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A SURPRISING PROPERTY OF SOME REGULAR POLYTOPES

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Abstract

The Platonic solids and some other polytopes have the property
that the adjacency information contained in the skeleton (graph) is
enough to determine the polytope completely. In particular, the
eigenmatrix of the adjacency matrix corresponding to the second
eigenvalue provides the coordinates of the vertices.

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1. INTRODUCTION

Let P be an m-dimensional convex polytope with n vertices. The _skeleton_ of P is the graph G whose vertices and edges are the vertices and edges of P. Since the skeleton retains none of the metric information about P, many polytopes have isomorphic skeletons. For instance, the skeletons of all three-dimensional parallelepipeds are isomorphic to that of the cube.

Godsil (1978) suggested the following way in which a graph G could produce some polytopes. Let G be a graph on n vertices and A be its adjacency matrix. Suppose that α is an eigenvalue of G (that is, an eigenvalue of A) with multiplicity m; then one can construct an n x m eigenmatrix Z of orthonormal eigenvectors of A : $AZ = \alpha Z$, $Z'Z = I$. (The prime ' indicates transposition.) The distinct rows of Z may be interpreted as coordinate matrices of points in m-dimensional euclidean space (or as the points themselves). The convex hull of these points, denoted by $C(\alpha)$, is the polytope associated with eigenvalue α. For instance, the largest eigenvalue of a connected graph is necessarily simple, so the associated polytope is generally an interval. However, if all vertices of G have the same valency k, then the largest eigenvalue equals k, the column of 1's (denoted by e) is a corresponding eigenvector, and the polytope C(k) degenerates to a single point. Except in this case, the polytope $C(\alpha)$ has dimension equal to the multiplicity of α. Since Z multiplied on the right by any orthogonal matrix still has the properties noted, $C(\alpha)$ is determined up to rotation and reflection.

When the graph G to which the above process is applied is the skeleton of a convex polytope P, the $C(\alpha)$ may be related to P in some way. If P is isomorphic to a polytope $C(\alpha)$ associated with an

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eigenvalue of its skeleton, we say that P is _self-reproducing_. In other words, the adjacency information about the polytope is sufficient to determine the polytope, up to isomorphism. (Polytope isomorphism includes, but is not limited to, nonsingular affine transformations. See Brondsted, 1983.)

The polygons provide a family of simple examples. The skeleton of an n-gon is an n-cycle. With appropriate numbering, its adjacency matrix has 1's next to the main diagonal and in the upper right and lower left corners and 0's elsewhere. The eigenvalues are given by the formula (see Cvetkovic, Doob and Sachs, 1980, p. 306, or Schwenk and Wilson, 1978, p. 318)

$$\alpha_k = 2 \cos((k-1)\theta), \quad \theta = 2\pi/n, \quad k = 1, 2, \ldots, [n/2]+1.$$

All have multiplicity 2 except the first and, when n is even, the last. The orthonormal eigenmatrix Z corresponding to the second eigenvalue has elements
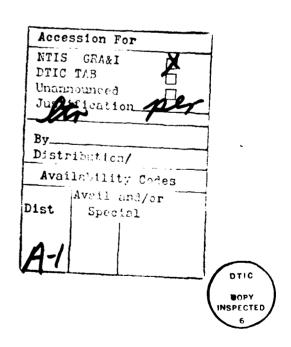
$$z_{j1} = \sqrt{2/n} \cos(j\theta), \quad z_{j2} = \sqrt{2/n} \sin(j\theta), \quad j = 1, \ldots, n.$$

It is obvious that the rows of Z are precisely the coordinates of the vertices of an n-gon centered at the origin. Thus we see that the n-gons (n > 2) are self-reproducing.

The objective of this paper is to show that the Platonic solids and some related polytopes are all self-reproducing. We will employ two methods of proof. One is to show that the matrix K, whose rows are the coordinates of the vertices of a polytope P, satisfies the eigenmatrix condition. The second is to show that the rows of an orthogonal eigenmatrix are coordinates of the vertices of (an isomorphic image of) the polytope P. In the case of the polygons, we have used the second method. We formalize the methods by stating two lemmas.

Lemma 1. Let K be the matrix of coordinates of the vertices of a convex polytope P, and let A be the adjacency matrix of its skeleton. If $AK = \alpha K$ and the multiplicity of α in the spectrum of A equals the dimension of P, then P is self-reproducing.

Proof. First note that the rank of K must equal the dimension of P, so the columns of K are independent. Then there is a nonsingular matrix T such that $Z = KT$ is an orthogonal eigenmatrix. The convex hull of the rows of Z is thus isomorphic to P, the convex hull of the rows of K.

Lemma 2. Let A be the adjacency matrix of the skeleton of a convex polytope P, and let Z be an orthogonal eigenmatrix corresponding to an eigenvalue α . If the rows of Z are coordinate matrices of the vertices of a polytope isomorphic to P, then P is self-reproducing.

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2. THE MAIN THEOREMS

In this section we prove the main results of the paper concerning the Platonic solids and some other regular polytopes.

Theorem 1. The Platonic solids are self-reproducing.

Proof. The tetrahedron, octahedron and cube are members of the three families of regular polytopes treated in Theorem 2. For the two remaining Platonic solids, we rely on Lemma 1 and the coordinates of central polyhedra reported by Coxeter (1973, pp. 52-53).

The icosahedron has twelve vertices whose coordinates are $(0, \pm \tau, \pm 1)$, $(\pm 1, 0, \pm \tau)$, $(\pm \tau, \pm 1, 0)$ where $\tau = (1 + \sqrt{5})/2$. Each vertex is adjacent to the five nearest vertices. Thus, row i of the equation $AK = \alpha K$, which we are testing, says that the sum of the coordinate matrices of the five neighbors of vertex i is a multiple of the coordinate matrix of vertex i. One can easily verify, for instance, that the five neighbors of $(\tau, 1, 0)$ are $(1, 0, \pm \tau)$, $(0, \tau, \pm 1)$ and $(\tau, -1, 0)$, with sum $\sqrt{5}(\tau, 1, 0)$. Next, note that the multiplicity of $\alpha_2 = \sqrt{5}$ in the spectrum of A is 3. (See Cvetkovic, Doob and Sachs, 1980, p. 310.)

The vertices of the dodecahedron are $(0, \pm \tau^{-1}, \pm \tau)$, $(\pm \tau, 0, \pm \tau^{-1})$, $(\pm \tau^{-1}, \pm \tau, 0)$ and $(\pm 1, \pm 1, \pm 1)$, and each vertex is adjacent to the three nearest vertices. One may verify that the neighbors of $(1, 1, 1)$ are $(\tau, 0, \tau^{-1})$, $(0, \tau^{-1}, \tau)$ and $(\tau^{-1}, \tau, 0)$, which sum to $(\tau + \tau^{-1})(1, 1, 1)$. The eigenvalue $\alpha_2 = \tau + \tau^{-1} = \sqrt{5}$ has multiplicity 3 in the spectrum of A (Cvetkovic, Doob and Sachs, 1980, p. 308).

The proofs are completed by observing that a vertex of either polyhedron may be carried to any other by a member of the rotation

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group of the polyhedron.

Theorem 2. The d-dimensional simplexes, cross-polytopes and orthotopes are self-reproducing, $d \geq$?.

Proof. (a) The skeleton of a d-dimensional simplex (including the tetrahedron) is the complete graph on $n = d+1$ vertices, with adjacency matrix $A = ee' - I$. Its spectrum consists of d (multiplicity 1) and -1 (multiplicity d). Let Z be an eigenmatrix corresponding to the eigenvalue -1: $AZ = -Z$, $Z'Z = I$. Then ZZ' is the projector of A corresponding to eigenvalue -1 (see Lancaster and Tismenetsky, 1985, p. 154): $ZZ' = I - (1/n)ee'$. From this equation it follows easily that the rows of Z, w_i', $i = 1,\ldots, n$, satisfy the equations

$$w_i'w_i = (n-1)/n, \quad \text{and} \quad w_i'w_j = -1/n \quad (i \neq j).$$

Thus $\|w_i - w_j\|^2 = w_i'w_i + w_j'w_j - 2w_i'w_j$ is the same for all pairs i,j ($i \neq j$), and the w_i are coordinates of $n = d+1$ equidistant points in d-dimensional space: i.e., vertices of a simplex. By Lemma 2, the d-dimensional simplex is self-reproducing.

(b) The coordinate matrix of the d-dimensional cross-polytope (generalized octagon) is given by

$$K_d = \begin{bmatrix} K_{d-1} & 0 \\ 0 & K_1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (d \geq 2)$$

Each vertex is adjacent to all other vertices except its antipode. Thus the skeleton is the complete d-partite graph with two vertices in

each partite set, and its 2d x 2d adjacency matrix is ($d \geq 2$)

$$A_d = \begin{bmatrix} A_{d-1} & E_{d-1} \\ E'_{d-1} & 0 \end{bmatrix} , \quad A_1 = 0$$

where E_d is a d x 2 matrix of 1's. By considering the complementary graph, it is easy to confirm that the spectrum of A_d consists of eigenvalues 2d-2 (simple), 0 (multiplicity d) and -2 (multiplicity d-1).

The last step is to show that $A_d K_d = 0$. Using partitioned matrix multiplication we find that one block of the product $A_d K_d$ is 0 and the remaining three are: $A_{d-1}K_{d-1}$ and $E'_{d-1}K_{d-1}$, both zero by induction; and $E_{d-1}K_1 = 0$ by direct calculation.

(c) The d-dimensional orthotope Y_d (including the cube) is formed as a prism raised on Y_{d-1} as base. Thus, the 2^d x d coordinate matrix is given by

$$K_d = \begin{bmatrix} K_{d-1} & e \\ K_{d-1} & -e \end{bmatrix} , \quad K_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad (d \geq 2)$$

The skeleton G_d of Y_d is the graph product $G_d = G_{d-1} \times G_1$, and the adjacency matrix is

$$A_d = \begin{bmatrix} A_{d-1} & I \\ I & A_{d-1} \end{bmatrix} , \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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Another way to express A_d is $A_d = I \times A_{d-1} + A_1 \times I$, where $\times$ denotes the Kronecker product. From this representation it follows that, if α is an eigenvalue of A_{d-1} then $\alpha+1$ and $\alpha-1$ are eigenvalues of A_d. (See Lancaster and Tismenetsky, 1985, pp. 406-413.) Hence the eigenvalues of A_d are

$$\alpha_k = d + 2 - 2k, \quad \text{multiplicity} \begin{pmatrix} d \\ k-1 \end{pmatrix}, \quad k = 1, 2, \ldots, d$$

By direct computation, one shows that $A_d e = de$ for all $d \geq 2$, and then the eigenmatrix equation $A_d K_d = (d-2)K_d$ is easily proved using induction and partitioned matrix multiplication.

3. QUESTIONS AND EXPLANATIONS

The facts proven above about Platonic solids and other regular polytopes immediately raise some questions, to which we can offer only partial answers.

Question 1. Why should a polytope of dimension d have an eigenvalue of multiplicity d? The answer seems to lie with the automorphism group of the polytope. Using the idea of the polytopes associated with eigenvalues, Godsil (1978) proved, but did not state, the following theorem. Babai (1978) proved it using vastly different methods. See Cvetkovic, Doob and Sachs (1980, Sec. 5.2) for a complete exposition.

Theorem A. Let G be a connected graph, and suppose that the distinct eigenvalues of G have multiplicities $m_1, \ldots m_s$. Then the automorphism group of G is isomorphic to a subgroup of

$$O(m_1) \times O(m_2) \times \ldots \times O(m_s)$$

where $O(m)$ is the group of orthogonal $m \times m$ real matrices.

Frucht (1936) identified the automorphism group of the skeleton for each of the Platonic solids. The results are as follows.

tetrahedron: S_4

cube and octahedron: $S_4 \times S_2$

dodecahedron and icosahedron: $A_5 \times S_2$

(Here S_n and A_n denote the symmetric and alternating groups on n symbols.) It is well known that S_4 and A_5 have no irreducible representation of degree less than 3; thus, Theorem A implies that the

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skeleton of each Platonic solid has an eigenvalue of multiplicity 3 at least.

Question 2. Why are the rows of an eigenmatrix corresponding to the second eigenvalue distinct? To set the stage for the partial answer, let $\alpha = \alpha_2$ be the next-to-largest eigenvalue of A , with multiplicity m, so an eigenmatrix has m columns and satisfies $AZ = \alpha Z$ and $Z'Z = I$.

Let us assume that G is vertex transitive, so that $Ae = ke$, where k is the valency of each vertex and consequently the largest eigenvalue. Then each column of Z is orthogonal to e : $e'Z = 0$. In other words, the sum of the rows of Z is 0. The vertex transitivity also implies that the rows of Z all have the same Euclidean length $\sqrt{m/n}$.

It is convenient to think in terms of the Laplacian matrix of G , $L = kI - A$. The eigenvalues of L (numbered from least to greatest) are related to those of A by $\lambda_i = k - \alpha_i$, and the eigenvectors of L corresponding to λ_i are precisely those of A corresponding to α_i.

A slight generalization of the Courant-Fischer theory and the Rayleigh quotient (see Lancaster and Tismenetsky, 1985, Section 8.2) allows us to identify the eigenvalue

(1) $\lambda_2 = (1/m) \min \{ \operatorname{tr}(X'LX) : X'X = I_m, e'X = 0 \}$.

It is known that the minimum occurs at an eigenmatrix Z. On the other hand, the expression being minimized in Eq. (1) can be transformed to

facilitate understanding of its meaning. In what follows, e_i is row i of the identity matrix.

(2) $\quad L = kI - A = \sum_i k e_i e_i' - \sum_{i,j} a_{ij} e_i e_j'$

But $\quad k = \sum_j a_{ij}$ for any i, so

(3) $\quad L = \sum_i \sum_j a_{ij} e_i (e_i' - e_j')$

(4) $\quad X'LX = \sum_i \sum_j a_{ij} y_i (y_i' - y_j') \qquad (e_i' X = y_i')$

(5) $\quad tr(X'LX) = \sum_i \sum_j a_{ij} tr(y_i(y_i' - y_j')) = \sum_i \sum_j a_{ij}(y_i' - y_j')y_i.$

Now use the symmetry of A to conclude that

(6) $\quad tr(X^T LX) = \frac{1}{2} \sum_i \sum_j a_{ij} ||y_i - y_j||^2.$

To visualize the situation (at least in the case of dimension m = 3), imagine that each row y'_i of X records the position of a unit mass in m-dimensional Euclidean space. These are to be located on the surface of a central sphere ($||y_i|| = \sqrt{m/n}$) in such a way that the center of mass is at the origin ($\Sigma y_i = 0$), but they must not all lie in a subspace of dimension less than m (rank X = m). Furthermore, if $a_{ij} \neq 0$ then the masses at y_i and y_j are joined by a spring. Finally, the locations are to be chosen so that the energy in the springs (Eq. 6) is minimized (Eq. 1).

From this formulation, it seems that superimposing some masses -- i.e., making some rows of X equal -- might be energetically

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advantageous: some springs would be brought to zero length. In the cases at hand, however, other strictures apply that prevent this from happening. Godsil (1978) and Powers (1981) showed that the set of vertices of G corresponding to equal rows of an eigenmatrix must form a set of imprimitivity of the automorphism group. Biggs (1974) showed that, in an edge- and vertex-transitive graph, any set of imprimitivity must contain only nonadjacent vertices: no spring may have zero length.

These incomplete explanations are the best available at the moment. The authors hope that they will be corrected or extended to theorems.

4. OTHER POLYTOPES, OTHER EIGENVALUES

To study other polytopes, we have been conducting computer experiments. We have written a program that accepts as input the adjacency list of a graph, computes an orthogonal eigenmatrix for each eigenvalue, and draws the given graph using the rows of two eigencolumns as coordinates of the vertices. When an eigenvalue has multiplicity 3 and the resulting drawing resembles a projection of a polytope, we have strong evidence that the polytope is self-reproducing, although this does not constitute a proof. In all the cases observed, it is the second eigenvalue that reproduces the polytope.

Most successes have been achieved with polytopes related to the Platonic solids in such a way as to preserve a suitably large group. For example, the truncated tetrahedron, octahedron and cube all seem to be self-reproducing. By direct computation, as in Theorem 1, we have shown that the cuboctahedron and the icosidodecahedron are self-reproducing. Since a polyhedron and its dual have the same group, one would expect the duals of these, the rhombic dodecahedron and the triacontahedron, to be self-reproducing. The computer results confirm the expectation for the former, but the latter appears to have a dodecahedron as the convex polytope associated with its second eigenvalue. Neither of these has a vertex-transitive graph. Among the semiregular polyhedra, besides those mentioned above, the snub cube and the small rhombicuboctahedron appear to be self-reproducing. The remaining semiregular polyhedra are too large for the presently used microcomputer. The 24-cell, a regular 4-dimensional polytope, also appears to be self-reproducing. Its second eigenvalue has multiplicity 4, and the drawings produced by the computer resemble

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closely the projections shown in Coxeter (1974), Plate VI, numbers
11-14.

Table 1: More self-reproducing polyhedra

Name	number of vertices
cuboctahedron	12
truncated tetrahedron	12
rhombic dodecahedron	14
snub cube	24
small rhombicuboctahedron	24
truncated cube	24
truncated octahedron	24
icosidodecahedron	30

Godsil's construction is valid for any eigenvalue, not just
the second. In Table 2 we have summarized the results of applying the
construction to all the eigenvalues of the Platonic solids.

Table 2: Polyhedra of eigenvalues

name	vertices	eigenvalue	multiplicity	points	convex hull
tetrahedron	4	3	1	1	point
		-1	3	4	tetrahedron
octahedron	6	4	1	1	point
		0	3	6	octahedron
		-2	2	3	triangle
cube	8	3	1	1	point
		1	3	8	cube
		-1	3	4	tetrahedron
		-3	1	2	interval
icosahedron	12	5	1	1	point
		$\sqrt{5}$	3	12	icosahedron
		-1	5	6	simplex
		$-\sqrt{5}$	3	12	icosahedron
dodecahedron	20	3	1	1	point
		$\sqrt{5}$	3	20	dodecahedron
		1	5	10	Petersen
		0	4	20	?
		-2	4	10	Petersen
		$-\sqrt{5}$	3	20	dodecahedron

Several comments are in order. First, note the number of points for the various polyhedra. As mentioned in Section 2, each point corresponds to a block of the automorphism group of the skelton of the polyhedron. The most commonly occurring blocks are pairs of antipodal vertices, frequently associated with the third eigenvalue. It is well known that identifying antipodal points in the graph of a Platonic solid yields a coloration of the graph (see Cvetkovic, Doob and Sachs, 1980, p.117). In particular this process applied to the dodecahedron graph yields the Petersen graph. Hence the presence of the two polytopes associated with the eigenvalues of the Petersen graph. A detailed study of these polytopes has been made. (Powers, 1986.)

Although Godsil's construction yields interesting results, an alternative is closer to the computer experiments and reveals a new feature. As before, suppose that G is a graph, Z is an eigenmatrix of its adjacency matrix corresponding to some eigenvalue, and the rows of Z are interpreted as points in m-dimensional Euclidean space. Let line segments join points corresponding to adjacent vertices of the original graph: if vertices i and j are adjacent in G, then a line segment joins points e_iZ and e_jZ. We call the result a framework corresponding to that eigenvalue.

For instance, the framework corresponding to the second eigenvalue of a 5-cycle is a pentagon, that of the third eigenvalue is a star or pentagram. Similarly, the frameworks corresponding to the last eigenvalues of the dodecahedron and icosahedron mark the edges of the great stellated dodecahedron and the great icosahedron respectively. This is not a great surprise, since Coxeter (1973, p.106) notes the fact that these are isomorphic pairs of polyhedra

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(under his definition of isomorphism). In all of the other cases of Table 2, the framework corresponding to an eigenvalue of multiplicity 3 is the skeleton of the convex polytope corresponding to that eigenvalue.

5. CONLCUSIONS

We conjecture that the three remaining regular polytopes, the 24-cell, the 120-cell and the 600-cell in four dimensions, are all self-reproducing. The skeletons of these polytopes are too large for the proof techniques used here. We hope to be able to prove that all sufficiently symmetric convex polytopes are self-reproducing.

The only necessary conditions uncovered -- eigenvalue multiplicity and distinct rows in the eigenmatrix -- are both linked to the automorphism group. It is surprising that vertex transitivity is not necessary, as the case of the rhombic dodecahedron indicates.

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