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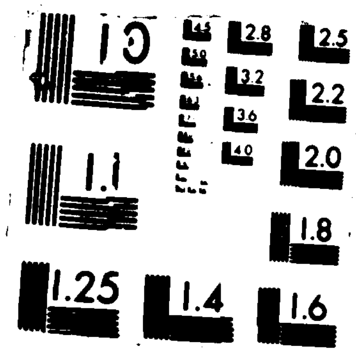
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RESEARCH REPORT

FUNCTIONAL RELATIONSHIPS BETWEEN RISKY AND RISKLESS MULTIATTRIBUTE UTILITY FUNCTIONS

Detlof von Winterfeldt

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December, 1979

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ABSTRACT

Expected utility theory and conjoint measurement theory form two major classes of models and assessment procedures to construct multi-attribute utility functions. In conjoint measurement theory a value function v is constructed which preserves preferences among riskless multi-attributed outcomes. The risky utility function u, constructed in the framework of expected utility theory, also preserves such riskless preferences. In addition, u is an appropriate guide for decisions under uncertainty since its expectation preserves risky preferences among gambles. Since both u and v are order preserving functions, they must be related by a strictly increasing transformation. However, u and v need not coincide or be related through any special functional forms, unless some simple decomposition forms are assumed. More restricted functional relationships obtain, if u and v are assumed to be either additive or multiplicative. In particular, u can be shown to be linearily, logarithmically, or exponentially related to v, depending on which function is additive and which is multiplicative. The paper proves such functional relationships based on the theory of functional equations, and techniques are described to assess the parameters of these functions. The results are discussed from a behavioral standpoint of interpretating the form and shape of multi-attribute utility functions and from a practical standpoint of simplifying multi-attribute utility assessment.

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INTRODUCTION

Multiattribute preferences can be modeled in two fundamentally different ways, depending on whether the decision problem involves uncertainties or not. The first modeling approach is based on conjoint measurement theory (see Krantz, Luce, Suppes, and Tversky, 1971) which applies to the problem of modeling riskless preferences for multiattributed outcomes. Conjoint measurement theory specifies the conditions under which a riskless <u>value function</u> v can be constructed which preserves the preference order among multiattributed outcomes and which can be expressed as a simple aggregate of single attribute value functions v_i. The best known conjoint measurement model forms are the additive and the multiplicative models:

$$v(x) = \sum_{i=1}^{n} v_i(x_i) \tag{1}$$

and

$$v(x) = \prod_{i=1}^{n} v_i(x_i), \tag{2}$$

where $x \in X$ is a multiattribute outcome, x_i is the level of x in the i-th attribute X_i, v_i is the i-th single attribute value function, and v is the overall value function. Other simple polynomial forms of v have been developed in Krantz and Tversky (1970) and Krantz et al (1971). Although v is an appropriate guide for preferences among sure things, nothing in conjoint measurement theory guarantees that the expectation of v is appropriate for selecting among gambles with multiattributed outcomes.

The second modeling approach, expected utility theory, explicitly addresses the problem of decision making under risk. Based on v. Neumann and Morgenstern's (1947) work which was later extended by Savage (1954) and others, this theory provides the rationale for constructing a utility function u which preserves the preferences for riskless outcomes and at the same time its expectation preserves the preferences among gambles for such outcomes. Applied to the multiattribute situation, several decomposition forms of u have been developed (see Keeney and Raiffa, 1976). The additive and the multiplicative forms are the best known:

$$u(x) = \sum_{i=1}^{n} u_i(x_i), \tag{3}$$

$$1 + k\,u(x) = \prod_{i=1}^{n} [1 + k\,u_i(x_i)]. \tag{4}$$

It is relatively trivial to establish the fact that u and v are related by a strictly increasing transformation (see Raiffa, 1969; Krantz et al, 1971; Keeney and Raiffa, 1976). But neither conjoint measurement theory, nor expected utility theory by themselves provide a rationale for any specific functional relationships between u and v. In principle, the shape and aggregation rule of u and v can be quite different. For example, v may be additive, while u is multiplicative or non-decomposable at all. All v_i's may be linear, while all u_i's may be non-linear, etc.

Establishing closed form functional relationships between u and v is, however, possible when special decomposition forms such as (1) - (4) are assumed. Scattered throughout the multiattribute literature are results which relate utility and value functions by some specific class of transformation, e.g., exponential or logarithmic functions (see Pollak, 1967; Krantz et al, 1971; Keeney and Raiffa, 1976; Dyer and Sarin, 1977). Keeney and Raiffa, for example, give a proof outline to show that an additive value function v and a multiplicative utility function u must be related exponentially:

$$u(x) = b \exp\{a\, v(x)\}. \tag{5}$$

Another way to relate u and v is by the uniqueness theorems of their respective measurement theoretic representations. For example, an additive conjoint measurement function v is unique up to a positive linear transformation. Consequently, any other additive and order preserving function v' must be related to v by

$$v'(x) = a \cdot v(x) + b. \tag{6}$$

for some $a > 0, b$. In particular, an additive utility function u should be related to an additive value function by a positive linear transformation.

Functional forms such as (5) and (6) have been proven by a variety of mathematical methods (differentiation methods in Pollak, constructive algebraic proofs in Krantz et al, risk attitude arguments in Keeney and Raiffa, and uniqueness theorems in Dyer and Sarin). No common framework or integrated presentation of these functional relations has been developed yet. This paper attempts to provide such an integrative framework through the theory of functional equations.

The interest in parametric functional relationships such as (5) and (6) is not merely theoretical. For the practitioner who has to assess multiattribute utility functions in an applied context such functional forms provide tools for simplifying and cross-checking utility and value functions. Simplifications in the construction of u are possible, by first constructing v and then exploring the functional form relating u and v through its parametric properties. Such a two step construction process has several advantages. First, v can usually be approximated by simple techniques such as Edwards' SMART procedure (1977), which involves similar judgmental processes as the theoretically feasible construction process of dual standard sequences (see Krantz et al, 1971). Second, the two step procedure avoids lengthy and tedious lottery assessments in the construction of u by assessing the parameters of the function $u = h(v)$ through a few simple questions involving risky outcomes.

Constructing the utility function u in the two step process has also theoretical advantages. The value function v and its single attribute functions v_i express purely riskless preference characteristics, such as marginally decreasing value, complementarity or substitution phenomena between attributes. Such considerations are usually compounded in u and its single attribute functions u_i with the risk attitudes of the decision maker. For example, it is quite conceivable that a single attribute utility function u_i appears risk neutral (linear) because marginal decreasing value is compensated by risk proneness. The two step construction process uncovers such anomalies by identifying riskless value aspects in the value function, and by incorporating "pure" risk considerations in the transformation $u = h(v)$.

The next section will show that u and v must be related by a strictly increasing transformation h, which under further technical assumptions must be continuous. Then four possible functional equations will be investigated and solved.

$$u = \sum_{i=1}^{n} u_i = h\left[\sum_{i=1}^{n} v_i\right] = h(v), \tag{7}$$

$$1+ku = \prod_{i=1}^{n} (1+ku_i) = h\left[\sum_{i=1}^{n} v_i\right] = h(v), \tag{8}$$

$$u = \sum_{i=1}^{n} u_i = h\left[\prod_{i=1}^{n} v_i\right] = h(v), \tag{9}$$

$$1+ku = \prod_{i=1}^{n} (1+ku_i) = h\left[\prod_{i=1}^{n} v_i\right] = h(v). \tag{10}$$

The solutions consist of a reduction to one of the fundamental Cauchy type functional equations (see Aczél, 1966). For certain standardizations of u and v these solutions turn out to be very simple, namely $u = v$ for (7), $1+ku = (1+k)^v$ for (8), $u = \ln v$ for (9) and $1 + ku = (\mathrm{sgn}k)(\mathrm{sgn}v)|v|^{\ln[1+(\mathrm{sgn}k)\cdot k]}$ for (10). Subsequent to proving these solutions, some behavioral and practical implications will be discussed.*

* Readers who are willing to accept the above solutions and do not want to bother with the rather technical proofs, are recommended to skip the following two sections.

Definitions and Preliminary Results

The value and utility functions v and u will be defined on the product set $X = X_1 \times X_2 \times \ldots \times X_i \times \ldots \times X_n$. x and y are typical elements of X. x_i and y_i characterize their respective levels in attribute X_i. P is the set of simple probability distributions over X with typical elements $p, q \in P$. A transitive and connected order relation $\succsim$ is defined on P with $p \succsim q$ being interpreted as "p is preferred to or indifferent to q." By considering degenerate probability distributions $\succsim$ can be reduced to X. Thus $p \succsim q$ where $p(x) = q(y) = 1$ for some $x, y \in X$ is written as $x \succsim y$, and interpreted as "x is preferred to or indifferent to y." u and v are defined as follows (for more details, see Fishburn, 1970):

<u>Definition 1</u> A function $u: X \to Re$ is a (v. Neumann and Morgenstern) utility function if for all $p, q \in P$

$$p \succsim q$$
$$\text{if and only if} \tag{11}$$
$$E(u \mid p) \geq E(u \mid q)$$

where $E(u \mid .)$ denotes the expectation of u with respect to some probability distribution. If u can be expressed as in (3) it is called additive. If u can be expressed as in (4) it is called multiplicative.

Note: From Definition 1 and the reduction of $\succsim$ to X it follows that u is order preserving, i.e., for all $x, y \in X$

$$x \succsim y$$
$$\text{if and only if} \tag{12}$$
$$u(x) \geq u(y)$$

<u>Definition 2</u> A function $v: X \to Re$ is a (conjoint measurement) value function, if for all $x,y \in X$

$$x \succsim y$$
$$\text{if and only if} \tag{13}$$
$$v(x) \geq v(y)\,.$$

If v can be expressed as in (1) it is called additive. If v can be expressed as in (2), it is called multiplicative. The axioms and representation theorems leading to definitions 1 and 2 are of no special interest here. Axiomatic foundations of utility functions can be found in v. Neumann and Morgenstern (1947), Savage (1954), and Fishburn (1970). Axioms for the additive and multiplicative decomposition forms of u are presented in Keeney and Raiffa (1976) and Fishburn (1970). Axiomatic foundations of conjoint measurement value functions are given in Krantz et al (1971), including the additive and the multiplicative forms as well as other simple polynomials.

Besides the assumptions implicit in definitions 1 and 2, the solutions to the functional equations (7)-(10) require that h be continuous. Since continuity is a rather abstract concept, the more natural assumption will be made in the following that u and v are defined onto some convex subset of Re, labelled I_u and I_v respectively. From the definitions of u and v and from this onto property it follows that h must be strictly increasing and continuous. Lemma 1 formalizes this implication.

<u>Lemma 1</u> Assume that $u: X \xrightarrow{\text{onto}} I_u$ is a utility function, where I_u is a convex subset of Re. Assume further that $v: X \xrightarrow{\text{onto}} I_v$ is a value function where I_v is a convex subset of Re. Then there exists a function $h: I_v \xrightarrow{\text{onto}} I_u$ which is strictly increasing and continuous.

Proof. The proof is trivial except for continuity. Define h by $u(x) = h[v(x)]$ for all $x \in X$. From the equality part of (12) and (13) it follows that h is well defined. From the inequality part of (12) and (13) it follows that h is strictly increasing. h is onto since both u and v are onto.

To prove continuity, consider the contrary. Then there exists at least one point $r \in I_v$ at which h is discontinuous. By the onto property of v and the definition of h, h(r) is defined. Consider first the case where r is not a boundary point of I_v, and define right and left hand limits as

$$\lim_{t \to r^-} h(t) = L ,$$
$$\lim_{t \to r^+} h(t) = R . \tag{14}$$

By the assumption that r is not a boundary point, we can define a sufficiently small e such that h(r+e) and h(r-e) exist. By the fact that h is strictly increasing

$$h(r-e) < L \leq h(r) \leq R < h(r+e). \tag{15}$$

This establishes boundaries for L and R and shows that both limits must exist. By the assumed discontinuity at r, however, at least one of the weak inequalities in (15) must be strict. By the convexity of I_u we therefore can find an a such that $m = aR + (1-a)L \neq h(r)$; $m \in I_u$. However, by (15) there exists no $s \in I_v$ such that $h(s) = m$, thus contradicting the onto property of h. Consequently, h must be continuous, at r.

A similar argument can be made for the cases where r is either a lower or an upper boundary point of I_V by only considering right hand or left hand limits. In each of these cases the assumption that h is discontinuous leads to a violation of the onto property of h. Thus h must be continuous everywhere in I_V.

In solving the functional equations (7)-(10) it is often convenient to transform u and v. The following simple lemmas state which transformations are admissable in the sense that they do not "destroy" any property of a value on utility functions. These lemmas are actually the necessary parts of the uniqueness theorems in expected utility theory and conjoint measurement theory. Since the proofs are very simple, only proof notes are given for each lemma.

<u>Lemma 2</u> If v is an additive value function with additive terms v_i, then $v' = av + b$, $a>0$, is also an additive value function with additive terms $v_i' = av_i + b_i$, $\sum_{i=1}^{n} b_i = b$.

Proof note. Since a is positive, v' is order preserving. Also, v' can be written as an additive function by using v_i'. Therefore v' is an additive value function.

<u>Lemma 3</u> If v is a multiplicative value function with multiplicative terms v_i, then $v' = (\operatorname{sgn} v)\cdot b\cdot |v|^a$, $a,b>0$, is also a multiplicative value function with multiplicative terms $v_i' = (\operatorname{sgn} v_i)\cdot b_i |v_i|^a$, where $\prod_{i=1}^{n} b_i = b$.

Proof note. Since a and b are positive, v' is order preserving. Also, v' can be written as a multiplicative function using terms v_i'. The sign conditions make sure that v and v' have identical signs everywhere.

<u>Lemma 4</u> If u is an additive utility function with additive terms u_i, then $u' = au + b$ is also an additive utility function with additive terms $u_i' = au_i + b_i$, where $\sum_{i=1}^{n} b_i = b$.

Proof note. Since a is positive, u' is order preserving over X. Since $E(u'|\ .) = a\ E(u|\ .) + b$, the expectation of u' is also order preserving over P. Finally u' can be expressed as an additive function using the u_i's.

<u>Lemma 5</u> If $k>0$ and $1+ku$ is a multiplicative utility function with multiplicative terms $1+ku_i$, then $u' = a\ (1+ku)$, $a>0$ is also a multiplicative utility function with multiplicative terms $u_i' = a_i(1+ku_i)$, $\prod_{i=1}^{n} a_i = a$. If $k<0$ and $-(1+ku)$ is a multiplicative utility function with multiplicative terms $(1+ku_i)$, then $u' = -a(1+ku), a>0$, is also a multiplicative utility function with multiplicative terms $a_i(1+ku_i)$, $\prod_{i=1}^{n} a_i = -a$.

Proof note. The two cases $k>0$ and $k<0$ must be considered separately to choose appropriate signs of the constants. (The case $k = 0$ is equivalent to u being additive; thus Lemma 3 applies.) If $k>0$, and u is the original utility function, then $1+ku$ and u' are also utility functions since a is positive and since the expectation operator is linear. Also, u' can be expressed multiplicatively by using the u_i's. The sign of the a_i's is arbitrary as long as their product is positive, thus maintaining the order preserving property of u'. If $k< 0$, and u is the original utility function, $1+ku$ produces an inverse ordering. Thus $-(1+ku)$ is again a utility function of the form $-(1+ku) = -\prod_{i=1}^{n} (1+ku_i)$. Since a is positive u' also is a utility function, which can be expressed multiplicatively by using u_i'. The scaling of the a_i's guarantees that the product of the single attribute u_i's is again negative.

A point worth noting here is that the multiplicative value functions could be transformed exponentially while still maintaining order preserving properties and multiplicativeness. Such transformations are not admissible for utility functions since they would destroy their expectation property.

The Basic Functional Equations and Their Solutions

With these results as a background, functional equations (7)-(10) will now be solved by a reduction to fundamental Cauchy type equations with some simple solution. In the simplest case, both u and v are additive:

Theorem 1 Assume that u and v are additive as in (1) and (3). Then there exist real numbers $a>0$, b such that

$$u = av + b \tag{14}$$

Proof. The uniqueness of u and v can be used to prove the theorem (see Dyer and Sarin, 1977). However, uniqueness is itself a rather difficult property to establish, and its proof is usually hidden in the constructive algebraic proofs of the representation theorems leading up to the definition of utility and value functions. This proof therefore uses the more fundamental route of reducing (7) to Cauchy's equation $h(x+y) = h(x) + h(y)$.

Consider some arbitrary $x^o = (x_1^o, x_2^o, \ldots, x_i^o, \ldots x_n^o)$ with $v_i(x_i^o) = v_i^o$, $v(x^o) = v^o$, $u_i(x_i^o) = u_i^o$, and $u(x^o) = u^o$. Define transformed value and utility functions by

$$v_i' = v_i - v_i^o, \tag{15}$$

$$v' = \sum_{i=1}^{n} v_i' \quad , \tag{16}$$

$$u_i = u_i - u_i^o, \tag{17}$$

$$u' = \sum_{i=1}^{n} u_i' \quad . \tag{18}$$

By Lemma 2 v' is an additive value function, and by Lemma 4 u' is an additive utility function. Furthermore if v and u are defined onto some real interval, so are v' and u'. Therefore, Lemma 1 applies and there exists a strictly increasing continuous function h with

$$\sum_{i=1}^{n} u_i'(x_i) = h[\sum_{i=1}^{n} v_i'(x_i)]. \tag{19}$$

Let $x_j = x_j^o$ for all $j \neq i$. By (15) and (17) $v_j'(x_j^o) = 0$ and $u_j'(x_j^o) = 0$, thus (19) simplifies to

$$u_i'(x_i) = h[v_i'(x_i)]. \tag{20}$$

Substituting the right hand side of (20) nto (19) gives

$$\sum_{i=1}^{n} h[v_i'(x_i)] = h[\sum_{i=1}^{n} v_i'(x_i)]. \tag{21}$$

In particular, let $x_i = x_i^o$ for $i = 3,4,\ldots., n$. This reduces (21) to

$$h[v_1'(x_1)] + h[v_2'(x_2)] = h[v_1'(x_1) + v_2'(x_2)]. \quad (22)$$

(22) is Cauchy's fundamental functional equation $h(x) + h(y) = h(x + y)$ (see Aczél, 1966), which for continuous h has the non-trivial solution

$$h(r) = ar. \quad (23)$$

Substituting (23) into (20) and (19) gives

$$u_i'(x_i) = av_i'(x_i), \quad (24)$$

$$\sum_{i=1}^{n} u_i'(x_i) = a \sum_{i=1}^{n} v_i'(x_i), \quad (25)$$

and

$$u' = a\, v'. \quad (25)$$

Resubstituting u and u for u' and v' gives the desired result

$$u = av + b,$$

where $b = u^o - av^o$. $a>0$ follows directly from the fact that h is strictly increasing.

Theorem 2 Assume that (1+ku) is multiplicative as in (4) and that v is additive as in (1). Then there exist $a>0$, $b>0$ such that

$$1+ku = b \exp\{(\operatorname{sgn} k)\, a\, v\} \quad (26)$$

Proof. The proof is based on a reduction of (8) to the functional equation $h(xy) = h(x) + h(y)$. Again, consider some arbitrary $x^o = (x_1^o, x_2^o, \ldots, x_i^o, \ldots, x_n^o)$ and define v_i^o, v^o, u_i^o and u^o, v_i' and v' as in the proof of Theorem 1. Define the transformed utility function u' by

$$u_i' = \frac{1}{(1+ku_i^0)} (1+ku_i), \tag{27}$$

$$u' = \begin{cases} \prod_{i=1}^{n} u_i' & \text{if } k>0 , \\ -\prod_{i=1}^{n} u_i' & \text{if } k>0 . \end{cases} \tag{28}$$

v' is an additive value function (Lemma 2). To establish that u' is a multiplicative utility function, observe that $[(1+ku_i(x_i)] > 0$ for all i = 1,2,..., n and $x_i \varepsilon X_i$. For if there was a y_i such that $1+ku_i(y_i) \leq 0$, then reversals of preferences must occur or all elements in X must be indifferent for that value of y_i. (See also Fishburn and Keeney, 1974.) Neither case is compatible with the additive value function. Therefore the sign scaling conditions of Lemma 5 are valid, and u' is a multiplicative utility function. In addition, since v and u are defined onto some convex subset of Re, so are u' and v'. Therefore Lemma 1 applies, and there exists a strictly increasing continuous function h such that

$$\prod_{i=1}^{n} u_i'(x_i) = h[(\text{sgn } k) \sum_{i=1}^{n} v_i'(x_i)] . \tag{29}$$

(Note: (29) is the compact form of the two functional equations $u' = h(v')$ for k>0 and $-u' = h[(-v')]$ for k<0). Evaluating (29) at $x_j = x_j^0$ for $j \neq i$ and noting that $u_j'(x_j^0) = 1$ and $v_j'(x_j^0) = 0$ reduces (29) to

$$u_i'(x_i) = h[(\text{sgn } k)\ v_i'(x_i)]. \tag{30}$$

Substituting the right hand side of (30) into (29) gives

$$\prod_{i=1}^{n}[h\ (\text{sgn } k)\ v_i'(x_i)] = h[(\text{sgn } k)\ \sum_{i=1}^{n} v_i'(x_i)]. \tag{31}$$

In particular, let $x_i = x_i^o$ for $i = 3,4,\ldots, n$. This reduces (31) to

$$h[(\text{sgn } k)\ v_1(x_1)]\ h\ [(\text{sgn } k)\ v_2(x_2)] =$$
$$h\{[(\text{sgn } k)\ v_1(x_1)] + [(\text{sgn } k)\ v_2(x_2)]\}. \tag{32}$$

(32) is the fundamental functional evaluation $h(xy) = h(x + y)$ (see Aczel, 1961), which for continuous h has the non-trivial solution

$$h(r) = \exp\ (a\ r). \tag{33}$$

Since h is strictly increasing, $a>0$.

Substituting (33) into the original functional equations (29) and (30) gives

$$u_i'(x_i) = \exp[(\text{sgn } k)\ a\ v_i'(x_i)], \tag{34}$$

and $$\prod_{i=1}^{n} u_i'(x_i) = \exp\ [(\text{sgn } k)\ a \sum_{i=1}^{n} v_i'(x_i)]. \tag{35}$$

Resubstituting u_i and v_i for u_i' and v_i' gives the desired results:

$$1+ku_i(x_i) = (1+ku_i^o) \exp\{(\text{sgn } k)\ a\ [v_i(x_i)-v_i^o]\}, \quad (36)$$

and
$$\prod_{i=1}^{n}[1+ku_i(x_i)] = \prod_{i=1}^{n}(1+ku_i^o)\cdot\exp\{(\text{sgn } k)\ a \sum_{i=1}^{n}[v_i(x_i)-v_i^o]\}, \quad (37)$$

which, after appropriate definitions of constants becomes

$$1 + ku = b \exp[(\text{sgn } k)\ a\ v],\ b,\ a>0. \quad (38)$$

Another way of writing (38) is somewhat more instructive:

$$u = (\text{sgn } k)\ b' \exp\{(\text{sgn } k)\ a\ v\} + c. \quad (39)$$

This form shows that k directly controls the shape of the exponential transformation. In utility theoretic terms, if $k<0$ then h is a constantly risk averse transformation, if $k>0$ it is a constantly risk seeking transformation. This result was previously shown by Keeney and Raiffa, 1976, (p.331).

The next theorem reverses the roles of u and v:

Theorem 3 If u is additive as in (3) and v is multiplicative as in (2), then there exist $a>0$, b, such that

$$u = a \ln v + b, \quad (40)$$

where $v>0$.

Proof. The proof reduces (9) to the functional equation $h(x) + h(y) = h(xy)$. Consider some arbitrary $x^o = (x_1^o, x_2^o, \ldots x_i^o, \ldots, x_n^o)$ and define v_i^o, v^o, u_i^o, u^o, u_i' and u' as in Theorem 1.

The transformed value function is defined by

$$v_i' = \frac{v_i}{v^o} \tag{41}$$

$$v' = \frac{v}{v^o} \tag{42}$$

u' is an additive utility function (Lemma 4). To establish that v_i' and v' exist and represent a multiplicative value function, it suffices to show that for all i, $x_i \varepsilon X_i$, $v_i(x_i)>0$. If there was an y_i such that $v_i(y_i) \leq 0$, then reversals of preferences must occur among elements in X or all elements in X must be indifferent for that value of y_i. (See also Fishburn and Keeney, 1974.) Both cases are incompatible with the additive utility function u. Therefore (41) and (42) are positive transformations and v' is a multiplicative value function (Lemma 3). Since both u and v are defined onto some convex subset of Re, so are u' and v'. Therefore there exist a strictly increasing continuous function h such that

$$\sum_{i=1}^{n} u_i'(x_i) = h\left[\prod_{i=1}^{n} v_i'(x_i)\right]. \tag{43}$$

Evaluating (43) at $x_j = x_j^o$ for $j \neq i$ and, noting that $u_j'(x_j^o) = 0$ and $v_j(x_j^o) = 1$, (43) can be reduced to

$$u_i'(x_i) = h[v_i'(x_i)]. \tag{44}$$

Substituting (44) into (43) gives

$$\sum_{i=1}^{n} h[v_i'(x_i)] = h[\prod_{i=1}^{n} v_i'(x_i)]. \tag{45}$$

In particular, letting $x_i = x_i^o$ for $i = 3,4,\ldots, n$ reduces (45) to

$$h[v_1'(x_1)] + h[v_2'(x_2)] = h[v_1'(x_1)v_2'(x_2)]. \tag{46}$$

(46) is the functional equation $h(x) + h(y) = h(xy)$ (see Aczél, 1966) which for continuous h and positive r has the solution

$$h(r) = a \ln r, \tag{47}$$

where $a>0$, since h is strictly increasing.

Substituting (47) into the original functional equations (44) and (45) gives

$$u_i'(x_i) = a \ln v_i'(x_i), \tag{48}$$

and

$$\sum_{i=1}^{n} u_i'(x_i) = a \ln[\prod_{i=1}^{n} v_i'(x_i)]. \tag{49}$$

Resubstituting u_i and v_i for u_i' and v_i' gives the desired results:

$$u_i = a \ln \frac{v_i}{v_i^o} + u_i^o \tag{50}$$

and

$$u = a \ln \frac{v}{v^o} + u^o, \tag{51}$$

which, after appropriate definitions of constants is

$$u = a \ln v + b. \tag{52}$$

Theorem 4 If both us and v are multiplicative as in (2) and (4), then there exist a, b>0 such that

$$1+ku = (\text{sgn } k)(\text{sgn } v)\ b|v|^a \tag{53}$$

Proof. The proof consists of reducing functional equation (10) to the fundamental equation $h(xy) = h(x)h(y)$. Although the route will be similar to the previous theorems, the situation here is more general since sign reversals and null region are allowed. Again some $x^o = (x_1^o, x_2^o, \ldots, x_i^o, \ldots, x_n^o)$ is picked, but this time zero multipliers must be avoided, i.e., $1+ku_i(x_i^o) \neq 0$; $v_i(x_i^o) \neq 0$. There is no problem with excluding zero multipliers, as long as there exist at least one x and one y in X such that $x \succ y$. Because if in one attribute the only realizable value z_i^o was a zero multiplier, all elements in X would be indifferent.

Let v_i^o, v^o, u_i^o, u^o be defined as in the previous theorem.

Transformed value and utility functions are defined as follows:

$$v_i' = \begin{cases} \dfrac{v_i}{v_i^o} & \text{if } v_i^o > 0, \\[2ex] -\dfrac{v_i}{v_i^o} & \text{if } v_i^o < 0. \end{cases} \tag{54}$$

$$v' = \prod_{i=1}^{n} v_i' . \tag{55}$$

$$u_i' = \begin{cases} \dfrac{1+ku_i}{1+ku_i^o} & \text{if } 1+ku_i^o > 0 , \\ -\dfrac{1+ku_i}{1+ku_i^o} & \text{if } 1+ku_i^o < 0 . \end{cases} \tag{56}$$

$$u' = \begin{cases} \prod_{i=1}^{n} u_i' & \text{if } k > 0 , \\ -\prod_{i=1}^{n} u_i' & \text{if } k < 0 . \end{cases} \tag{57}$$

The above definitions guarantee that all single attribute transformations are positive, thus the sign scaling conditions of Lemmas 3 and 5 are met. Consequently u' and v' are again proper multiplicative value and utility functions. In addition, since u and v are defined onto some convex subset of Re, so are u' and v'. Thus Lemma 1 applies and there exists a strictly increasing continuous function h such that

$$\prod_{i=1}^{n} u_i'(x_i) = h[(\operatorname{sgn} k) \prod_{i=1}^{n} v_i'(x_i)]. \tag{58}$$

(Note: (58) is the compact form of the two functional equations $u' = h(v')$ for $k>0$ and $-u' = h(-v')$ for $k<0$.) Evaluating (58) at $x_j = x_j^o$ for $j \neq i$, and noting that $u_j'(x_j^o) = v_j'(x_j^o) = 1$ reduces (58) to

$$u_i'(x_i) = h[(\operatorname{sgn} k)\, v_i'(x_i)]. \tag{59}$$

Substituting the right hand side of (59) into (58) yields

$$\prod_{i=1}^{n} h[(\operatorname{sgn} k)\ v_i'(x_i)] = h[(\operatorname{sgn} k) \prod_{i=1}^{n} v_i'(x_i)]. \quad (60)$$

In particular, let $x_i = x_i^o$ for i=3,4,..., n. This reduces (60) to

$$h[(\operatorname{sgn} k)\ v_1(x_1)]\ h[(\operatorname{sgn} k)\ v_2(x_2)] =$$
$$h\{[(\operatorname{sgn} k)\ v_1(x_1)][(\operatorname{sgn} k) v_2(x_2)]\}. \quad (61)$$

(62) is the fundamental functional equation $h(x)h(y) = h(xy)$ (see Aczél, 1966) which, for continuous h has the nontrivial solution

$$h(r) = (\operatorname{sgn} r)\ |r|^a. \quad (62)$$

Since h must be strictly increasing $a > 0$. (See Aczél, 1966.)

Substituting (62) into the original functional equations (58) and (59) gives:

$$u_i'(x_i) = (\operatorname{sgn} k)[\operatorname{sgn} v_i'(x_i)]|v_i'(x_i)|^a \quad (63)$$

and $$\prod_{i=1}^{n} u_i'(x_i) = (\operatorname{sgn} k)[\operatorname{sgn} \prod_{i=1}^{n} v_i'(x_i)]|\prod_{i=1}^{n} v_i'(x_i)|^a. \quad (64)$$

Resubstituting u_i and v_i for u_i' and v_i' gives the desired results:

$$1+ku_i = (1+ku_i^o)(\operatorname{sgn} k)(\operatorname{sgn} \frac{v_i}{v_i^o})|v_i^o|^{-a}\ |v_i|^a, \quad (65)$$

$$\prod_{i=1}^{n} 1+ku_i = \prod_{i=1}^{n} (1+ku_i^o)(\mathrm{sgn}\ k)(\mathrm{sgn} \prod_{i=1}^{n} \frac{v_i}{v_i^o}) | \prod_{i=1}^{n} v_i^o |^{-a} | \prod_{i=1}^{n} v_i |^a, \quad (66)$$

which, after appropriate definitions of signs and constant terms gives

$$1+ku = (\mathrm{sgn}\ k)(\mathrm{sgn}\ v)\ b\ |v|^a \quad (67)$$

where $b,\ a > 0$.

Scaling Procedures and Examples

The solutions of the functional equations presented in Theorems 1-4 are unique up to the specification of two parameters. By Lemmas 2-5, however, we can transform value and utility functions using two free parameters. Thus it is possible to use standardization conventions in the construction of u and v such that both functions go through two arbitrary fixed points. Such conventions will "consume" the two free parameters in the functional equations and produce special forms of functional equations which do not depend any more on a and b.

For the case in which both u and v are additive this is possible by choosing elements x^o, $x^1 \in X$ such that $x^1 \succ x^o$, and by defining $u(x^o) = v(x^o) = 0$ and $u(x^1) = v(x^1) = 1$. Solving for a and b in the solution (14) of Theorem 1 gives the results $a = 1$ and $b = 0$; therefore

$$u = v. \quad (68)$$

In other words, if u and v are standardized a priori to assume values zero and one at identical points in X, they must be identical everywhere.

Using the same standardizations in the case where u is multiplicative and v is additive gives the solutions $b = 1$ and $a = (\mathrm{sgn}\ k)\ \ln(1+k)$. Substituting a and b in the solution (26) of Theorem 2 gives

$$1+ku = (1+k)^{v} . \tag{69}$$

Figure 1 gives examples of such functions, which only depend on the "interaction" parameter k in the multiplicative model. In the literature k typically is found to lie somewhere between -.95 and -.1 (see Keeney and Raiffa, 1976). For such negative values h is concave, which is interpreted as risk aversion in value. For $k \to -1$ the risk aversion increases, i.e., the function becomes more concave. For $k \to 0$ the function approaches $u = v$, which corresponds to the result that for $k \to 0$ u goes over into the additive model. For $k > 0$ the function h becomes convex with increasing convexity (risk proneness) as k gets larger.

The range between 0 and 1 is of particular interest for analyzing the curvature of h, since typically both u and v would be standardized by selecting x^{0} as the worst alternative and x^{1} as the best. Figure 2 shows a blowup of Figure 1 in the local range between 0 and 1. For k values between -.5 and +1 the functional relationship between u and v is almost linear, but for $k = -.99$ concavity (risk aversion) is substantial, and for $k = 50$ convexity (risk proneness) is significant.

Insert Figures 1 and 2 about here

In the case where u is additive and v is multiplicative the above standardization conventions cannot be used since $v > 0$ everywhere. However, applying the same conventions to u and defining $v(x^o) = 1$ and $v(x^1) = e$ gives the convenient solutions $a = 1$ and $b = 0$. Using these values in (52), the solution of Theorem 3 gives

$$u = \ln v. \tag{70}$$

When both u and v are multiplicative, standardizations are constrained by the fact that $(1+ku)$ and v must obtain zero values at identical points. This fixes a point of u as a function of v and makes it necessary to standardize separately for the cases where k is positive and where k is negative. If k is positive, the following conventions will be used: $u(x^o) = 0$, $v(x^o) = 1$; $u(x^1) = 1$, $v(x^1) = e$. These are the same conventions used above, but there is an additional restriction: x^o must not be a natural zero point (i.e., a zero multiplier), and for any zero multiplier z^o, $x^1 \succ x^o \succ z^o$. These conventions give solutions $b = 1$ and $a = \ln(1+k)$. Consequently the solution (67) of Theorem 4 reduces to

$$1+ku = (\operatorname{sgn} v)|v|^{\ln(1+k)}, \quad k > 0. \tag{71}$$

FIGURE 1

GLOBAL PROPERTIES OF FUNCTIONS RELATING u AND v IF u IS MULTIPLICATIVE AND v IS ADDITIVE (Standardization conventions as in text).

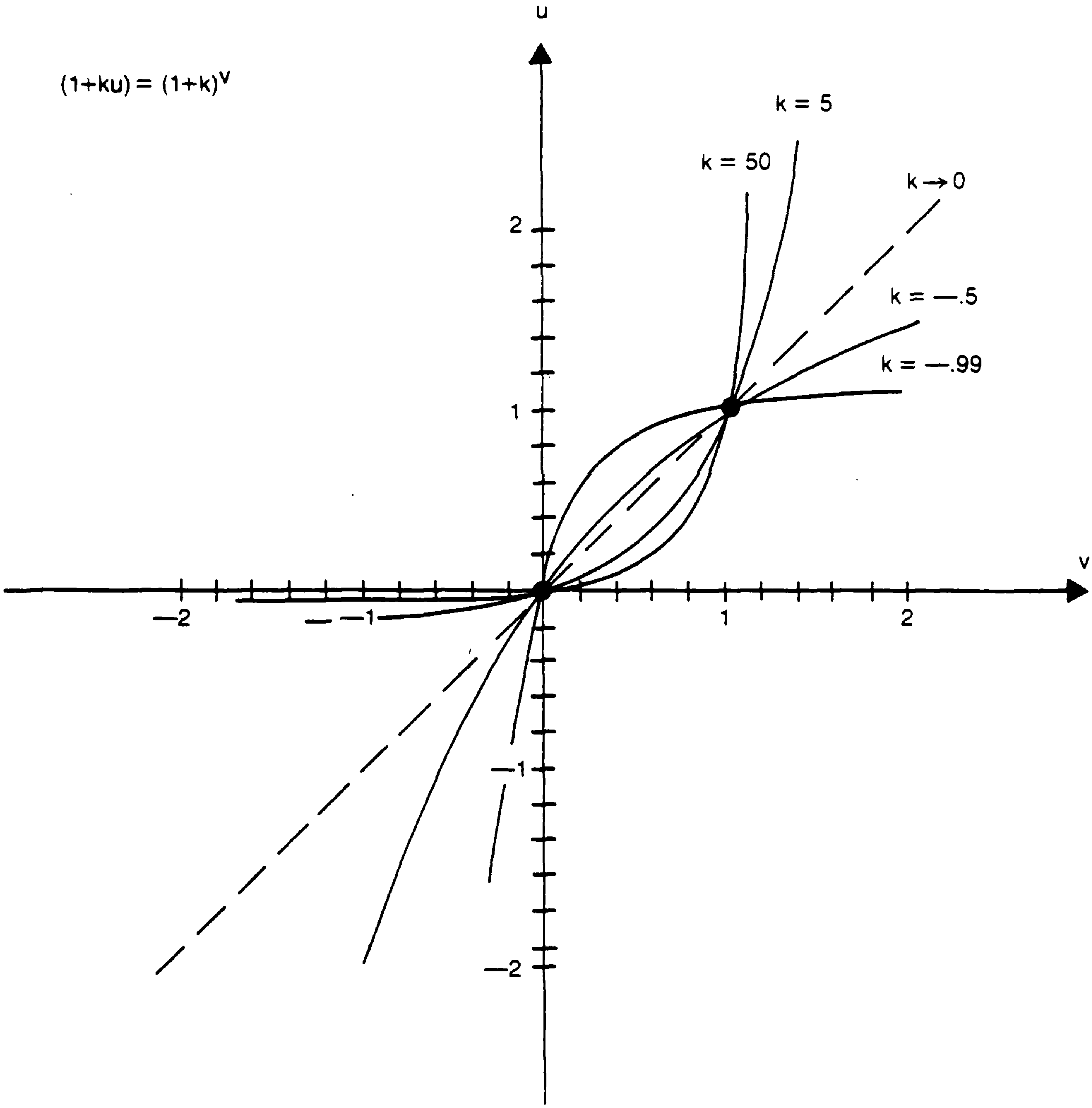


FIGURE 2

LOCAL PROPERTIES OF FUNCTION RELATING u AND v IF u IS MULTIPLICATIVE and v IS ADDITIVE (Standardization conventions as in text)

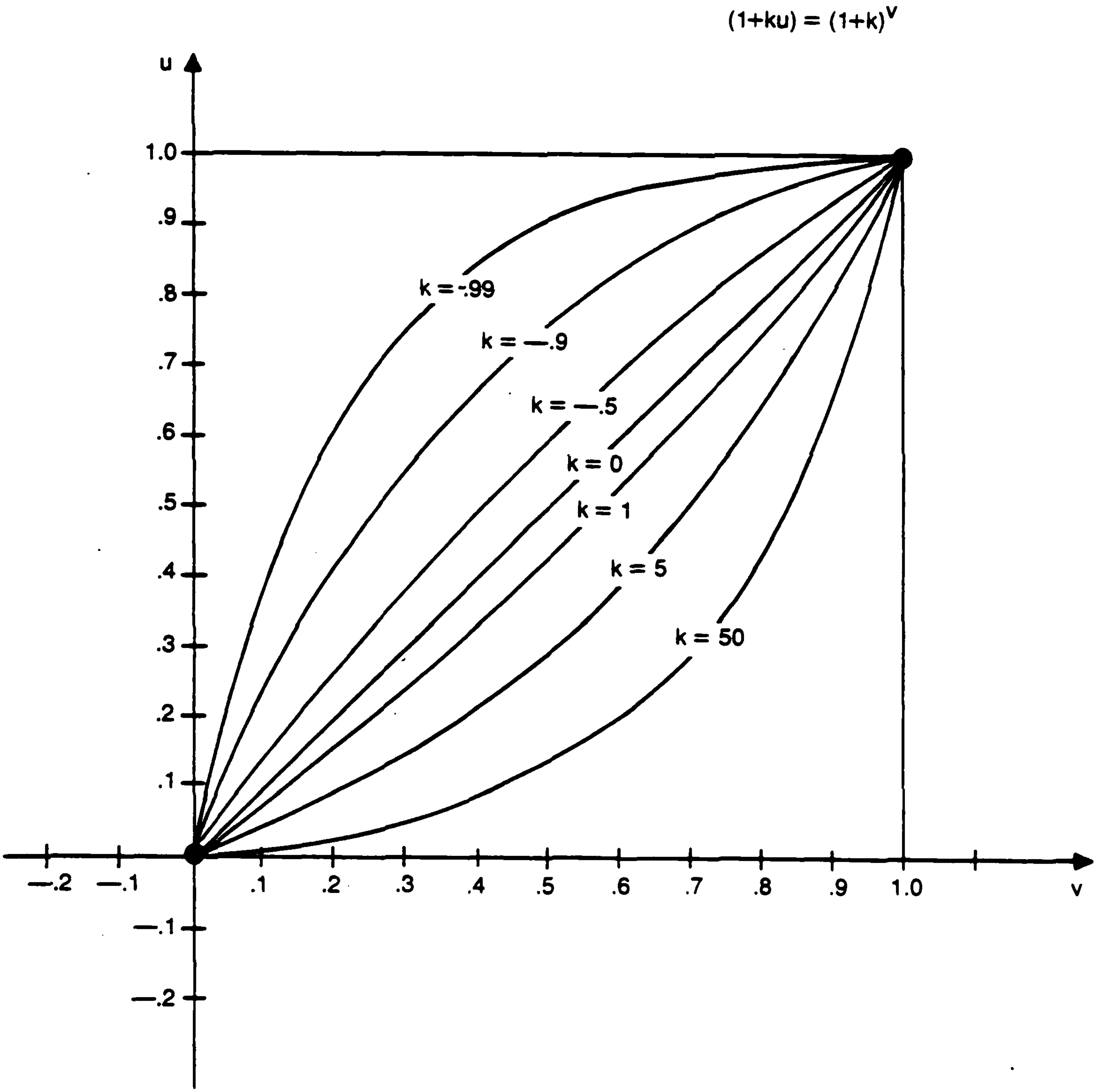


Figure 3 gives examples of this functional relationship which again only depends on k. $v = 0$ is in an inflection point for all k, and the positive and negative sides are mirror images up to a multiplicative transformation. This fact follows from the character of negative multipliers which should maintain the essential properties of the utility function in its negative and positive part. $k = e - 1$ gives the linear relationship. For larger k, h is concave in negative values of v and convex in positive ones. For smaller k this trend reverses. A final observation is the natural limit of the concavity of h when k tends towards zero. The dotted line represents this limit, which is in fact the function $u = \ln v$, as would be predicted since u becomes additive if k goes to zero.

Insert Figure 3 about here

Figure 4 gives again the blowup for the standard range of u between 0 and 1. Here it appears that k values between 1 and 5 tend to produce close to linear functions.

Insert Figure 4 about here

When k is negative the following standardization conventions will be used: $u(x^o) = 0$, $v(x^o) = -1$; $u(x^1) = -1$, $v(x^1) = -e$. These are similar to the conventions for positive k, but applied to the negative ranges of u and v. Here the additional assumption must be made that x^o

FIGURE 3

GLOBAL PROPERTIES OF FUNCTIONS RELATING u AND v IS BOTH ARE MULTIPLICATIVE (Standardization conventions for k > 0 as in text)

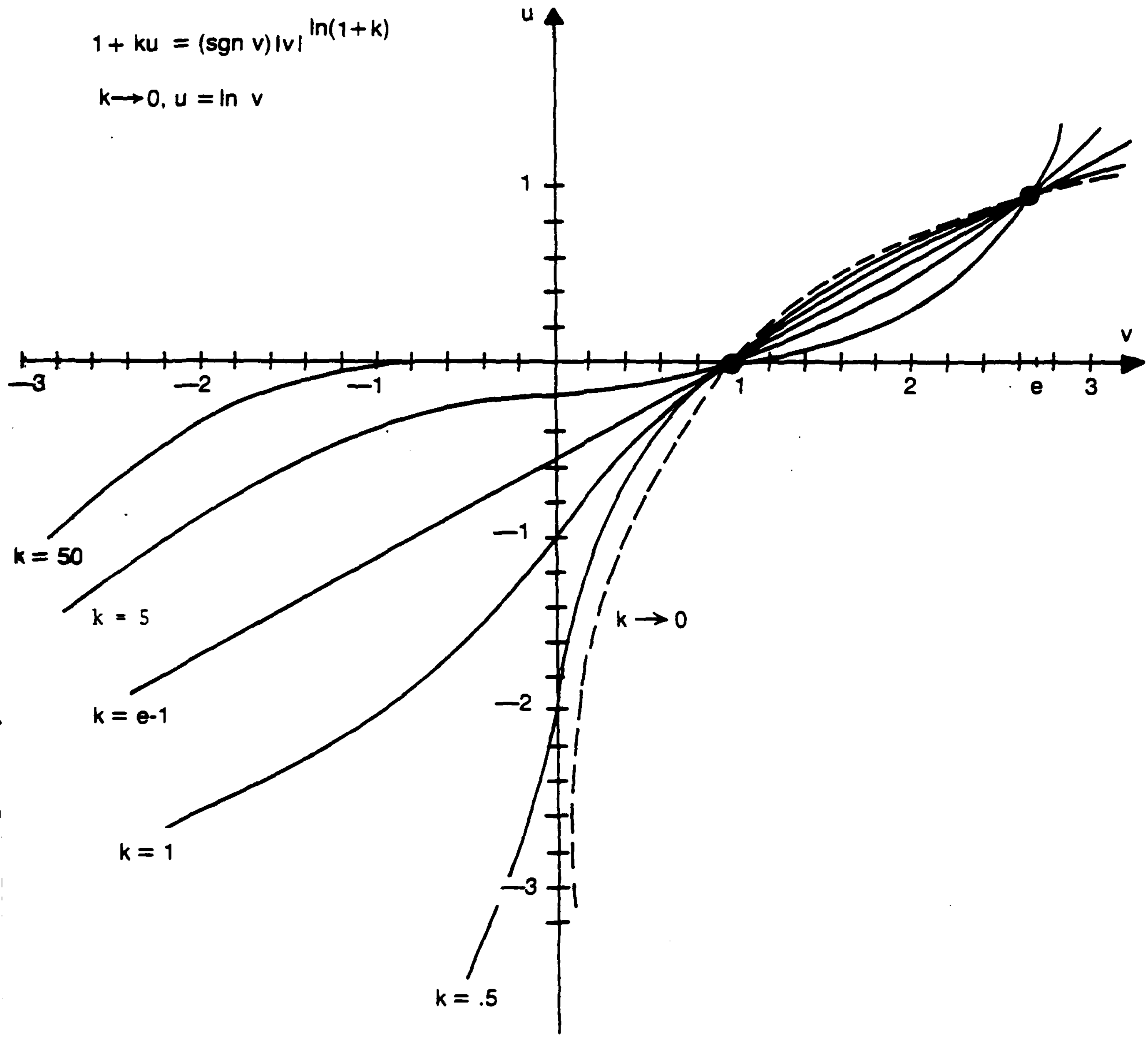
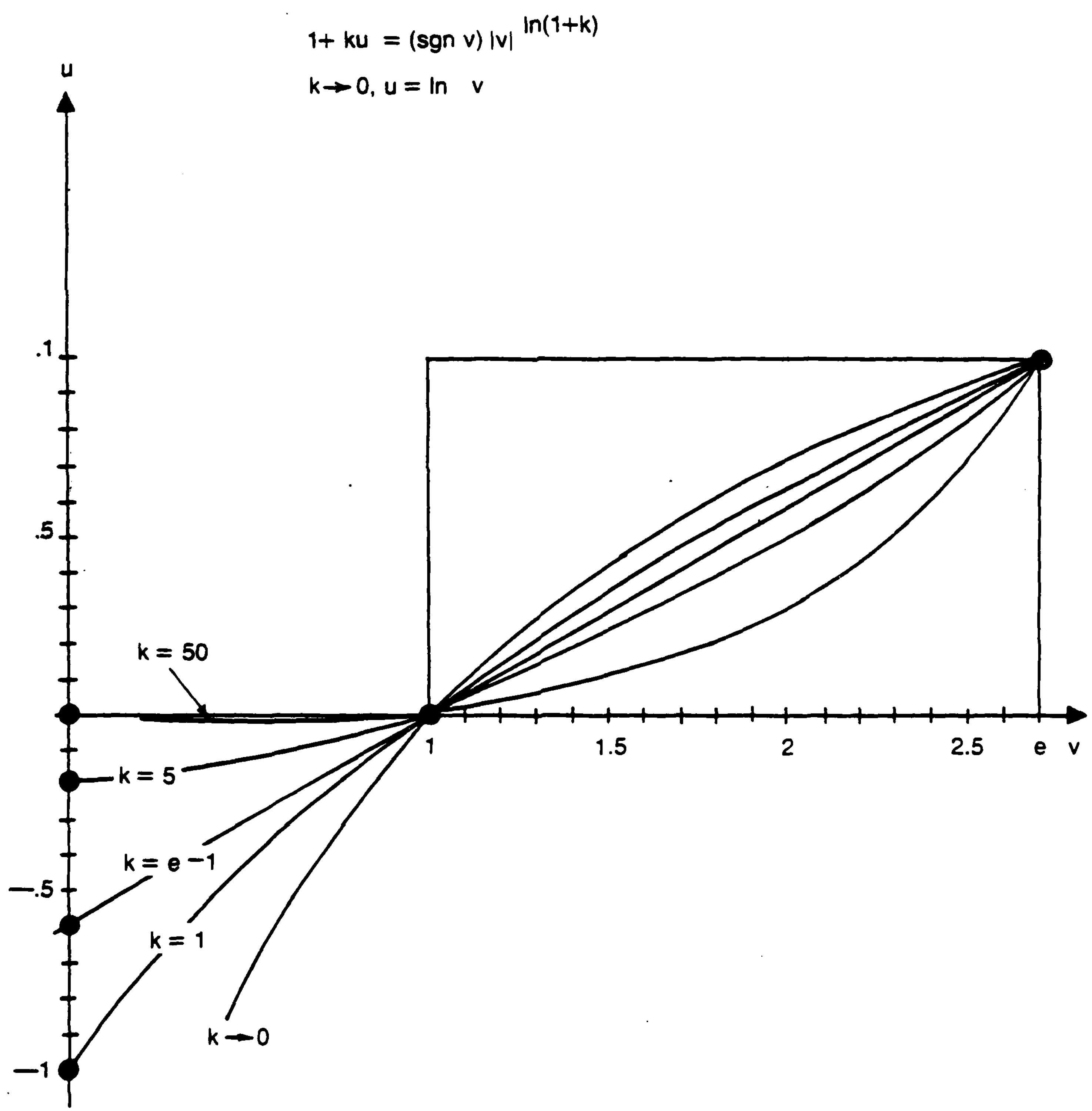


FIGURE 4

LOCAL PROPERTIES OF FUNCTIONS RELATING u AND v IF BOTH ARE MULTIPLICATIVE (Standardization conventions for k > 0 as in text)



is not a zero multiplier, and that for any zero multiplier z^o $z^o \succ x^o \succ x^1$. Using these standardization conventions, the solutions for a and b are b = 1 and a = ln (1-k). Substituting these solutions into (53), the solution to Theorem 5 gives

$$1+ku = -(\text{sgn } v)\ |v|^{\ln(1-k)},\ k < 0 \tag{72}$$

Figures 5 and 6 present examples of this functional relationship both for the global range of u and v and the local range within the standardized values. v = o is again an inflection point, and the positive and negative segments are mirror images up to a multiplicative transformation. In this case, however, h is convex (risk seeking) in the negative values of k and concave (risk prone) in the positive values. The local curvature of h within the standardized range is close to linear for most of the range of negative k's.

Insert Figures 5 and 6 about here

Behavioral and Applied Implications

Behavioral implications. To explore the behavioral meaning of the mathematical results it is helpful to consider a simple commodity bundle evaluation example. Let each X_i denote a commodity with unit price p_i. $x = (x_1, x_2, \ldots, x_i, \ldots, x_n)$ is a commodity bundle with amounts x_i. The market price for such a bundle would be

$$p(x) = \sum_{i=1}^{n} p_i x_i . \tag{73}$$

FIGURE 5

GLOBAL PROPERTIES OF FUNCTIONS RELATING u AND v IF BOTH ARE MULTIPLICATIVE (Standardization conventions for k < 0 as in text)

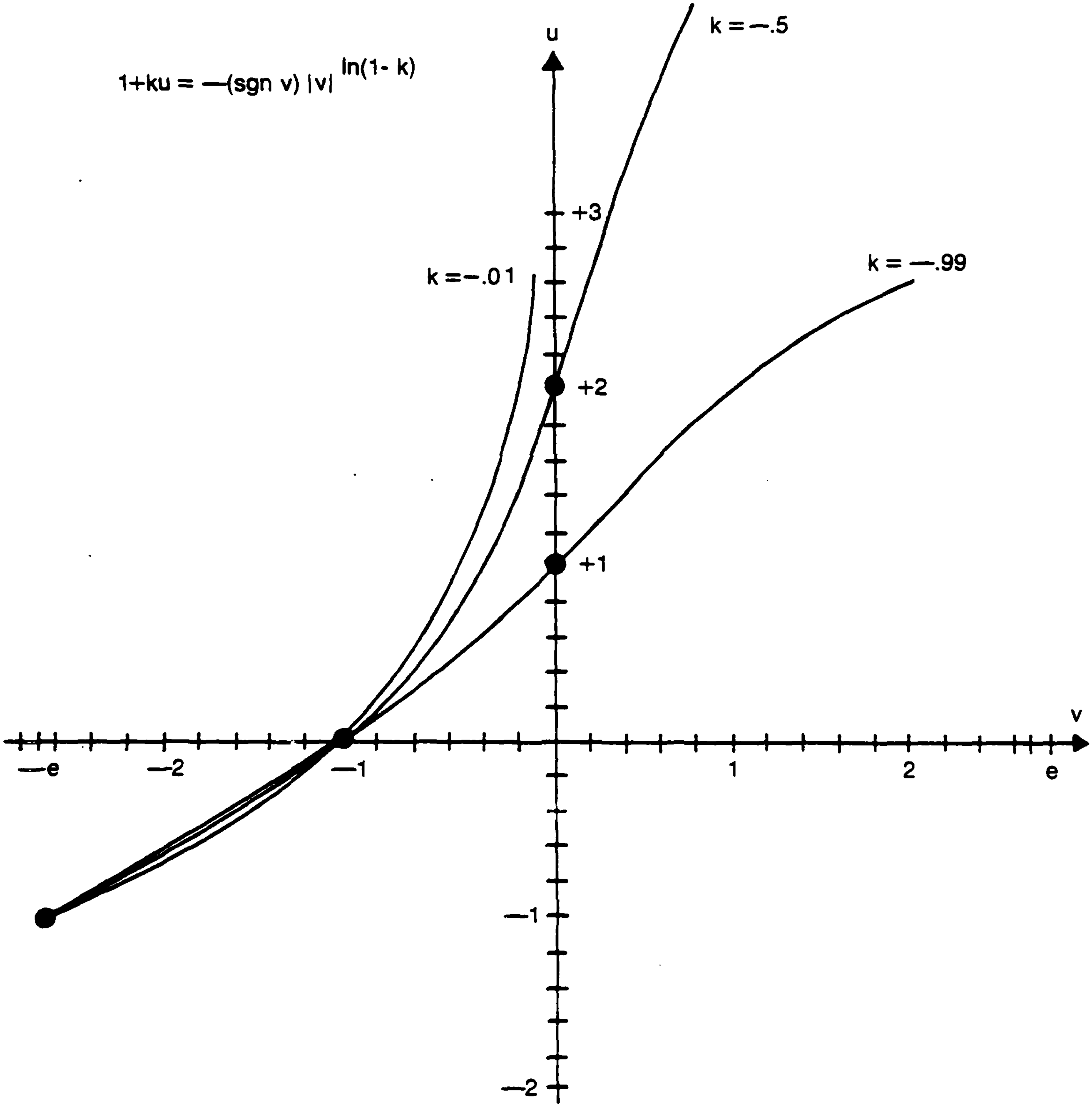
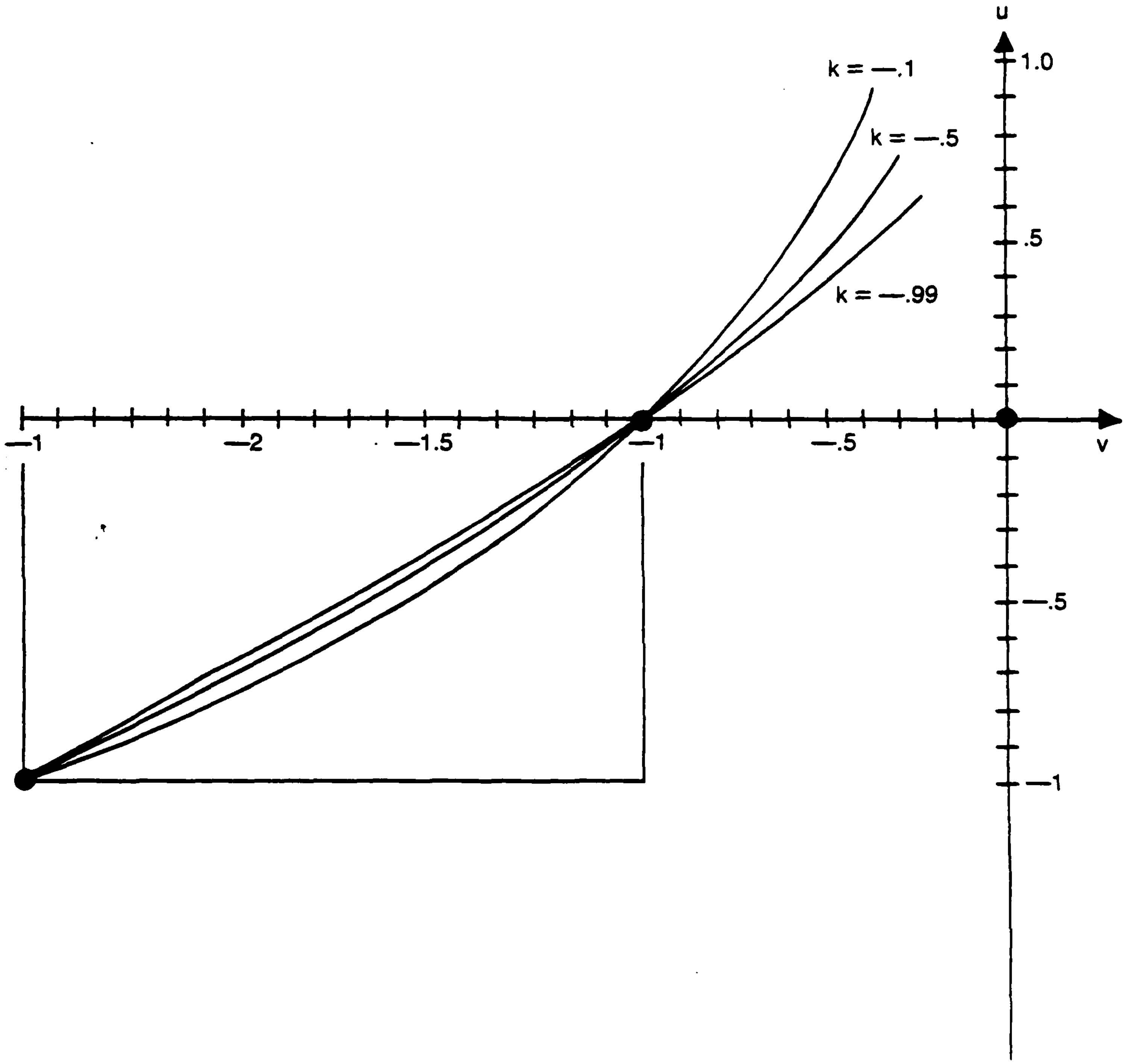


FIGURE 6

LOCAL PROPERTIES OF FUNCTIONS RELATING u AND v IF BOTH ARE MULTIPLICATIVE (Standardization conventions for k>0 as in text)



(73) is also a simple additive value model in which preferences for (riskless) commodity bundles are dictated by their respective prices. The implications of linear value functions $v_i = p_i x_i$ and linear indifference curves are obvious.

Assuming that an analyst only knows that (73) is an appropriate conjoint measurement value representation for some decision maker and that the v. Neumann and Morgenstern utility function u is additive, theorem 1 implies the strong results:

$$u(x) = \sum_{i=1}^{n} p_i x_i, \tag{74}$$

$$u_i(x_i) = p_i x_i. \tag{75}$$

That, of course, means that the decision maker is <u>risk neutral</u>.

To extend this line of argument to nonlinear value functions, assume that all v_i's are marginally decreasing as described by a logarithmic function

$$v_i(x_i) = \ln x_i, \tag{76}$$

and that v is additive. Assuming again that u is additive implies

$$u_i = \ln x_i. \tag{77}$$

The single attribute utility functions in (77) <u>appear</u> risk averse, but risk aversion in this particular context means nothing more than marginally decreasing value. There is absolutely no element of the colloquial notion of "avoiding to take chances" implied by the shape of the u_i's. On the contrary, I would consider a decision maker whose preferences follow (76) and (77) risk neutral in value.

This risk neutrality in value comes, in fact, as no surprise considering the structural assumption behind the additivity of u. This assumption, called the marginality assumption (Fishburn, 1965), requires the decision maker to be indifferent among all gambles with identical single attribute (marginal) probability distributions. To illustrate, consider the following two gambles for commodity bundles:

	GAMBLE 1		GAMBLE 2	
	HEADS	TAILS	HEADS	TAILS
GASOLINE	16 G	0	0	16 G
GROUND BEEF	10 P	0	10 P	0

According to the marginality assumption these two gambles should be indifferent. But an experimental study (v. Winterfeldt , 1979) revealed that subjects generally prefer the right option, because it has more balanced outcomes. Similar results were obtained by Delbeke and Fauville (1974) and in applied studies (Keeney and Raiffa, 1976). This is, of course, a form of risk aversion but to distinguish it from the usual single attribute risk aversion it has been called multiattribute risk aversion (see also Richard, 1975). The marginality assumption is therefore equivalent to multiattribute risk neutrality, and single attribute risk neutrality <u>in value</u>. It does not, however, require linear utility functions as example (77) shows.

Disentangling risk attitude from marginal value considerations becomes more interesting if we consider the riskless model $v(x) = \sum_{i=1}^{n} \ln x_i$ in connection with a multiplicative utility function. The standardized form of Theorem 2 implies

$$1+ku(x) = (1+k)^{\sum_{i=1}^{n} \ln x_i}. \tag{78}$$

Setting all $x_j = x_j^o$ except for some i gives

$$1+ku_i(x_i) = (1+k)^{\ln x_i}. \tag{79}$$

Now assume further than $k = e-1$, which yields

$$u_i = \frac{1}{e-1} x_i - \frac{1}{e-1}. \tag{80}$$

which is a <u>linear</u> utility function. This example shows that it is theoretically possible to have linear utility functions, not because the decision maker is risk neutral, but because his risk proneness compensates his marginal decreasing value function. The example also shows that the marginality tests could conceivably be violated in spite of linear utility functions. Or, in other words, a decision maker may be multivariate risk averse (prone) in spite of single attribute risk neutrality. Of course, this single attribute risk neutrality would only be an appearance when u is multiplicative and v is additive, since it may be based on strong risk aversion or proneness in value.

Multiattribute risk aversion and proneness can be related to the interaction parameter k in the multiplicative model (see Keeney and Raiffa, 1976). If $k < 0$, the decision maker must be multiattribute risk averse, i.e., he would prefer the more balanced gamble in the marginality test. If $k > 0$, the decision maker must be multiattribute risk prone, i.e., he would prefer the more imbalanced gamble in the marginality test.

Theorems 3 and 4 do not contribute substantively to the behavioral understanding of riskless and risky evaluation phenomena. The case where u is additive and v is multiplicative (in the non-genuine sense of having no null zones or sign reversals) appears to be very rare. In fact, from a strictly measurement theoretic point of view it cannot happen at all, since additivity of u implies the existence of an additive v. In the conjoint measurement sense a multiplicative v without null zones or sign reversals is indistinguishable from an additive v. To maintain that v is multiplicative in such a case requires a priori reasoning which lies outside the arguments of conjoint measurement theory. For example, in evaluating cars, the attributes safety and performance may be considered multiplicative factors a priori, even if there are not preference reversals or null multipliers in the attribute ranges under consideration. Such multiplicative versions of a theoretically additive conjoint measurement model may be preferable for face validity reasons. Consider the case of a non-genuine multiplicative v where all v_i's are exponential in the form $v_i(x_i) = e^{x_i}$, and where u is additive. By Theorem 3

$$u = \ln x = \sum_{i=1}^{n} x_i . \tag{81}$$

(81) demonstrates again the possibility of a linear utility function with curvilinear value functions.

Genuinely multiplicative value and utility functions are very rare and if they occur they are striking phenomena as Krantz et al note (Krantz et al, 1971). The behavioral interpretations for functional relationships between u and v in this case are less clear than in the previous cases. Assume that v is multiplicative in the form

$$v(x) = \prod_{i=1}^{n} x_i . \tag{82}$$

If u is also multiplicative, the standardized solutions of Theorem 4 implies

$$u_i(x_i) = (\operatorname{sgn} k)\, \frac{1}{k}\, (\operatorname{sgn} x_i)\ |x_i|^{\ln[1 + (\operatorname{sgn} k)k]} \tag{83}$$

which is risk neutral if $k = e - 1$, risk prone in positive x_i if $k > e-1$ and risk averse in positive x_i if $k < e-1$. The negative x_i's have the opposite risk attitude when compared with the positive x_i's.

Applied implications. The main applied argument for using Theorems 1-4 is that v can be constructed with simpler methods than u. u can be obtained from v on the basis of a priori reasoning about the aggregation form of u and v, and on the basis of a few risky questions to assess k.

Why is v easier to construct than u? In (conjoint measurement) theory a special sequence of indifference judgments is required to construct v. This procedure is called "dual standard sequence" or "lock step procedure" (see Krantz, et al, 1971 and Keeney and Raiffa, 1976). This procedure is not substantially simpler than the lottery methods required to construct u. But simple rating and weighting techniques, notably Edwards' SMART procedure (1977), can approximate dual standard sequences, since this involves similar cognitive processes. It would be much more difficult to justify a SMART approximation of u, since risk attitude and cognitive processes particular to lottery methods are involved in the direct assessment of u.

Assuming that SMART or some similar methods are fair or good approximations of v, Theorems 1-4 provide a simple basis of transforming v into u and of crosschecking the construction of value and utility functions. If the considerations of Theorem 1 apply, SMART can be used directly to take expectations, since u = v. If the conditions of Theorem 2 are met, the analyst has to assess the interaction parameter k in addition to v. Often this can be done by a priori reasoning about multiattribute risk attitude (to determine the sign of k) and by a few exploratory questions about the degree of multivariate risk attitude (to determine the size of k). Sometimes translating v into u may not even be worth the effort, e.g., if k is between -.5 and +1. Figures 1 and 2 show that in such cases the exponential transformations are almost linear, and expected utilities would be very difficult to distinguish from expected values taken over v.

Similar methods can be applied in the cases where the conditions of Theorems 3 and 4 are met. In all cases it would be good practice to construct both u and v and cross check their implications, when the transformation of v is assessed.

This last argument leads up to a development of methods for performing sensitivity analyses, based on Theorems 1-4. In particular, these theorems give some simple deterministic bounds for modelling errors, i.e., from using an additive SMART type model when the true model is multiplicative. By now, the steps to perform such sensitivity analysis should be obvious. Further experimental and numerical studies of the functional forms in the four theorems may prove useful for applications of decision analysis.

REFERENCES

1. J. AZCEL, Lectures on Functional Equations and Their Applications, Academic Press, New York, 1966.

2. L. DELBEKE and J. FAUVILLE, "An Empirical Test of Fishburn's Additivity Axiom," Acta Psychologica 38, 1-20 (1974).

3. J.S. DYER and R.K. SARIN, "Measurable Multiattribute Value Functions," Discussion Paper No. 66, Management Science Study Center, University of California, Los Angeles, 1977.

4. W. EDWARDS, "How to Use Multiattribute Utility Measurement for Social Decision Making," IEEE Transactions on Systems, Man, and Cybernetics 7, 326-340 (1977).

5. P.C. FISHBURN, Utility Theory for Decision Making, Wiley, New York, 1970.

6. P.C. FISHBURN and R.L. KEENEY, "Generalized Utility Independence and Some Implications", Operations Research 23, 928-940 (1974).

7. R.L. KEENEY and H. RAIFFA, Decisions with Multiple Objectives: Preferences and Value Tradeoffs, Wiley, New York, 1976.

8. D.H. KRANTZ and A. TVERSKY, "Conjoint-Measurement Analysis of Composition Rules in Psychology," Psychological Review 78, 151-169 (1970).

9. D.H., KRANTZ, R.D., LUCE, P. SUPPES, and A. TVERSKY, Foundations of Measurement, Vol. I, Academic Press, New York, 1971.

10. J. VON NEUMANN and O. MORGENSTERN, Theory of Games and Economic Behavior, 2nd ed., Princeton University Press, Princeton, N.J., 1947.

11. R.A. POLLAK, "Additive von Neumann and Morgenstern Utility Functions." Econometrica 35, 485-494 (1967).

12. H. RAIFFA, "Preferences for Multi-Attributed Alternatives," Memorandum No. 5868, The Rand Corporation, Santa Monica, 1969.

13. S.F. RICHARD, "Multivariate Risk Aversion, Utility Independence, and Separable Utility Functions," Management Science 22, 12-21 (1975).

14. L.J. SAVAGE, The Foundations of Statistics, Wiley, New York, 1954.

15. D. v.WINTERFELDT, "Additivity and Expected Utility in Risky Multiattribute Preferences," submitted to Journal of Mathematical Psychology, 1979.

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Expected utility theory and conjoint measurement theory form two major classes of models and assessment procedures to construct multiattribute utility functions. In conjoint measurement theory a value function v is constructed which preserves preferences among riskless multiattributed outcomes. The risky utility function u, constructed in the framework of expected utility theory, also preserves such riskless preferences. In addition, u is an appropriate guide for decisions under uncertainty since its expectation preserves risky preferences among gambles.

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Since both u and v are order preserving functions, they must be related by a strictly increasing transformation. However, u and v need not coincide or be related through any special functional forms, unless some simple decomposition forms are assumed. More restricted functional relationships obtain, if u and v are assumed to be either additive or multiplicative. In particular, u can be shown to be linearily, logarithmically, or exponentially related to v, depending on which function is additive and which is multiplicative. The paper proves such functional relationships based on the theory of functional equations, and techniques are described to assess the parameters of these functions. The results are discussed from a behavioral standpoint of interpretating the form and shape of multiattribute utility functions and from a practical standpoint of simplifying multiattribute utility assessment.

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