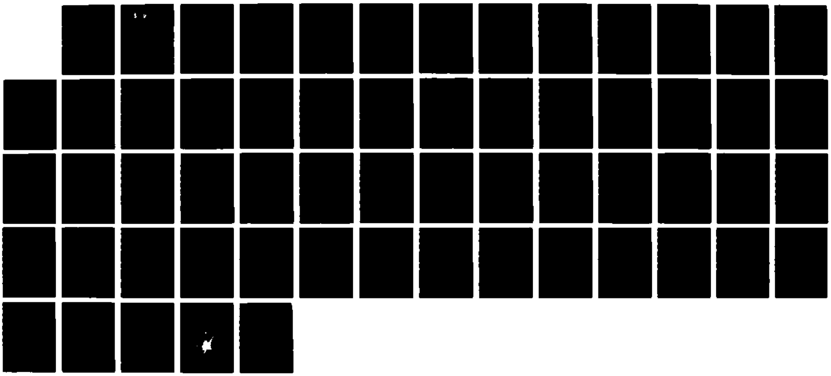
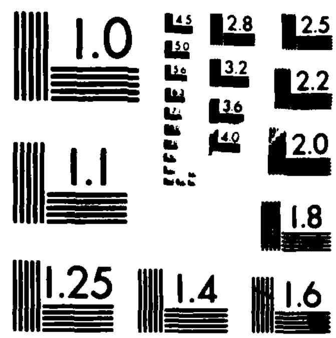
AD-A179 989 SOME PROBLEMS IN THE CONTROL OF LARGE SPACE STRUCTURES (U) VIRGINIA POLYTECHNIC INST AND STATE UNIV BLACKSBURG DEPT OF E.. L MEIROVITCH 16 DEC 86 AFOSR-TR-87-0426 1/1

UNCLASSIFIED AFOSR-83-0017 F/G 12/9 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

DTIC FILE COPY (2)

SECURITY CLASSIFICATION OF THIS PAGE

AD-A179 989

S E L E C T E APR 2 8 1987 D

1a. REPORT SECURITY CLASSIFICATION
Unclassified

2a. SECURITY CLASSIFICATION AUTHORITY

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE

...INGS

...LABILITY OF REPORT
...ublic release. Distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)

5. MONITORING ORGANIZATION REPORT NUMBER(S)
AFOSR-TR- 87-0426

6a. NAME OF PERFORMING ORGANIZATION
Virginia Polytechnic Institute and State University

6b. OFFICE SYMBOL (If applicable)

7a. NAME OF MONITORING ORGANIZATION
AFOSR/NA

6c. ADDRESS (City, State and ZIP Code)
Department of Engineering Science & Mechanics
Blacksburg, VA 24061

7b. ADDRESS (City, State and ZIP Code)
Bolling ABF
Washington, DC 20332

8a. NAME OF FUNDING/SPONSORING ORGANIZATION
AFOSR

8b. OFFICE SYMBOL (If applicable)
NA

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR 83-0017

8c. ADDRESS (City, State and ZIP Code)
Bolling AFB
Washington, DC 20332

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2302	B1	

11. TITLE (Include Security Classification) (U) Some Problems in the Control of Large Space Structures

12. PERSONAL AUTHOR(S)
Leonard Meirovitch

13a. TYPE OF REPORT
Final Report

13b. TIME COVERED
FROM ______ TO ______

14. DATE OF REPORT (Yr., Mo., Day)
1986, December 16

15. PAGE COUNT
3+2 Appendixes

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
Control of traveling waves, direct feedback control, pole placement

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Work during this period has been concerned with: 1) control of traveling waves in structures and 2) further developments in the control of distributed structures. It has been demonstrated that, in modal control of traveling waves, the control forces tend to concentrate in the immediate vicinity of disturbances and tend to vanish at points removed from disturbances. In direct feedback control of distributed structures, if one insists on placing the poles associated with the controlled modes, the possibility of destabilizing the uncontrolled modes exists.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION
Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL
Dr. Anthony K. Amos

22b. TELEPHONE NUMBER (Include Area Code)
202 767-4937

22c. OFFICE SYMBOL
NA

DD FORM 1473, 83 APR

EDITION OF 1 JAN 73 IS OBSOLETE.

SECURITY CLASSIFICATION OF THIS PAGE

AFOSR-TR- 87-0426

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY

DEPARTMENT OF ENGINEERING SCIENCE AND MECHANICS

Final Scientific Report on

SOME PROBLEMS IN THE CONTROL OF LARGE SPACE STRUCTURES

Covering the Period

1 January 1986 - 30 June 1986

on the

Research Grant AFOSR-83-0017

December 16, 1986

Principal Investigator: Leonard Meirovitch

University Distinguished Professor

Abstract

Work during this period has been concerned with two aspects: 1) control of traveling waves in structures and 2) further developments in the control of distributed structures.

In modal control of traveling waves, the question can be raised whether actuator forces at points removed from a given disturbance can begin working before the arrival of the disturbance. This question is prompted by the fact that modal forces begin acting at $t = 0$. However, the modal forces are not the actual forces, although the actual actuator forces are linear combinations of the modal forces. It is demonstrated that these combinations are such that the control forces tend to concentrate in the immediate vicinity of the disturbance and tend to vanish at points removed from the disturbance (Ref. 1).

One problem in the control of distributed structures is that control implementation must be carried out by discrete actuators. In using direct feedback, whereby the sensors and actuators are collocated and the actuator input depends only on the sensor output at the same location, asymptotic stability can be virtually guaranteed. Problems arise when one desires to place the closed-loop poles. It appears that there is some incompatibility between direct feedback and pole placement. In particular, in placing the poles for a number of controlled modes, the possibility of destabilizing uncontrolled modes exists (Ref. 2). This problem arises from the insistence on placing the poles associated with the controlled modes and would not arise in direct feedback alone, i.e., without specifying the location of the poles in advance.

References (Enclosed)

1. Meirovitch, L. and Bennighof, J. K., "Modal Control of Traveling Waves in Flexible Structures," Journal of Sound in Vibration, 1986.

2. Meirovitch, L., "Some Problems Associated with the Control of Distributed Structures," Journal of Optimization Theory and Applications (to appear).

CONTROL OF TRAVELING WAVES IN FLEXIBLE STRUCTURES*

by

L. Meirovitch and J. K. Bennighof

Department of Engineering Science and Mechanics

Virginia Polytechnic Institute and State University

Blacksburg, VA 24061

SUMMARY

This paper is concerned with the control of a traveling wave in a structure by the independent modal-space control method. It is demonstrated that the control forces tend to concentrate in the immediate vicinity of the disturbance, and there are virtually no control forces acting at any point of the structure before the arrival of the disturbance. Two numerical examples are included, one for a string in transverse vibration and one for a beam in bending. Satisfactory control was achieved in spite of the fact that only a finite number of modes was retained for control.

1. INTRODUCTION

Modal control implies controlling the motion of a flexible structure by controlling its modes. To carry out the control task, it is necessary first to derive the modal equations of motion, design the

* Supported in part by the AFOSR Research Grant 83-0017.

modal control forces and finally synthesize the actuator forces from the modal control forces by means of a linear transformation [1].

Because the modes of vibration of a structure form a complete set, a disturbance in the structure can be described to any degree of accuracy by a linear combination of these modes by merely increasing the number of terms [2]. Modal control amounts to determining and implementing modal control forces designed to suppress the modes excited. But, the modes of a structure are global functions, i.e., they are defined over the entire domain of the structure. Moreover, any given modal control force is only an abstract force, translating into an actual force distributed over the entire domain and having the shape of the mode in question multiplied by the mass distribution. However, distributed control implies an infinite-dimensional controller, so that practical considerations dictate implementation of modal control by a finite-dimensional controller. This, in turn, implies controlling a finite number of modes only, raising questions on the effect of modal truncation on the performance of modal control.

In structures likely to exhibit disturbances in the form of traveling waves, which tend to be localized in nature, the question can be raised as to the suitability of representing local disturbances by a finite number of modes, and more importantly of controlling such disturbances by a finite number of modal control forces. In particular, if the control is to be implemented by a finite number of actuators located throughout the structure, the question can be raised whether these actuators will start working in certain parts of the structure before the disturbance has arrived yet. On the other hand, one can conceive of the situation in which the actuator forces, which represent linear combinations of all the modal control forces, combine in such a way that they become significant only in the neighborhood of the disturbance and tend to reduce to zero in areas removed from the disturbance.

This paper is concerned with the control of a disturbance in the form of a traveling wave by the independent modal-space control (IMSC) method. It is shown that the only force actuators activated are those in the immediate vicinity of the disturbance, and that there are virtually no control forces acting at any point of the structure before the arrival of the disturbance. Two numerical examples are included, one for a string in transverse vibration and one for a beam in bending. Satisfactory control was achieved in each case in spite of the fact that only a relatively small number of modes were retained for control.

2. INDEPENDENT MODAL-SPACE CONTROL

Consider a distributed parameter system whose behavior is governed by the partial differential equation of motion [2]

$$L u(P,t) + m(P)\frac{\partial^2 u(P,t)}{\partial t^2} = f(P,t) \qquad (1)$$

subject to the boundary conditions $B_i u(P,t) = 0$, $i = 1,2,\ldots,p$. Here, L is a linear, self-adjoint differential operator of order $2p$, $u(P,t)$ is the displacement, a function of the position P and time t, $m(P)$ is the distributed mass and $f(P,t)$ is the distributed force. The B_i's are also linear differential operators. The solution of the associated eigenvalue problem consists of a denumerably infinite set of eigenvalues Λ_r and the corresponding eigenfunctions $\phi_r(P)$ $(r = 1,2,\ldots)$. The eigenvalues are the squares of the natural frequencies ω_r of the system, $\Lambda_r = \omega_r^2$, and the eigenfunctions are orthogonal and can be normalized so as to satisfy $\int_D m(P)\phi_r(P)\phi_s(P)dD = \delta_{rs}$, $\int_D \phi_r(P)L\phi_s(P)dD = \Lambda_r\delta_{rs} = \omega_r^2\delta_{rs}$ where δ_{rs} is the Kronecker delta.

By the expansion theorem [2], the displacement of the structure can be expressed in terms of its modes by

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P)u_r(t) \tag{2}$$

where $u_r(t)$ are the modal displacements. Using the standard approach, we obtain the decoupled modal ordinary differential equations of motion

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t), \quad r = 1,2,\ldots \tag{3}$$

where

$$f_r(t) = \int_D \phi_r(P)f(P,t)dD, \quad r = 1,2,\ldots \tag{4}$$

are the modal forces. In the independent modal-space control method (IMSC), each modal force depends only on the corresponding modal displacement and velocity [1]. Hence, for linear feedback,

$$f_r(t) = f_r[u_r(t),\dot{u}_r(t)] = -\,g_r u_r(t) - h_r\dot{u}_r(t) \tag{5}$$

where g_r and h_r are modal control gains.

Implementation of modal control without spillover requires a distributed control having the expression

$$f(P,t) = \sum_{r=1}^{\infty} m(P)\,\phi_r(P)\,f_r(t) \tag{6}$$

If the control is to be carried out by means of m discrete actuators, the distributed control force can be written as

$$f(P,t) = \sum_{j=1}^{m} F_j(t)\delta(P - P_j) \tag{7}$$

and, from Eq. (4), each modal force is given by

$$f_r(t) = \int_D \phi_r(P) \sum_{j=1}^{m} F_j(t)\delta(P - P_j)dD = \sum_{j=1}^{m} F_j(t)\phi_r(P_j) \tag{8}$$

Letting $\underset{\sim}{f}$ be the vector of modal forces and $\underset{\sim}{F}$ the vector of actual forces, we can write

$$\underset{\sim}{f} = B\underset{\sim}{F} \tag{9}$$

where the matrix $B = [B_{rj}] = [\phi_r(P_j)]$ is known as the modal participation matrix. We can obtain the vector of actual forces from the vector of modal forces by writing

$$\underset{\sim}{F} = B^{+}\underset{\sim}{f} \tag{10}$$

where B^{+} is the pseudo-inverse of B. If there are as many discrete actuators as controlled modes, then B is a square matrix. Then, assuming that B is nonsingular, Eq. (10) reduces to

$$\underset{\sim}{F} = B^{-1}\underset{\sim}{f} \tag{11}$$

To generate the modal forces, we need the modal displacements and velocities. We can extract them from the displacement and velocity profiles using the expansion theorem

$$\begin{aligned} u_r(t) &= \int_D M(P)\phi_r(P)u(P,t)dD \\ \dot{u}_r(t) &= \int_D M(P)\phi_r(P)\dot{u}(P,t)dD \end{aligned} \tag{12}$$

If we use n discrete sensors, then we can interpolate between the sensor measurements to obtain the approximations $\hat{u}(P,t)$ and $\hat{\dot{u}}(P,t)$. Then, we compute $\hat{u}_r(t)$ and $\hat{\dot{u}}_r(t)$ by inserting $\hat{u}(P,t)$ and $\hat{\dot{u}}(P,t)$ in Eqs. (12). Alternatively, we note that at the sensor locations P_i,

$$u(P_i,t) = \sum_{r=1}^{\infty} \phi_r(P_i)u_r(t) = \underset{\sim}{\phi}^T(P_i)\underset{\sim}{u}(t) \quad i = 1,2,\dots,n \tag{13}$$

where $\underset{\sim}{\phi}^T(P_i)$ is the infinite-dimensional vector of eigenfunctions evaluated at $P = P_i$ and $\underset{\sim}{u}(t)$ is the infinite-dimensional modal vector. Introducing the measurement vector $\underset{\sim}{y}(t)$ with components $y_i(t) = u(P_i, t)$, we can rewrite Eqs. (13) as

$$\underset{\sim}{y}(t) = \Phi^T\underset{\sim}{u}(t) \tag{14}$$

Then, if we truncate the modal vector $\underset{\sim}{u}(t)$ so that its dimension is equal to the number of sensors, we can estimate the modal displacement vector from

$$\hat{\underset{\sim}{u}}(t) = (B_s^T)^{-1}\underset{\sim}{y}(t) \tag{15a}$$

where B_s is a square truncated matrix Φ and it represents the sensor participation matrix. This is equivalent to using the lowest n modes to represent the displacement profile. Similarly, the estimated modal velocity vector is

$$\hat{\dot{\underset{\sim}{u}}}(t) = (B_s^T)^{-1}\dot{\underset{\sim}{y}}(t) \tag{15b}$$

Finally, the modal equations of motion become

$$\ddot{u}_r + \omega_r^2 u_r = f_r = -g_r\hat{u}_r - h_r\hat{\dot{u}}_r \tag{16}$$

In this paper, we use gains that minimize the performance functional

$$J = \int_0^{\infty}\int_D [m(P)\dot{u}^2(P,t) + u(P,t)Lu(P,t) + Rf^2(P,t)]dD\,dt \tag{17}$$

Using the expansion theorem, the minimization can be carried out for each mode independently resulting in the gains [1]

$$g_r = -\omega_r^2 + \omega_r(\omega_r^2 + \frac{1}{R})^{1/2}$$

$$h_r = [-2\omega_r^2 + \frac{1}{R} + 2\omega_r(\omega_r^2 + \frac{1}{R})^{1/2}]^{1/2} \tag{18}$$

If we have damping in the system, the partial differential equation of motion becomes

$$Lu(P,t) + C\frac{\partial u(P,t)}{\partial t} + m(P)\frac{\partial^2 u(P,t)}{\partial t^2} = f(P,t) \tag{19}$$

where C is a differential operator. If, for some constants α_1 and α_2, we have

$$C = \alpha_1 L + \alpha_2 m(P) \tag{20}$$

then the modal equations of motion become

$$\ddot{u}_r + (\alpha_1\omega_r^2 + \alpha_2)\dot{u}_r + \omega_r^2 u_r = f_r \tag{21}$$

so that the equations remain uncoupled. This special case of damping is known as proportional damping.

3. MODAL CONTROL OF TRAVELING WAVES

In this paper, we examine the possibility of using IMSC to control traveling waves in flexible structures. In each case, we begin with initial conditions describing a single, localized traveling disturbance in the structure. Because the modes of a distributed system form a set that is complete in energy, any disturbance can be expressed as a linear combination of the modes, provided a sufficiently large number of modes is included.

We consider first the wave motion in a second-order system, such as a string in transverse vibration, a bar in axial vibration, or a shaft in torsional vibration. Then, we consider traveling waves in fourth-order systems, such as a beam in bending vibration. In the first case, the waves travel through the system without changing shape; in the second, the system is dispersive, so that the wave changes shape as it travels. For both types of systems, we assume that there is internal damping present in the system, and that this damping is proportional to the local rate of strain in the material. In each case, we consider first the globally optimal solution to the control problem obtained by using distributed actuators. Although implementation of control by means of distributed actuators may not be within the state of the art, the globally optimal solution is valuable because it provides a benchmark against which any other design can be measured. Then, we consider control of the wave motion using a finite number of discrete actuators, in which case only a limited number of the lower modes is controlled. A comparison of the results obtained using discrete actuators with the globally optimal solution demonstrates the effectiveness of IMSC in controlling waves, even when only a small number of discrete actuators is used.

i. Second-Order Systems

We consider a second-order system in the form of a string in transverse vibration. Assuming that the system is undamped, the free vibration is governed by the partial differential equation [Ref. 3].

$$-T\frac{\partial^2 u(x,t)}{\partial x^2} + m\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \qquad (22)$$

where $u(x,t)$ is the transverse displacement, T is the tension and m is the mass per unit length. Here the differential operator L is equal to

$- T\, \partial^2/\partial x^2$. It is assumed in Eq. (22) that both T and m are constant.

If the string is of infinite length, it is easy to show that the solution of Eq. (22) can be written in the form [3]

$$u(x,t) = F_1(x - \nu t) + F_2(x + \nu t) \tag{23}$$

where F_1 and F_2 are wave profiles traveling to the right and to the left, respectively, with the wave velocity

$$\nu = \sqrt{T/m} \tag{24}$$

The transverse velocity is

$$\dot{u}(x,t) = - \nu F_1'(x - \nu t) + \nu F_2'(x + \nu t) \tag{25}$$

where primes denote differentiation with respect to the corresponding arguments.

If the string is finite and fixed at both ends, then $u(x,t)$ must satisfy the *boundary conditions*

$$u(0,t) = u(L,t) = 0 \tag{26}$$

The natural frequencies are

$$\omega_r = r\pi\sqrt{T/mL^2}\ , \quad r = 1,2,... \tag{27}$$

and the associated normalized eigenfunctions are

$$\phi_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{r\pi x}{L}\ , \quad r = 1,2,... \tag{28}$$

According to the expansion theorem, Eq. (3), the displacement of the string can be represented by a linear combination of the eigenfunctions of the form (2). Alternatively, at any instant in time, the motion can be described by Eq. (23) in terms of traveling waves, as long as the boundary conditions are satisfied. These boundary conditions determine *how the wave is reflected at the ends of the string.*

Next, we add an external distributed force and distributed damping that is proportional to the local strain rate, so that the partial differential equation of motion becomes

$$- T \frac{\partial^2 u(x,t)}{\partial x^2} - C \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} + m \frac{\partial^2 u(x,t)}{\partial t^2} = f(x,t) \tag{29}$$

where C is assumed to be constant. Because damping is of the proportional type [2], the eigenfunctions of the damped system are the same as the eigenfunctions of the undamped system, although the eigenvalues are different. Hence, inserting Eq. (2) with P = x into Eq. (29), multiplying by $\phi_s(x)$, integrating over the length of the string and making use of the orthogonality relations, we obtain the independent ordinary differential equations of motion

$$\ddot{u}_r(t) + (C\omega_r^2/T)\dot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t), \quad r = 1,2,... \tag{30}$$

where $f_r(t) = \int_0^L \phi_r(x)f(x,t)\, dx$ is the rth modal force. Here we note that the damping factor is proportional to the natural frequency, $\zeta_r = (C/2T)\omega_r$ $(r = 1,2,...)$. Hence, we expect the higher modes to decay more rapidly than the lower modes, which is confirmed by the observed behavior.

If the string has the initial displacement profile

$$u(x,0) = \begin{cases} \frac{1}{2}\left(1 - \cos \frac{2\pi x}{\lambda}\right), & 0 \le x \le \lambda \\ 0, & \lambda \le x \le L \end{cases} \tag{31}$$

where λ is the wave length, then the initial modal displacements are

$$u_r(0) = \int_0^L m u(x,0)\phi_r(x)\, dx = \frac{2\sqrt{2mL}}{r\pi}\left[\frac{1 - \cos r\pi\lambda/L}{4 - r^2\lambda^2/L^2}\right] \tag{32}$$

If this initial disturbance is to travel to the right, then the transverse velocity must be

$$\dot{u}(x,0) = \begin{cases} -\sqrt{\frac{T}{m}} \sin \frac{2\pi x}{\lambda}, & 0 \le x \le \lambda \\ 0, & \lambda \le x \le L \end{cases} \tag{33}$$

and the modal velocities are

$$\dot{u}_r(0) = \int_0^L m\dot{u}(x,0)\phi_r(x)\, dx = 2\sqrt{2TL}\left[\frac{\sin r\pi\lambda/L}{4 - r^2\lambda^2/L^2}\right] \tag{34}$$

Figure 1 shows the motion of the string with the above initial conditions and with $\lambda = 0.1L$. Here, eighty modes were used to model the string. The value of C was chosen so as to give 0.1% damping in the fundamental mode, and no control forces were applied. The effect of the damping is to decrease the energy in the highest modes rapidly, so that the disturbance profile loses its initial sharpness and its amplitude decreases.

Next, we consider a distributed control force with a control effort weighting factor of R = 0.2 in the performance index, Eq. (17). The results are shown in Fig. 2. We observe from Fig. 2 that the control is localized at the wave, although the control force is a linear combination of modal forces and each of the modal forces is distributed over the entire domain. This demonstrates that IMSC can control localized disturbances quite satisfactorily, because the control force tends to concentrate around the disturbance and it travels with the wave.

We also observe from Fig. 2 that the optimal control force is very nearly equal to a scalar multiple of the velocity, which is consistent with the fact that energy dissipation is the control objective. This control force causes the wave to essentially retain its shape as the amplitude decreases. It turns out that we can vary the rate of decay of the wave by varying R. In this example, we selected the value of R so as to be able to monitor the effect of the controls on the system as the wave travels. In general, R represents a penalty on the control and is chosen by the analyst so as to produce desired system performance.

Figure 3 shows results obtained by using nine discrete actuators and nineteen discrete sensors, all equally spaced, to control the lowest nine modes of the string. The sensors measure the actual displacement and velocity of the string at each sensor location. Then, these measurements are used in conjunction with Eqs. (14) and (15) to estimate the corresponding modal displacements and velocities. The use of more sensors than actuators allows much of the motion due to uncontrolled modes to be filtered out. The modal control forces are calculated from the estimated modal displacements and velocities using the gains prescribed by Eqs. (18) and the actual actuator forces are calculated using Eq. (11). In this example, we continue to model the lowest eighty modes, so that we expect to see residual energy in uncontrolled modes, observation spillover from uncontrolled modes and control spillover into uncontrolled modes, at least to some degree. Here, we still consider the effects of damping as in the previous two cases. The use of discrete actuators causes the wave to lose its initial smooth shape with time, although the disturbance can still be identified as it travels. Examining the plot corresponding to $t = 0$, it is clear that, as long as the disturbance in the system is still localized, the control force accompanies the disturbance. Comparing the rate of energy dissipation with the damped but uncontrolled case of Fig. 1, we observe that controlling only the lowest nine modes increases the energy dissipation substantially. In the time increment between $t = 0$ and $t = 0.4$, damping causes a 50% loss of energy in the uncontrolled case, while the discrete actuator controls dissipate an additional 14% by operating on the lowest nine modes. As time progresses, the controls become essentially inactive, indicating that motion in the lowest nine modes has been annihilated. The strain rate damping then causes the remaining energy to decay quickly. Hence, we conclude that the use of IMSC to control only the lowest nine modes with discrete actuators and sensors is effective in controlling this traveling wave.

In Fig. 4, we have plots of the modal contributions $\underset{\sim}{b}_r f_r(t)$ to the actuator force vector $\underset{\sim}{F}(t)$ at $t = 0$, where the vectors $\underset{\sim}{b}_r$ are the columns of $B^{\dagger}$ in Eq. (10), or of B^{-1} in Eq. (11). In this case, B^{-1} was used. Also, in each of these plots, we have sketched the corresponding mode shape, to give an idea what the contributions would have been if the actuators were distributed devices, instead of point actuators. The last plot represents $\sum_{r=1}^{m} \underset{\sim}{b}_r f_r(t)$ at $t = 0$, which is recognized from Eq. (9) as the actual actuator force vector $\underset{\sim}{F}(t)$ at $t = 0$. This figure brings out the fact that, although the modal forces are active at points far away from the disturbance, these modal forces tend to cancel out at these points. Hence, the actual forces, as exerted by the actuators, tend to be concentrated in the vicinity of the disturbance.

ii. FOURTH-ORDER SYSTEMS

The motion of beams in undamped free vibration is governed by the fourth-order partial differential equation [3]

$$EI \frac{\partial^4 u(x,t)}{\partial x^4} + m \frac{\partial^2 u(x,t)}{\partial t^2} = 0 \tag{35}$$

where EI is the bending stiffness and m is the mass per unit length, both assumed to be constant. Here, $L = EI\ \partial^4/\partial x^4$. Equation (33) admits a solution in the form of the wave motion

$$u(x,t) = \cos \frac{2\pi}{\lambda} (x - \nu t) \tag{36}$$

where λ is the wavelength and

$$\nu = \frac{2\pi}{\lambda} \sqrt{EI/m} \tag{37}$$

is the wave velocity. Hence, if a given wave profile is resolved into sinusoidal components by Fourier analysis, each wave component will travel with a different velocity. It follows that the wave profile changes shape as it travels, so that the beam is dispersive [3].

If the beam is of length L with pinned ends, then the displacement must satisfy the boundary conditions

$$u(0,t) = u(L,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = 0 \tag{38}$$

The natural frequencies are

$$\omega_r = (r\pi)^2 \sqrt{\frac{EI}{mL^4}} \qquad r = 1,2,... \tag{39}$$

and the associated normalized eigenfunctions are the same as for the string, Eq. (28).

In the presence of distributed damping proportional to the local strain rate and a distributed control force, the partial differential equation of motion becomes [4]

$$EI \frac{\partial^4 u(x,t)}{\partial x^4} + C \frac{\partial^5 u(x,t)}{\partial x^4 \partial t} + m \frac{\partial^2 u(x,t)}{\partial t^2} = f(x,t) \tag{40}$$

where C is the damping coefficient. The modal equations of motion can be obtained by the same approach as for the string, and the equations are identical to Eqs. (30). However, in this case the damping factor for each mode is proportional to the square of the mode number, because it is still proportional to the natural frequency. Hence, for a beam with strain rate damping, the higher modes decay much faster than the lower modes.

We propose to control traveling waves on this pinned-pinned beam. To this end, we assume that the initial conditions are such as to produce a traveling wave. For a given initial displacement profile producing a wave traveling to the right, the velocity profile is not as easy to obtain as for the string, because waves of different wavelengths travel with different velocities along the beam. We can determine the velocity profile for a given displacement profile by finding the sinusoidal components of the displacement profile and stipulating that each component travels with its own velocity, depending on the wavelength (see Eq. (37)), and that all components travel in the same direction. If a disturbance profile has zero amplitude at the boundaries, the beam can be regarded as being infinite. Hence, we consider an infinite beam and denote the coordinate along the length of the beam by y. If the wave profile is initially even in y and is denoted by u(y,0), upon taking the Fourier transform and then the inverse Fourier transform of the wave profile we have

$$u(y,0) = \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty u(y',0) \cos \omega y \, dy \right] \cos \omega y \, d\omega \qquad (41)$$

For a beam of stiffness EI and mass m, a wave component of the form

$$u(y,t) = \cos \omega(y - \nu t) \qquad (42)$$

travels with the wave velocity $\nu = \omega\sqrt{EI/m}$. If all wave components are traveling to the right, then we have

$$u(y,t) = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty u(y,0)\cos\omega y\,dy\right]\cos\omega(y - \nu t)d\omega \tag{43}$$

and hence

$$\dot{u}(y,0) = \frac{2\sqrt{EI/m}}{\pi}\int_0^\infty \left[\int_0^\infty u(y,0)\cos\omega y\,dy\right]\omega^2\sin\omega y\,d\omega \tag{44}$$

We choose a wave profile even in y and defined for positive y by

$$u(y,0) = \begin{cases} 1 - 6(\frac{2y}{\lambda})^2 + 8\,(\frac{2y}{\lambda})^3 - 3(\frac{2y}{\lambda})^4, & 0 \le y \le \frac{\lambda}{2} \\ 0\,, \quad y \ge \frac{\lambda}{2} \end{cases} \tag{45}$$

Then, upon carrying out the integrations in Eq. (44), we obtain

$$\dot{u}(y,0) = \frac{24\sqrt{EI/m}}{\pi\lambda^2}\left\{-12(\frac{2y}{\lambda}) + 4(\frac{2y}{\lambda})\ln\left[\frac{1-(\frac{2y}{\lambda})^2}{(\frac{2y}{\lambda})^2}\right]^2 + [1 + 3(\frac{2y}{\lambda})^2]\ln\left[\frac{1+\frac{2y}{\lambda}}{1-\frac{2y}{\lambda}}\right]^2\right\} \tag{46}$$

and we observe from Eq. (44) that $\dot{u}(y,0)$ must be odd in y. Because $\dot{u}(y,0)$ approaches zero asymptotically, we start the wave at the center of the beam, so that the boundary conditions (38) are largely satisfied. Hence, to describe the transformation from x to y, we let $y = x - 1/2$.

Once again we choose a beam of unit length, mass, and stiffness with $\lambda = 0.1$ and the damping constant C so as to produce about 0.1% damping in the lowest mode. Again, we are modeling eighty modes to represent the motion of the beam. Figure 5 shows the uncontrolled motion of the beam with the initial conditions described earlier. In the time increment between t = 0 and t = 0.004, we notice that over 98% of the energy disappears as the higher modes decay rapidly. Also, the dispersive nature of the beam becomes obvious by observing how the wave spreads out as it travels. The dispersion is more obvious in the absence of damping, because a number of wavelets break away instantly ahead of the wave. We observe that, because of this dispersion, the traveling wave quickly becomes much less localized and with only a small amount of damping it has the appearance of a beam vibrating in several of its lowest modes.

In Fig. 6, we use distributed actuators to control all eighty of the modeled modes with R = 0.0005. Here again we observe that the globally optimal control is localized at the wave and is nearly a scalar multiple of the velocity.

In Fig. 7, nineteen sensors and nine actuators are again used to control the wave, as in the case of the string. Again, R = 0.0005. If we compare the controlled displacement and velocity profiles in Fig. 7 with those in Fig. 3, where discrete sensors and actuators are used to control the string, we observe that the beam profiles are much smoother, indicating less participation of the higher modes. This is because there is less control spillover into the higher modes due to the stiffness of the beam. Also, because the higher modes decay so much faster, control spillover into the higher modes is dissipated rapidly. In essence, the beam acts like a low-pass filter.

Comparing the rate of energy dissipation when discrete actuators are used with the uncontrolled case, it is clear that IMSC is effective in controlling the wave, even in time intervals so short that the wave has traveled only a short distance. Again we notice that the actuators are only active in the immediate vicinity of the disturbance.

4. CONCLUSIONS

The numerical examples presented here demonstrate the effectiveness of IMSC in controlling traveling waves in structures governed by second- and fourth-order partial differential equations, in spite of the fact that only a limited number of actuators were used. In general, a great deal of higher-mode participation is needed to describe a highly localized disturbance, but, as shown here, a small amount of material damping dissipates a great deal of the energy in the higher modes, even when the lower modes have negligible damping. Note that the traveling single wave is not an ordinary occurrence, as it takes an unusual excitation to produce it. It was used here for the purpose of investigating the control force distribution in the case of a disturbance in the form of a traveling wave. As demonstrated, the control tends to be concentrated in the vicinity of the disturbance, and there is no significant control action in areas where there is no disturbance.

5. REFERENCES

1. L. MEIROVITCH and H. BARUH, "Control of Self-Adjoint Distributed-Parameter Systems," Journal of Guidance,, Control, and Dynamics, Vol. 5, Jan.-Feb. 1982, pp. 60-66.

2. L. MEIROVITCH, Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, The Netherlands, 1980.

3. L. MEIROVITCH, Analytical Methods in Vibrations, The MacMillan Co., New York, 1967.

4. R. W. CLOUGH and J. PENZIEN, Dynamics of Structures, McGraw-Hill, New York, 1975.

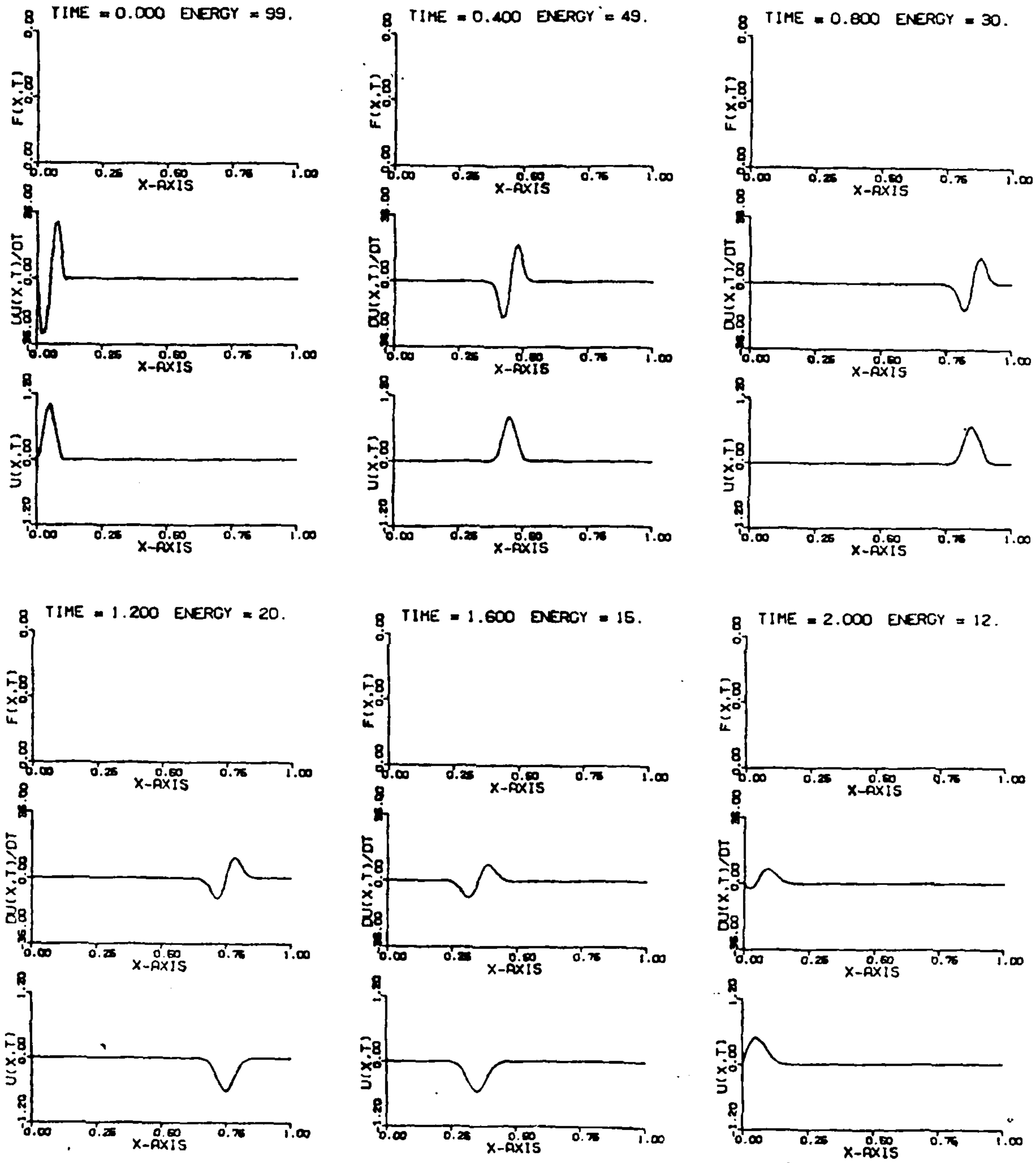


Figure 1. Uncontrolled Wave Motion in a String.

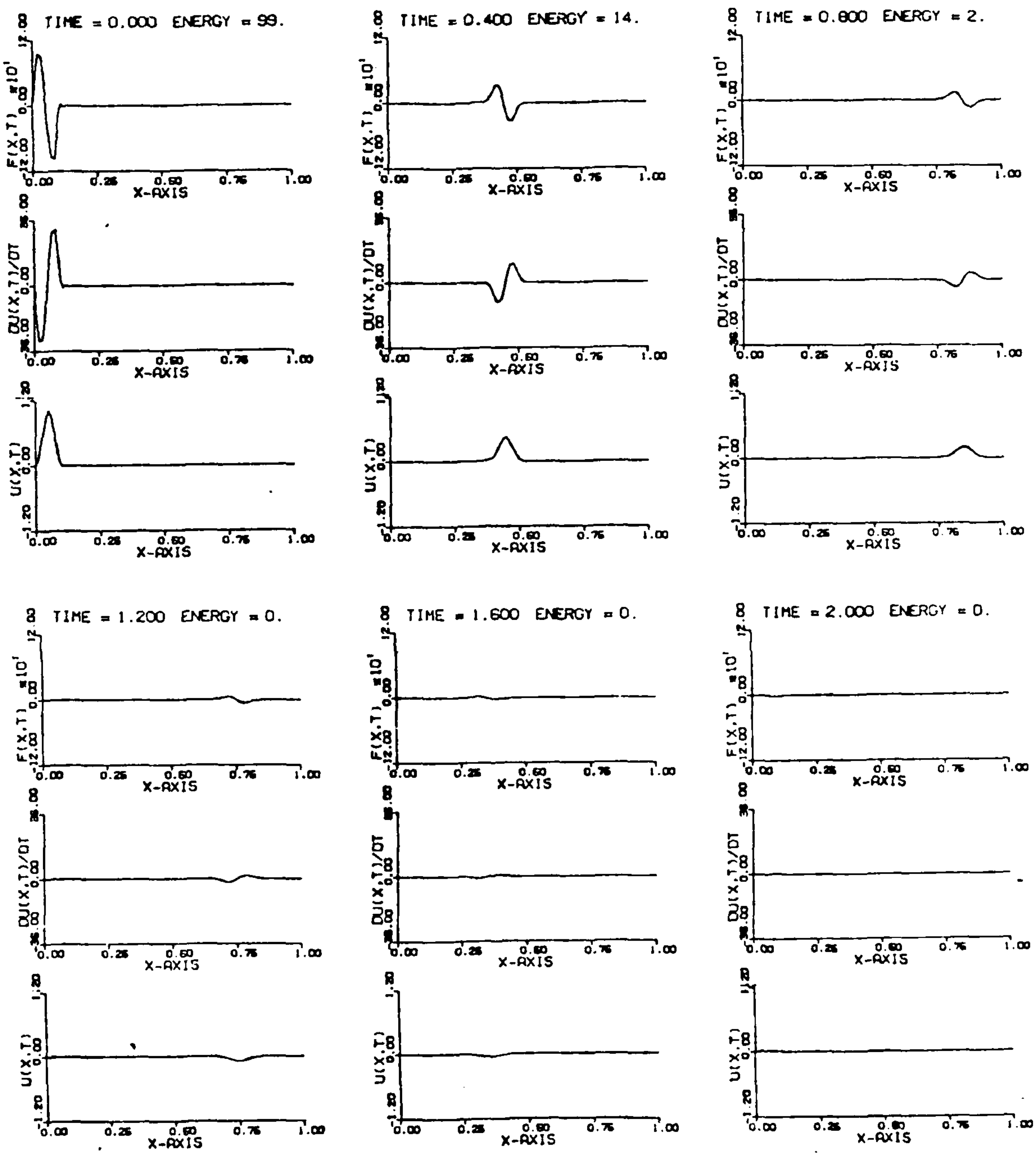


Figure 2. Distributed Control of Wave Motion in a String with R = 0.2.

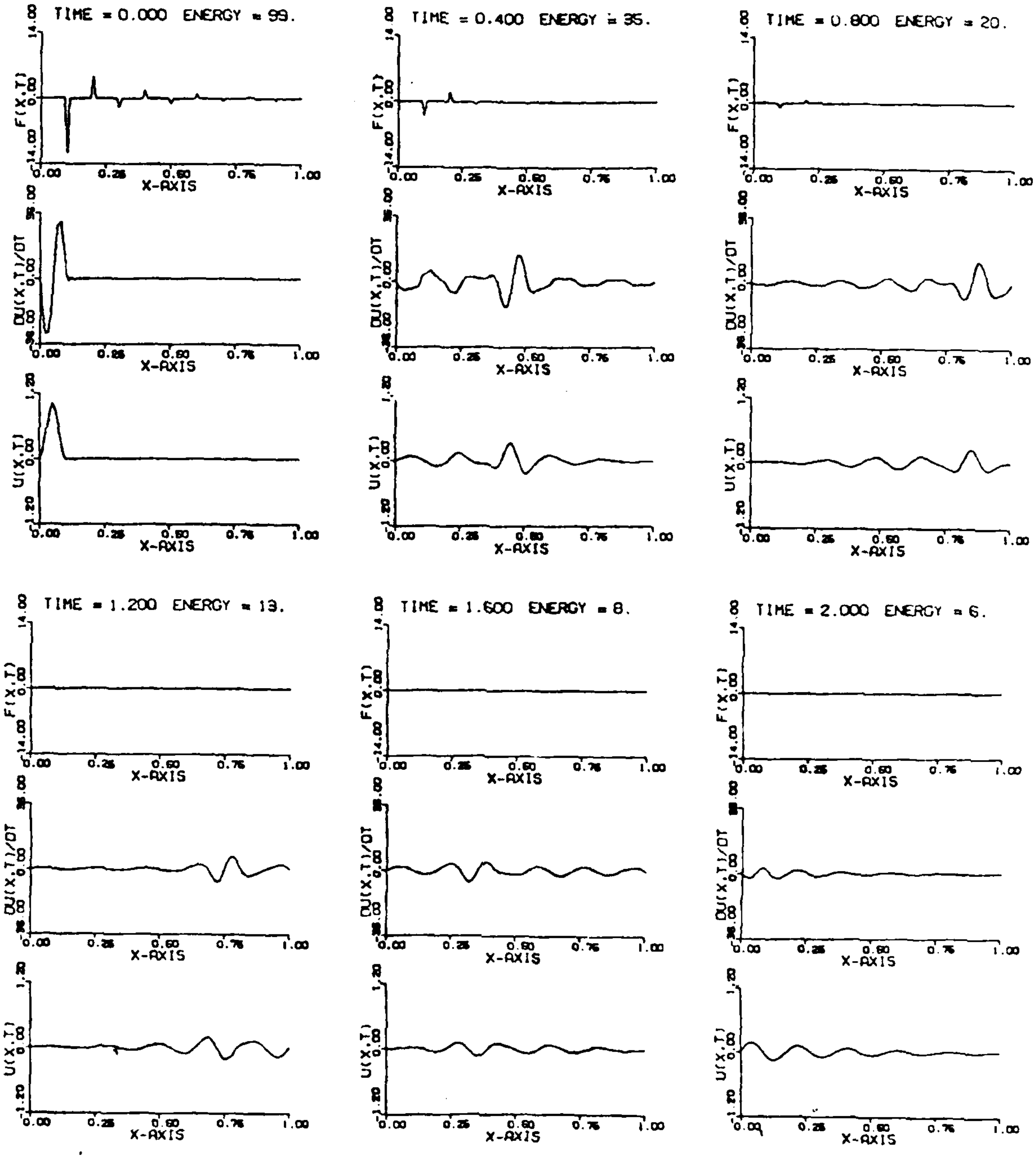


Figure 3. Discrete-Actuator Control of Wave Motion in a String with R = 0.002.

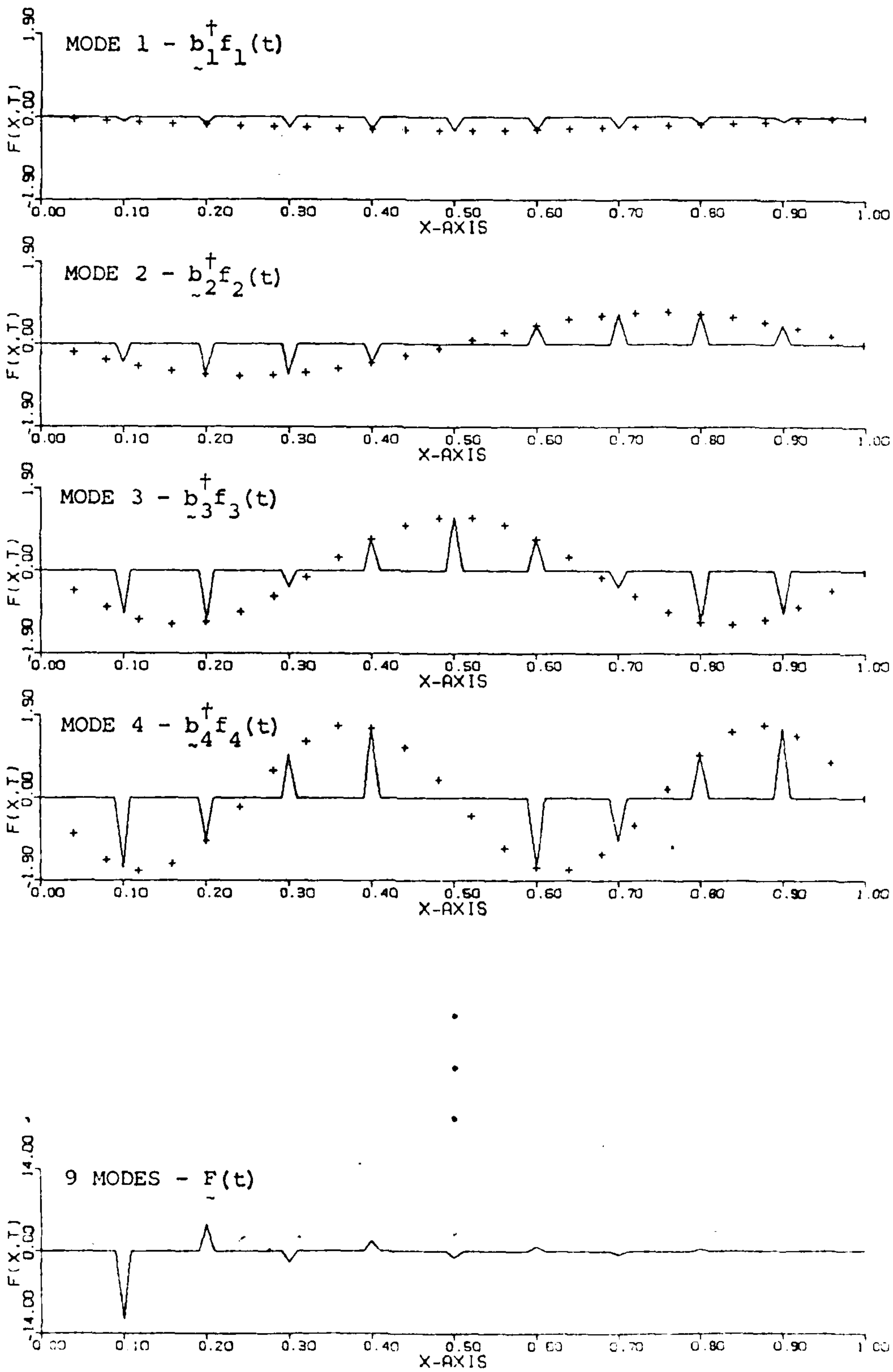


Figure 4. Modal Contributions to the Actuator Forces and the Resulting Sum, the Actual Actuator Forces, for the String at t = 0.

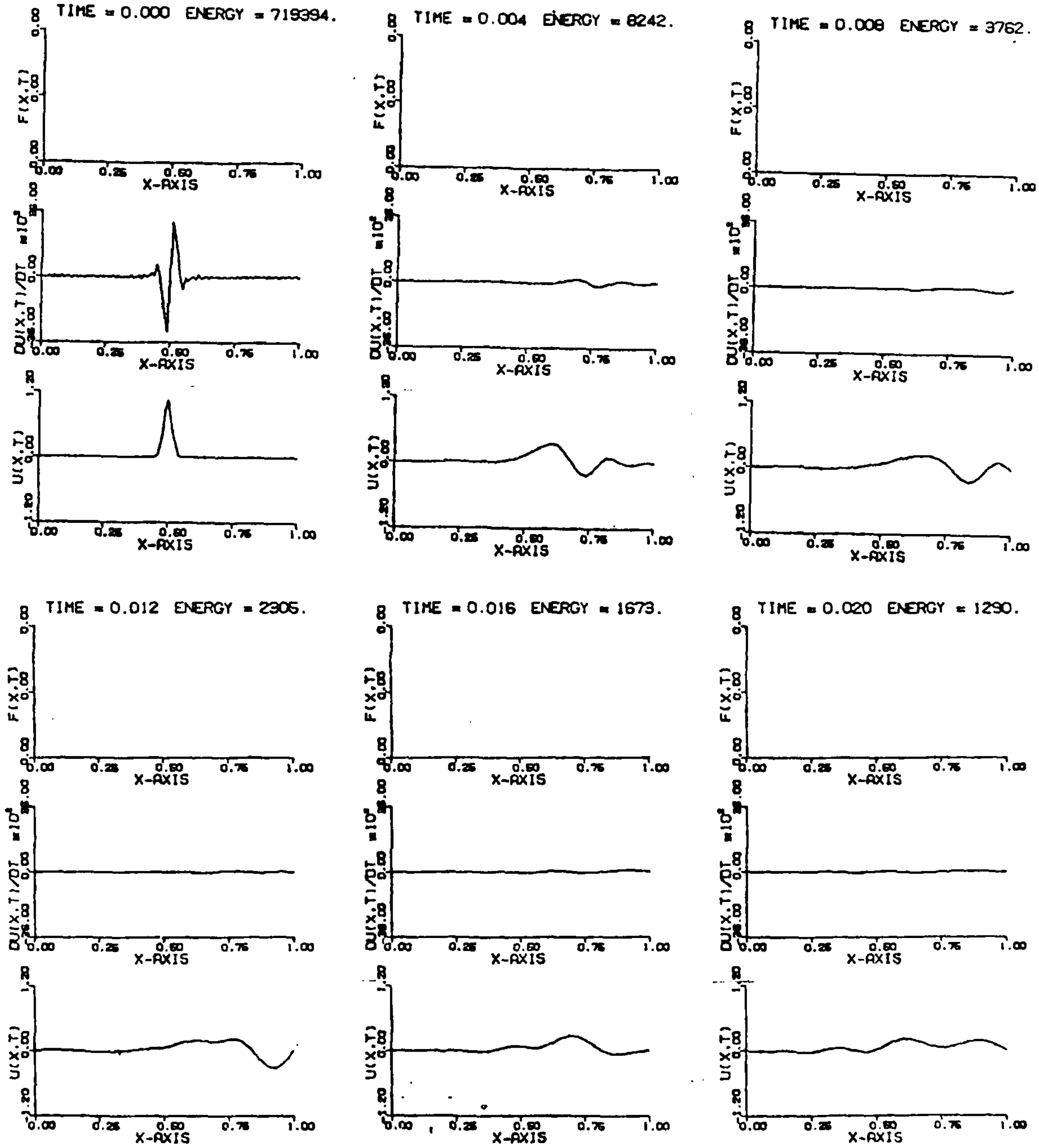


Figure 5. Uncontrolled Wave Motion in a Beam.

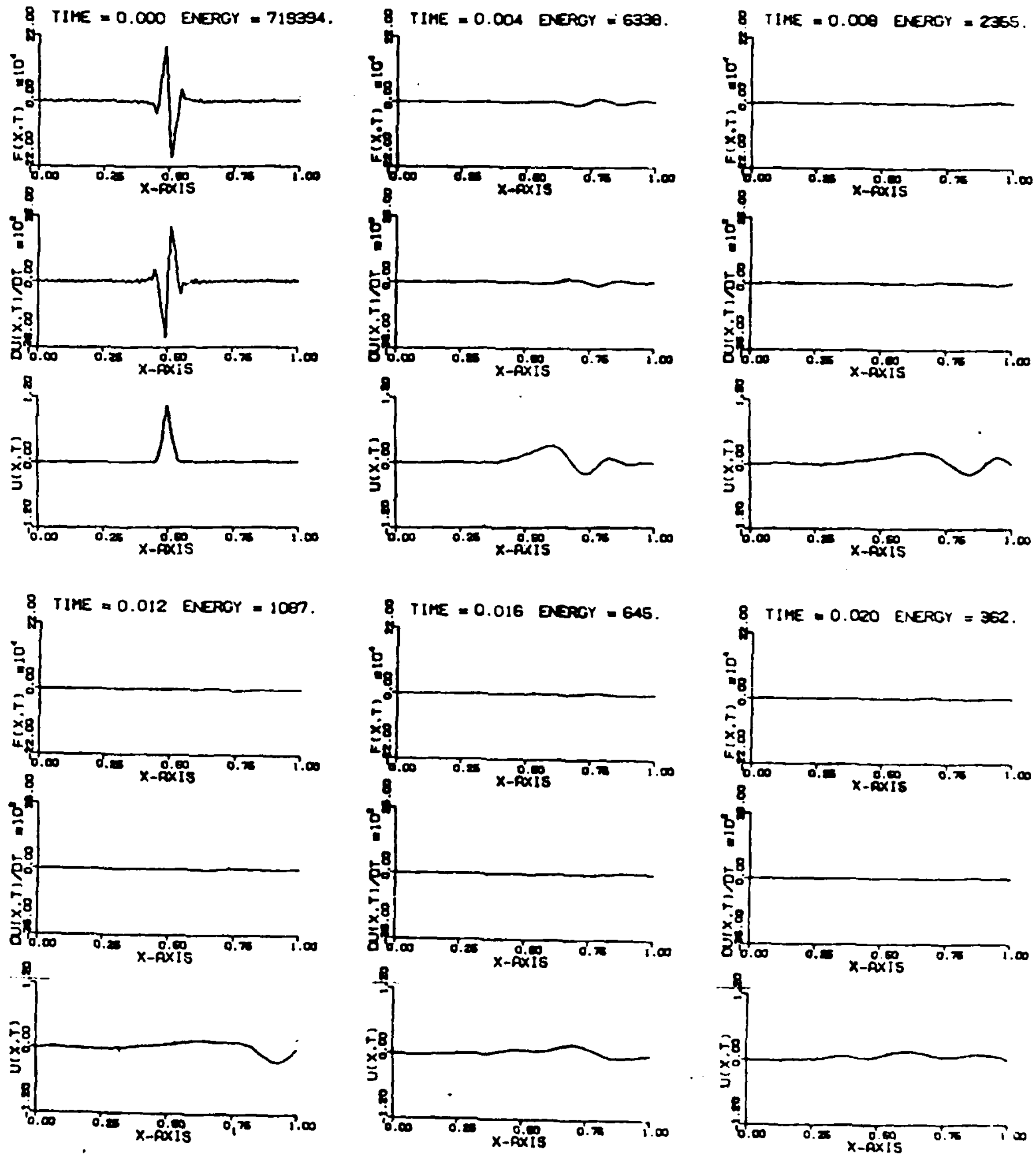


Figure 6. Distributed Control of Wave Motion in a Beam with
R = 0.0005.

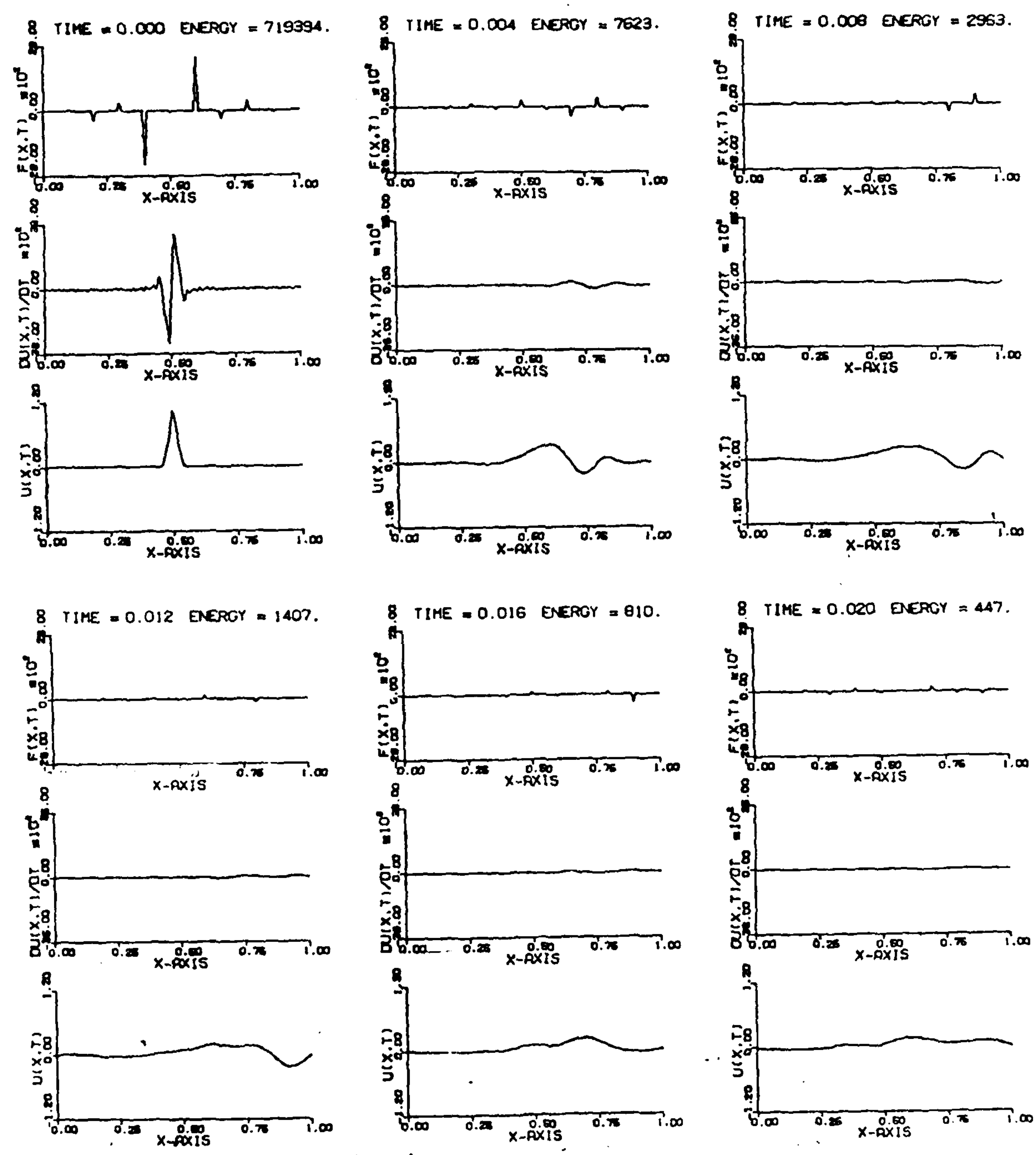


Figure 7. Discrete-Actuator Control of Wave Motion in a Beam with R = 0.0005.

JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS:

SOME PROBLEMS ASSOCIATED WITH THE CONTROL OF DISTRIBUTED STRUCTURES[1]

Leonard Meirovitch[2]

1 Sponsored by the Air Force Office of Scientific Research Grant AFOSR-83-0017 monitored by Dr. A. K. Amos, whose support is fully appreciated. Presented at the Optimal Control and Calculus of Variations Meeting, Oberwolfach, West Germany, June 15-21, 1986.

2 University Distinguished Professor, Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061

Abstract: Control of structures can be carried out conveniently by modal control, whereby the structure is controlled by controlling its modes. Modal control requires estimation of the modal states for feedback, which can present a problem. One approach that does not require modal state estimation is direct feedback control, which implies collocated sensors and actuators. This paper examines some problems encountered in direct feedback control of distributed structures in conjunction with pole placement. A perturbation technique permits the computation of control gains for multi-input systems. The paper demonstrates that the difficulties experienced in using direct feedback in conjunction with pole placement are endemic to the approach.

Key Words. distributed structures, modal control, direct feedback control, collocated sensors and actuators, pole placement, perturbation technique.

1. INTRODUCTION

Structures represent distributed-parameter systems, described by partial differential equations (Ref. 1). In some form or another, control of structures is carried out by modal control, whereby the structure is controlled by controlling its modes. Control of the entire infinity of modes requires in general a distributed actuator and a distributed sensor. If the control is such that the modes are coupled, then determination of the control gains is not possible. However, a solution is possible if the modes are controlled independently. Indeed, the independent modal-space control method is able to produce a globally optimal solution by preserving the independence of the modal equations (Ref. 2). It is shown in Ref. 2 that the optimal independent modal-space control can be implemented approximately by means of discrete sensors and actuators.

Modal control requires estimation of the modal states for feedback. This can present a problem, particularly for two- and three-dimensional structures. One approach that does not require modal state estimation is direct feedback control, whereby the control is carried out by collocated sensors and actuators. Direct feedback implies a gain matrix consisting of two diagonal submatrices, one for displacement and the other for velocity feedback. The fact that the off-diagonal gains are zero can be regarded as placing constraints on the controls. As a result determination of control gains by pole allocation or by optimal control experiences difficulties.

This paper examines some of the problems encountered in the control of distributed structures, concentrating on the problem of using direct feedback control in conjunction with pole placement. A perturbation technique permits the computation of control gains for multi-input control. The paper demonstrates that difficulties experienced in using direct feedback in conjunction with pole placement to control distributed structures are endemic to the approach and are not merely mathematical in nature. The difficulties can be attributed to the insistence on selecting the closed-loop poles in advance, as no problem exists if the control gains are selected first and the closed-loop eigenvalues are computed later.

2. MODAL EQUATIONS

We are concerned with the problem of controlling a distributed structure whose behavior is governed by the partial differential equation (pde) (Ref. 1)

$$\mathcal{L}w(P,t) + m(P)\ddot{w}(P,t) = f(P,t), \quad P\epsilon D \tag{1}$$

where $w(P,t)$ is the displacement of a typical point P inside domain D and at time t, $\mathcal{L}$ is a homogeneous, self-adjoint, positive definite differential operator, referred to as stiffness operator, $m(P)$ is the mass density and $f(P,t)$ is a distributed control force. The displacement $w(P,t)$ is subject to given boundary conditions to be satisfied at every point of the boundary S of D.

The open-loop eigenvalue problem has the form

$$\mathcal{L}\phi(P) = \omega^2 m(P)\phi(P), \quad P\epsilon D \tag{2}$$

where $\phi(P)$ is subject to given boundary conditions. The solution of Eq. (2) consists of a denumerably infinite set of eigenvalues ω_r^2, where ω_r are the natural frequencies, and associated eigenfunctions ϕ_r $(r = 1, 2,\ldots)$. The eigenfunctions are orthogonal and can be normalized so that

$$(\phi_s, m\phi_r) = \delta_{rs}, \quad (\phi_s, \mathcal{L}\phi_r) = \omega_r^2\delta_{rs}, \quad r,s = 1,2,\ldots \tag{3a,b}$$

where (,) denotes an inner product. Using the expansion theorem (Ref. 1), the displacement $w(P,t)$ can be expressed as the linear combination

$$w(P,t) = \sum_{r=1}^{\infty} \phi_r(P)q_r(t) \tag{4}$$

where $q_r(t)$ $(r = 1,2,\ldots)$ are generalized coordinates ordinarily known as <u>modal coordinates</u>. Similarly, we can expand the distributed force $f(P,t)$ in the series

$$f(P,t) = \sum_{r=1}^{\infty} m(P)\phi_r(P)f_r(t) \tag{5a}$$

where

$$f_r(t) = (\phi_r(P),f(P,t)), \quad r = 1,2,\ldots \tag{5b}$$

are known as <u>modal forces</u>. Then, inserting Eqs. (4) and (5) into Eq. (1), multiplying through by $\phi_s(P)$, integrating over the domain D and considering Eqs. (3), we obtain the <u>modal equations</u>

$$\ddot{q}_r(t) + \omega_r^2 q_r(t) = f_r(t), \quad r = 1,2,\ldots \tag{6}$$

We refer to control of a distributed structure by using Eqs. (6) to control the modes of the structure as <u>modal control</u>.

3. MODE CONTROLLABILITY AND OBSERVABILITY

It will prove convenient to cast the modal equations in state form. To this end, we define the rth modal state vector $\underset{\sim}{x}_r(t) = [q_r(t)\ \dot{q}_r(t)]^T$. Then, adjoining the identities $\dot{q}_r(t) = \dot{q}_r(t)$ $(r = 1,2,\ldots)$, Eqs. (6) can be written in the state form

$$\dot{\underset{\sim}{x}}_r(t) = A_r\underset{\sim}{x}_r(t) + \underset{\sim}{B}_r f_r(t), \quad r = 1,2,\ldots \tag{7}$$

where

$$A_r = \begin{bmatrix} 0 & 1 \\ -\omega_r^2 & 0 \end{bmatrix}, \quad \underset{\sim}{B}_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad r = 1,2,\ldots \tag{8a,b}$$

are coefficient matrices. Next, we define the modal controllability matrix

$$\mathbf{C}_r = [\underset{\sim}{B}_r \quad A_r\underset{\sim}{B}_r] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad r = 1,2,\ldots \tag{9}$$

and state that <u>the distributed system is modal-state controllable if and only if each and every controllability matrix $\mathbf{C}_r$ is of full rank 2</u>, which is clearly the case. This, of course, implies that each and every modal control $f_r(t)$ is nonzero, in which case the application of the controllability criterion is a trival formality. Note that an infinity of modal controls $f_r(t)$ is tantamount to an actual distributed control function $f(P,t)$, as indicated by Eq. (5a).

Next, we assume that the modal states are related to the modal measurements $y_r(t)$ by

$$y_r(t) = \underset{\sim}{C}_r^T \underset{\sim}{x}_r(t), \quad r = 1,2,\ldots \tag{10}$$

where in the case of displacement measurements $\underset{\sim}{C}_r^T = [1 \ 0]$ and in the case of velocity measurements $\underset{\sim}{C}_r^T = [0 \ 1]$. The modal observability matrix is defined as

$$\mathbf{O}_r = [\underset{\sim}{C}_r \quad A_r^T\underset{\sim}{C}_r], \quad r = 1,2,\ldots \tag{11}$$

and it permits us to state that <u>the distribued system is modal-state observable if and only if each and every observability matrix $\mathbf{O}_r$ is of full rank 2</u>. For displacement measurements

$$\mathbf{O}_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad r = 1,2,\ldots \tag{12a}$$

and for velocity measurements

$$\mathbf{O}_r = \begin{bmatrix} 0 & -\omega_r^2 \\ 1 & 0 \end{bmatrix}, \quad r = 1,2,\ldots \tag{12b}$$

so that the system is in general observable with either displacement measurements or velocity measurements. Notable exceptions are semi definite systems, which admit rigid-body modes with zero eigenvalues. Indeed, semidefinite systems are not observable with velocity measurements alone. Note that an infinity of modal displacement or modal velocity observations implies distributed displacement measurement $w(P,t)$ or distributed velocity measurement $\dot{w}(P,t)$, respectively.

4. FEEDBACK CONTROL

Let us consider the distributed linear feedback control

$$f(P,t) = -\mathcal{G}(P)w(P,t) - \mathcal{H}(P)\dot{w}(P,t) \tag{13}$$

where $\mathcal{G}(P)$ and $\mathcal{H}(P)$ are control gain operators. Inserting Eq. (13) into Eq. (1), we obtain the closed-loop pde

$$\mathcal{L}^* w(P,t) + \mathcal{H}(P)\dot{w}(P,t) + m(P)\ddot{w}(P,t) = 0, \quad P \varepsilon D \tag{14}$$

where

$$\mathcal{L}^* = \mathcal{L} + \mathcal{G} \tag{15}$$

is a closed-loop stiffness operator. Retracing the steps leading from Eq. (1) to Eqs. (6), we obtain the closed-loop modal equations, which can be written in the compact form

$$\ddot{\underset{\sim}{q}}(t) + H\dot{\underset{\sim}{q}}(t) + (\Lambda + G)\underset{\sim}{q}(t) = \underset{\sim}{0} \tag{16}$$

where $\underset{\sim}{q}(t)$ is the infinite-dimensional modal configuration vector, Λ is the infinite-order diagonal matrix of eigenvalues and G and H are square control gain matrices of infinite order with entries given by

$$g_{sr} = (\phi_s, \mathcal{G}\phi_r), \quad h_{sr} = (\phi_s, \mathcal{H}\phi_r), \quad r,s = 1,2,\ldots \tag{17a,b}$$

In the general case, the matrices G and H are not diagonal, so that the effect of feedback control is to couple the modal equations. Physically, the term $g_{sr}q_r(t)$ implies a generalized spring force and the term $h_{sr}\dot{q}_r(t)$ a generalized damping force. Hence, the fact that the matrices G and H are not diagonal implies that the feedback control provides nonproportional stiffness and damping (Ref. 1), respectively. We refer to the case in which G and H are not diagonal as <u>coupled modal control</u>. Note that in this case the matrices G and H may not be even symmetric.

Before the behavior of the closed-loop system can be established, it is necessary to determine the gain operators $\mathcal{G}$ and $\mathcal{H}$ or the gains matrices G and H. However, there are no algorithms capable of producing the operators $\mathcal{G}$ and $\mathcal{H}$ or the infinite-order matrices G and H. Hence, distributed feedback control realized through coupled modal control is not possible.

Next, we introduce the 2∞ -dimensional modal state vector $\underset{\sim}{x}(t) = [\underset{\sim}{q}^T(t) \mid \dot{\underset{\sim}{q}}^T(t)]^T$, so that Eq. (16) can be rewritten in the state form

$$\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t) \tag{18}$$

where

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -(\Omega^2 + G) & -H \end{array}\right] \tag{19}$$

in which $\Omega = \text{diag}\,[\omega_r]$. The problem of determining the control gain matrices G and H remains. In this regard, one can consider pole allocation and optimal control. In the pole allocation method, the problem reduces to the solution of a set of nonlinear algebraic equations (Ref. 3), which is not feasible for infinite-dimensional systems. Similarly, for optimal control using a quadratic performance index, one is faced with the solution of a matrix Riccati equation of order 2∞, which is not possible.

5. <u>INDEPENDENT MODAL-SPACE CONTROL</u>

There is one special case in which distributed feedback control is possible, namely the one in which the operators $\mathcal{G}$ and $\mathcal{H}$ satisfy the eigenvalue problems

$$\mathcal{G}\phi_r(P) = g_r m(P)\phi_r(P), \mathcal{H}\phi_r(P) = h_r m(P)\phi_r(P), \quad r = 1,2,... \tag{20a,b}$$

which imply that $\mathcal{G}$ and $\mathcal{H}$ are such that

$$(\phi_s, \mathcal{G}\phi_r) = g_r\delta_{rs}, \; (\phi_s, \mathcal{H}\phi_r) = h_r\delta_{rs}, \quad r,s = 1,2,... \tag{21a,b}$$

In this case the closed-loop modal equations reduce to the independent set

$$\ddot{q}_s(t) + h_s\dot{q}_s(t) + (\lambda_s + g_s)q_s(t) = 0, \quad s = 1,2,... \tag{22}$$

Because of the independence of the closed-loop modal equations, this type of control is called independent modal-space control (IMSC). It is characterized by modal control forces of the form

$$f_s(t) = -g_s q_s(t) - h_s\dot{q}_s(t), \quad s = 1,2,... \tag{23}$$

In open-loop response problems, the coordinates $q_s(t)$ corresponding to independent equations of motion are called natural. Because IMSC guarantees the independence of the closed-loop equations, we refer to IMSC as natural control.

The fact that both the open-loop and closed-loop modal equations are independent has very important implications. Indeed, this implies that the open-loop eigenfunctions ϕ_s are closed-loop eigenfunctions as well. Hence, in natural control, the control effort is directed entirely to altering the eigenvalues, leaving the eigenfunctions unaltered. In this regard, it should be recalled that the stability of a linear system is determined by the system eigenvalues, with the eigenfunctions playing no role, so that in natural control no control effort is used unnecessarily.

The question remains as to how to determine the modal gains g_s and h_s ($s = 1,2,...$). Two of the most widely used techniques are pole allocation and optimal control:

i. Pole allocation

In the pole allocation method, the closed-loop poles are selected in advance and the gains are determined so as to produce these poles. In the IMSC, the procedure is exceedingly simple. Denoting the closed-loop eigenvalue associated with the sth mode by $-\alpha_s + i\beta_s$, the solution of Eqs. (22) can be written as

$$q_s(t) = c_s e^{(-\alpha_s + i\beta_s)t}, \quad s = 1,2,... \tag{24}$$

Inserting Eqs. (24) into Eqs. (22) and separating the real and imaginary parts, we obtain the modal gains

$$g_s = \alpha_s^2 + \beta_s^2 - \lambda_s, \; h_s = 2\alpha_s, \quad s = 1,2,... \tag{25}$$

To guarantee asymptotic stability, however, it is only necessary to impart the open-loop eigenvalues some negative real part and it is not necessary to alter the frequencies. This can be achieved by letting $\beta_s = \sqrt{\lambda_s} = \omega_s$ (s = 1,2,...), where, ω_s is the sth natural frequency of the open-loop system. Hence, the frequency-preserving control gains are

$$g_s = \alpha_s^2, \; h_s = 2\alpha_s, \quad s = 1,2,... \tag{26}$$

ii. Optimal control

In optimal control, the closed-loop poles are determined by minimizing a given performance index. Consistent with previous developments, we are interested in constant gains and, to this end, we consider the performance functional

$$J = \int_0^\infty [(\dot{w},m\dot{w}) + (w,\mathcal{L}w) + (f,rf)]dt \tag{27}$$

where the various quantities are as defined in Eq. (1), except for r = r(P) which is a weighting function assumed to satisfy (Ref. 2)

$$(f,rf) = \sum_{r=1}^{\infty} R_r f_r^2 \tag{28}$$

where R_r are modal weights. Inserting Eqs. (4) and (28) into Eq. (27) and recalling Eqs. (3), we obtain

$$J = \sum_{r=1}^{\infty} J_r \tag{29a}$$

where

$$J_r = \int_0^{\infty} (\dot{q}_r^2 + \omega_r^2 q_r^2 + R_r f_r^2)dt, \quad r = 1,2,... \tag{29b}$$

are modal performance indices. Because in IMSC the modal control f_r is independent of any other modal control, it follows that

$$\min J = \min \sum_{r=1}^{\infty} J_r = \sum_{r=1}^{\infty} \min J_r \tag{30}$$

so that the minimization can be carried out independently for each mode.

The minimization of J_r leads to a 2×2 matrix Riccati equation that can be solved in closed form (Ref. 4), yielding the modal control gains

$$\begin{aligned} g_r &= -\omega_r^2 + \omega_r(\omega_r^2 + R_r^{-1})^{1/2} \\ h_r &= [-2\omega_r^2 + R_r^{-1} + 2\omega_r(\omega_r^2 + R_r^{-1})^{1/2}]^{1/2} \end{aligned}, \quad r = 1,2,... \tag{31}$$

Because no constraint has been imposed on the control function $f = f(P,t)$, the solution defined by Eqs. (5), (23) and (31) is globally optimal, and is unique because the solution to the linear optimal control problem is unique (Ref. 5).

It should be pointed out that the solution presented above requires distributed sensors and actuators. Indeed, inserting Eqs. (23) into Eq. (5a), we obtain the distributed feedback control force

$$f(P,t) = -\sum_{r=1}^{\infty} m(P)\phi_r(P)[g_r q_r(t) + h_r \dot{q}_r(t)] \tag{32}$$

Equation (32) indicates that control implementation requires the entire infinity of modal displacements $q_r(t)$ and modal velocities $\dot{q}_r(t)$ $(r = 1,2,...)$ for feedback. This, in turn, implies a distributed sensor. Note that, inserting Eq. (4) into Eq. (13) and comparing the results with Eq. (32), we can verify Eqs. (20). At this point, we observe that the gain operators $\mathcal{G}$ and $\mathcal{H}$ are never determined explicitly, nor is it necessary to do so, as the determination of the modal gains g_r and h_r $(r = 1,2,\ldots)$ is sufficient to produce the feedback control density function $f(P,t)$.

6. CONTROL BY POINT ACTUATORS

As pointed out in Sec. 5, globally optimal control of a distributed structure requires a distributed actuator. On the assumption that distributed actuation is not feasible, we seek control by means of a finite number p of discrete actuators acting at the points $P = P_i$ $(i = 1,2,\ldots,p)$ of the structure. Discrete actuators can be treated as distributed by writing

$$f(P,t) = \sum_{i=1}^{p} F_i(t)\delta(P - P_i), \quad P\varepsilon D \tag{33}$$

where $F_i(t)$ are force amplitudes and $\delta(P - P_i)$ are spatial Dirac delta functions. Introducing Eq. (33) into Eq. (5b), we obtain the relation between the modal forces and the actuator forces in the form

$$f_r(t) = (\phi_r(P),f(P,t)) = \sum_{i=1}^{p} F_i(t) \int_D \phi_r(P)\delta(P - P_i)dD = \sum_{i=1}^{p} \phi_r(P_i)F_i(t), \quad r = 1,2,\ldots \tag{34}$$

which can be written in the compact form

$$\underset{\sim}{f}(t) = \Phi \underset{\sim}{F}(t) \tag{35}$$

where $\underset{\sim}{f}(t)$ is the infinite-dimensional modal vector, Φ is the $\infty \times p$ modal participation matrix and $\underset{\sim}{F}(t)$ is the p-vector of actuator forces.

Considering the feedback control

$$\underset{\sim}{F}(t) = -G\underset{\sim}{q}(t) - H\dot{\underset{\sim}{q}}(t) \tag{36}$$

where this time G and H are $p \times \infty$ control gain matrices, the closed-loop state equations can once again be written in the form (18), but this time the coefficient matrix is

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -(\Omega^2 + \Phi G) & -\Phi H \end{array}\right] \tag{37}$$

The difficulties cited in Sec. 4 in conjunction with the determination of the gain matrices G and H remain. Some of these difficulties can be reduced by controlling a finite number of modes. This raises the question of control spillover into the uncontrolled modes (Ref. 6), particularly if the number of controlled modes is small.

7. DIRECT FEEDBACK CONTROL

One problem that can prove troublesome in modal control is the estimation of the modal states for feedback. To this end, one can consider a Luenberger observer (Ref. 7), but the question of observation spillover is potentially more serious than the problem of control spillover, as it can lead to instability (Ref. 6). Hence, a procedure not requiring modal state estimation appears desirable.

One approach not requiring modal state estimation is direct feedback control, whereby the sensors are collocated with the actuators and a given actuator force is a linear function of the sensor output at the same point. We consider p discrete actuators acting at the points $P = P_i$ $(i = 1,2,\ldots,p)$, where the force amplitudes are

$$F_i(t) = -g_i w(P_i,t) - h_i \dot{w}(P_i,t), \quad i = 1,2,\ldots,p \tag{38}$$

in which g_i and h_i $(i = 1,2,\ldots,p)$ are actual control gains. Clearly, the gains must be positive. As before, the discrete actuators can be regarded as distributed by writing

$$f(P,t) = -\sum_{i=1}^{p} [g_i w(P,t) + h_i \dot{w}(P,t)]\delta(P - P_i), \quad P \in D \tag{39}$$

To make the connection with Eq. (13), we can regard $\mathcal{G}$ and $\mathcal{H}$ as operators having the expressions

$$\mathcal{G}(P) = \sum_{i=1}^{p} g_i \delta(P - P_i), \mathcal{H}(P) = \sum_{i=1}^{p} h_i \delta(P - P_i), \quad P \varepsilon D \qquad (40a,b)$$

so that, inserting Eqs. (40) into Eqs. (17), we obtain the entries of the control gain matrices G and H in the explicit form

$$g_{sr} = \sum_{i=1}^{p} g_i \phi_s(P_i)\phi_r(P_i), \; h_{sr} = \sum_{i=1}^{p} h_i \phi_s(P_i)\phi_r(P_i), \quad r,s, = 1,2,\ldots (41a,b)$$

The state equations remain in the form (18) and the coefficient matrix A remains in the form (19).

Once again the problem is that of determining the control gains. The problem is different here because there is only a finite number of gains g_i and h_i $(i = 1,2,\ldots,p)$ and the system is infinite-dimensional. There is no computational algorithm permitting the computation of the control gains in conjunction with either pole allocation or optimal control, so that one must consider modal truncation. Even for the truncated model, the situation remains questionable. The reason for this is that pole allocation and optimal control most likely will require gain matrices with entries independent of each other while direct feedback control implies that the entries of G and H are not independent, as can be see from Eqs. (41). In fact, there is some question whether arbitrary pole placement is possible for direct feedback control. Moreover, because the entries of G and H are not independent, there is some question whether optimal control is possible in the presence of constraints on the control gains.

8. A PERTURBATION APPROACH TO POLE ALLOCATION FOR DIRECT FEEDBACK

Application of the pole allocation method to multi-input control can cause serious difficulties. Moreover, the method is suitable for discrete systems only. In this section, we present an approach suitable for distributed systems, and in the process we reveal some limitations of the pole allocation method.

The eigenvalue problem corresponding to the closed-loop equation, Eq. (18), is

$$A\underset{\sim}{u} = \lambda \underset{\sim}{u} \tag{42}$$

where A is given by Eq. (19). We propose to determine the control gains by a perturbation approach. To this end, we assume that A can be expressed in the form

$$A = A_0 + A_1 \tag{43}$$

where

$$A_0 = \left[\begin{array}{c|c} 0 & I \\ \hline -\Omega^2 & 0 \end{array}\right], \quad A_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline -G & -H \end{array}\right] \tag{44a,b}$$

in which A_1 is "small" relative to A_0 in some sense, so that the open-loop matrix A_0 represents the unperturbed coefficient matrix and A_1 is the perturbation due to closing of the loop.

The zero-order eigenvalue problem, i.e., the unperturbed open-loop eigenvalue problem is characterized by the eigenvalues $\lambda_{0j} = i\omega_j$ $(j = 1,2,\ldots)$, where $i = \sqrt{-1}$ and ω_j are the natural frequencies, and by the right and left eigenvectors

$$\underset{\sim}{u}_{0j} = \left[\begin{array}{c} \underset{\sim}{e}_j \\ \hline i\omega_j \underset{\sim}{e}_j \end{array}\right], \quad \underset{\sim}{v}_{0j} = \frac{1}{2}\left[\begin{array}{c} \underset{\sim}{e}_j \\ \hline -(i/\omega_j)\underset{\sim}{e}_j \end{array}\right], \quad j = 1,2,\ldots \tag{45a,b}$$

in which $\underset{\sim}{e}_j$ is a standard unit vector. The two sets of eigenvectors satisfy the biorthonormality relations $\underset{\sim}{v}_{0k}^T \underset{\sim}{u}_{0j} = \delta_{jk}$, $\underset{\sim}{v}_{0k}^T A_0 \underset{\sim}{u}_{0j} = i\omega_j \delta_{jk}$ $(j,k = 1,2,\ldots)$. Equations (45) specify only one half of the right and left eigenvectors. The other half consists of the complex conjugates $\bar{\underset{\sim}{u}}_{0j}$ and $\bar{\underset{\sim}{v}}_{0j}$ corresponding to the eigenvalue $-i\omega_j$. Then, the first-order perturbation solution of Eq. (42) can be expressed as (Ref. 8)

$$\lambda_j = \lambda_{0j} + \lambda_{1j} = i\omega_j + \lambda_{1j}, \quad \underset{\sim}{u}_j = \underset{\sim}{u}_{0j} + \underset{\sim}{u}_{1j}, \quad j = 1,2,\ldots \qquad (46a,b)$$

where

$$\lambda_{1j} = \underset{\sim}{v}_{0j}^T A_1 \underset{\sim}{u}_{0j}, \quad \underset{\sim}{u}_{1j} = \sum_{k=1}^{\infty} \left(\frac{\underset{\sim}{v}_{0k}^T A_1 \underset{\sim}{u}_{0j}}{\lambda_{0j} - \lambda_{0k}} \right) \underset{\sim}{u}_{0k}, \quad j = 1,2,\ldots;\ k \neq j \qquad (47a,b)$$

Inserting Eqs. (45) into Eq. (47a), we obtain

$$\lambda_{1j} = \frac{i}{2\omega_j} \underset{\sim}{e}_j^T (G + i\omega_j H) \underset{\sim}{e}_j^T, \quad j = 1,2,\ldots \qquad (48)$$

so that

$$\mathrm{Re}\,\lambda_{1j} = -\frac{1}{2} \underset{\sim}{e}_j^T H \underset{\sim}{e}_j = -\frac{1}{2} h_{jj} = -\frac{1}{2} \sum_{i=1}^{p} h_i \phi_j^2(P_i), \quad j = 1,2,\ldots \qquad (49a)$$

$$\mathrm{Im}\,\lambda_{1j} = \frac{1}{2\omega_j} \underset{\sim}{e}_j^T G \underset{\sim}{e}_j = \frac{1}{2\omega_j} g_{jj} = \frac{1}{2\omega_j} \sum_{i=1}^{p} g_i \phi_j^2(P_i), \quad j = 1,2,\ldots \qquad (49b)$$

Introducing the notation

$$\lambda_{1j} = -\alpha_j + i\Delta\omega_j, \quad \phi_j^2(P_i) = b_{ji} \qquad (50a,b)$$

Eqs. (49) can be rewritten as

$$\sum_{i=1}^{p} b_{ji} h_i = 2\alpha_j, \quad \sum_{i=1}^{p} b_{ji} g_i = 2\omega_j \Delta\omega_j, \quad j = 1,2,\ldots \qquad (51a,b)$$

Equations (51) represent two infinite sets of algebraic equations. Because the two sets are similar in nature, we confine our discussion to Eqs. (51a). We note from Eqs. (50b) that all b_{ji} $(i = 1,2,\ldots,p;\ j = 1,2,\ldots)$ are positive. If the gains h_i $(i = 1,2,\ldots,p)$ are selected in advance, and if we recall that they must be positive, we conclude that all $\alpha_j (j = 1,2,\ldots)$ are positive, which guarantees that, to the first approximation, direct feedback leads to asymptotic stability.

In pole placement, however, the closed-loop poles rather than the gains are selected in advance. If all the poles are to be placed, which implies that all $\alpha_j (j = 1,2,\ldots)$ must be selected in advance, then Eqs. (51a) represent an infinite set of equations and p unknowns, namely h_i $(i = 1,2,\ldots,p)$. Clearly, no solution is possible, so that we consider placing only a finite number of poles. Physically, this presents no problem as higher modes are seldom excited. In general, the object is to place a larger number of poles than the number p of actuators. Hence, let us assume that we wish to place the first n poles, $n \geq p$, and write Eqs. (51a) in the matrix form

$$B\underset{\sim}{h} = 2\underset{\sim}{\alpha} \tag{52}$$

where B is an $n \times p$ matrix, $\underset{\sim}{h}$ is the p-dimensional gain vector and $\underset{\sim}{\alpha}$ is the n-dimensional vector of preselected pole shifts along the real axis. A least-squares solution of Eq. (52) yields

$$\underset{\sim}{h} = 2B^{\dagger}\underset{\sim}{\alpha}, \quad B^{\dagger} = (B^T B)^{-1} B^T \tag{53a,b}$$

where $B^{\dagger}$ is the pseudo-inverse of B. Then, the shifts of the remaining poles along the real axis can be obtained from Eqs. (51a) corresponding to $j = n + 1, n + 2, \ldots.$ For stability, $\alpha_j (j = n + 1, n + 2, \ldots)$ must all be nonnegative.

The fact that all α_j $(j = n + 1, n + 2, \ldots)$ must be nonnegative implies that all the gains h_i $(i = 1,2,\ldots,p)$ must be positive. Indeed, if some h_i are negative, then the left sides of Eqs. (51a) corresponding to $j > n$ represent indefinite forms, so that some α_j, $j > n$, can be negative, which implies destabilization of some of the higer modes. Yet, the solution (53a) cannot guarantee that all the components of $\underset{\sim}{h}$

are positive for any choice of $\underset{\sim}{\alpha}$. It follows that in direct feedback control the poles cannot be placed arbitrarily. This fact can be explained easily if we recognize that direct feedback is a special type of control in which a given actuator force depends only on the state at the same location, as expressed by Eqs. (38). As a result, the gain matrix contains no cross-products. The zero entries in the gain matrix can be regarded as constraints on the control, limiting the freedom to choose the poles. Hence, direct feedback control and pole allocation are incompatible.

It must be stressed that the difficulties encountered above do not exist when the control gains are selected first and the closed-loop poles are computed subsequently, so that the problem lies not with direct feedback control but with pole allocation used in conjunction with direct feedback to control a *reduced number of modes*.

The question remains as to whether the incompatibility between direct feedback and pole allocation is caused by the perturbation technique or is more endemic in nature. We address this question later in this paper.

9. SECOND-ORDER PERTURBATION EFFECTS

The analysis of Sec. 8 was based on linear approximation. In reality, the poles are likely to differ from the ones based on the first-order approximaton, but the question is whether the difference is significant. To explore this question, we turn to the second-order perturbation in the closed-loop poles. It can be shown that the second-order perturbation in the eigenvalues has the form (Ref. 8)

$$\lambda_{2j} = \sum_{k=1}^{\infty} \frac{(\underset{\sim}{v}_{0k}^T A_1 \underset{\sim}{u}_{0j})(\underset{\sim}{v}_{0j}^T A_1 \underset{\sim}{u}_{0k})}{\lambda_{0j} - \lambda_{0k}}$$

$$= i \sum_{k=1}^{\infty} \frac{1}{\omega_k - \omega_j} \left(- \frac{\omega_j}{2\omega_k} \underset{\sim}{e}_k^T H \underset{\sim}{e}_j + \frac{i}{2\omega_k} \underset{\sim}{e}_k^T G \underset{\sim}{e}_j\right)\left(- \frac{\omega_k}{2\omega_j} \underset{\sim}{e}_j^T H \underset{\sim}{e}_K + \frac{i}{2\omega_j} \underset{\sim}{e}_j^T G \underset{\sim}{e}_k\right),$$

$$j = 1,2,\ldots;j \neq k \quad (54)$$

so that

$$\text{Re } \lambda_{2j} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{\omega_k - \omega_j} \left(\frac{1}{\omega_k} \underset{\sim}{e}_k^T H \underset{\sim}{e}_j \cdot \underset{\sim}{e}_j^T G \underset{\sim}{e}_k + \frac{1}{\omega_j} \underset{\sim}{e}_j^T H \underset{\sim}{e}_k \cdot \underset{\sim}{e}_k^T G \underset{\sim}{e}_j\right)$$

$$= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{\omega_k - \omega_j} \left(\frac{1}{\omega_k} h_{kj} g_{jk} + \frac{1}{\omega_j} h_{jk} g_{kj}\right), \quad j = 1,2,\ldots;j \neq k \quad (55a)$$

$$\text{Im } \lambda_{2j} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{\omega_k - \omega_j} \left(\underset{\sim}{e}_k^T H \underset{\sim}{e}_j \cdot \underset{\sim}{e}_j^T H \underset{\sim}{e}_k - \frac{1}{\omega_j \omega_k} \underset{\sim}{e}_k^T G \underset{\sim}{e}_j \cdot \underset{\sim}{e}_j^T G \underset{\sim}{e}_k\right)$$

$$= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{\omega_k - \omega_j} \left(h_{kj} h_{jk} - \frac{1}{\omega_j \omega_k} g_{kj} g_{jk}\right), \quad j = 1,2,\ldots;\ j \neq k \quad (55b)$$

From Eqs. (55a), it appears that some Re λ_{2j} can be positive. Because Re λ_{2j} involves only off-diagonal entries of the modal gain matrices G and H, however, it is not likely to exceed α_j in magnitude, so that in general stability is not really threatened. Moreover, if the control does not involve displacement feedback, then all Re λ_{2j} are zero. Hence, a second-order perturbation solution is not expected to be significantly different from the first-order solution.

A similar analysis can be carried out in conjunction with Eqs. (55b). For positive definite structures, such an analysis is not very significant, as it involves only the imaginary part of the closed-loop poles, which does not affect the structure stability. For positive semidefinite systems, however, the open-loop poles associated with the rigid-body modes are zero, so that the rigid-body modes are not controllable with velocity feedback alone. Still, because Eqs. (55b) are concerned only with the imaginary part of the eigenvalues, the conclusion involving the nature of the second-order perturbation solution remains the same.

From the above discussion, we must conclude that the incompatibility between direct feedback control and pole placment for control of distributed structures has deeper roots and is not merely caused by the perturbation approach.

10. NUMERICAL EXAMPLE

Let us consider the problem of controlling the cantilever beam shown in Fig. 1 by means of three equally-spaced actuators, $x_i = iL/3$ $(i = 1,2,3)$. The eigenfunctions are given by (Ref. 1)

$$\phi_r(x) = A_r[\cos \beta_r x - \cosh \beta_r x + \frac{\sin \beta_r L - \sinh \beta_r L}{\cos \beta_r L + \cosh \beta_r L}(\sin \beta_r x - \sinh \beta_r x)],$$

$$r = 1,2,... \quad (56)$$

where $\beta_r L$ are the roots of the characteristic equation $\cos \beta_r L \cosh \beta_r L = -1$. Normalizing the eigenfunctions so that $\int_0^L m\phi_r^2\, dx = 1$, we obtain $A_1 = 0.99803\ m^{-1/2}$, $A_2 = 0.99803\ m^{-1/2}$, $A_3 = 0.99802\ m^{-1/2}$, $A_4 = 1.0230\ m^{-1/2}$, $A_5 = 1.0177\ m^{-1/2}$, $A_6 = 1.0143\ m^{-1/2}$, ... Moreover, the roots of the characteristic equation are $\beta_1 L = 1.87510$, $\beta_2 L = 4.69409$, $\beta_3 L = 7.85476$, $\beta_4 L = 10.99550$, $\beta_5 L = 14.13720$, $\beta_6 L = 17.27879$, ..., and note that as the mode number increases the roots approach odd multiples of $\pi/2$.

Letting r = 3 and using Eq. (50b), we obtain the matrix

$$B = m^{-1}\begin{bmatrix} 0.1092 & 0.1192 \times 10 & 0.3984 \times 10 \\ 0.1385 \times 10 & 0.7119 & 0.3984 \times 10 \\ 0.2076 \times 10 & 0.1651 \times 10 & 0.3984 \times 10 \end{bmatrix} \quad (57)$$

so that, from Eq. (53a), we obtain the control gains

$$\begin{aligned} h_1 &= (-1.2274\ \alpha_1 + 0.6000\ \alpha_2 + 0.6276\ \alpha_3)m \\ h_2 &= (0.9036\ \alpha_1 - 2.5720\ \alpha_2 + 1.6686\ \alpha_3)m \\ h_3 &= (0.2654\ \alpha_1 + 0.7530\ \alpha_2 - 0.5164\ \alpha_3)m \end{aligned} \tag{58}$$

It is clear that, because the gains must be positive, the poles cannot be placed arbitrarily. We recall that α_1, α_2 and α_3 must also be positive. To develop a feel for the restrictions on the pole placement, let us imagine a three-dimensional space defined by α_1, α_2 and α_3. The pole shifts must be such that $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_3 > 0$, which restricts the placement to the positive one eighth of the three-dimensional space. Then, we consider a typical equation from the set (58) and write it in the form

$$h = a\alpha_1 + b\alpha_2 + c\alpha_3 \tag{59}$$

For $h = 0$, Eq. (59) represents a plane through the origin of the three-dimensional space α_1, α_2, α_3. Hence, the inequality $h > 0$ implies that the acceptable points lie in one half of the space. Denoting by S_0 the space defined by $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_3 > 0$ and by S_1 the space corresponding to $h > 0$, we conclude that the closed-loop poles must be such that α_1, α_2 and α_3 lie in the intersection of S_0 and S_1. In our case, there are three inequalities, $h_i > 0$ $(i = 1,2,3)$, to be satisfied. Denoting the associated spaces by S_i $(i = 1,2,3)$, we conclude that α_1 α_2 and α_3 must lie in the intersection of the spaces S_0, S_1, S_2 and S_3. This intersection defines a cone with the vertex at the origin of the space α_1, α_2, α_3 (Fig. 2). Whereas this region may provide many choices, it is obvious that a choice of α_1, α_2 and α_3 cannot be made arbitrarily. In fact, it can be verified by inspecting Eqs. (58) that it is very easy to choose values of α_1, α_2 and α_3 such that h_1, h_2, or

h_3 becomes negative. The reason for this is that the cone has a narrow base. For values of α_1, α_2 and α_3 corresponding to points lying outside the cone, the first three modes are asymptotically stable, but some of the higher modes are likely to be destablized.

As an illustration of the case in which arbitrarily chosen poles destabilize the higher modes, let us consider the shifts in the first three poles

$$\alpha_1 = 3\alpha, \ \alpha_2 = 2\alpha, \ \alpha_3 = \alpha \tag{60}$$

Inserting Eqs. (60) into Eqs. (58), we obtain the control gains

$$h_1 = -1.8546 \ \alpha m, \ h_2 = -0.7646 \ \alpha m, \ h_3 = 1.7858 \ \alpha m \tag{61}$$

To determine the shift in the poles 4,5 and 6, we refer to Eq. (50b) and compute

$$\begin{aligned} b_{41} &= 0.1605 \ m^{-1}, \ b_{42} = 0.1405 \ m^{-1}, \ b_{43} = 1.046 \ m^{-1} \\ b_{51} &= 1.017 \ m^{-1}, \ b_{52} = 1.036 \ m^{-1}, \ b_{53} = 1.036 \ m^{-1} \\ b_{61} &= 1.911 \ m^{-1}, \ b_{62} = 1.920 \ m^{-1}, \ b_{63} = 1.029 \ m^{-1} \end{aligned} \tag{62}$$

Then, inserting Eqs. (61) and (62) into Eqs. (51a), we obtain

$$\alpha_4 = 2.9257 \ \alpha, \ \alpha_5 = -1.6563 \ \alpha, \ \alpha_6 = -6.3490 \ \alpha \tag{63}$$

so that modes 5 and 6 are destablized by the choice (60).

One suitable choice, i.e. one lying inside the cone, is that in which the shifts in the first three poles are

$$\alpha_1 = \alpha, \ \alpha_2 = 2\alpha, \ \alpha_3 = 3\alpha \tag{64}$$

In this case, the control gains become

$$h_1 = 1.8551 \ \alpha m, \ h_2 = 0.7651 \ \alpha m, \ h_3 = 0.2223 \ \alpha m \tag{65}$$

Because $h_i > 0$ $(i = 1,2,3)$, it follows from Eqs. (51a) that all the expressions on the left side represent positive definite quadratic forms, so that all the closed-loop poles are shifted to the left of the imaginary axis. Inserting Eqs. (64) and (65) into Eqs. (51a), we obtain

$$\alpha_4 = 0.3190\ \alpha \quad , \quad \alpha_5 = 1.4547\ \alpha \quad , \quad \alpha_6 = 2.6212\ \alpha \tag{66}$$

indicating that now the modes 4, 5 and 6 are damped adequately in comparison to the first three modes.

It will prove of interest to examine the accuracy of the pole-placement technique based on the perturbation scheme. To this end, we propose to solve the closed-loop eigenvalue problem for the successful choice, i.e., for the case in which the gains are given by Eqs. (65). Because the solution of the eigenvalue problem is strictly a numerical problem, we must assign values to the system parameters. For convenience, we choose $\alpha = 1$, $m = 1$, $EI = 1$, $L = 1$, where EI is the bending stiffness. Using Eqs. (41b), in conjunction with the gains given by Eqs. (65), we obtain

$$H = \begin{bmatrix} 2.0000 & 0.5407 & 0.6956 & 0.1049 & 0.6833 & -2.4547 \\ & 4.0000 & 1.4311 & 1.5707 & -1.9965 & -3.4627 \\ & & 6.0000 & 0.2487 & -3.2446 & -2.7830 \\ & & & 0.6379 & -0.6891 & -1.1943 \\ \text{symm.} & & & & 2.9093 & 1.2780 \\ & & & & & 5.2424 \end{bmatrix} \tag{67}$$

On the other hand, because we are only using velocity feedback, $G = 0$. Moreover, the matrix of natural frequencies is

$$\Omega = \text{diag}[3.516 \quad 22.034 \quad 61.697 \quad 120.901 \quad 199.860 \quad 298.557] \tag{68}$$

The eigensolution was obtained by truncating A to a 4×4, a 5×5 and a 6×6 matrix. The corresponding closed-loop eigenvalues are displayed in Table I. Comparing the values in Eqs. (64) and (66) with the corresponding ones in Table I, we conclude that the results obtained by the perturbation approach are accurate to the fourth significant figure. It is also easy to verify that truncation of the matrix A does not affect the eigenvalues materially. Hence, the perturbation approach to the computation of the control gains for pole allocation in conjunction with direct feedback control gives sufficiently accurate results, at least in this particular example.

11. CONCLUSIONS

Control of distributed structures requires distributed actuators and sensors. Practical considerations dictate that control implementation be carried out by means of discrete actuators and sensors. Moreover, it is impossible to control or estimate the entire infinity of modes, so that control must be limited to a finite number of modes. Problems of modal control and estimation remain when the natural frequencies are closely spaced, as is often the case with two- and three-dimensional structures.

One approach not requiring modal state estimaton is direct feedback control, in which an actuator at a given point of a structure generates a force input depending on the sensor output at the same point. For linear control, the gain matrix consists of two diagonal submatrices. The question remains as to how to produce the control gains. Two widely used techniques are pole allocation and optimal control. The diagonal nature of the gain matrix characterizing direct feedback control is likely to cause difficulties.

In the pole allocation method, the closed-loop poles are selected first and the gains matching these poles are computed subsequently. There are two factors that may limit the freedom to choose closed-loop poles in direct feedback. In the first place, the gain matrix has a special nature, characterized by the off-diagonal entries being equal to zero, which can be intepreted as placing constraints on the gains. In the second place, the control gains must be such that the uncontrolled modes are not destabilized. We recall that for a distributed structure there are always uncontrolled modes.

This paper develops a perturbation approach to the computation of control gains corresponding to given closed-loop poles, whereby in the first approximation the problem reduces to the solution of linear algebraic equations for the control gains. The approach reveals an inherent difficulty in the use of pole placement in conjunction with direct feedback control. In particular, whereas in computing gains for a discrete system in which all the modes are controlled the problem can be regarded as solved provided controllability is satisfied, here the gains are constrained by the requirement that the higher modes not be destabilized. This can be guaranteed by requiring that all the gains be positive. Hence, physical considerations dictate that the only admissible solutions of the algebraic equations for the control gains are those in which all the components of the solution vector are positive. Because this cannot be guaranteed for any preselected closed-loop poles, it follows that the closed-loop poles cannot be chosen arbitrarily. If we envision a space defined by the real part of the closed-loop poles, then the admissible controls lie in a certain cone-shaped subregion of constraint of that space.

The question can be raised as to whether it is possible to draw such sweeping conclusions from a first-order perturbation analysis. The answer must be affirmative. Indeed, for small real parts of the closed-loop poles, the first-order perturbation yields accurate results. As the real parts increase in magnitude, the constraints on the control gains remain, so that the nature of the problem does not change. The likely outcome of a higher-order perturbation is to affect the boundaries of the cone of constraint, in the sense that the boundaries become curved surfaces tangent to the hyperplanes of constraint at the origin, but cannot negate the existence of such subdomains of constraint. It should be pointed out that, in the absence of displacement feedback, a second-order perturbation does not affect the real parts of the eigenvalues.

The ideas presented in this paper are demonstrated via a numerical example in which an attempt is made to control a cantilever beam by means of three point actuators while placing three poles. Placing the poles so that the real parts lie outisde the cone of constraint yields instability, thus showing that poles cannot be placed arbitrarily. On the other hand, placing the poles so that the real parts lie inside the cone yields stability. Then, using the computed gains to generate the matrix of coefficients A, the closed-loop eigenvalue problem corresponding to the stable case is solved "exactly," i.e., without the use of a perturbation analysis. The first six computed eigenvalues agree to the fourth significant figure with those achieved by the perturbation approach to pole placement.

12. REFERENCES

1. MEIROVITCH, L., Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, The Netherlands, 1980.
2. MEIROVITCH, L. and SILVERBERG, L. M., "Globally Optimal Control of Self-Adjoint Distributed Systems," Optimal Control Applications and Methods, Vol. 4, 1983, pp. 365-386.
3. PORTER, B. and CROSSLEY, T. R., Modal Control-Theory and Applications, Taylor and Francis, London, 1972.

4. MEIROVITCH, L. and BARUH, H., "Control of Self-Adjoint Distributed-Parameter Systems, Journal of Guidance, Control, and Dynamics, Vol. 5, No. 1, 1982, pp. 60-66.

5. CURTAIN, R. F. and PRITCHARD, A. J., "The Infinite-Dimensional Riccati Equation," Journal of Mathematical Analysis and Applications, Vol. 47, 1974, pp. 43-57.

6. BALAS, M. J., "Active Control of Flexible Systems," Journal of Optimization Theory and Applications, Vol. 24, No. 3, 1978, pp. 415-436.

7. BROGAN, W. L., Modern Control Theory, QPI Publishers, New York, 1974.

8. WILKINSON, J. H. The Algebraic Eigenvalue Problem, Oxford University Press, London, 1965.

TABLE I - Closed-Loop Eigenvalues from Truncated A

	A is 4×4		A is 5×5		A is 6×6	
r	Re λ_r	Im λ_r	Re λ_r	Im λ_r	Re λ_r	Im λ_r
1	-1.00068	± 3.37175	-1.00071	± 3.37173	-1.00075	± 3.37175
2	-2.00108	±21.94475	-2.00113	±21.94574	-2.00141	±21.94696
3	-2.99873	±61.59982	-2.99991	±61.60840	-3.00046	±61.61201
4	-0.31851	±120.88660	-0.31857	±120.88730	-0.31889	±120.88810
5			-1.45320	±199.80580	-1.45360	±199.80940
6					-2.61968	±298.48120

List of Figures

1. Figure 1 - A Cantilever Beam Controlled by Three Actuators

2. Figure 2 - Region of Admissible Controls

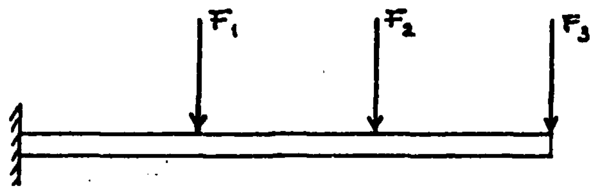


Fig. 1 A Cantilever Beam Controlled by Three Actuators

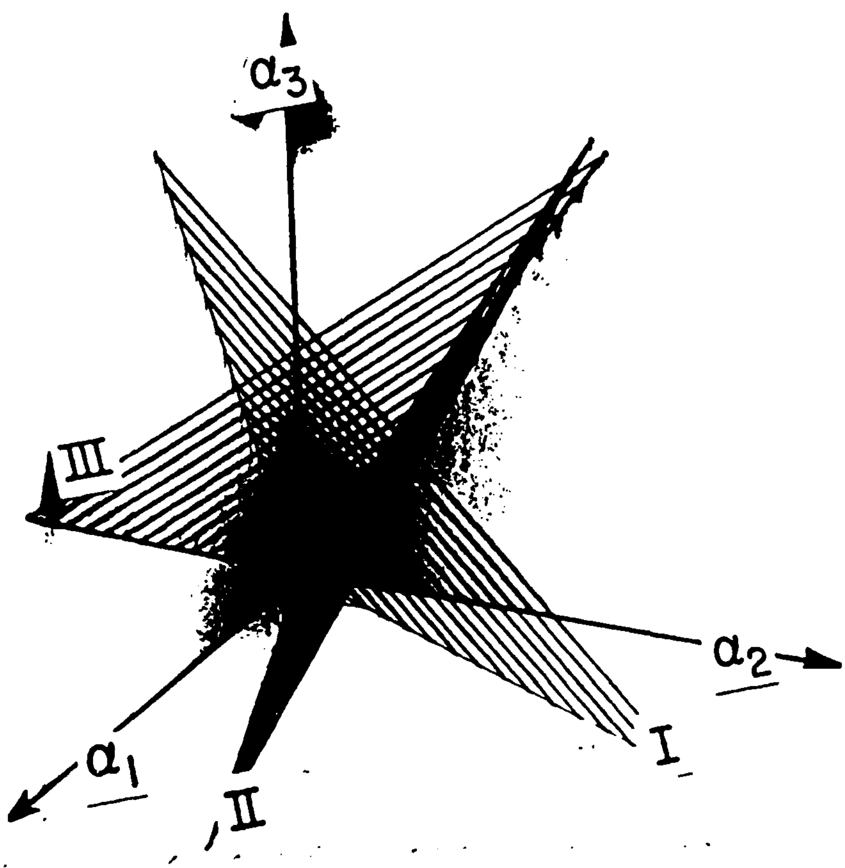


Fig. 2 Region of Admissible Controls

END
5-87
DTIC