta(p)$ as defined in (3.8b). Substituting for the K_v from (3.8a) in (3.8b) it follows that ζ is zero for $p'_v = j\omega_{v0}$, independently of the values of the other p_1 . Hence, all c_v divide ζ , i.e., since the c_v are distinct, $d_1 = \zeta/c$ is a polynomial in p . Equation (3.9a) then follows by straightforward algebraic manipulation, where c' is as defined in (3.9b).

$$F = (c'/c) + (d_1/g) ; \quad c' = \sum_{v=1}^n (K_v c) c_v \quad (3.9a, b)$$

Clearly, c'/c is a reactance function. Next, we note that for $p = j\omega$, ω real, $\operatorname{Re}(d_1/g) = \operatorname{Re} F \geq 0$ at all those points where $c(j\omega)g(j\omega) \neq 0$ and thus, by continuity, where $g'(j\omega) \neq 0$. Due to (3.9) and $F = d/gc$ the inequalities $\deg_i d \leq \deg_i(gc)$ imply that $\deg_i d_1 \leq \deg_i g$ for each $i = 1$ to k . Hence, invoking Theorem 3.8.1 it follows that d_1/g is a positive function. The proof is thus complete.

Note that the above result can be easily extended to the case with simple poles at $p_1 = \infty$ as follows.

Theorem 3.8.2b: If $F = d/g$ is a rational positive function such that the polynomial g is scattering Hurwitz and $\deg_i d > \deg_i g$ for some of the $i \in I$, then there exist nonnegative constants K_i such that F can be written in the form

$$F = F_0 + \sum_{i=1}^k K_i p_i \text{ where } F_0 = d_0/g \text{ is a positive function with}$$

$$\deg_i d_0 \leq \deg_i g, \quad i = 1 \text{ to } k.$$

Proof: Assume first $\deg_i d > \deg_i g$ for $i = 1$. Let $A(p')$ and $B(p')$ be the respective leading coefficients of d and g , when considered as polynomials in p_1 with the coefficients written as polynomials in p' . For any p'_0 with $\operatorname{Re} p'_0 > 0$ the function $F_1(p_1) = F(p_1, p'_0)$ is a positive function in the variable p_1 only. Furthermore, since due to Theorem 3.3.13, $A(p'_0) \neq 0$, $B(p'_0) \neq 0$ the degrees of d and g in p_1 remain unaltered due to the substitution $p' = p'_0$. Hence, $F_1(p_1)$ is a positive function having a pole at infinity, necessarily simple, with residue $K_1(p'_0)$. Therefore, $K_1(p'_0)$ is real and positive, thus in particular finite and therefore holomorphic for all $\operatorname{Re} p'_0 > 0$. This implies, in view of a standard result in the theory of functions of complex variables, that K_1 is a constant. The same holds for the remaining $i \in I$; altogether we find that the K_i , defined correspondingly to K_1 , are nonnegative

constants for $i = 1$ to k .

Consider next the rational function $F_0 = d_0/g$ defined

via $F_0 = F - \sum_{i=1}^k K_i p_i$. Clearly, $\operatorname{Re} F_0(j\omega) = \operatorname{Re} F(j\omega)$ where ω is

any real k -tuple such that $g(j\omega) \neq 0$. Furthermore, $\deg_i d_0 \leq \deg_i g$ for $i = 1$ to k . This follows, e.g., for the case $i = 1$ by writing the polynomials d and g appearing in $F = d/g$ in the same way as above and taking into account that $K_1 = A(p')/B(p')$. Invoking Theorem 3.8.1, it then follows that F_0 is a positive function.

3.9. Conclusions

The artifice of sequentially almost complete and sequentially infinite sets, which proved to be very useful in the present context, have been introduced. Properties of widest-sense Hurwitz polynomials, self-paraconjugate Hurwitz polynomials, strict sense Hurwitz polynomials and scattering Hurwitz polynomials have been studied. Several properties of multivariable positive rational functions have been investigated in this context. Reactance Hurwitz polynomials and immittance Hurwitz polynomials have been introduced. They fall out as the appropriate polynomials occurring as the numerators and denominators of (rational) reactance functions and positive functions respectively. The hierarchical relationship between the several classes of multivariable Hurwitz polynomial thus delineated is diagrammatically shown in Figure 3.1, in which an arrow (single or double) points to subclasses of polynomials, whereas double arrows originate from classes formed by products of elements of classes to which they point. A nontrivial result, which proves to be very useful in theoretical tests for the property of positivity of holomorphic functions and is formulated in terms of the behavior of its real part on the distinguished boundary of the domain of holomorphy has been derived. Finally, in view of its validity in the one-variable case, it seems plausible to conjecture that given any positive function with a self-paraconjugate Hurwitz factor in its denominator it is always possible to extract a reactance from it, thus leaving a positive function with scattering Hurwitz denominator only. A partial result in this direction has been included.

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CHAPTER 4

REALIZATION OF STRUCTURALLY PASSIVE MULTIDIMENSIONAL DIGITAL FILTERS

4.1. Introduction:

Various synthesis schemes such as the Darlington synthesis scheme for synthesizing lossless transfer functions as a cascade interconnection of most elementary lossless building blocks such as inductors, capacitors, gyrators etc. in the continuous time domain have now become classical in the network theoretic literature. The corresponding problem in the discrete time domain, namely that of synthesizing a discrete lossless bounded (or positive) transfer function as a structurally passive interconnection of elementary lossless building blocks was first resolved via transformation from prototype problems in the continuous time domain, and the resulting class of filter structures are now known as the wave digital filters [1]. Recently, however, successful attempts to derive these and similar other discrete domain results without making explicit use of tools of classical network theory have been made. Notable among these are the orthogonal filters [2], and the class of filters described in [3], [4] and in related other publications.

In view of interest in the synthesis of multidimensional (k-D) structurally passive digital filters, the problem of synthesis of k-D lossless two-port transfer scattering matrix via the bisection of a prescribed two-port into a cascade connection of two lossless two-port sections of smaller "degree" has been addressed in the continuous time domain in [5]. An attempt to develop a self consistent theory for the synthesis of k-D structurally passive digital filters independent of the continuous time methods have already been initiated in [6] by discussing the discrete domain stability

properties of a class of multidimensional polynomials. The present report addresses the problem of synthesizing a k -D discrete lossless bounded matrix as the transfer function of a structurally passive two-port digital filter directly in the discrete domain. Our approach is to bisect the prescribed discrete lossless two-port into a cascade interconnection of two discrete lossless two-ports as shown in figure 4.1. Necessary and sufficient conditions as to the feasibility of the bisection is obtained. It falls out that in the one-dimensional (1-D) case the aforementioned bisection is always feasible. Our discussion in the 1-D context thus yields yet another algorithm for the structurally passive synthesis of 1-D lossless digital filter transfer functions, previously not discussed in the literature.

4.2. Notation, Terminology and Problem Formulation:

We first explain the notation to be used in the rest of the paper in the following. Notations such as a, b, c will denote polynomials: $a=a(\underline{z}), b=b(\underline{z}), c=c(\underline{z})$ in k -variables $\underline{z} = (z_1, z_2, \dots, z_k)$. Notations such as n_{a1} or $\deg_1 a$ will denote the partial degree of a in the variable z_1 . The compact notation:

$\underline{z}^{\underline{n}} a \triangleq z_1^{n_{a1}} z_2^{n_{a2}} \dots z_k^{n_{ak}}$ will also be used.

Finally, $\tilde{a} \triangleq a^*(z_1^{*-1}, z_2^{*-1}, \dots, z_k^{*-1}), \tilde{a} \triangleq \tilde{a} \cdot \underline{z}^{\underline{n}} a$, where $*$ denotes complex conjugation. Corresponding notations for various polynomials other than the polynomial a will also be used.

A k -D discrete lossless two-port is characterized [6] by an associated transfer function matrix H as in (4.1) or by a transmission matrix T as in (4.2).

$$[H]_{11} = b/\tilde{a}, [H]_{12} = d\tilde{c}z^{\underline{n}}a/\tilde{a} \quad (4.1a,b)$$

$$[H]_{21} = c/\tilde{a}, [H]_{22} = -d\tilde{b}z^{\underline{n}}a/\tilde{a} \quad (4.1c,d)$$

$$[T]_{11} = da/c, [T]_{12} = b/c, \quad (4.2a,b)$$

$$[T]_{21} = d\tilde{b}z^{\underline{n}}a/c, [T]_{22} = \tilde{a}/c \quad (4.2c,d)$$

where a, b, c are polynomials such that $\tilde{a}$ is scattering Schur [6], $\deg_i b \leq \deg_i a, \deg_i c \leq \deg_i a$ for all $i=1$ to k , d is a unimodular complex constant i.e., $|d| = 1$ and

$$a\tilde{a} = b\tilde{b} + c\tilde{c}, \quad (4.3)$$

Note that (4.1) can be regarded as a discrete k -D

counterpart of Belevitch canonical form for the representation of lossless bounded two-port scattering matrices, well known in classical network theory.

In more specific terms the problem dealt with in the present report can be described as follows. Given T as in (4.2), two unimodular complex constants d', d'' with $d = d'd''$, and the polynomial factorization $c = c'c''$, along with two sets of integers $\underline{n}' = (n'_1, n'_2, \dots, n'_k)$ and $\underline{n}'' = (n''_1, n''_2, \dots, n''_k)$ such that $\deg_i c' \leq n'_i$, $\deg_i c'' \leq n''_i$ and $n_{ai} = n'_i + n''_i$ for all $i = 1$ to k , we seek a factorization $T = T'T''$, where T' and T'' are also discrete lossless two-port transmission matrices with associated polynomials (a', b', c') and (a'', b'', c'') respectively. In addition, the requirements $\deg_i a' \leq n'_i$ and $\deg_i a'' \leq n''_i$ needs to be satisfied. Thus, both T' and T'' are also required to have representations similar to those expressed in (4.2). In particular, the polynomial triples (a', b', c') and (a'', b'', c'') are also required to satisfy the condition that $\hat{a}', \hat{a}''$ are scattering Schur, $\deg_i b' \leq \deg_i a'$, $\deg_i b'' \leq \deg_i a''$ for all $i = 1$ to k and (4.4) holds true. The discrete lossless two-ports with associated transmission matrices T' and T'' resulting from the factorization of the transmission matrix T is shown in figure 4.1.

$$a'\hat{a}' = b'\hat{b}' + c'\hat{c}' \quad (4.4a)$$

$$a''\hat{a}'' = b''\hat{b}'' + c''\hat{c}'' \quad (4.4b)$$

It then easily follows by considering representations of T' and T'' such as that expressed in (4.2) for T that the condition $T = T'T''$ is equivalent to the conditions expressed in (4.5a) and (4.5b) in the following.

$$a = a'a'' + d' b' b'' z^{-\underline{n}} a'' \quad (4.5a)$$

$$b = d'a'b'' + b'\hat{a}'' \quad (4.5b)$$

The above considerations motivate the following definition.

Definition 4.2.1: The pair of polynomial two-tuples (a', b') and (a'', b'') is said to be a solution to the algebraic equation if equations (4.4) and (4.5) along with the degree restrictions $\deg_i a' \leq n_i'$ and $\deg_i a'' \leq n_i''$ for $i=1$ to k are satisfied.

We note that in the above definition the degree restrictions on the polynomials a' and a'' are expressed as weak inequalities rather than equalities as is required by the solution to the original problem. Also, the restrictions that the polynomials a' and a'' be scattering Schur polynomials are not imposed at all.

Definition 4.2.2: A polynomial triple (a'', b'', b') is said to satisfy the fundamental equation if (4.6) along with (4.7) holds true.

$$d' \cdot ab'' - ba'' = -b'c''\bar{c}''z^{\bar{n}''} \quad (4.6)$$

$$\deg_i a'' \leq n_i'' \text{ and } \deg_i b' \leq n_i' \quad (4.7)$$

Note that equation (4.6) is obtained by eliminating the polynomial a' from (4.5a,b) and (4.4b). Obviously then any solution of the algebraic equation also satisfies the fundamental equation.

4.3. Solution to the Algebraic equation:

Clearly, any solution to the problem of factorization of T into $T'T''$ is also a solution to the algebraic equation. The following theorem shows that the scattering Schur properties of $\tilde{a}'$ and $\tilde{a}''$ and the degree requirements on a' and a'' are automatically satisfied by any solution to the fundamental equation, and therefore, any solution to the algebraic equation is also a solution to the problem of factorization of T into $T'T''$.

Theorem 4.3.1: If the pair of polynomial two-tuples (a', b') and (a'', b'') constitute a solution to the algebraic equation then the polynomials $\tilde{a}'$ and $\tilde{a}''$ are scattering Schur and $\deg_i a' = n_i'$, $\deg_i a'' = n_i''$ for all $i = 1$ to k .

Proof: Consider the rational function defined as:

$$\psi = (\tilde{a}'\tilde{a}'')/\tilde{a} = ((a'a'')/\tilde{a})z^p \quad (4.8)$$

$$\begin{aligned} \text{where } p &= (p_1, p_2, \dots, p_k), \\ \text{and } p_i &= n_{a_i} - (n_{a', i} + n_{a'', i}) \end{aligned} \quad (4.9)$$

Since $\tilde{a}$ is a scattering Schur polynomial, $n_i' + n_i'' = n_{a_i}$, and factors of a scattering Schur polynomial are also scattering Schur, the denominator polynomial of ψ is also scattering Schur.

Furthermore, straightforward algebraic manipulation of equations (4.4b) and (4.5) yield the following.

$$\psi = (a''/c'')(\tilde{a}''/\tilde{c}'')[1 - d'(B/\tilde{a})(b''/a'')] \quad (4.10)$$

Since it follows from (4.3) that $|B/\tilde{a}| \leq 1$ and $|b''/a''| \leq 1$ for $|z_i| = 1$ for $i = 1$ to k , an examination of (4.10) yields that $\operatorname{Re} \psi \geq 0$ for $|z_i| = 1$, wherever ψ is well defined. Thus, by

invoking a result proved in [6] it follows that ψ is a discrete positive function. Consequently, the numerator polynomial of ψ , in irreducible form, is a widest sense Schur polynomial. This, however, implies that $n_{ai} = n_{a,i} + n_{a-i}$ for all $i=1$ to k . The last equality along with the facts that $n_i' \geq n_{a,i}$, $n_i'' \geq n_{a-i}$ and $n_{ai} = n_i' + n_i''$ together imply that $n_i' = n_{a,i}$ and $n_i'' = n_{a-i}$.

The widest sense Schur property of δ' has already been established. Next, if for some $\underline{z}_0$ on the distinguished boundary of the polydisc $|z_i| \leq 1$, $i=1$ to k we have $a'(\underline{z}_0) = 0$ then from (4.4a) it follows that $b'(\underline{z}_0) = 0$, which in turn due to (4.5a) imply that $a(\underline{z}_0) = 0$. Consequently, if δ' , and thus a , had a sequentially almost complete set [6] of zeros on the distinguished boundary then a would also have a sequentially almost complete set of zeros there, which is impossible if δ scattering Schur. Therefore, δ cannot have sequentially almost complete set of zeros on the distinguished boundary. The scattering Schur property of δ is thus established in view of results in [5]. Similar arguments hold for δ'' .

A lossless two-port is said to be an allpass if the polynomial b associated with it is identically equal to zero.

We will need the following result as a preparation for the rest of the discussions to follow.

Theorem 4.3.2: Any discrete lossless two-port transmission matrix T can be factored as $T = T_f T_o T_r$, where T_f , T_o , T_r are also discrete lossless two-port transmission matrices such that T_f and T_r are allpass and if T_o has representation in terms of polynomials a , b , c as in (4.2) then the polynomial a is relatively prime with b as well as $b z^{-a}$.

In physical terms the above factorization amounts to

extraction of discrete lossless two-port sections from the front and rear end of the prescribed transmission matrix. Thus, without loss of generality it will be assumed in all forthcoming discussions that the polynomial a is relatively prime with b as well as with $Bz^{\frac{n}{a}}$.

4.4. Solution to the fundamental equation:

We will need the following lemmas.

Lemma 4.4.1: If the polynomial a is relatively prime with b

as well as $\underline{B}z^{\underline{n}-a}$ then neither a nor δ can have a factor in common with the polynomial $\underline{z}^{\underline{n}''}c''\bar{c}''$.

Proof: Since δ is scattering Schur δ cannot have z_i as a factor for any i . Also by rewriting equation (4.3) along with $c=c'c''$ in the form of (4.11)

$$a\delta = b(\underline{B}z^{\underline{n}-a}) + (\underline{z}^{\underline{n}'}c'\bar{c}')(\underline{c}''\bar{c}''\underline{z}^{\underline{n}''}) \quad (4.11)$$

it can be seen that if δ or a had a factor in common with the polynomial $(\underline{c}''\bar{c}''\underline{z}^{\underline{n}''})$ then it would also have a factor in common with the $b(\underline{B}z^{\underline{n}-a})$ i.e., in common with either b or $\underline{B}z^{\underline{n}-a}$ both of which is ruled out by the fact that a is relatively prime with b and $\underline{B}z^{\underline{n}-a}$.

Lemma 4.4.2: If the polynomial triple $(\alpha'', \beta'', \beta')$ is a solution to the fundamental equation then $\deg_i \beta'' \leq n_i'$. Furthermore, there exists a polynomial α' given by (4.12) such that the

polynomial triple $(\beta''\underline{z}^{\underline{n}''}, \bar{\alpha}''\underline{z}^{\underline{n}''}, -\alpha'd')$ is also a solution to the fundamental equation. Also, we have that $\deg_i \alpha' \leq n_i'$ for all $i=1$ to k .

$$\alpha' = (\bar{\alpha}''c'\bar{c}' + \underline{z}^{-\underline{n}''}\beta'b)/\delta \quad (4.12)$$

Proof: The fact that $\deg_i \beta'' \leq n_i'$ follows directly from the fundamental equation for the triple $(\alpha'', \beta'', \beta')$. Next, by straightforward algebraic manipulations with the fundamental equation for $(\alpha'', \beta'', \beta')$ and (4.11) and (4.12) we get the following.

$$\underline{z}^{\underline{n}''} (a\underline{\alpha}'' - d' b \underline{\beta}'') = (\underline{z}^{\underline{n}''} c'' \underline{c}') (\underline{\alpha}'' \underline{c}' c' + \underline{z}^{-\underline{n}''} \underline{\beta}' b) \underline{z}^{\underline{n}''} a / \hat{a} \quad (4.13)$$

Since the left hand side of (4.13) is a polynomial, due to lemma 4.4.1, $\hat{a}$ must divide $(\underline{\alpha}'' \underline{c}' c' + \underline{z}^{-\underline{n}''} \underline{\beta}' b) \underline{z}^{\underline{n}''} a$. Thus, α' in (4.12) is a polynomial. The fact that $\deg_i \alpha' \leq n'_i$ then follows by considering the degree restrictions on c' , c'' , α'' , b , and β' and a .

Lemma 4.4.3: If the polynomial a is relatively prime with b as well as $\underline{b} \underline{z}^{-\underline{a}}$, and $(\alpha''_1, \beta''_1, \beta'_1)$ and $(\alpha''_2, \beta''_2, \beta'_2)$ are two polynomial triples satisfying the fundamental equation then the rational function given in (4.14) is a constant.

$$(\alpha''_1 \beta''_2 - \beta''_1 \alpha''_2) / (\underline{z}^{\underline{n}''} c'' \underline{c}'') \quad (4.14)$$

Proof: By multiplying the fundamental equations for $(\alpha''_1, \beta''_1, \beta'_1)$ and $(\alpha''_2, \beta''_2, \beta'_2)$ respectively by α''_2 and $(-\alpha''_1)$ and adding the resulting equations one obtains equation (4.15).

$$d' (\alpha''_1 \beta''_2 - \alpha''_2 \beta''_1) = (\beta'_1 \alpha''_2 - \alpha''_1 \beta'_2) (\underline{z}^{\underline{n}''} c'' \underline{c}'') / a \quad (4.15)$$

Since the lefthand side of (4.15) is a polynomial, by invoking lemma 4.4.1 it then follows that a must divide the polynomial $P = (\beta'_1 \alpha''_2 - \alpha''_1 \beta'_2)$. Since $\deg_i P \leq n'_i + n''_i - n_{ai} = \deg_i a$ for all $i=1$ to k we have that P/a is a constant. The result then follows by noting that the expression in (4.14), in view of (4.15), is equal to (Pd'/a) .

Lemma 4.4.4: If the polynomial a is relatively prime with b as well as $\underline{b} \underline{z}^{-\underline{a}}$, and $(\alpha'', \beta'', \beta')$ is a polynomial triple satisfying the fundamental equation then the expression given in (4.16) is a constant.

$$\alpha'' \beta'' - \beta'' \beta' = c'' c'$$

Proof: Follows from lemma 4.4.2 and lemma 4.4.3.

Lemma 4.4.5: If the polynomial triple $(\alpha'', \beta'', \beta')$ is a solution to the fundamental equation then there exists an α' as given by lemma 4.4.2 such that $(p\alpha'' + qz^{\frac{n}{2}}\beta'', p\beta'' + qz^{\frac{n}{2}}\tilde{\alpha}'', p\beta' - qd\alpha')$ is also a solution to the fundamental equation, where p and q are arbitrary complex numbers.

Proof: Obviously follows from lemma 4.4.2.

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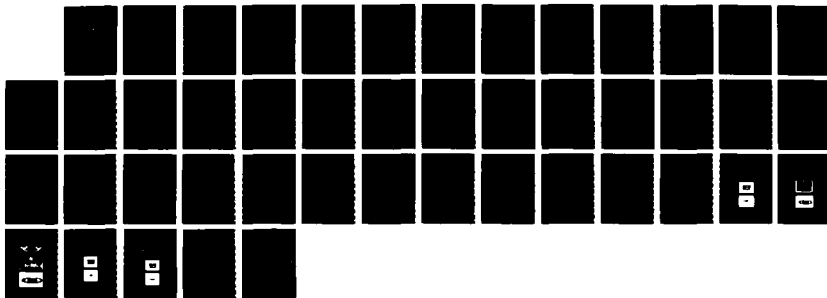
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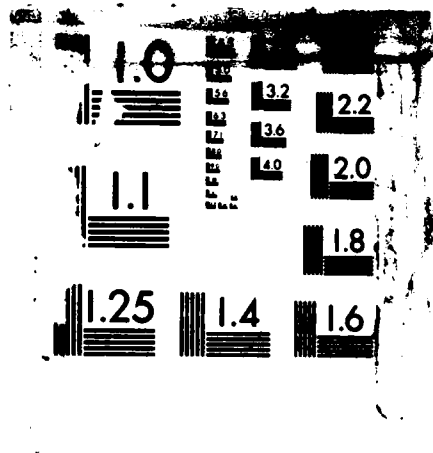
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4.5. Factorization of the discrete lossless two-port transmission matrix:

Two polynomial triples $(\alpha_1'', \beta_1'', \beta_1')$ and $(\alpha_2'', \beta_2'', \beta_2')$ each satisfying the fundamental equation will be said to be linearly dependent if there exists constants p and q not simultaneously zero such that $p\alpha_1'' + q\alpha_2'' = p\beta_1'' + q\beta_2'' = p\beta_1' + q\beta_2' = 0$.

Also, a solution $(\alpha'', \beta'', \beta')$ to fundamental equation will be said to be nonsingular if $\alpha''\bar{\alpha}'' \neq \beta''\bar{\beta}''$.

The following two theorems constitute the major results of this report.

Theorem 4.5.1: Assuming that the polynomial a is relatively prime with b as well as $\bar{b}z^{-n}$, the problem of factorization of discrete lossless two-port transmission matrix T admits a solution if and only if there exists a nonsingular solution $(\alpha'', \beta'', \beta')$ to the fundamental equation.

Proof: Necessity is obvious. If $(\alpha'', \beta'', \beta')$ is a nonsingular solution to the fundamental equation then due to lemma 4.4.5, $a'' = p\alpha'' + q\bar{z}^{-n}\beta''$, $b'' = p\beta'' + q\bar{z}^{-n}\bar{\alpha}''$, $b' = p\beta' - qd\alpha'$ is a solution to the fundamental equation. Straightforward algebraic manipulation then yields that

$$(a''\bar{\alpha}'' - b''\bar{\beta}'')/c''\bar{c}'' = (|p|^2 - |q|^2)K \quad (4.17)$$

$$\text{where } K = (\alpha''\bar{\alpha}'' - \beta''\bar{\beta}'')/c''\bar{c}'' \quad (4.18)$$

Since due to lemma 4.4.3 and nonsingularity of $(\alpha'', \beta'', \beta')$, K is a nonzero constant, by proper choice of p and q in the right hand side of (4.17) it is possible to have $(a''\bar{\alpha}'' - b''\bar{\beta}'') = c''\bar{c}''$.

Furthermore, there exists a' such that $(\bar{b}''z^{-n}, \bar{\alpha}''z^{-n}, -a'd)$, by virtue of lemma 4.4.2, satisfies the fundamental

equation. It can then be verified via routine algebraic manipulation that the pair of two-tuples (a', b') and (a'', b'') satisfies the algebraic equation, and thus, due to theorem 4.3.1, is a solution to the problem of factorization of T .

Theorem 4.5.2: Assuming that the polynomial a is relatively prime with b as well as with $Bz^{\frac{n}{2}-a}$, the problem of factorization of discrete lossless two-port transmission matrix T admits a solution if and only if there exists two linearly independent polynomial triples $(\alpha_i'', \beta_i'', \beta_i')$, $i=1,2$ each of which satisfy the fundamental equation.

Proof: Necessity is obvious. If one of the solutions $(\alpha_i'', \beta_i'', \beta_i')$, $i=1,2$ is nonsingular then sufficiency follows from theorem 4.5.1. If both solutions are singular then the triple (a'', b'', b') obtained as: $a'' = p\alpha_1'' + q\beta_1'$, $b'' = p\beta_1'' + q\beta_2'$, $b' = p\beta_1' + q\beta_2'$, where p and q are complex numbers, satisfies the fundamental equation. Algebraic manipulation then yields that

$$(a''\tilde{a}'' - b''\tilde{b}'')/c''\tilde{c}'' = L + \tilde{L} \quad (4.19)$$

$$L = p^*q(\alpha_2''\tilde{\alpha}_1'' - \beta_2''\tilde{\beta}_1'')/c''\tilde{c}'' \quad (4.20)$$

By invoking lemmas 4.4.2 and 4.4.3 it then follows that L in (4.20) is a constant, and thus, $\tilde{L} = L^*$. Furthermore, by following arguments similar to those in [5] it can be proved via the use of results in [7] that $L \neq 0$ if $p \neq 0$ and $q \neq 0$. (The details of this derivation is left out of here for the sake of brevity). Consequently, by proper choice of p and q in (4.19) and (4.20) it is possible to have $a''\tilde{a}'' - b''\tilde{b}'' = c''\tilde{c}''$. The rest of the proof follows by imitating the last paragraph in the proof of theorem 4.5.1.

The fundamental equation (4.6), when considered as a set of linear simultaneous equations involving the coefficients of the polynomials a'' , b'' , b' , along with the upper bounds on their degrees, turns out to be overdetermined in general.

More explicitly, we note that the unknown polynomials a'' , b'' and b' contain a total of u unknown coefficients, whereas the total number of linear simultaneous equations can easily be found to be equal to e , u and e being as given in (4.21) and (4.22) below.

$$u = 2 \prod_{i=1}^k (n_i'' + 1) + \prod_{i=1}^k (n_i' + 1) \quad (4.21)$$

$$e = \prod_{i=1}^k (2n_i'' + n_i' + 1) \quad (4.22)$$

Since for $k > 1$ we have $e > u$ in a generic situation a solution to the problem may not exist.

Furthermore, in order for the digital network so synthesized to be 'computable' it may not contain delay free loops arising from cascading of two elementary sections. In spite of the fact that it is known [8] that this problem can always be circumvented, at least in the one-dimensional case, by incorporating digital equivalents of unit elements it is of interest to note that by properly utilizing the flexibility in the choice of p and q in (4.20) it is always possible to avoid the occurrence of such delay free loops in the filter structure. This point is further elaborated in the following section.

4.6. One-dimensional synthesis as a special case:

In the one-dimensional case i.e., if $k = 1$, a closer examination of (4.21) and (4.22) reveals that we have $u-e=2$, and, therefore, there are two more unknown coefficients than the number of linear equations in the set of linear simultaneous equations which determine the solution to the fundamental equation. Thus, there are at least two linearly independent solutions of the fundamental equation, and in view of theorem 4.5.2, the problem of factorization of T always admits of a solution. Consequently, structurally passive synthesis for T is achieved by performing a sequence of further factorizations of T' and T'' into discrete lossless transmission matrices of progressively lower complexity, until a stage is reached when each of the resulting transmission matrices cannot be factorized any further. This latter situation corresponds to the case that each of the two-ports resulting from the decomposition satisfy $\deg a = 1$, $\deg c \leq 1$ and $\deg b \leq 1$. However, if the specified two-port transmission matrix T has real coefficients for its numerator and denominator polynomials and realization involving only real multipliers are sought then the constituent two-ports may also be of the type $\deg a = 2$, $\deg c = 2$, and $\deg b \leq 2$. Two-port sections of the above types will be called elementary sections and can in turn be realized in structures possibly other than the cascade structure by exploiting synthesis techniques as discussed, for example, in [4].

To address the issue of absence of delay free loops at the junction of the two-ports associated with T' and T'' it may be noted that the only restriction governing the choice of the numbers p and q is that the right hand side of (4.17) or (4.19) be equal to one. This flexibility in the choice of p and q may thus be exploited to make $b''(0) = 0$, which ensures

the absence of the delay free loops of the type mentioned above. Furthermore, since $\hat{a}$ is Schur and thus $\hat{a}(0) \neq 0$, it follows from (4.5b) that if $b(0)=0$ and $b''(0)=0$ then $b'(0)=0$. This fact guarantees that the original two-port can be decomposed into cascade interconnection of elementary two-ports in such a way that the b -polynomial associated with each of the constituent two-ports, except possibly the one at the extreme left, is equal to zero for $z=0$. Absence of delay free loops from each junction is thus guaranteed.

Realizations for elementary sections with $\deg a=1$, $\deg c \leq 1$, $\deg b \leq 1$ and $b(0)=0$ as interconnections of Gray-Markel sections and delays are shown in figures 4.4 and 4.5. Gray-Markel sections of two different kinds used in these figures are shown in figures 4.2 and 4.3. An elementary section with $\deg c = 2$, $\deg a = 2$, $\deg b \leq 2$ and $b(0)=0$ is shown in figure 4.6. Thus, an arbitrary lossless two-port can indeed be synthesized as a cascade interconnection of these elementary sections only. It turns out that elementary sections just referred to are exactly the same as those discussed in the literature [1],[2],[3].

4.7 Conclusions:

A simple algorithm involving the examination of rank of a set of linear simultaneous equations for studying the synthesizability of an arbitrary multidimensional lossless two-port in a cascade structure has been derived via factorization of the associated transmission matrix T . It turns out that under a generic situation synthesis in a cascade structure may not be feasible. In the special case of one-dimension the algorithm provides a new method of realizing structurally passive filters directly in the digital domain. The problem of multidimensional synthesis, though not necessarily in cascade structure, can also be addressed via factorizations of hybrid matrix or the transfer function matrix associated with the lossless two-port. The class of multidimensional lossless two-ports thus synthesizable along with the class identified in the present study would thus broaden the whole class of synthesizable multidimensional structurally passive lossless two-ports. It may be remarked that even though in the $k > 1$ case synthesis may not be feasible for an arbitrary discrete lossless T , the possibility of synthesis for special classes of discrete lossless T is by no means ruled out. This is especially true in view of synthesizability of certain classes [9], [10] of two-dimensional continuous time systems arising in studies of lumped-distributed networks. The class of multidimensional discrete lossless two-port transmission matrices T , which admits of such synthesis remains, however, to be identified.

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CHAPTER 5

MULTIDIMENSIONAL INTERPOLATION AND DECIMATION SCHEMES FOR FAST IMPLEMENTATION

5.1. Introduction

The processes of sampling rate increase and sampling rate reduction commonly referred to as interpolation and decimation, are required whenever it is necessary to change from one sampling rate to another. The fact that many commonly encountered one-dimensional signal processing tasks such as speech processing, single side band frequency multiplexing require the processes of sampling rate change is now well known [3]. Similarly a large number of multidimensional signal processing [5] tasks also require the operation of digital interpolation and decimation. Such application areas include transmission of television pictures [13], antenna beamforming [5], target tracking [1], astronomical data processing [17] geophysical signal processing [16], and medical tomography [9]. We point out exactly how the specific problem of multidimensional sampling rate alteration enters into some of these applications. In radio astronomical observations it is often desirable to estimate the radio brightness of the sky at intermediate points from observations made by directing the measuring antennas at a regular array of points in the sky [17]; in the problem of transmission of television pictures efficient coding schemes require that the time-varying image signal be known between two successive picture frames [13]; whereas in X-ray computed tomography the problem of interpolation manifests itself when a higher

resolution in reconstruction is necessitated [9]. Similar other examples of need of multidimensional interpolation as well as decimation schemes can be drawn from these and other application areas mentioned earlier. A feature common to all of these multidimensional processing tasks is the enormous amount of computational requirement. This fact becomes especially prohibitive to practical implementation if real time or adaptive applications are called for.

On the other hand, recent advent of VLSI technology has the potential to make such computation intensive multidimensional signal processing tasks by increasing the throughput rate via utilization of new concepts such as parallelism, pipelining, concurrency modularity of implementation etc. [8]. The more recent optical technologies [2] provide yet another potential means for highspeed implementation of many multidimensional signal processing algorithms. In this context, the need for reconsidering existing signal processing algorithms as well as that of designing algorithms for previously intractable problems have already been recognized in general [11], and both new computational schemes and their implementations in hardware for solving specific multidimensional signal processing tasks are now beginning to emerge [6] [12]. It is in this perspective that the general problem of designing an algorithm for interpolation and decimation of a broad class of multidimensional signals is investigated in the present report.

For the type of applications under consideration, it is important to understand the processes of interpolation and decimation from the point of view of digital signal

processing rather than from numerical analysis standpoint. For example, linear interpolation is not satisfactory in most digital signal processing tasks. In classical numerical analysis, the inadequacies of polynomial interpolation schemes such as the multidimensional Lagrange type interpolation method lead to the use of higher order polynomials, the inappropriateness of which has been pointed out in [3] in the one-dimensional context. In the present study we consider the problems of interpolation and decimation based on the frequency domain description of the multidimensional signal. It then turns out that, as has already been discussed in the literature for one-dimensional signals [3], the problem of interpolation and decimation of multidimensional signals can also be interpreted as linear filtering operations in the frequency domain. More importantly, the filtering scheme leads to computational structures, which is highly modular and derive full advantages of parallel and pipeline implementation. Performance of the computational algorithm so designed is also examined by experimenting with both real and synthetic two-dimensional signals.

In section 5.2 the notation, terminology and the sampling scheme to be used for the rest of the report is introduced. The problems of interpolation and decimation are formulated, and the filtering schemes leading to their solution are discussed in section 5.3. The fact that such computational schemes can be implemented by exploiting the concepts of both parallelism and pipelineability forms the contents of section 5.4. Some filter design examples of interpolators and decimators and their performances on image data are presented in section 5.5 and in section 5.6 conclusions are drawn.

5.2. Multidimensional Periodic Sampling

Of the several ways to generalize [5] one-dimensional (1-D) periodic sampling schemes to multidimensions (N-D) ($N \geq 2$), the most straightforward, although not the most efficient [15], is periodic sampling in rectangular cartesian coordinates, which we will simply call rectangular sampling.

In what follows underlined characters will be used to denote column vectors and the notation "''" will be used to denote the transpose of a vector. For example,

$\underline{T}' = (T_1, T_2, \dots, T_N)$, where $\underline{T}$ is a column vector. Similar notations will be used for variables such as $\underline{t}$, $\underline{q}$, $\underline{\omega}$, $\underline{W}$, $\underline{n}$, $\underline{k}$ etc. the later physical meanings of which will be made clear when the context arises.

If $x_a(\underline{t})$, is a multidimensional continuous signal, the discrete signal $x(\underline{n})$ obtained from it by rectangular sampling is given by:

$$x(\underline{n}) = x_a(n_1 T_1, n_2 T_2, \dots, n_N T_N) \quad (5.2.1)$$

where $T_1, T_2, \dots, T_N$ are positive constants known as the sampling intervals or periods in the respective sampling directions.

The N-D Fourier transform $X_a(\underline{q})$ of the continuous signal $x_a(\underline{t})$, and its inverse are given in (5.2.2a,b).

$$X_a(\underline{q}) = \int_{-\infty}^{\infty} x_a(\underline{t}) e^{-j(\underline{q} \cdot \underline{t})} d\underline{t}; \quad x_a(\underline{t}) = (1/4\pi)^2 \int_{-\infty}^{\infty} X_a(\underline{q}) e^{j(\underline{q} \cdot \underline{t})} d\underline{q} \quad (5.2.2a,b)$$

The discrete sequence $x(\underline{n})$ obtained by sampling $x_a(\underline{t})$ at spatial locations $t_i = n_i T_i$, $i=1$ to N can then be obtained

as in (5.2.3a), whereas the Fourier transform of $x(\underline{n})$ is given by $X(\underline{\omega})$ in (5.2.3b).

$$x(\underline{n}) = (1/4\pi^2) \int_{-\infty}^{\infty} X_a(\underline{\omega}) e^{-j(\underline{\omega} \cdot \underline{n})} d\underline{\omega}; \quad x(\underline{\omega}) = \sum_{\underline{n}} x(\underline{n}) e^{j(\underline{\omega} \cdot \underline{n})}. \quad (5.2.3a,b)$$

We assume that the signal $x_a(\underline{t})$ is bandlimited. More specifically, we assume that the Fourier transform $X_a(\underline{\omega})$ have a support which is contained in the hypercube $|\omega_i| < W_i$ for $i=1$ to N in the N -dimensional frequency space i.e., $X_a(\underline{\omega})=0$ for $|\omega_i| \geq W_i$, $i=1$ to N . It then follows from the multidimensional version of the Nyquist sampling theorem [5] that the continuous signal $x_a(\underline{t})$ can be recovered from the discretized multidimensional signal $x(\underline{n})$ according as equation (5.2.4), in which $T_i < \pi/W_i$.

$$x_a(\underline{t}) = \sum_{\underline{n}} x(\underline{n}) \prod_{i=1}^N [\sin(\pi/T_i)(t_i - n_i T_i)] / (\pi/T_i)(t_i - n_i T_i) \quad (5.2.4)$$

Equations (5.2.1) and (5.2.4) taken together, form the basis of the multidimensional sampling theorem in rectangular cartesian coordinates.

5.3. Multidimensional Sampling Rate Conversion

The process of sampling rate conversion is one of converting the sequence $x(\underline{n})$ obtained from the sampling of the bandlimited signal $x_a(\underline{t})$ with periods T_i , to another sequence $y(\underline{m})$ obtained from sampling $x_a(\underline{t})$ with periods T'_i for each $i = 1$ to N . Conceptually at least, the most straightforward way to perform this conversion is to reconstruct $x_a(\underline{t})$ (or the low-pass filtered version of it) from the samples of $x(\underline{n})$ and then resample $x_a(\underline{t})$ (assuming that it is sufficiently bandlimited for the new sampling rate) with periods T'_i , $i=1$ to N to give $y(\underline{m})$. For any $\underline{m}$, the value of $y(\underline{m})$ can be then obtained as:

$$y(\underline{m}) = x_a(\underline{t}) \quad \text{for } t_i = m_i T'_i, \quad i=1 \text{ to } N \quad (5.3.1)$$

By substituting (5.3.1) into (5.2.4) and renaming the variables $\underline{n}$ as $\underline{k}$, we have:

$$y(\underline{m}) = \sum_{\underline{k}} x(\underline{k}) \prod_{i=1}^N \frac{\sin(\pi((m_i T'_i / T_i) - k_i))}{\pi((m_i T'_i / T_i) - k_i)} \quad (5.3.2)$$

5.3.1 Interpolation of Multidimensional Signals

If the sampling rate in the i -th dimension is increased by an integer factor L_i , then the new sampling period T'_i for $i=1$ to N are given by

$$T'_i / T_i = 1 / L_i \quad (5.3.3)$$

This process of increasing the sampling rate (interpolation) of a signal $x(\cdot)$ by L_i implies that we must interpolate $(L_i - 1)$ new sample values between each pair of sample values of $x(\cdot)$. By substituting (5.3.3)

into (5.3.2), (5.3.4a) is obtained, in which $h_i(s_i)$ for integer values of s_i , $i=1$ to N are given in (5.3.4b).

$$y(\underline{m}) = \sum_{\underline{k}} x(\underline{k}) \prod_{i=1}^N h_i(m_i - k_i L_i); \quad h_i(s_i) = \sin(\pi s_i / L_i) / (\pi s_i / L_i) \quad (5.3.4a,b)$$

An alternative formulation of equation (5.3.4a), as given in (5.3.6), can be obtained via the introduction the change of variables:

$$k_i = \lfloor m_i / L_i \rfloor - n_i, \text{ for } i=1 \text{ to } N \quad (5.3.5)$$

where $\lfloor u \rfloor$ denotes the integer less than or equal to u .

$$\begin{aligned} y(\underline{m}) &= \sum_{\underline{n}} x(\lfloor \underline{m} / \underline{L} - \underline{n} \rfloor) \prod_{i=1}^N h_i(m_i - \lfloor m_i / L_i \rfloor \cdot L_i + n_i L_i) \\ &= \sum_{\underline{n}} \prod_{i=1}^N h_i(n_i L_i + m_i \ominus L_i) x(\lfloor \underline{m} / \underline{L} - \underline{n} \rfloor) \end{aligned} \quad (5.3.6)$$

where $m_i \ominus L_i$ denotes the value of m_i modulo L_i for $i=1$ to N and the notation $\lfloor \underline{m} / \underline{L} - \underline{n} \rfloor$ is taken to mean the N -tuple of integers $\underline{s} = (s_1, s_2, \dots, s_N)$ with $s_i = \lfloor m_i / L_i \rfloor - n_i$, $i=1$ to N . Equation (5.3.6) expresses the output $y(\cdot)$ in terms of the input $x(\cdot)$ and the set of one-dimensional sequences $h_i(\cdot)$ $i=1$ to N , as given in (5.3.4b). Thus, in a compact notation $y(\underline{m})$ can be written as in (5.3.7a), where $g_{\underline{m}}(\underline{n})$ is as expressed in (5.3.7b) for all N -tuple of integers $\underline{m}$ and $\underline{n}$.

$$y(\underline{m}) = \sum_{\underline{n}} g_{\underline{m}}(\underline{n}) x(\lfloor \underline{m} / \underline{L} - \underline{n} \rfloor); \quad g_{\underline{m}}(\underline{n}) = \prod_{i=1}^N h_i(n_i L_i + m_i \ominus L_i) \quad (5.3.7a,b)$$

Note that $g_{\underline{m}}(\underline{n})$ is periodic in m_i with period L_i for each $i=1$ to N . Furthermore, by referring to (5.3.7b) it follows that $g_{\underline{m}}(\underline{n})$ is a product separable function, and each of its factors are periodic in m_i with respective periods L_i i.e.,

$$g_{\underline{m}}(\underline{n}) = \prod_{i=1}^N g_{m_i}(n_i) ; g_{m_i}(n_i) = h_i(n_i L_i + m_i \otimes L_i) \quad (5.3.8a, b)$$

where (5.3.8b) holds for each $i=1$ to N . Thus, by using (5.3.8a), (5.3.7a) can be written as (5.3.9).

$$y(\underline{m}) = \sum_{n_N=-\infty}^{\infty} g_{m_N}(n_N) \dots \sum_{n_1=-\infty}^{\infty} g_{m_1}(n_1) x(\lfloor \underline{m}/L - \underline{n} \rfloor) \quad (5.3.9)$$

An alternative formulation of equation (5.3.9), which yields the implementation of the equation in digital filtering terms is given in equation (5.3.10).

$$\begin{aligned} & (i) \\ y^{(i)}(m_1, \dots, m_i, v_{i+1}, \dots, v_N) \\ &= \sum_{n_i=-\infty}^{\infty} g_{m_i}(n_i) y^{(i-1)}(m_1, \dots, m_{i-1}, \lfloor m_i/L_i \rfloor - n_i, v_{i+1}, \dots, v_N) \\ & \quad (0) \end{aligned} \quad (5.3.10)$$

for $1 \leq i \leq k$, where $y^{(0)}(.) = x(.)$

Note that (5.3.10) together with (5.3.9) yields that $y^{(N)}(.) = y^{(0)}(.)$. Thus, output $y(\underline{m})$, as given in (5.3.9), can also be computed via the recursive construction of the intermediate signals $y^{(i)}(.)$, for $1 \leq i \leq N$. Furthermore, for each i in $1 \leq i \leq N$, (5.3.10) can be interpreted as the input-output equation of a one-dimensional spatially varying filter operating on the i -th dimension of the intermediate signal $y^{(i-1)}(.)$, whose impulse response is periodically space varying with a period equal to L_i . A closer examination of (5.3.10) reveals that (5.3.10) can be digitally implemented in N stages as shown in Figure 5.1, where the i -th stage represents a set of one-dimensional filters operating on the i -th dimension of intermediate signal $y^{(i-1)}(.)$, one for each value of the indices $m_1, \dots, m_{i-1}, v_{i+1}, \dots, v_k$, to produce the next

intermediate signal $y^{(i)}(.)$. In the special case, when $N=2$

i.e., for two-dimensional image signals, for example, (5.3.10) can be written as follows.

$$y^{(1)}(m_1, v_2) = \sum_{n_1=-\infty}^{\infty} g_{m_1}(n_1) x([m_1/L_1] - n_1, v_2) \quad (5.3.11)$$

$$\begin{aligned} y(m_1, m_2) &= y^{(2)}(m_1, m_2) \\ &= \sum_{n_2=-\infty}^{\infty} g_{m_2}(n_2) y^{(1)}(m_1, [m_2/L_2] - n_2) \end{aligned} \quad (5.3.12)$$

Equation (5.3.11) represents a one-dimensional interpolator which interpolates the rows of the given input image $x(\cdot)$. This row interpolation is performed on each row of the input signal $x(\cdot)$ and the results are stored in an intermediate image signal $y^{(1)}(m_1, v_2)$. The second step is an implementation of (5.3.12), which amounts to performing the operation of interpolation on the columns of the intermediate image $y^{(1)}(\cdot)$. The entire process, therefore, can be implemented in two stages of 1-D interpolations. It must be noted that by rewriting in (5.3.9) the order in which the summations over different indices are considered, it is also possible to perform the column interpolations first and the row interpolations next.

5.3.2 Decimation of Multidimensional Signals

The process of reducing the sampling rate (decimation) of $x(\underline{n})$ by an integer factor M_i in the i -th dimension, is considered next. If $T'_i/T_i = M_i/1$ for $i = 1$ to N then the new sampling rate is given by $F'_i = 1/T'_i = 1/M_i T_i = F_i/M_i$.

In order to lower the sampling rate and to avoid aliasing at this lower rate, it is necessary to filter the signal $x(\underline{n})$ with a digital low-pass filter whose unit impulse

response is denoted by $h(\underline{n})$. The sampling rate reduction is then achieved by forming the sequence $y(\underline{m})$ by saving for each i in $1 \leq i \leq N$ only every M_i -th sample in the i -th dimension of the filtered output.

If $h(\cdot)$ denotes the N -D impulse response of the ideal low-pass filter then it follows that the signal at the output of the low-pass filter is given by :

$$w(\underline{n}) = \sum_{\underline{k}} h(\underline{k}) x(\underline{n} - \underline{k}) \quad (5.3.13)$$

Furthermore, if $y(\underline{m})$ is obtained by considering every M_i -th sample in the i -th dimension of the signal $w(\underline{n})$ then $y(\underline{m})$ is given by (5.3.14), where the notation $x(\underline{M} \cdot \underline{m} - \underline{k})$ is taken to mean $x(M_1 m_1 - k_1, M_2 m_2 - k_2, \dots, M_N m_N - k_N)$.

$$y(\underline{m}) = [w(\underline{n})]_{n_i = m_i M_i} = \sum_{\underline{k}} h(\underline{k}) x(\underline{M} \cdot \underline{m} - \underline{k}) \quad (5.3.14)$$

If the frequency response of the ideal low-pass filter having impulse response $h(\underline{n})$ is given by (5.3.15) then $h(\underline{n})$ is product separable [5] and can be written as in (5.3.16).

$$H(\underline{\omega}) = \begin{cases} 1 & \text{for } |\omega_i| < \pi/M_i, \text{ for } i = 1 \text{ to } N \\ 0, & \text{otherwise} \end{cases} \quad (5.3.15)$$

$$h(\underline{n}) = \prod_{i=1}^N h'_i(n_i); \quad h'_i(n_i) = \sin(\pi n_i) / \pi n_i \text{ for } i = 1 \text{ to } N \quad (5.3.16a, b)$$

Making use of (5.3.16) in (5.3.14) it follows that the output signal from the entire decimator $y(\underline{m})$ can be written as in (5.3.17).

$$y(\underline{m}) = \sum_{k_N} h'(k_N) \dots \sum_{k_1} h'(k_1) x(\underline{M} \cdot \underline{m} - \underline{k}) \quad (5.3.17)$$

Furthermore, as for the interpolator we note that the output signal $y(\underline{m})$ can be computed via the recursively defined intermediate signals $y^{(i)}(.)$, $1 \leq i \leq N$ as given in (5.3.18) below, where $y^{(0)}(.) = x(.)$. We then have $y(\underline{m}) = y^{(N)}(\underline{m})$.

$$y^{(i)}(m_1, \dots, m_i, v_{i+1}, \dots, v_N) =$$

$$\sum_{k_i} h'(k_i) y^{(i-1)}(m_1, \dots, m_{i-1}, M_i m_i - k_i, v_{i+1}, \dots, v_N) \quad (5.3.18)$$

Clearly, for each i equation (5.3.18) represents the operation of performing 1-D decimation in the i -th direction of the intermediate signal $y^{(i-1)}(.)$, as a result of which the next intermediate signal $y^{(i)}(.)$ is obtained, and the computation of $y^{(i)}(.)$ in (5.3.18) for values of $i = 1, 2, \dots, N$ correspond to the implementation of N stages of such 1-D decimators in succession. For two-dimensional signals, for example, i.e., if $N=2$ the entire operation can be interpreted as first decimating the rows of the discrete signal and then decimating the columns of the resulting signal obtained from the output of the first stage.

5.4. High Speed Implementation of Interpolators and Decimators

We note that (5.3.10) and (5.3.18) are the basic equations for implementation for N-D interpolators and decimators, which can be implemented in N decoupled stages. Each such stage, in fact, consists of a set of 1-D interpolation or decimation filters. In this section the fast computational scheme associated with (5.3.10) and (5.3.18) that results in this highspeed implementation will be discussed. It will be shown that the set of 1-D interpolators as well as the decimators just referred to can be implemented by using certain types of 1-D interpolators or decimators known to be the polyphase filters [3] as our basic module of implementation. Only nonrecursive implementations of this basic module will be sought in the present study.

We first consider the i -th stage of implementation of the interpolator. Similar considerations apply to each such stage. The impulse responses $g_{m_i}(n_i)$ of the 1-D filters mentioned in the previous paragraph, by virtue of equations (5.3.8b), are periodically shift varying in m_i with a period L_i . Due to this, for any fixed set of values of $m_1, \dots, m_{i-1}, v_{i+1}, \dots, v_N$ the computation of $y^{(i)}(.)$ can be carried out via the use of L_i different shift-invariant filters in such a way that each filter provides every L_i -th sample of $y^{(i)}(.)$ in the i -th dimension. Consequently, for a given value of i as shown in Figure 5.3, the entire filtering operation represented by (5.3.10) can be implemented as a parallel interconnection of L_i different shift invariant filters having $y^{(i-1)}(.)$ at its input, followed by the process of sampling rate expansion by a factor of L_i i.e., insertion of (L_i-1)

zeros between two consecutive samples in the direction m_i . Finally, for each $j=1$ to (L_i-1) the output from j -th such shift invariant filter is shifted j space units, and the resulting signals are added to obtain $y^{(i)}(.)$.

Similar implementational considerations also apply to each of the N stages of the decimator. By making the substitution $k_i = r_i M_i + \rho_i$ in (5.3.18) it follows after some algebraic manipulations that (5.4.1) to (5.4.3) hold true.

$$\begin{aligned}
 y^{(i)} &= (m_1, \dots, m_i, v_{i+1}, \dots, v_N) \\
 &= \sum_{\rho_i=0}^{M_i-1} \sum_{r_i=-\infty}^{\infty} p_{\rho_i}(r_i) y_{\rho_i}^{(i-1)}(m_1, \dots, m_{i-1}, m_i - r_i, v_{i+1}, \dots, v_N)
 \end{aligned} \tag{5.4.1}$$

$$\begin{aligned}
 \text{where } y_{\rho_i}^{(i-1)}(m_1, \dots, m_{i-1}, m_i - r_i, v_{i+1}, \dots, v_N) \\
 = y^{(i-1)}(m_1, \dots, m_{i-1}, m_i M_i - \rho_i, v_{i+1}, \dots, v_N)
 \end{aligned} \tag{5.4.2}$$

$$\text{and } p_{\rho_i}(r_i) = h'(m_i M_i + \rho_i) \text{ for } \rho_i = 0, 1, \dots, (M_i-1) \tag{5.4.3}$$

Thus, for fixed integer values of $m_1, \dots, m_{i-1}, v_{i+1}, \dots, v_N$ (5.3.18) or equivalently (5.4.1) to (5.4.3) can be implemented as a parallel connection M_i 1-D filters, one for each value of ρ_i , and having impulse responses as given in (5.4.3). Note that the input $y^{(i-1)}(.)$ to the ρ_i -th such filter is obtained from $y^{(i-1)}(.)$ by shifting the samples by an amount ρ_i and then by considering only every M_i -th sample. The resulting scheme for implementation is shown in Figure 5.4.

It is important to notice that for interpolation as well as for decimation the computation of $y^{(i)}(.)$ in (5.3.10) and in (5.4.1) for different values of $m_1, \dots, m_{i-1},$

$v_{i+1}, \dots, v_N$ are completely independent of one another, and thus can be carried out concurrently. For each value of the set of $(N-1)$ tuples just mentioned it would thus be necessary to have an identical copy of the filter described in Figures 5.3 or 5.4 the required number of such copies is obviously determined by the size of the

support of the signal $y^{(i-1)}(.)$.

Furthermore, in spite of the fact that as shown in Figure 5.1 and 5.2 the entire scheme is to be implemented in N decoupled stages, a thorough examination of the computational scheme reveals that it is at least in principle, possible to begin partial computation of

intermediate signals $y^{(i+1)}(.)$, $y^{(i+2)}(.)$, ... etc. even before the computation of $y^{(i)}(.)$ is completed, thus potentially resulting in further speedup in arithmetic. To exemplify this situation in the 2-D case, it may be noticed that since the copies of row interpolators all operate in parallel - each on one row of the input image - if each row interpolator is made to sequentially process the rows from the same edge of the picture frame then as computation proceeds, the columns of the intermediate signal $y^{(i)}(.)$ begin to make themselves available to the column interpolator of the succeeding stage even before the computation of $y^{(i)}(.)$ is completed.

In what follows the implementation of the 1-D shift invariant filters $g_m(n_i)$ for the interpolator or the $p_p(r_i)$ for the decimators will be sought in direct form nonrecursive (FIR) structures. Other structures can be also used, but our choice is motivated by the particular nature of the interpolation and decimation filters, which

makes the design of FIR structure especially simple as well as by the fact that such structures can be conveniently implemented in systolic (pipeline) architecture [2] [10]. Figure 5.5 shows an example of such an 1-D nonrecursive structure, which occurs in each branch of the polyphase filters for the case when the filter order is equal to N_0 . Since the design of such 1-D filters is extensively documented [4] in the literature we shall not undertake the discussion of this issue here, but only highlight the fact that by multidimensional sampling rate alterations can be performed at a higher speed by exploiting in parallel several copies of the type of 1-D filter structures discussed in existing signal processing literature. Reduced computation time may thus be achieved at the expense of increased amount of hardware requirement. Furthermore, it may be noticed that for the implementation schemes under consideration the entire interpolator or decimator have a high degree of modularity.

5.5. Design examples in two-dimensions.

It follows from equations (5.3.8b) and (5.4.3) that the impulse responses $g_{m_i}(n_i)$ and $p_{p_i}(n_i)$ which constitute the respective branches of the 1-D polyphase filter structures for the interpolator and the decimator are space shifted versions of impulse response of an ideal low-pass filter. In the following design examples the window method of designing FIR filters was used for realization of the impulse responses just mentioned. The Kaiser window was used for each of the following examples, in which δ =ripple in the passband and stopband from ideal response; ω_c =highest frequency of interest in the input signal; ω_p =passband edge frequency; ω_s =stopband edge frequency; and N_0 = required filter order. The details of design can be found in [14].

Interpolator: $L=2$.

The following choices are made $\delta=0.1$, $\omega_c=.65\pi$, $\omega_p=\omega_c/2L$, $\omega_s=\pi/L$. Then $N_0=5$. The resulting polyphase filter was used in each of the two stages of implementation. The performance of the interpolator was tested for three different test signals: (S1) $x(n_1, n_2)=\sin(r)/r$, where $r=\sqrt{u^2+v^2}$ with $u, v=0, \pm 1, \pm 2, \dots, \pm 25$ as shown in Figure 5.6a; (S2) the "Jet" image of size 64 x 64 pixels, with 4 bits/pixel resolution as shown in Figure 5.6b; and (S3) the "Saturn" image of size 64 x 64 pixels, with resolution 4 bits/pixel as shown in Figure 5.6c. The interpolated signals are shown in Figures 5.7a, 5.7b and 5.7c respectively.

$L=3$. Again $\delta=0.1$, $\omega_c=.65\pi$, $\omega_p=\omega_c/2L$, $\omega_s=\pi/0.8L$, $N_0=5$, and Figures 5.8a, 5.8b, and 5.8c show the result of the interpolation scheme corresponding to input signals S1, S2 and S3 respectively.

Decimation: $M=2$.

The following parameters were chosen for each branch of the polyphase filters of the basic module for each stage. $\delta=0.7$, $\omega_c=.65\pi$, $\omega_p=\omega_c/1.3M$, $\omega_s=\pi/M$. Consequently $N_0=9$. The composite filtering scheme was tested on the following set of test signals. (T1) $x(n_1, n_2) = \sin(r)/r$, where $r = \sqrt{u^2 + v^2}$, $u, v = 0, \pm 1, \pm 2, \dots, \pm 75$ as shown in Figure 5.9a; (T2) the 128x128 pixel, 4 bits/pixel "Jet" data as displayed in Figure 5.9b and (T3) the 128x128 pixel, 4 bits/pixel "Saturn" data as displayed in Figure 5.9c. The respective decimated signals are shown in Figures 5.10a, 5.10b, and 5.10c.

$M=3$. Here $\delta=0.1$, $\omega_c=.65\pi$, $\omega_p=\omega_c/2L$, $\omega_s=\pi/.8L$. Consequently $N_0=5$. The input signals are: (U1) same as in (T1) above; (U2) the 192x192 pixel, 4 bits/pixel "Jet" data of Figure 5.8b and (U3) the 192x192 pixel, 4 bits/pixel "Saturn" data of Figure 5.8c. The corresponding decimated signals are as shown in Figures 5.10a, 5.10b and 5.10c respectively.

All programs were written in a DEC10 computer in sequential mode as opposed to parallel/ pipeline modes suggested in the present study.

5.6. Conclusion

Motivated by practical applications, a highspeed computing scheme for the interpolation and decimation of a broad class of multidimensional signals, which can potentially derive advantages from parallel, pipelined and modular implementation has been proposed. Only interpolation or decimation by integer factors are discussed, but sampling rate alterations by non-integer rational factors can also be performed via the techniques discussed by cascading in two successive stages interpolators and decimators of the type discussed here. The computational scheme is based on the frequency domain description of the signal, and uses one-dimensional interpolation or decimation filters as a basic module for implementation. Only nonrecursive implementation of this basic module has been considered in the present study. Apart from the ease of design and convenience of pipelineability, the choice of such structure may have advantages in applications such as image processing, where linear phase is a highly desirable characteristic of the processing scheme. The signal has been assumed to be sampled according to the rectangular cartesian sampling scheme. Other sampling geometries as multidimensional ($N > 2$) generalizations of the hexagonal sampling scheme [15] prove to be more economical in terms of the number of samples per unit volume required to represent a bandlimited signal. Interpolation and decimation schemes for multidimensional signals sampled in such geometries and their implementation in currently emerging highspeed architectures remain, however, to be investigated.

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CHAPTER 6

CONCLUSIONS

Simple and rigorous proofs of results on tests for the property that a multivariable polynomial be devoid of zeros in the closed unit polydisc are given. The proof technique rests on a complete formulation of the fact that the zeros of a polynomial are continuous functions of its coefficients. It is shown that all other stability related results can be derived in this manner.

New classes of multivariable polynomials arising in studies of passive multidimensional systems have been identified and their properties have been studied. In particular, polynomials occurring as the numerators and denominators of multivariable reactance functions and positive functions are characterized. Related properties of these and other classes of multivariable Hurwitz polynomials are also studied. A nontrivial test for the property of positivity of rational functions, holomorphic in a domain, in terms of their behavior on the distinguished boundary is formulated.

The problem of structurally passive synthesis of multidimensional digital filters as a cascade interconnection of more elementary building blocks has been addressed via the factorization of the associated discrete lossless two-port transmission matrix. Necessary and sufficient conditions for the factorization to be feasible are obtained. In particular, it is shown that in one-dimension the factorization can always be performed, and as a consequence, known filter structures fall out as special cases of the results developed. Thus, an alternative algorithm for synthesizing one dimensional structurally passive digital filters is also obtained.

The problem of sampling rate alteration of deterministic multidimensional signals is addressed on the basis of frequency domain description of the signal. It is shown that the problem can be formulated in such a way that solutions can be obtained via filtering techniques known for one-dimensional signals. Fast non-recursive implementations of such interpolation and decimation schemes are investigated. The resulting algorithms can be potentially be implemented in a combination of parallel and pipelined architecture. Experiments with digitized images are also reported to demonstrate the performance of the designed interpolation and decimation schemes.

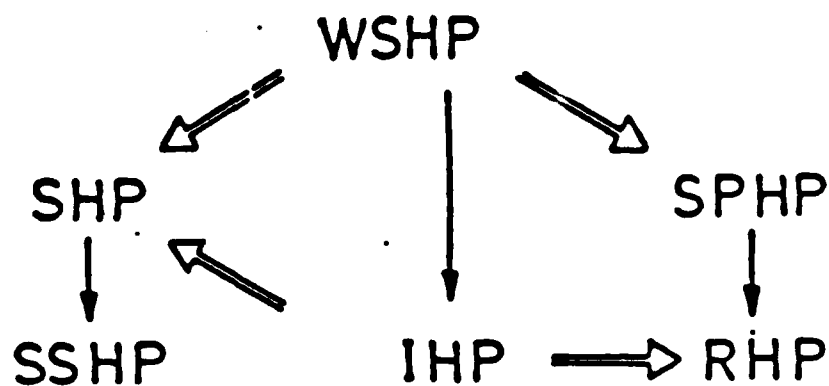
The fundamental results developed in the present report open up ways of investigation into a large number of problems of both theoretical and practical importance in the area of multidimensional signal processing. Efficient test procedures for the various classes of polynomials identified in chapter 3, namely the scattering Hurwitz, reactance Hurwitz and the immittance Hurwitz polynomials etc. are lacking and needs to be developed. Since polynomials of this type, particularly the scattering Hurwitz polynomials, enter into the description of passive systems in a fundamental manner, this should prove to be an important step in designing various types of multidimensional filters. In this context, a detailed study into the properties of discrete domain counterparts of the various multidimensional polynomials discussed in chapter 3 and their testing procedures also remain to be carried out. In the area of synthesis of structurally passive multidimensional digital filter design the problem has been addressed only in the context of synthesis in cascade type structure via the factorization of the transmission matrix associated with a multidimensional lossless two-port. Other possibilities of investigating synthesizability of lossless multidimensional two-ports in

structures other than the cascade structure also exist, e.g., via the factorization of the hybrid matrix or the transfer function matrix itself. The need for this investigation is strongly felt in view of the result established in chapter 4 that in a generic situation multidimensional lossless two-ports may not be synthesizable as cascade interconnection of most elementary passive building blocks. Moreover, exactly how these synthesis schemes can be utilized in special cases of practical interest when the frequency response of the filter is required to have certain symmetries, for example, spherical symmetry (image processing applications) or planar or conical symmetry (direction finding applications) remains to be investigated. Attention has only been restricted to the quarter plane type recursive schemes so far. Other recursive schemes such as the symmetric or the asymmetric half plane or the fully recursive half plane recursive schemes and multidimensional generalizations thereof also needs to be considered.

The close relationship between passive filtering and modelling of stationary or nearly stationary stochastic processes is well known for one-dimensional signals. The results of the present investigation can thus be potentially utilized towards resolving problems in the domain of modelling of random fields e.g., in (spectral) estimation, and prediction problems associated with multidimensional signals. Multidimensional extensions of various time/space varying adaptive filtering schemes can also prove to be an important topic of future research in this context.

Since generic multidimensional signal processing tasks are severely computation intensive the feasibility of implementation of the algorithms resulting from the above studies in high-speed architectures is also an important area of investigation. For example, in the specific problem of fast image interpolation and decimation dealt with in chapter 5

details of issues relating to implementation in systolic VLSI and/or optical architectures remain to be studied and can form a topic of future research.



WSHP = widest-sense Hurwitz polynomial

SHP = scattering-Hurwitz polynomial

SPHP = self-paraconjugate Hurwitz polynomial

SSHP = strict-sense Hurwitz polynomial

RHP = reactance Hurwitz polynomial

IHP = immittance Hurwitz polynomial

Figure 3.1

Hierarchical relationship between various classes of multidimensional stable polynomials.

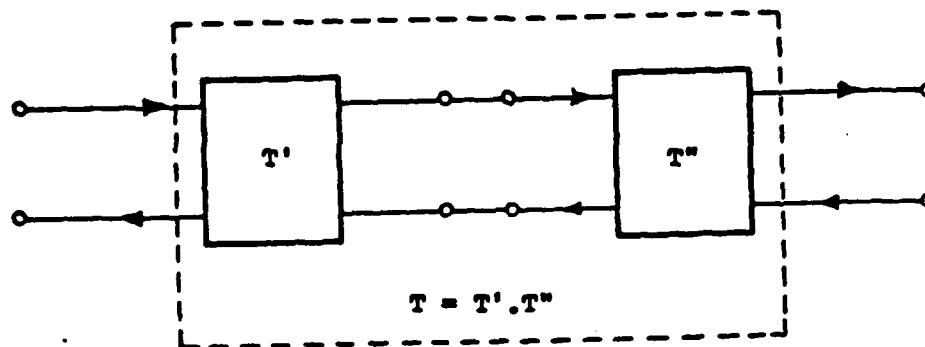


Figure 4.1
Cascade decomposition of a multidimensional
lossless two-port

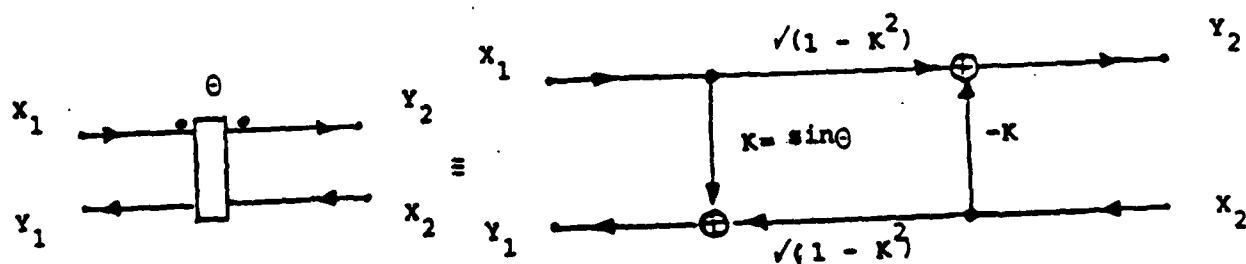


Figure 4.2
First kind of Gray-Markel section

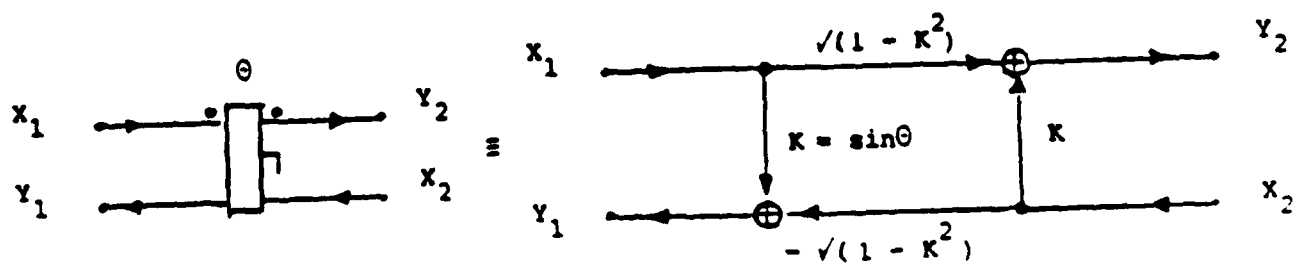


Figure 4.3
Second kind of Gray-Markel section

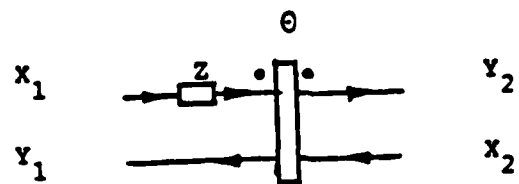


Figure 4.4
Elementary section of type 1.

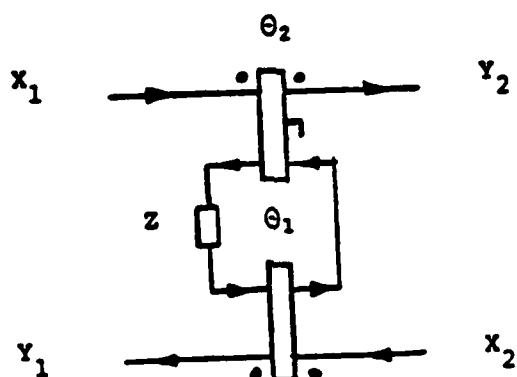


Figure 4.5
Elementary section of type 2.

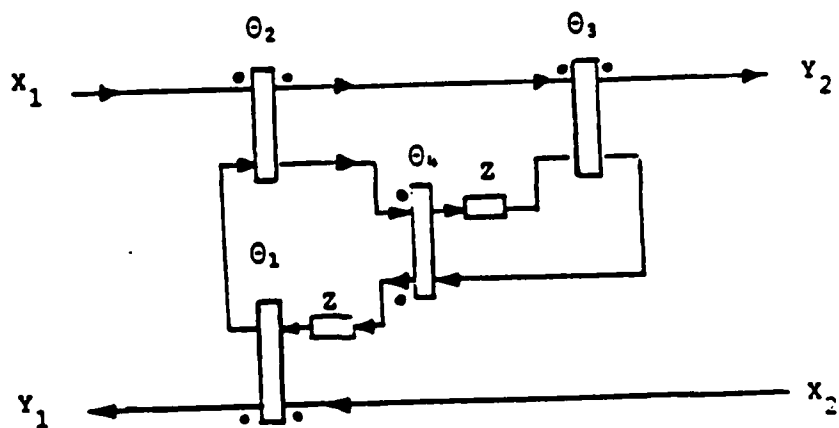


Figure 4.6
Elementary section of type 3.

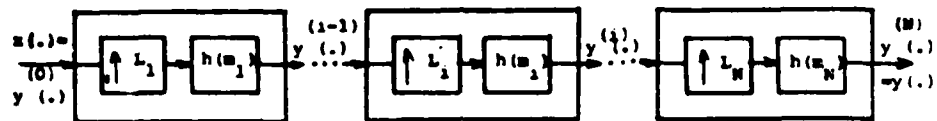


Figure 5.1 Block diagram of N-D interpolation

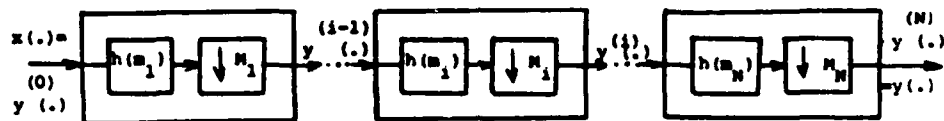


Figure 5.2 Block diagram of N-D decimator

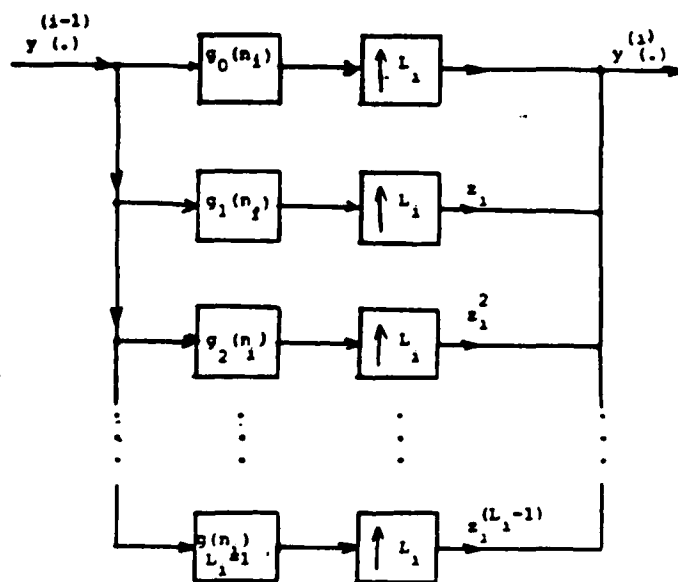


Figure 5.3 i-th Stage of interpolator and its polyphase structure.

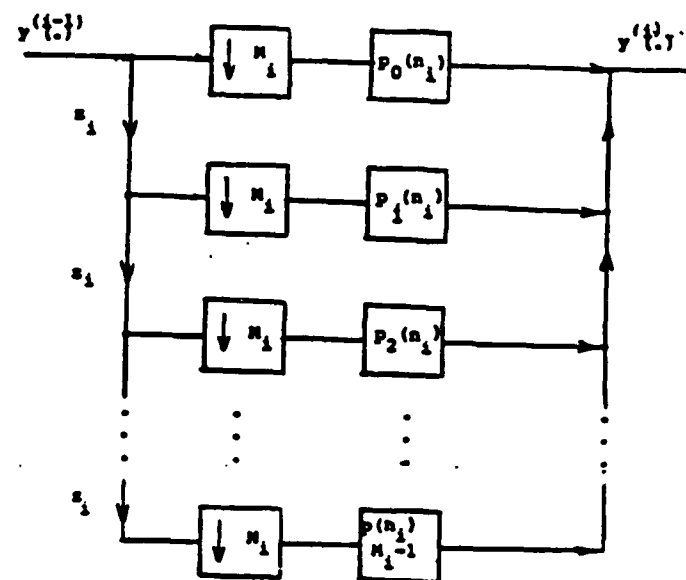


Figure 5.4 i -th Stage of decimator and its polyphase structure.

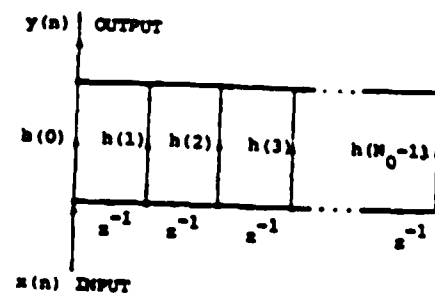
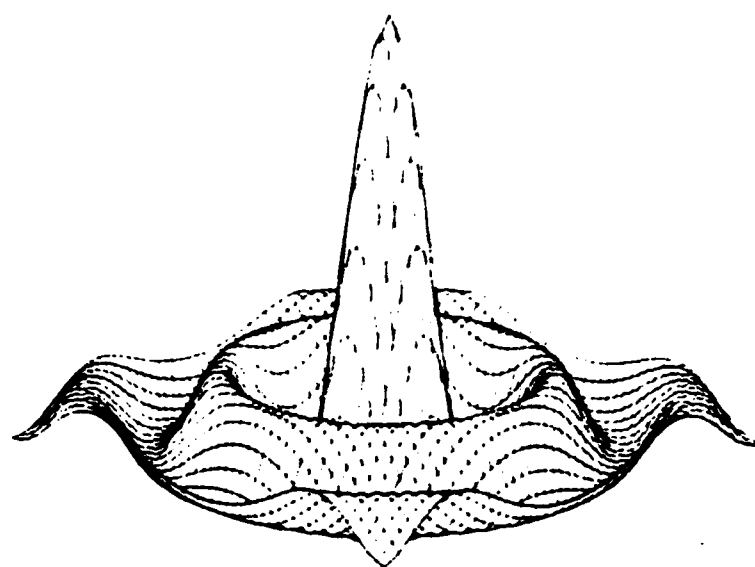
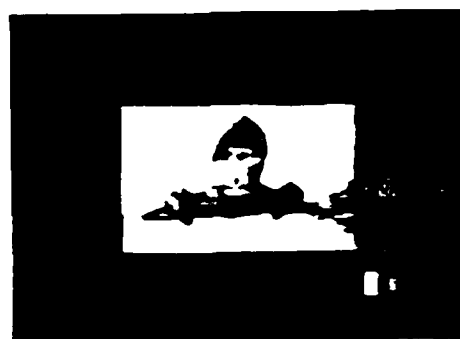


Figure 5.5 Typical implementation corresponding to $g_{m_i}(n_i)$ or $p_0(r_i)$.



(S1)

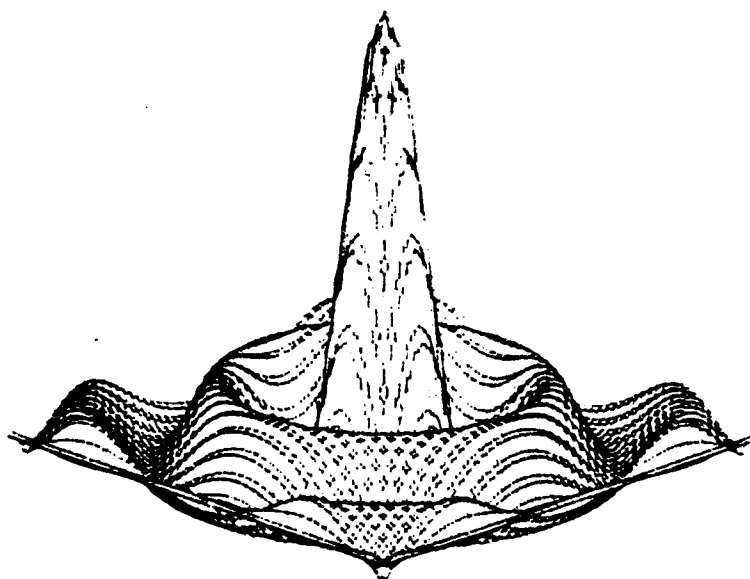


(S2)

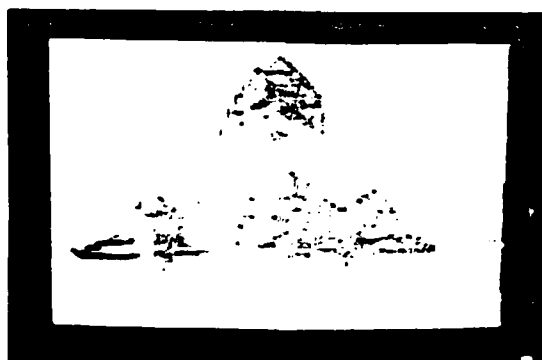


(S3)

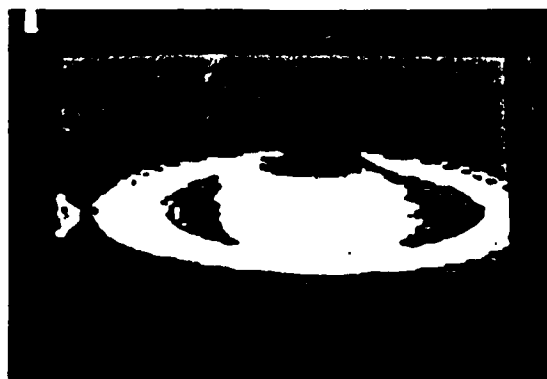
Figure 5.6 Illustration of test input signals (S1), (S2), (S3).



(T1)

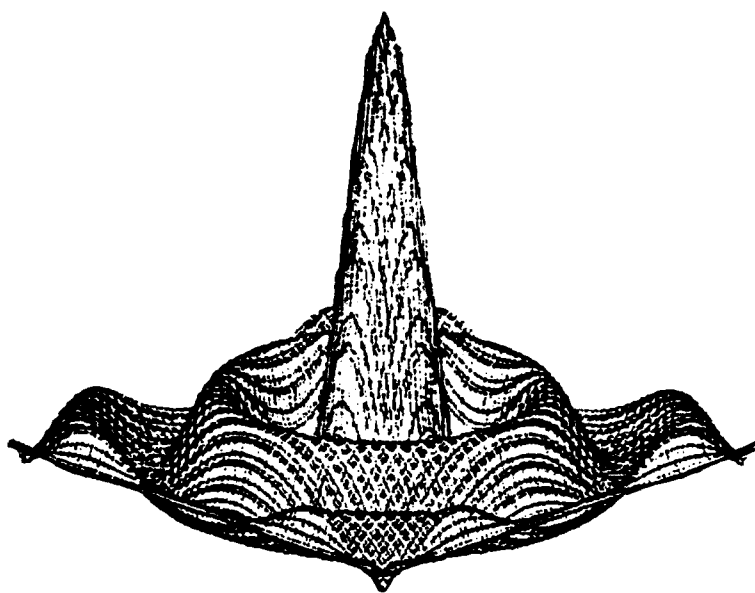


(T2)



(T3)

Figure 5.7 Interpolated (S1), (S2), (S3). $L=2$.



(U1)



(U2)



(U3)

Figure 5.8 Interpolated (S1), (S2), (S3). $L=3$

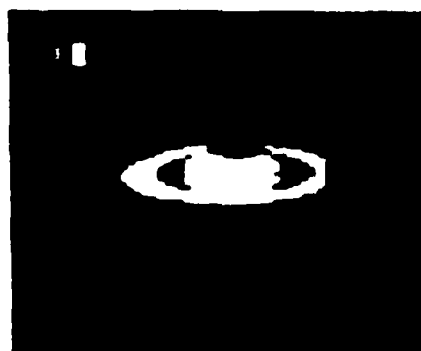
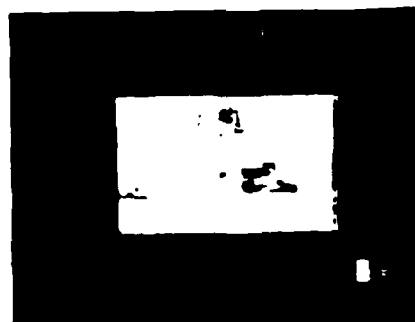
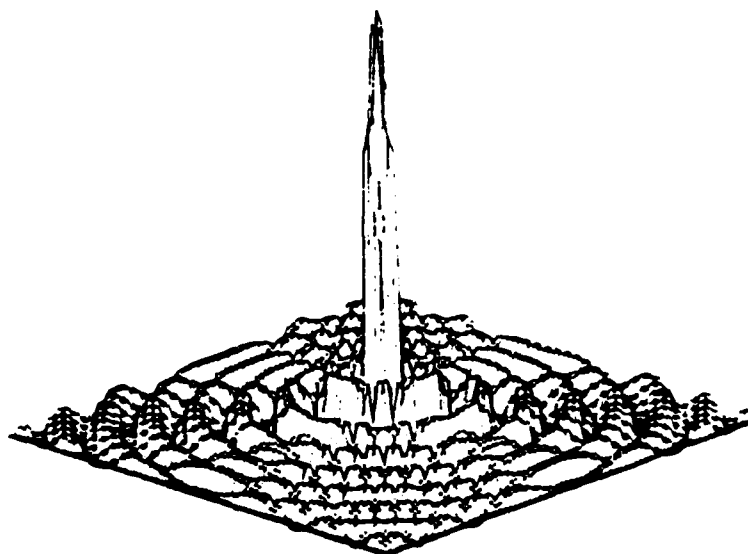


Figure 5.9 Decimated (T1), (T2), (T3). $M=2$.

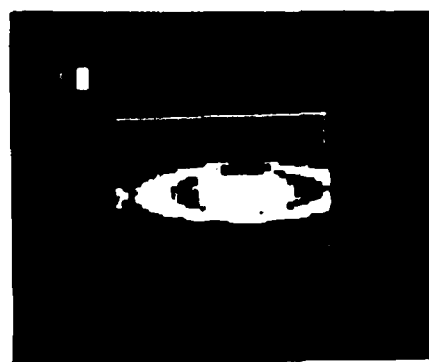
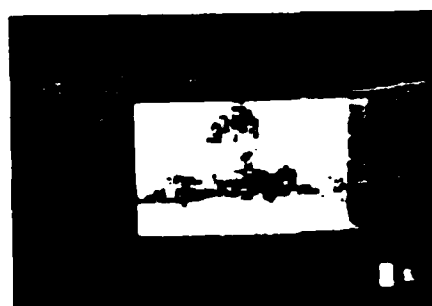
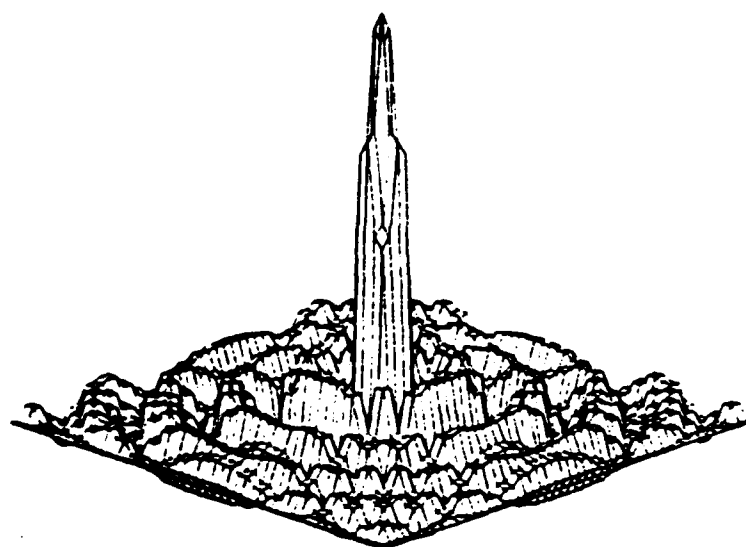
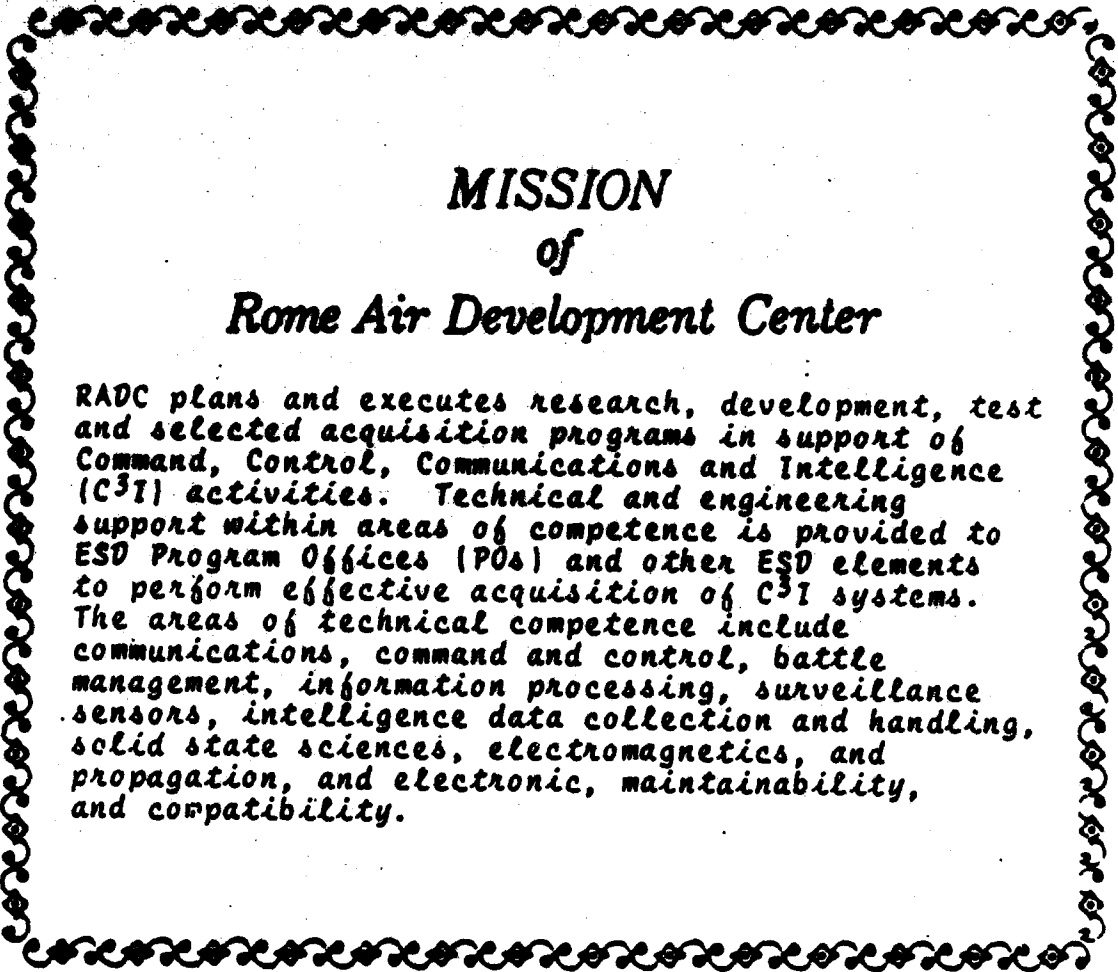


Figure 5.10 Decimated (U1), (U2), (U3). $M=3$.



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5-87

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