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INTRODUCTION AND TERMINOLOGY 2-EXTENDABILITY IN
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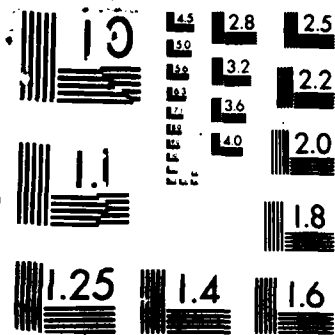
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1. INTRODUCTION AND TERMINOLOGY

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2-EXTENDABILITY IN 3-POLYTOPES

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by

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1. Introduction and Terminology

Suppose G is a graph with p points and let n be a positive integer such that $p \geq 2n + 2$. Graph G is said to be n -extendable if every matching of size n in G extends to a perfect matching. (We will abbreviate the term "perfect matching" to "p.m." hereafter.) A graph G is called bicritical if $G - u - v$ has a perfect matching, for all pairs of points $u, v \in V(G)$. In [LP2] a canonical decomposition theory for graphs in terms of their maximum (or, when present, perfect) matchings is discussed at length. Bicritical graphs play an important roll in this theory. In particular, those bicritical graphs which are 3-connected (the so-called bricks) currently represent the "atoms" of the decomposition theory in that no further decomposition of these graphs has been obtained as yet. Indeed at present we seem far from an understanding of the structure of bicritical graphs or even that of bricks.

Although interesting in its own right, the study of n -extendability became more important when in [Plu1] it was shown that every 2-extendable non-bipartite graph is a brick and that, for $n \geq 2$, every n -extendable graph is also $(n - 1)$ -extendable. Thus we have available for study a nested set of subcollections of bicritical graphs.

The results of the present paper will be presented in two parts. In Section 2, we present some new procedures for constructing infinite

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families of bricks by means of constructing families of non-bipartite 2-extendable graphs.

In Section 3, we focus our attention on extending matchings in planar graphs. Planar 1-extendable graphs abound; for example, every cubic 3-polytopal graph has this property. On the other hand, it has been shown [Plu2] that no planar graph is 3-extendable. Between these extremes, then, lies the class of planar 2-extendable graphs.

We will further restrict our attention in Section 3 to cubic 3-connected planar graphs (or cubic polytopal graphs, as they are sometimes called). Which of these graphs are 2-extendable? Our main result states that any cubic polytopal graph which is cyclically 4-connected, but has no quadrilateral face, is 2-extendable. (Clearly, if such a graph has no quadrilateral face, it can have no cycle of length 4 at all.) In particular, cubic polytopal graphs with cyclic connectivity at least 5 must be 2-extendable.

2. Some families of bicritical and 2-extendable graphs

The complete graphs on an even number of points, $\{K_{2r}\}_{r=1}^{\infty}$ are trivially bicritical, as are the *wheel graphs* with even total number of points. (A **wheel graph** is obtained by joining every point of a cycle to a common point (or "hub") not on the cycle.) The first non-trivial class of graphs proven to be bicritical seems to be the class of **Halin graphs**. (See [LP1]. These graphs have also been called **based polyhedra** by Rademacher [R1] and **roofless polyhedra** by Pólya.) These graphs are sometimes indicated by $T \cup C$ where T is a tree on an even number of points in which each non-endpoint has minimum degree 3, and C is a cycle through the endpoints of tree T so that $T \cup C$ is planar.

For $i = 1, 2$, let G_i be a graph containing a point v_i of degree 3. Further suppose that the neighbors of v_i in G_i are $\{x_i, y_i, z_i\}$. Let us denote by $G_1(v_1v_2)G_2$ (or simply $G_{1,2}$ when the v_i 's are understood) the graph obtained from G_1 and G_2 by deleting points v_1 and v_2 and then inserting the lines x_1x_2, y_1y_2 and z_1z_2 . We shall have occasion to call this operation **3-joining**. (See Figure 2.1.)

2.1. THEOREM. Suppose G_1, G_2, v_1, v_2 and $G_{1,2}$ are as given above. Then:

- (a) if G_1 and G_2 are bicritical, so is $G_{1,2}$.
- (b) if G_1 and G_2 are 2-extendable and non-bipartite, so is $G_{1,2}$.

PROOF. Let $G'_i = G_i - v_i$, for $i = 1, 2$. To prove (a), let u_1 and u_2 be two distinct points in graph $G_{1,2}$. First suppose $u_1, u_2 \in V(G'_1)$. Let

FIGURE 2.1.

P_1 be a p.m. of $G_1 - u_1 - u_2$ and without loss of generality let v_1x_1 be the line of P_1 covering point v_1 . Then let P_2 be a p.m. of G_2 containing line v_2x_2 . (Remember that every bicritical graph is 1-extendable.) Then $P_1 \cup P_2 - v_1x_1 - v_2x_2 + x_1x_2$ is a p.m. for $G_{1,2} - u_1 - u_2$.

Now suppose $u_1 \in V(G'_1)$ and $u_2 \in V(G'_2)$. Let P_i be a p.m. for $G_i - u_i - v_i$ for $i = 1, 2$. Then $P_1 \cup P_2$ is a p.m. in $G_{1,2}$.

To prove (b), let e_1 and e_2 be two independent lines in $G_{1,2}$.

Before proceeding, let us note that since each G_i is 2-extendable, $|V(G_i)| \geq 6$ and hence $\{x_i, y_i, z_i\}$ is a cutset in G_i . Moreover, by Theorem 3.2 of [Plu1], each G_i is 3-connected. But then by Theorem 2.2 of [Plu3] each cutset $\{x_i, y_i, z_i\}$ is independent in G_i .

Now suppose that $\{e_1, e_2\} \subseteq V(G'_1)$. Let P_1 be a p.m. of G_1 containing e_1 and e_2 . Since $\{x_1, y_1, z_1\}$ is independent, without loss of generality we may assume that x_1v_1 is the line of P_1 covering point v_1 . Then if P_2 is a p.m. of G_2 containing line x_2v_2 , the matching $P_1 \cup P_2 - x_1v_1 - x_2v_2 + x_1x_2$ is a p.m. for $G_{1,2}$ containing e_1 and e_2 .

Secondly, suppose $e_1 \in E(G'_1)$ and $e_2 \in E(G'_2)$. Now line e_1 meets at most one of the points x_1, y_1 and z_1 , since $\{x_1, y_1, z_1\}$ is independent in G_1 , so among the pairs $\{x_1, x_2\}$, $\{y_1, y_2\}$ and $\{z_1, z_2\}$ we can choose one - say $\{x_1, x_2\}$ - such that neither x_1 nor x_2 is covered by either e_1 or e_2 . For $i = 1, 2$, let P_i be a p.m. for G_i containing e_i and x_iv_i . Then $P_1 \cup P_2 - x_1v_1 - x_2v_2 + x_1x_2$ is a p.m. for $G_{1,2}$ containing e_1 and e_2 .

Thirdly, suppose $e_1 = x_1x_2$ and $e_2 \in E(G'_2)$. Let P_1 be a p.m. for G_1 containing x_1v_1 and let P_2 be a p.m. for G_2 containing x_2v_2 and e_2 . Then $P_1 \cup P_2 - x_1v_1 - x_2v_2 + x_1x_2$ is a p.m. for $G_{1,2}$ containing e_1 and e_2 .

Finally, suppose $\{e_1, e_2\} \subseteq \{x_1x_2, y_1y_2, z_1z_2\}$, say without loss of generality that $e_1 = x_1x_2$ and $e_2 = y_1y_2$. Now since each G_i is non-

bipartite, it is bicritical by Theorem 4.2 of [Plu1] and hence $G_i - x_i - y_i$ has a p.m. P_i . Moreover, P_i must contain line $v_i z_i$. But then $P_1 \cup P_2 - v_1 z_1 - v_2 z_2 + z_1 z_2 + e_1 + e_2$ is a p.m. for $G_{1,2}$ containing lines e_1 and e_2 .

It only remains to show that $G_{1,2}$ is non-bipartite. But since G_1 and G_2 are both non-bipartite, they are bicritical (again using Theorem 4.2 of [Plu1]) and hence by part (a) of the present theorem, $G_{1,2}$ is also bicritical. But no bicritical graph is bipartite and this completes the proof. ■

Since the operation of 3-joining preserves the properties of being cubic, 3-connected and planar, it may be used to obtain infinite families of bicritical polytopal graphs and infinite families of 2-extendable cubic 3-polytopal graphs. For example, let G_1 and G_2 be two copies of the dodecahedron and let v_i be any point in G_i for $i = 1, 2$. Then $G_{1,2}$ is again 2-extendable and, since it is non-bipartite, it is also bicritical.

Let us point out that the complete graph on four points, K_4 , is a much smaller starting graph for generating bicritical graphs by 3-joining than is the dodecahedron, but K_4 is not 2-extendable. Note that if one joins two copies of K_4 together via a 3-joining, the resulting 6-point graph (the so-called *triangular pyramid*) is not 2-extendable either.

It is also interesting to point out that, although 3-joining preserves 2-extendability in non-bipartite graphs, this operation when applied to bipartite graphs, *never* preserves 2-extendability! More particularly, see Corollary 2.3 which follows the next theorem.

A set of lines $L = \{e_1, \dots, e_k\} \subseteq E(G)$ is a **cyclic k -cut** in a connected graph G if $G - L$ consists of two components each of which contains a cycle.

2.2. THEOREM. *Suppose G is a 2-extendable graph with a cyclic 3-cut $L = \{e_1, e_2, e_3\}$. Then if H_1 and H_2 are the components of $G - L$, neither H_1 nor H_2 is bipartite.*

PROOF. Suppose $L = \{e_1, e_2, e_3\}$ is a cyclic 3-cut and that $e_1 = x_1 x_2$, $e_2 = y_1 y_2$ and $e_3 = z_1 z_2$ where $\{x_i, y_i, z_i\} \subseteq V(H_i)$. Let $H'_i = H_i - x_i - y_i - z_i$ for $i = 1, 2$.

First we claim that L is a matching. For suppose not. Without loss of generality, we may suppose that e_1 and e_2 are adjacent at point $x_1 = y_1$. Then since H_1 contains a cycle, it must contain a point $u \notin \{x_1, y_1, z_1\}$. But then $\{x_1, z_1\}$ is a cutset in G . (We point out that z_1 may or may not be different from $x_1 = y_1$ here.) But this contradicts the fact that G must be 3-connected by Theorem 3.2 of [Plu1] and the claim is proved.

Suppose that $V(H'_1) = \emptyset$. Then since G is 3-connected, H_1 must

be a triangle on points x_1, y_1 and z_1 . But then let f be any line in H_2 incident with x_2, y_2 or z_2 . Without loss of generality, suppose f is incident with x_2 . Then lines $\{f, y_1 z_1\}$ do not extend to a p.m. in G , a contradiction.

Thus we may assume both $V(H'_1) \neq \emptyset$ and $V(H'_2) \neq \emptyset$. However, then for both $i = 1, 2$, the set $\{x_i, y_i, z_i\}$ is a point cutset in G and again by Theorem 2.2 of [Plu3], each is independent. But then since $\text{mindeg } G \geq 3$, it follows immediately that $|V(H'_i)| \geq 2$, for $i = 1, 2$.

Suppose now that H_2 is bipartite. There are then two cases to treat, depending upon the parity of $|V(H_2)|$.

(a) Suppose first that $|V(H_2)|$ is even. Let the bipartition of H_2 be (A_2, B_2) . Then among the points x_2, y_2, z_2 some two must belong to the same color class. Without loss of generality, suppose that $x_2, y_2 \in A_2$. Now let P_1 be a p.m. of G containing lines e_1 and e_2 . By parity, P_1 must match z_2 into H'_2 . Hence $|A_2| = |B_2| + 2$.

But now since $|V(H'_2)| \geq 2$, and $\{x_2, y_2, z_2\}$ is an independent cutset, there must be two independent lines joining x_2 and y_2 to two points in $V(H'_2)$. Call these lines f_1 and f_2 . Let P_2 be a p.m. of G containing lines f_1 and f_2 . Then by parity, since H_2 is even, P_2 must match z_2 into H'_2 as well. So P_2 restricted to H_2 is a p.m. of H_2 and hence $|A_2| = |B_2|$, a contradiction. Hence $|V(H_2)|$ cannot be even.

Suppose, therefore, that $|V(H_2)|$ is odd. As before, we may assume that (A_2, B_2) is the bipartition of H_2 with $x_2, y_2 \in A_2$.

First suppose that $z_2 \in A_2$ as well. Let P_3 be a p.m. for G containing e_1 and e_2 — and hence by parity — e_3 as well. Thus $|A_2| = |B_2| + 3$. Once again, as in Case (a) above, there must be two independent lines f_1 and f_2 matching x_2 and y_2 into H'_2 . Let P_4 be a p.m. of G containing f_1 and f_2 . Then by parity, P_4 contains e_3 . But then $|A_2| = |B_2| + 1$, a contradiction.

Thus we may assume that $z_2 \in B_2$. Let P_5 be a p.m. of G containing e_1 and e_2 and — by parity — e_3 as well. Thus $|A_2| = |B_2| + 1$. But as before, without loss of generality, we may assume there are two independent lines f_1 and f_2 matching x_2 and y_2 into H'_2 . Let P_6 be a p.m. for G containing lines f_1 and f_2 . Then, by parity, line $e_3 = z_1 z_2 \in P_6$. Then $|A_2| = |B_2| - 1$, a contradiction.

Thus H_2 is not bipartite and, similarly, neither is H_1 . ■

2.3. COROLLARY. *If G is a 2-extendable bipartite graph, then G is cyclically 4-connected.* ■

We can in fact be a bit more precise about just which sets of two

independent lines extend to p.m.'s in a graph obtained from two bipartite 2-extendable graphs by 3-joining. To this end we provide the following result.

2.4. THEOREM. *Let G_1 and G_2 be 2-extendable bipartite graphs and suppose $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ are both points of degree 3. Then:*

- (a) *the 3-join graph $G_1(v_1v_2)G_2 = G_{1,2}$ is 1-extendable and bipartite and*
- (b) *if e_1 and e_2 are two independent lines in $G_{1,2}$ then they extend to a p.m. of $G_{1,2}$ if and only if at least one of e_1 and e_2 is not a join line.*

PROOF. Suppose the bipartition of G_i is $A_i \cup B_i$ where $v_i \in A_i$ for $i = 1, 2$. Then $|A_i| = |B_i|$ and $(B_1 \cup A_2 - v_2) \cup (A_1 \cup B_2 - v_1)$ is a bipartition of $G_{1,2}$.

Now let e_1 and e_2 be independent lines in $G_{1,2}$. If $\{e_1, e_2\} \subseteq E(G'_1)$, if $e_1 \in E(G'_1)$ and $e_2 \in E(G'_2)$, or if e_1 is a join line and $e_2 \in E(G'_2)$, say, then $\{e_1, e_2\}$ extends to a p.m. of $G_{1,2}$ by Theorem 2.1(b).

So suppose $e_1 = x_1x_2$ and $e_2 = y_1y_2$ are both join lines. Further, suppose that P is a p.m. for $G_{1,2}$ containing e_1 and e_2 . Then since $|V(G'_i)|$ is odd for $i = 1, 2$, P must contain $e_3 = z_1z_2$ as well. But $\{x_1, y_1, z_1\} \subseteq B_1$ and hence P must match $B'_1 = B_1 - \{x_1, y_1, z_1\}$ onto $A_1 - v_1$ which is impossible since $|B'_1| = |A_1 - v_1| - 2$. Thus there is no p.m. for $G_{1,2}$ containing e_1 and e_2 . This completes the proof of part (b).

It remains only to show that every join line extends to a p.m. of $G_{1,2}$. Let us consider the join line $e = x_1x_2$. Now each G_i is 1-extendable by Theorem 2.2 of [Plu1]. Thus let P_i be a p.m. for $G_i - x_i - v_i$ for $i = 1, 2$. Then $P = P_1 \cup P_2 - x_1v_1 - x_2v_2 + x_1x_2$ is a p.m. for $G_{1,2}$. ■

We now present a second construction. Again for $i = 1, 2$, let G_i be a graph containing a point v_i of degree 3. Suppose once again that the neighbors of v_i in G_i are $\{x_i, y_i, z_i\}$. Let us denote by $G_1\text{hex}(v_1, v_2)G_2$ (or simply $G_1\text{hex}G_2$ when the v_i 's are understood) the graph obtained from G_1 and G_2 by deleting v_1 and v_2 , adding a hexagon on 6 new points, $a_1a_2b_1b_2c_1c_2a_1$, and adding lines x_ia_1 , y_ib_1 and z_ic_1 for $i = 1, 2$. We will call $G_1\text{hex}G_2$ a **hex-join** of G_1 and G_2 . (See Figure 2.2.)

We then have the following result which is parallel to Theorem 2.1.

2.5. THEOREM. *Suppose G_1, G_2, v_1, v_2 and $G_1\text{hex}G_2$ are as given above. Then:*

- (a) *If G_1 and G_2 are bicritical, so is $G_1\text{hex}G_2$, and*

FIGURE 2.2.

(b) If G_1 and G_2 are 2-extendable and non-bipartite, then so is $G_1 \text{ hex } G_2$.

PROOF. (a). The proof here is in much the same spirit as the proof of part (a) of Theorem 2.1. Hence we will treat only one representative case and leave the rest to the reader. (Again we adopt the labeling shown in Figure 2.2. The reader should observe that the symmetry displayed in Figure 2.2 substantially reduces the number of cases which need to be treated.)

Let us suppose that $u \in \{a_1, b_1, c_1\}$. Without loss of generality, suppose that $u_1 = a_1$. Also suppose $v \in \{a_2, b_2, c_2\}$. By symmetry we need treat only two subcases, namely when $v = a_2$ and when $v = b_2$.

If $v = a_2$, let P_1 be a p.m. of $G - y_1 - v_1$ and let P_2 be a p.m. of $G_2 - y_2 - v_2$. Then $P_1 \cup P_2 + y_1b_1 + y_2b_2 + z_1c_2$ is a p.m. for $(G_1 \text{ hex } G_2) - u - v$. If $v = b_2$, let P_1 be a p.m. for $G_1 - y_1 - v_1$ and P_2 , a p.m. for $G_2 - x_2 - v_2$. Then $P_1 \cup P_2 + y_1b_1 + x_2a_2 + c_1c_2$ is a p.m. for $(G_1 \text{ hex } G_2) - u - v$.

Similarly, the proof of part (b) here mimics that of Theorem 2.1. We therefore present only two representative cases, one in which the bicriticality of G_1 and G_2 is used (and hence the assumption that each is non-bipartite) and one in which it is not used.

So first let $e_1 = x_1a_1$ and $e_2 = y_1b_1$. We seek a p.m. for

$G_1 \text{ hex } G_2$ containing e_1 and e_2 . Since G_1 is bicritical, there is a p.m. P_1 for $G_1 - x_1 - y_1$ and it must contain line z_1v_1 . Similarly, there is a p.m. P_2 for $G_2 - x_2 - y_2$ and it must contain line z_2v_2 . But then $P_1 \cup P_2 - z_1v_1 - z_2v_2 + x_1a_1 + y_1b_1 + z_1c_1 + x_2a_2 + y_2b_2 + z_2c_2$ is a p.m. for $G_1 \text{ hex } G_2$ containing e_1 and e_2 .

Finally, let $e_1 = a_1a_2$ and $e_2 = b_1b_2$. Then let P_1 be a p.m. for G_1 containing z_1v_1 and P_2 be a p.m. for G_2 containing z_2v_2 . Then $P_1 \cup P_2 - z_1v_1 - z_2v_2 + z_1c_1 + z_2c_2 + a_1a_2 + b_1b_2$ is a p.m. for $G_1 \text{ hex } G_2$ containing e_1 and e_2 . ■

3. The main result

Recall from the previous section that the construction procedures called 3-joining and hex-joining preserve the properties of 3-regularity, 3-connectivity, planarity, bicriticality and 2-extendability. On the other hand, since each of these operations automatically inserts a cyclic cutset of size 3, cyclic connectivity is *not* necessarily preserved.

A question which arose early in the studies culminating in this paper was whether or not a cubic 3-connected planar graph (hereafter called a **simple 3-polytopal graph**) of sufficiently high cyclic connectivity must be 2-extendable. (For more information on polytopal graphs, the reader is referred to the classical book of Grünbaum [G1]. Suffice it to say, for our purposes, that the 3-connected planar graphs are called polytopal because they are just the skeleta of 3-polytopes by a celebrated theorem of Steinitz [S1].)

Examples showing that cyclic 3- and 4-connectivity are not sufficient to insure 2-extendability in cubic 3-polytopal graphs are presented at the end of this section.

We now present our main result.

3.1. THEOREM. *If G is a cubic 3-polytopal graph which is cyclically 4-connected and has no faces of size 4, then G is 2-extendable.*

PROOF. Before proceeding, we would point out that the hypotheses of this theorem imply that G cannot have any 3-cycles or 4-cycles.

Now suppose G satisfies the hypotheses of the theorem, but is not 2-extendable. So let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be two independent lines in G which do not lie in a p.m. for G . Thus graph $G' = G - x_1 - y_1 - x_2 - y_2$ has no perfect matching and hence, by Tutte's classical theorem on perfect matchings, there is a set $S' \subseteq V(G)$ such that $c_o(G' - S') > |S'|$,

where $c_o(G' - S')$ denotes the number of odd components of $G' - S'$. Then since $|V(G')|$ is even, parity dictates that $c_o(G' - S') \geq |S'| + 2$.

Suppose in fact that $c_o(G' - S') \geq |S'| + 3$ and hence again by parity that $c_o(G' - S') \geq |S'| + 4$. Now G is 1-extendable, by a result of Plesník [Ple1] (and independently by a result of Little, Grant and Holton [LGH1, LGH2]). So line $e_2 = x_2y_2$ lies in a p.m. for G and hence $G'' = G - x_2 - y_2$ has a p.m. But then in G'' we have a set $S'' = S' \cup \{x_1, y_1\}$ such that $c_o(G'' - S'') = c_o(G' - S') \geq |S'| + 4 = |S''| + 2$. But then by Tutte's Theorem, graph G'' has no p.m., a contradiction. Thus $c_o(G' - S') = |S'| + 2$.

Let $S = S' \cup \{x_1, y_1, x_2, y_2\}$. We claim that $G - S$ has no even components. For suppose C_e were such an even component. Then since G is 3-connected, there must be at least 3 (and hence by parity, at least 4) lines from C_e to S . These lines, together with e_1 and e_2 , imply that no more than $3|S| - 8$ lines are sent from S to the odd components of $G - S$. But viewing these lines from $G - S$, each odd component must send at least 3 lines to S and hence there are at least $3(|S| - 2) = 3|S| - 6$ of these lines, so we have a contradiction. Thus $G - S$ has no even components.

Let N denote the number of lines joining S to $G - S$. Note that since G is 3-connected, each odd component of $G - S$ must send at least 3 lines to S and hence $N \geq 3(|S| - 2) = 3|S| - 6$. So we have the inequality $3|S| - 6 \leq N \leq 3|S| - 4$. Accordingly, there are three cases to consider.

Case 1. Suppose $N = 3|S| - 6$. So in S there exists precisely one more line e_3 , in addition to lines e_1 and e_2 , and each odd component of $G - S$ sends exactly 3 lines up to S . Thus up to a relabeling of the three e_i , G has the appearance of one of the three graphs shown in Figure 3.1.

Henceforth we will denote by $C_1, C_2, \dots, C_{|S|-2}$ the odd components of $G - S$.

Suppose now that all the C_i 's are singletons. Then a well-known variation on Euler's formula relating the number of points, lines and faces of any planar graph yields $\sum_i (6 - i)f_i = 12$, where f_i denotes the number of faces containing i lines in their boundary. Hence $3f_3 + 2f_4 + f_5 \geq 12$ and since $f_3 = f_4 = 0$, we must have $f_5 \geq 12$. But $G - e_1 - e_2 - e_3$ is bipartite, and it then follows that $f_5 \leq 6$, a contradiction.

Hence we may assume that there exists one of the C_i — say C_1 — with $|V(C_1)| \geq 3$.

Claim 1. Component C_1 contains a cycle.

For suppose not. Then it must be a tree with at least 2 endpoints and hence it must send at least 4 lines to S , a contradiction.

Claim 2. Subgraph $G_1 = G[V(G) - V(C_1)]$ contains a cycle.

FIGURE 3.1.

Suppose not. Then G_1 is a forest containing at least 3 lines, so it must contain at least 2 endpoints. But then G_1 must send at least 4 lines down to C_1 , again a contradiction.

So we have shown that the 3 lines joining C_1 to G_1 are a *cyclic* cutset of size 3, contradicting the hypothesis that G is cyclically 4-connected.

Case 2. Suppose $3|S| - 5$. But then since there are exactly $|S| - 2$ odd components in $G - S$ and each must send at least 3 lines to S by the 3-connectivity of G , we must have one odd component of $G - S$ sending exactly 4 lines to S and all the rest sending exactly 3. But G is cubic, so it is impossible for any odd component of $G - S$ to send an even number of lines to S and we have a contradiction.

Case 3. So we may assume that $3|S| - 4 = N$. So there must be exactly two lines in the induced subgraph $G[S]$ and they must be e_1 and e_2 . Since no odd component of $G - S$ can send exactly 4 lines to S by parity, but all odd components must send at least 3 lines to S by 3-connectivity, we must have exactly one odd component of $G - S$, without loss of generality say it is C_1 , sending at least 5 lines to S . (So component C_1 must contain at least 3 points.) But then we have $3|S| - 4 = N \geq 5 + 3(|S| - 3) = 3|S| - 4$, and it follows that we must have *exactly one* odd component which sends *exactly* 5 lines to S , this odd

FIGURE 3.2.

component must contain at least 3 points and all other odd components of $G - S$ send exactly 3 lines to S .

Let C_2 be any other odd component of $G - S$ different from C_1 . Suppose component C_2 is not a singleton.

By the arguments of Claims 1 and 2 of Case 1 above, component C_2 and subgraph $G[V(G) - V(C_2)]$ both contain cycles.

But then we have a cyclic cutset joining C_2 and G_2 containing exactly 3 lines, contradicting the hypothesis that G is cyclically 4-connected.

Thus component C_1 is an odd component sending exactly 5 lines to S and all the remaining odd components of $G - S$, namely, $C_2, \dots, C_{|S|-2}$, are *singletons* incident with exactly 3 lines from S . (See Figure 3.2.)

Claim 3. Component C_1 contains at least 5 points.

Suppose not. Then $|V(C_1)| = 3$. But then C_1 must be a path of length 2, since we know graph G contains no triangles. Let the two adjacent lines of C_1 be denoted by e_5 and e_6 . As before, using the Euler formula to do a face count, we have $3f_3 + 2f_4 + f_5 \geq 12$ and, since $f_3 = f_4 = 0$, we have $f_5 \geq 12$. But $G - e_1 - e_2 - e_5 - e_6$ is bipartite and hence G has $f_5 \leq 8$, a contradiction and Claim 3 is proved.

Claim 4. Component C_1 contains a cycle.

If not, it must be a tree with at least 2 endpoints and hence sends at least 4 lines to S . If it had at least 3 endpoints, it would have to send at least 6 lines to S , a contradiction. Thus tree C_1 contains exactly 2 endpoints and hence must be a path. But since C_1 sends exactly 5 lines to S , it must be a path of length 2, contradicting Claim 3.

Let G' denote the graph obtained from G by contracting component C_1 to a single point u_1 . Of course G' is planar since G is and G' has a single point u_1 of degree 5 and all others of degree 3. It is possible that by contracting C_1 to a point we have introduced parallel lines in G' . However, if $p' = |V(G')|$, $q' = |E(G')|$ and r' denotes the number of faces in any imbedding of graph G' in the plane, using Euler's formula, we have $\sum_i (6-i)f'_i = 6r' - 2q' = 6r' - 3p' - 2 = 6(2+q'-p') - 3p' - 2 = 16$, since in G' we have $2q' = 3p' + 2$. So, in particular, in G' we have $4f'_2 + 3f'_3 + 2f'_4 + f'_5 \geq 16$.

Now since induced subgraph $G[S]$ contains only 2 lines, there can be at most 4 odd faces in G' . Hence $f'_3 + f'_5 \leq 4$. Thus $2f'_2 + f'_3 + f'_4 \geq 6$. But G has no faces of size 3 or 4, so all triangular or quadrilateral faces in G' must contain u_1 in their boundary. Thus in G' we also have But since $\deg u_1 = 5$ in G' , we also have $f'_3 + f'_4 \leq 5$. Hence $f'_2 \geq 1$. This implies that in G we have $|V(L) \cap S| \leq 4$ where L denotes the set of lines joining component C_1 to $G[V(G) - V(C_1)]$. There are thus only 2 possible values for $|V(L) \cap S|$ and we now proceed to treat each.

Case 3.1. Suppose $|V(L) \cap S| = 3$.

Let v_1, v_2 and v_3 be the 3 points of S adjacent to points of C_1 . Note that if any of these v_i 's is adjacent to 3 points of C_1 , then the other two together form a cutset of G of size 2, contradicting the hypothesis that G is 3-connected. So we must have 2 of the v_i 's incident with 2 lines to C_1 and the third v_i incident with 1 line to C_1 . Without loss of generality, assume that v_1, v_2 are each adjacent to 2 points of C_1 and v_3 is adjacent to 1 point of C_1 .

Note that since G is 3-connected, $\{v_1, v_2, v_3\}$ is an independent set. Now let f_i be the line joining v_i to a point not in C_1 for $i = 1, 2$, and let f_3 be the line joining v_3 to C_1 . Then $\{f_1, f_2, f_3\}$ is a cutset in G separating $G_2 = G[V(C_1) \cup \{v_1, v_2\}]$ from $G_3 = G[V(G) - V(C_1) - \{v_1, v_2\}]$. (See Figure 3.3.) Moreover, G_2 contains a cycle since it contains component C_1 .

We claim that G_3 also contains a cycle. Suppose not. Then G_3 is a forest containing the 2 lines e_1 and e_2 and hence is a forest containing at least 2 endpoints. Thus G_3 sends at least 4 lines to G_2 , a contradiction.

Thus $\{f_1, f_2, f_3\}$ is a *cyclic* cutset of size 3 in G , contradicting the hypothesis that G is cyclically 4-connected.

Case 3.2. Suppose $|V(L) \cap S| = 4$.

Let v_1 be the point of S adjacent to 2 points of C_1 and v_2, v_3, v_4 be the rest of $V(L) \cap S$. Let w be the point adjacent to v_1 which lies outside of $V(C_1)$.

FIGURE 3.3.

First suppose v_1 is adjacent to one of v_2, v_3 or v_4 ; say, without loss of generality, to v_2 via line e_1 . Let the lines joining v_i to C_1 be f_i for $i = 2, 3, 4$. Finally, let g be the line incident with v_2 , where $g \neq e_1$ or f_2 . Then $\{g, f_3, f_4\}$ is a cutset in G separating $J_1 = G[V(C_1) \cup \{v_1, v_2\}]$ from the rest of G . Let us denote $G[V(G) - V(J_1)]$ by J_2 . Note first that J_1 contains a cycle since C_1 does.

Claim 5. Subgraph J_2 contains a cycle.

Suppose not. We know that J_2 contains line e_2 and hence is a forest with at least 2 endpoints. So J_2 sends at least 4 lines to J_1 , a contradiction.

Thus $\{g, f_3, f_4\}$ is a *cyclic* cutset in G , contradicting the hypothesis that G is cyclically 4-connected.

So we may assume that v_1 is adjacent to none of the points v_2, v_3, v_4 ; that is, $w \notin \{v_2, v_3, v_4\}$.

Now contract the subgraph $G[V(C_1) \cup \{v_1\}]$ to a single point c_1 and

FIGURE 3.4.

call the resulting graph G''' . (See Figure 3.4.)

The graph G''' has all points of degree 3, with the single exception of point c_1 which has degree 4. Let p''' , q''' and r''' denote the number of points, lines and faces of graph G''' respectively. Then doing an Euler face count in G''' , we have $\sum_i (6-i)f_i''' = 6r''' - 2q''' = 6r''' - (3p''' + 1) = 14$. So in particular, we have $4f_2''' + 3f_3''' + 2f_4''' + f_5''' \geq 14$.

But $w \notin \{v_2, v_3, v_4\}$, so $f_2''' = 0$, and so in G''' we have:

$$3f_3''' + 2f_4''' + f_5''' \geq 14 \quad (A)$$

We also know that since $\deg c_1 = 4$ in G''' ,

$$f_3''' + f_4''' \leq 4. \quad (B)$$

Now either $w \in S$ or w is a singleton component of $G - S$ different from C_1 .

Suppose that $w \in S$. Then any triangle or quadrilateral in G''' must use one of the lines e_1 or e_2 , so

$$f_3''' + f_5''' \leq 4. \quad (C)$$

But then if we compute $(A) - ((B) + (C))$ we obtain $f_3''' + f_4''' \geq 6$ which contradicts inequality (B).

So we may suppose that w is a singleton component of $G - S$.

Let line $c_1 w$ be denoted by h in G''' . Let the 4 faces at point c_1 be denoted F_1, F_2, F_3 and F_4 as shown in Figure 3.5.

FIGURE 3.5.

FIGURE 3.6.

Note that since lines e_1 and e_2 are independent, at most one of F_2 and F_3 is a triangle.

3.2.1. First suppose that F_4 is a triangle.

3.2.1.1. Suppose that face F_1 is also a triangle. (See Figure 3.6.) Then if F_3 is a triangle, the points $\{v_2, v_3\}$ are a cutset in G , a contradiction. Thus F_3 is not a triangle and by symmetry, neither is F_2 . But then let $h_2 = v_2x$, $x \notin \{c_1, w\}$, $h_3 = v_3c_1$ and $h_4 = v_4y$, $y \notin \{c_1, w\}$. Then $\{h_2, h_3, h_4\}$ is a set of 3 independent lines separating cycle $v_2c_1wv_2$, for

FIGURE 3.7.

example, from a subgraph H''' of G''' containing 3 points of degree 2 in H''' , namely x, v_3 , and y . Moreover, since G is 3-connected, subgraph H''' must be connected and hence $\{h_2, h_3, h_4\}$ is a *cyclic* 3-cut, a contradiction.

3.2.1.2. So suppose face F_1 is not a triangle. Since e_1 and e_2 are independent, at most one of F_2 and F_3 is a triangle. Hence $f_3''' \leq 2$.

If $f_3''' = 1$, then $f_4''' \leq 3$ and $f_5''' \leq 4$. But then $2f_3''' + f_4''' + f_5''' \leq 9$ and combining this inequality with inequality (A), we obtain $f_3''' + f_4''' \geq 5$ which contradicts inequality (B).

So we may conclude that $f_3''' = 2$ and hence exactly one of F_2 and F_3 is a triangle.

3.2.1.2.1. Suppose F_2 is a triangle. In particular, suppose line e_1 joins points v_2 and v_3 . Then e_2 is not incident with v_2 , so v_2 must be adjacent to a point x in $G - S$, where $x \neq w$. But x cannot be adjacent to w , so face F_1 cannot be a quadrilateral. So $f_4''' \leq 1$. It then follows that $f_3''' + f_4''' + f_5''' \leq 6$ and hence $f_3''' + f_4''' \geq 6$, contradicting inequality (B).

3.2.1.2.2. Now suppose F_3 is a triangle. (And face F_2 is not a triangle.) So we may assume that line e_1 joins points v_3 and v_4 .

We know that F_1 is not a triangle.

If face F_1 is not a quadrilateral, we get the same contradiction that we obtained in Subcase 3.2.1.2.1, so suppose F_1 is a quadrilateral. (See Figure 3.7.)

Thus line e_2 must join point v_2 to a point y in S and, in addition,

FIGURE 3.8.

y is adjacent to w .

So $f_3''' = 2$, $f_4''' \leq 2$ and $f_5''' \leq 2$. But then $2f_3''' + f_4''' + f_5''' \leq 8$ and combining this inequality with inequality (A), we obtain $f_3''' + f_4''' \geq 6$ which contradicts inequality (B).

3.2.2. Suppose now that F_4 is not a triangle. (And by symmetry, we may also suppose that F_1 is not a triangle either.) Thus $f_3''' \leq 1$. Now if $f_3''' = 1$, then $f_4''' \leq 3$ and $f_5''' \leq 3$, while if $f_3''' = 0$, then $f_4''' \leq 4$ and $f_5''' \leq 4$. But in either case, $2f_3''' + f_4''' + f_5''' \leq 8$ and once again combining this inequality with inequality (A), it follows that $f_3''' + f_4''' \geq 6$ and again we have a contradiction of inequality (B).

This completes the proof of the theorem. ■

The following corollary is now immediate.

3.2. COROLLARY. *If G is a cubic, 3-connected, planar graph and, in addition, is cyclically 5-connected, then G is 2-extendable.* ■

We conclude with several remarks as to the sharpness of Theorem 3.1. First we note that there are graphs which satisfy the hypotheses of Theorem 3.1, but not those of Corollary 3.2. Such a graph is displayed in Figure 3.8.

We now observe that our theorem is best possible in the sense that we cannot weaken the assumption that the cyclic connectivity is 4 to the assumption that it is only 3 in Theorem 3.1. The graph in Figure 3.9 is cubic, 3-connected and planar without any triangles or quadrilaterals, but it is *not* 2-extendable. (Lines e_1 and e_2 do not extend to a p.m.)

FIGURE 3.9.**FIGURE 3.10.**

Finally, in Figure 3.10 we exhibit a graph which is cubic, 3-connected, planar and cyclically 4-connected, but not 2-extendable. (Lines e_1 and e_2 do not extend.) Of course, by Theorem 3.1, such a graph must contain a quadrilateral and the example we display contains two such.

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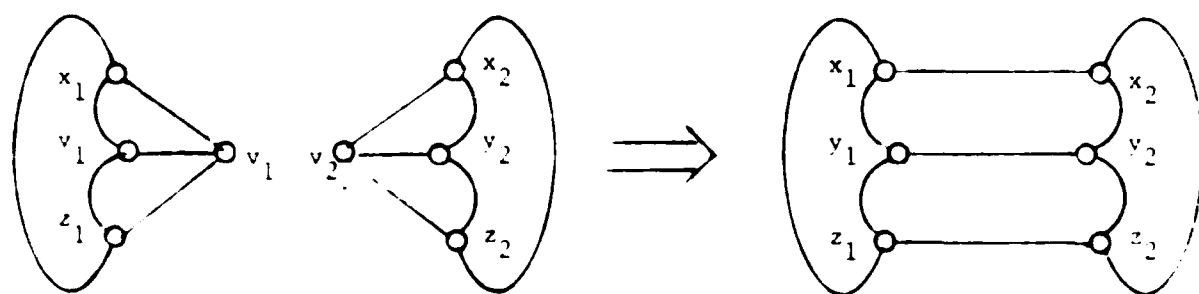


Figure 2.1

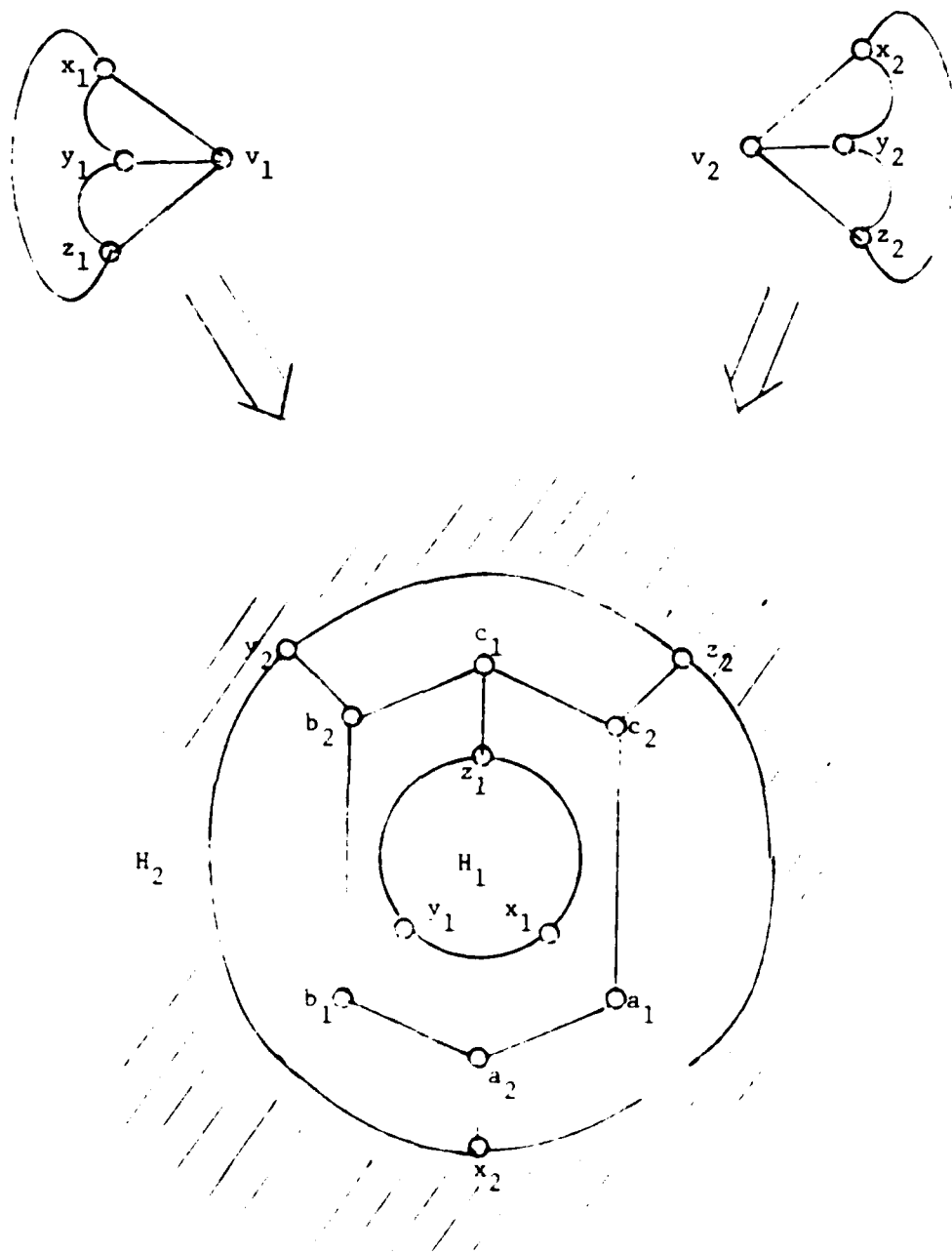


Figure 2.2

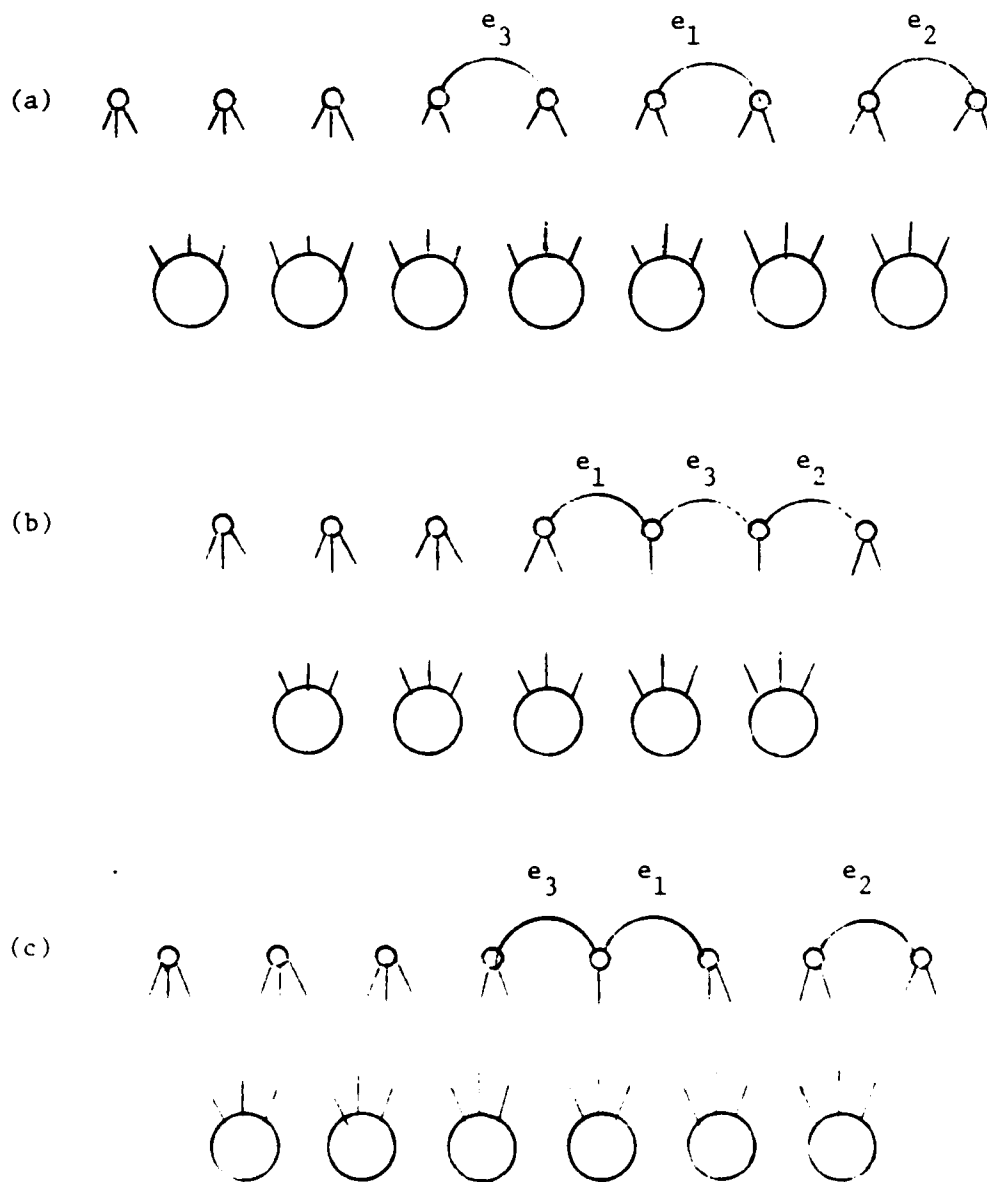


Figure 3.1

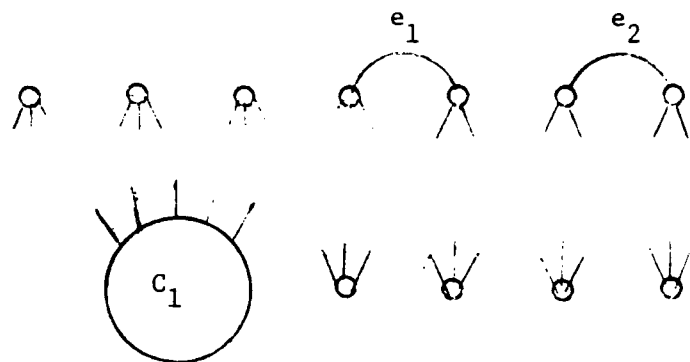


Figure 3.2

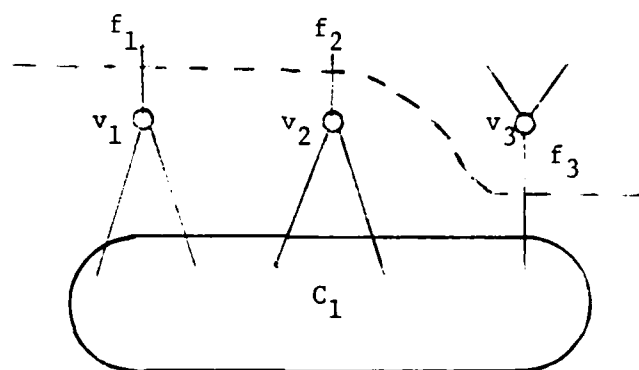


Figure 3.3

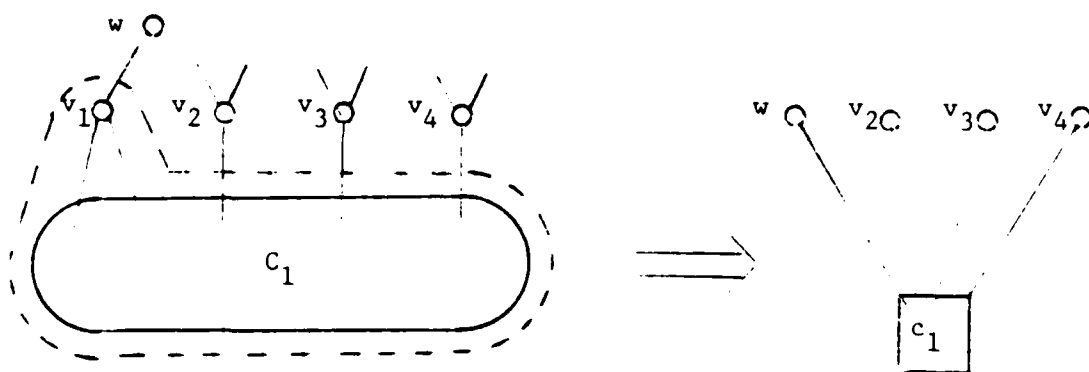


Figure 3.4

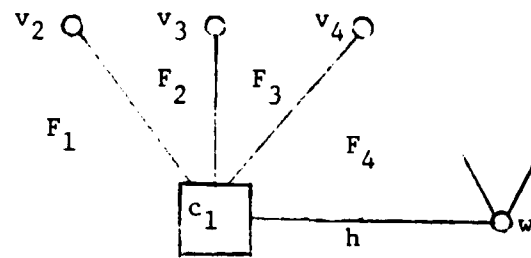


Figure 3.5

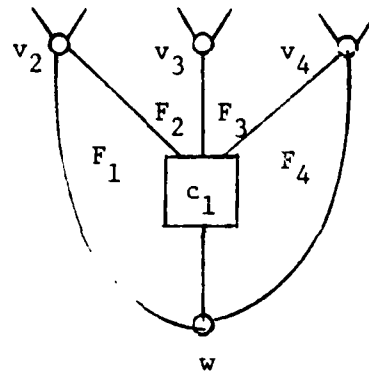


Figure 3.6

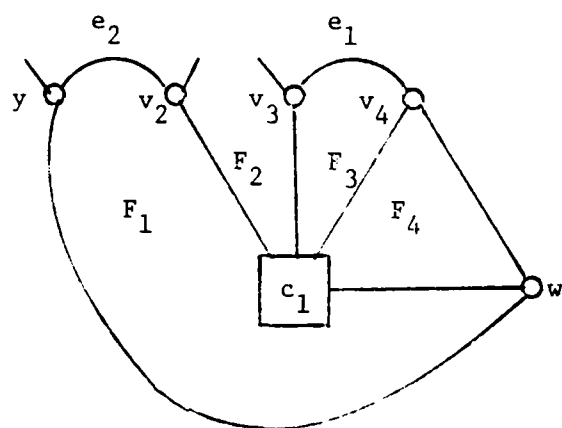


Figure 3.7

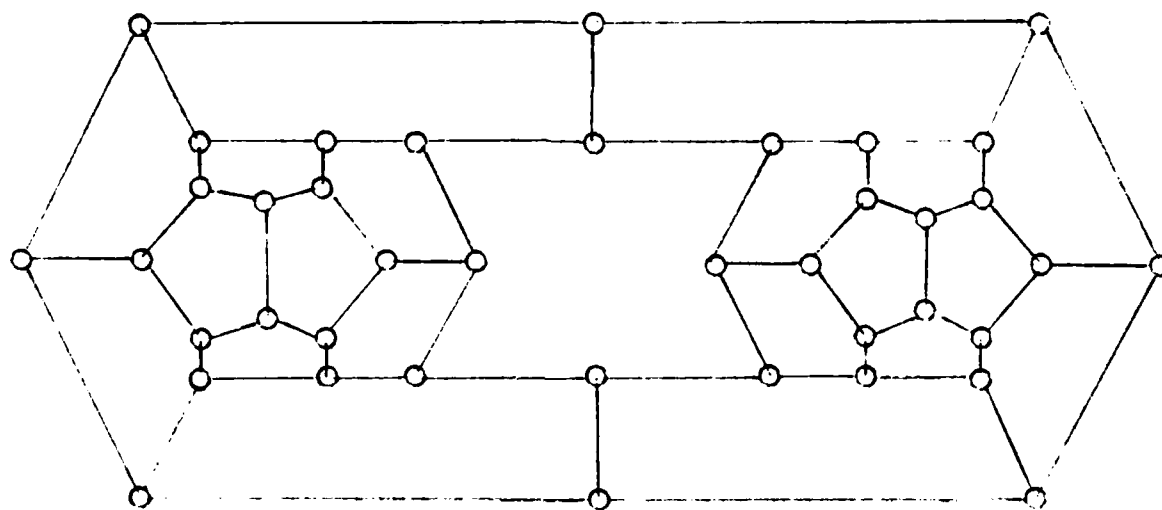


Figure 3.8

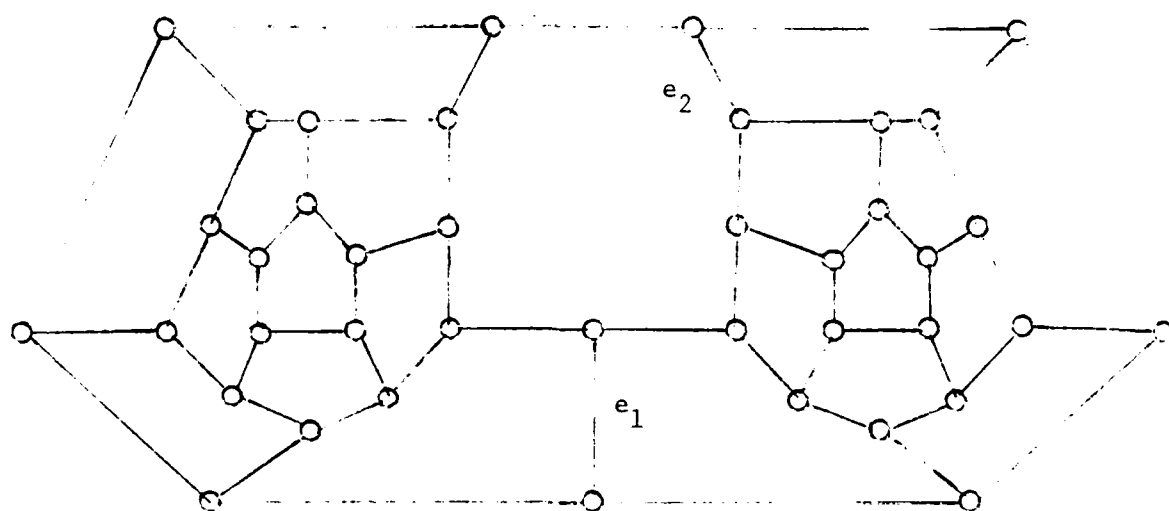


Figure 3.9

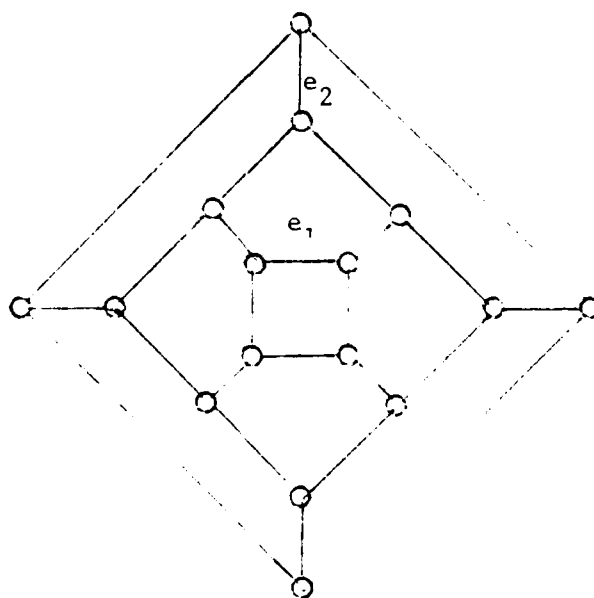


Figure 3.10

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