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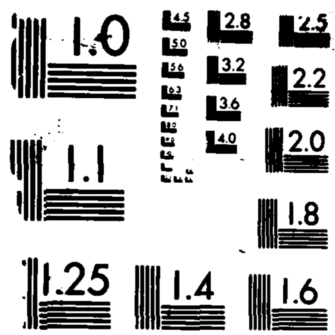

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TECHNICAL REPORT RD-GC-86-8

THE OPTIMUM HARMONIC CONTENT FOR DISCRETE FOURIER SERIES REPRESENTATION OF A FINITE DISCRETE DATA SET

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JUNE 1986

U.S. ARMY MISSILE COMMAND
Redstone Arsenal, Alabama 35898-5000

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It is well known that any real continuous-data function can be represented by a Fourier series of infinite terms, provided a certain set of conditions are met. In practice, the infinite series is truncated to contain only a finite number of terms. Better approximation is obtained if more terms are included in the series. This last statement is not exactly true for the case of a real discrete-data function. For this case, there is an optimum truncation for its Fourier series representation. This fact has not been widely recognized. The purpose of this report is to discuss this fact.

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I. INTRODUCTION

It is well known that any real continuous-data function can be represented by a Fourier series of infinite terms, provided a certain set of conditions are met. In practice, the infinite series is truncated to contain only a finite number of terms. Better approximation is obtained if more terms are included in the series. This last statement is not exactly true for the case of a real discrete-data function. For this case, there is an optimum truncation for its Fourier series representation. This fact has not been widely recognized by practicing engineers. This report discusses this important and interesting fact.

II. DISCUSSION

A real continuous-data function $f(x)$ for $a < x < b$ can be represented by a Fourier series of the form

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k \frac{2\pi}{D} x) + \sum_{k=1}^{\infty} B_k \sin(k \frac{2\pi}{D} x) \tag{1}$$

where $D = b - a$, provided that $f(x)$ is finite for x in (a,b) and has a finite number of discontinuities. In practice, the infinite series is truncated after the N-th harmonic for a chosen N to make the series finite. This gives the approximation

$$f(x) \approx A_0 + \sum_{k=1}^{N} A_k \cos(k \frac{2\pi}{D} x) + \sum_{k=1}^{N} B_k \sin(k \frac{2\pi}{D} x). \tag{2}$$

The approximation is improved if more harmonics are included in equation (2), that is, if a larger N is chosen; however, this is not entirely true for the case of a real discrete-data function. For the case of a real discrete-data function, there is an optimum truncation for the Fourier series representation.

Consider a real discrete-data function $f(n)$ for $n_1 < n < n_2$ where n, n_1 and n_2 are integers. This function can be represented approximately by a discrete-data Fourier series of finite harmonics, namely,

$$f(n) = A_0 + \sum_{k=1}^{K} A_k \cos(k \frac{2\pi}{N} n) + \sum_{k=1}^{K} B_k \sin(k \frac{2\pi}{N} n) \tag{3}$$

where $N = n_2 - n_1$ and

$$A_0 = \frac{1}{N} \sum_{n=1}^{N} f(n) \tag{4}$$

$$A_k = \frac{2}{N} \sum_{n=1}^{N} f(n)\cos(k \frac{2\pi}{N} n) \qquad (5)$$

$$B_k = \frac{2}{N} \sum_{n=1}^{N} f(n)\sin(k \frac{2\pi}{N} n) \qquad (6)$$

Note that N is also an integer which is the number of data points representing f(n) in the interval from n_1 to n_2. The following interesting and important fact exists:

> For a set of N data points representing f(n), the optimum finite Fourier series consists of the constant term, the fundamental, and all harmonics up to and including $\frac{N}{2}$.

Thus, for the optimum representation, (3) becomes

$$f(n) = A_0 + \sum_{k=1}^{N/2} A_k\cos(k \frac{2\pi}{N} n) + \sum_{k=1}^{N/2} B_k\sin(k \frac{2\pi}{N} n). \qquad (7)$$

III. VERIFICATION

Three of the ways this fact can be verified are by sampling process, exponential form of the discrete Fourier series, and solution of simultaneous equations.

A. Sampling Process

Review the sampling process by considering a sampled signal as shown in Figure 1. The following quantities can be established:

Record length = D

Fundamental frequency $F = \frac{1}{D}$

Number of data points = N

Sampling period $d = \frac{D}{N}$

Sampling frequency $f_S = \frac{1}{d} = \frac{N}{D} = NF$

Folding frequency $f_F = \frac{1}{2}f_S = \frac{NF}{2}$

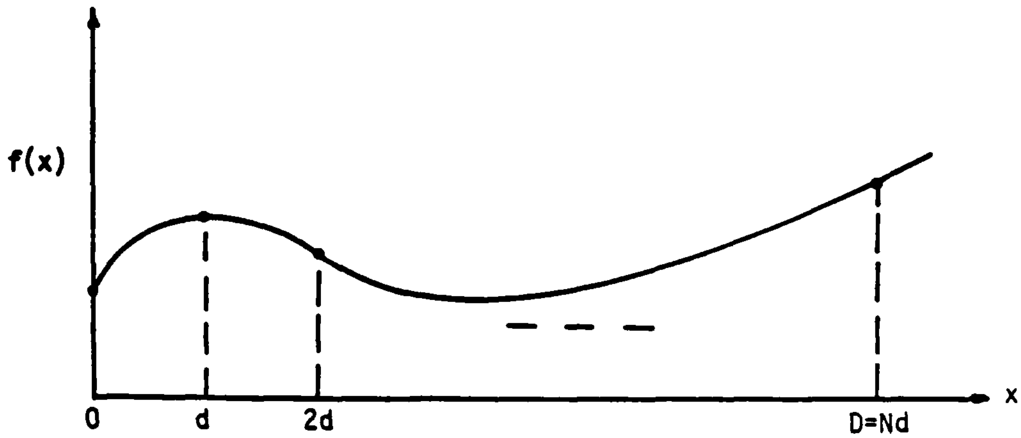
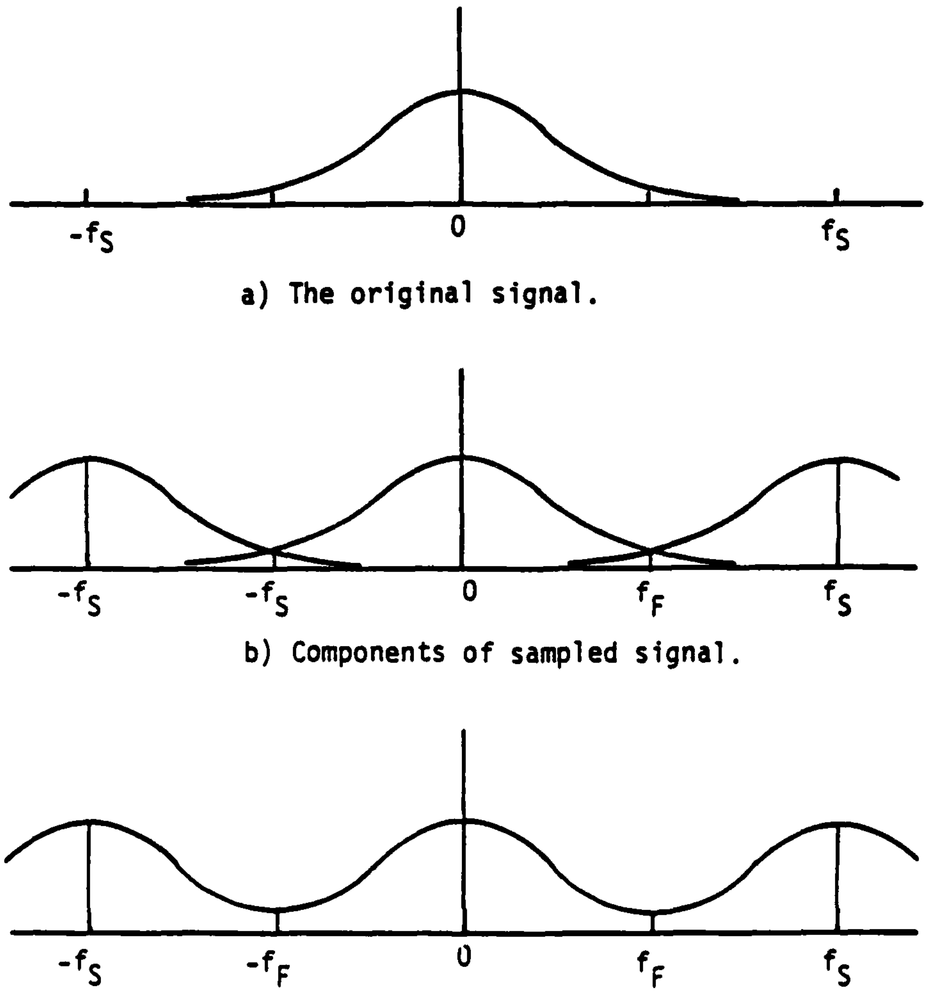


Figure 1. Sampled signal.

Thus, the folding frequency is the harmonic $\frac{N}{2}$ of the Fourier series. Figure 2 shows the frequency spectra of a continuous-data signal from which the discrete-data signal is obtained by sampling, components of the sampled signal, and the sampled signal. Note that the spectrum of the sampled signal is periodic with a period equal to f_S, the sampling frequency.

For practical purposes, the high frequency end of the sampled signal is at f_F, the folding frequency. Thus, the entire practical spectrum for the sampled signal spans from zero frequency to the harmonic $\frac{N}{2}$. There is no useful frequency component above the folding frequency. In fact, if the original analog signal contains harmonics above f_F, they are folded back to the low frequency region and are added to low frequency components, causing the aliasing effect. The accuracy of Fourier coefficients for the base band components can be impaired by the aliasing effect. The effect of aliasing is less if the sampling frequency is higher, implying a larger number of data points for a given length of record.

The discrete-data Fourier coefficients are not necessarily equal to the Fourier coefficients of the original analog signal due to the picket-fence effect. However, the finite Fourier series representation of the signal may agree well with the original analog signal at sampling points.



a) The original signal.

b) Components of sampled signal.

c) The resultant of sampled signal.

Figure 2. __Frequency spectra.__

B. Exponential Form

Next, consider the exponential form of the discrete Fourier series. It is well known[1] that the discrete Fourier series representation of a periodic sequence f(n) of period N needs only contain N complex exponential terms, namely,

$$f(n) = \sum_{k=0}^{N-1} F(k)\, e^{j \frac{2\pi}{N} nk} \tag{8}$$

where the F(k)'s are Fourier coefficients. One must be careful to note that equation (8) contains only the constant term and all harmonics up to $\frac{N}{2}$ inclusive, but not up to the N-th one. This is due to the fact that equation (8) is equivalent to

$$f(n) = \sum_{k=-N/2}^{N/2} F(k)\, e^{j \frac{2\pi}{N} nk} \tag{9}$$

and

$$F(k) = F(-k)^{*} = F(N-k)^{*} \tag{10}$$

where "*" denotes complex conjugate. With the help of Euler's formulas, equation (9) can be reduced to the form

$$f(n) = A_0 + \sum_{k=1}^{N/2} A_k \cos(k \frac{2\pi}{N} n) + B_k \sin(k \frac{2\pi}{N} n) \tag{11}$$

which does not have harmonica above $\frac{N}{2}$.

[1]A.V. Oppenheim and R. W. Schafer, Digital Signal Processing, Prentic-Hall, 1975, p. 88.

C. Simultaneous Equations

Finally, consider equation (11) for n = 1 to N which gives a set of N equations. On the right-hand side of each of these equations there are N coefficients to be determined. They are A_0, A_k for k = 1 to $\frac{N}{2}$ and B_k for k = 1 to $\frac{N}{2}-1$. Note that $B_{N/2} = 0$. Hence there are N equations to be solved for N unknowns, giving a set of unique solutions. In other words, the sequence f(n) can be perfectly represented by a finite discrete Fourier series containing the constant term and all harmonics up to $\frac{N}{2}$ inclusive, where N is the number of data points.

IV. RESULTS

A numerical example is given to illustrate the fact discussed. Consider a set of 36 data points representing the errors of a resolver at 36 equally spaced angular positions in the range from 0 to 360 degrees, as shown in Table 1. The coefficients for the corresponding discrete Fourier series are shown in Table 2. Notice that $B_{N/2} = B_{18} = 0$, and values of the coefficients are symmetrical about the $\frac{N}{2} = 18$ point. Table 3 shows the Fourier representation RMS error. Notice the error is minimum when the series contains only the constant term and all harmonics up to $\frac{N}{2}$ inclusive. In fact, this minimum value should be zero. The discrepancy is caused by the finite word-length effect of the computer used. Figure 3 depicts the RMS error versus the included harmonics.

TABLE 1. Resolver Error Data in 10 Degree Increments.

1	-136
2	-290
3	-179
4	55
5	83
6	73
7	122
8	375
9	446
10	277
11	116
12	163
13	322
14	264
15	73
16	68
17	226
18	234
19	21
20	-114
21	-69
22	183
23	153
24	80
25	124
26	331
27	352
28	177
29	28
30	28
31	185
32	102
33	-75
34	-99
35	112
36	62

RMS ERROR BEFORE COMPENSATION= 193.64206671

TABLE 2. **Fourier Coefficients of the Resolver Error Characteristics.**

HARMONICS K	COS COEFF A(K)	SIN COEFF B(K)	EXP COEFF C(K)
0	107.583333333		107.583333333
1	-80.6304477067	28.2839909441	85.4473711772
2	-129.756430444	-48.9906110651	138.696832025
3	-.332407750167	-10.5587235027	10.5639546061
4	46.2699631468	-58.4540750561	74.5505759887
5	4.3099044445	-2.03415835927	4.76582380616
6	5.86111111122	-3.51221413757	6.83288164731
7	.375601920506	-4.26831052972	4.28480473076
8	131.183525698	-42.6963543421	137.956863142
9	-1.83333333333	-.333333333333	1.86338998124
10	1.80414316845	-6.43221895044	6.68044708074
11	2.07084910547	-6.74962152478	7.06015628335
12	-8.58333333522	-4.18578945159	9.54957823553
13	.820506239261	2.27492947856	2.41837437569
14	-5.21437939322	1.56652300156	5.44460712731
15	-2.83425891661	.892056836222	2.97132781856
16	-2.95348884389	-14.2644932841	14.5670472299
17	-7.94641400322	-1.87602745122	8.16486218549
18	3.61111111111	0	3.61111111111
19	-7.94641400322	1.87602745122	8.16486218549
20	-2.95348884389	14.2644932841	14.5670472299
21	-2.83425891661	-.892056836222	2.97132781856
22	-5.21437939322	-1.56652300156	5.44460712731
23	.820506239261	-2.27492947856	2.41837437569
24	-8.58333333522	4.18578945159	9.54957823553
25	2.07084910547	6.74962152478	7.06015628335
26	1.80414316845	6.43221895044	6.68044708074
27	-1.83333333333	.333333333333	1.86338998124
28	131.183525698	42.6963543421	137.956863142
29	.375601920506	4.26831052972	4.28480473076
30	5.86111111122	3.51221413757	6.83288164731
31	4.3099044445	2.03415835927	4.76582380616
32	46.2699631468	58.4540750561	74.5505759887
33	-.332407750167	10.5587235027	10.5639546061
34	-129.756430444	48.9906110651	138.696832025
35	-80.6304477067	-28.2839909441	85.4473711772

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TABLE 3. Fourier Representation RMS Error Using Different Numbers of Harmonic Terms.

```
RMS ERROR USING 10 HARMONICS= 15.3418581462 ARCSEC
RMS ERROR USING 11 HARMONICS= 14.5068848482 ARCSEC
RMS ERROR USING 12 HARMONICS= 12.8394893114 ARCSEC
RMS ERROR USING 13 HARMONICS= 12.7251019045 ARCSEC
RMS ERROR USING 14 HARMONICS= 12.1287404572 ARCSEC
RMS ERROR USING 15 HARMONICS= 11.9453736057 ARCSEC
RMS ERROR USING 16 HARMONICS= 6.04917499303 ARCSEC
RMS ERROR USING 17 HARMONICS= 1.80555555516 ARCSEC
RMS ERROR USING 18 HARMONICS= 1.80555555588 ARCSEC
RMS ERROR USING 19 HARMONICS= 6.04917499675 ARCSEC
RMS ERROR USING 20 HARMONICS= 11.945373609 ARCSEC
RMS ERROR USING 21 HARMONICS= 12.1287404605 ARCSEC
RMS ERROR USING 22 HARMONICS= 12.7251019076 ARCSEC
RMS ERROR USING 23 HARMONICS= 12.8394893146 ARCSEC
RMS ERROR USING 24 HARMONICS= 14.506884853 ARCSEC
RMS ERROR USING 25 HARMONICS= 15.3418581508 ARCSEC
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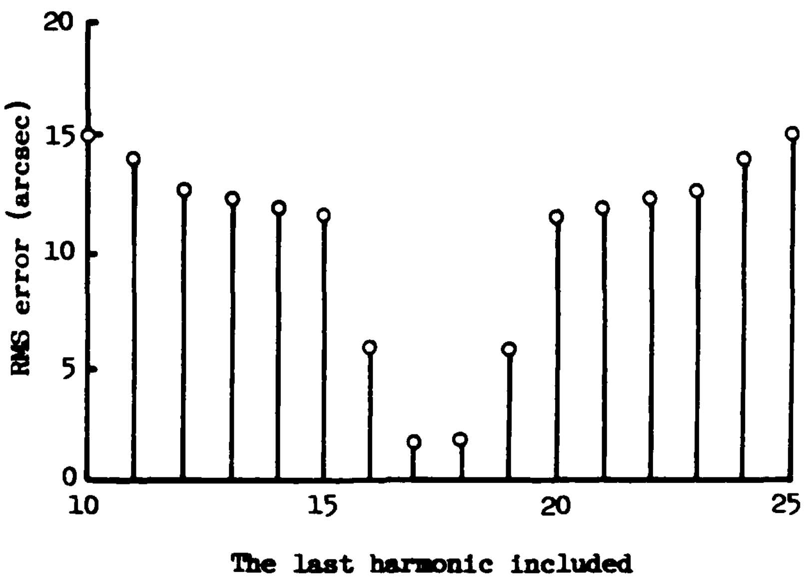

Figure 3. RMS error versus harmonics included.

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