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Abstract (cont.)

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**Extracting Qualitative Dynamics from  
Numerical Experiments**

Kenneth Man-kam Yip

**Abstract**

One central problem in Qualitative Physics is the qualitative prediction of long-time behavior of physical dynamic systems. The machinery developed for qualitative reasoning – qualitative state vector, quantity space, and limit analysis – are largely applicable to systems which are piecewise well-approximated by low-order linear systems or by first order nonlinear differential equations. Typical nonlinear systems – those commonly encountered in Physics – exhibit a far richer spectrum of dynamical behavior. I propose to look at the simplest non-trivial form of conservative systems, the area-preserving maps, to provide a new source of examples for investigation into the fundamental issues of descriptive language, style of reasoning, and representation techniques. I study how physicists explore the dynamics of these systems by judicious use of numerical experiments. To automate the experimenting process, two key problems – (1) automatic experiment control, and (2) result interpretation – have to be solved. Knowledge of qualitative dynamics and bifurcations provides a strong constraint on the type of dynamical behavior possible. This constraint can be exploited to solve the problems. I discuss an approach to the control problem based on this idea. Finally, I describe the main result of this paper: an implemented program which solves the interpretation problem by using techniques from computational geometry and computer vision.

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EVERYTHING SHOULD BE MADE AS SIMPLE AS POSSIBLE,  
BUT NOT SIMPLER.  
- A. Einstein

## 1 INTRODUCTION

Qualitative Physics is a young field. Progress is made when researchers formalize and implement their understanding of how certain qualitative reasoning tasks, such as prediction of future behavior, and explanation of how the behavior comes about, are being performed in particular problem domains. Two domains, among others, have received much attention: circuit analysis and design in the engineering domain, and simple boilers and fluid flow in commonsense physics. Early works in Qualitative Physics primarily dealt with incremental deviation from equilibrium states where time evolution is not explicitly considered [4]. More recent works attempt to extend DeKleer's qualitative algebra and incremental analysis to handle time-varying behavior [5,15,14,11].

A common characteristic of the examples examined so far is this: the physical dynamic systems studied in these disciplines can often be divided into different operating regions each of which can be modeled to a first approximation by simple low order linear differential equations, or by first order nonlinear differential equations. The behavior of linear systems is particularly simple: the complete input-output behavior can be summarized in a single system transfer function. Consequently, if the response to one type of input is known, no more information is needed to determine responses for other input signals.

The situation in a nonlinear system is completely different: essential changes in the qualitative behavior of the system may occur as the amplitude of the input signal changes, or as the starting conditions are varied. More importantly, nonlinear systems have a far richer spectrum of dynamical behavior. Simple equilibrium points, periodic and quasiperiodic motion, limit cycles, chaotic motion as unpredictable as a sequence of coin tosses - these are some of the behavior found in a typical nonlinear system.

Unfortunately, these nonlinear characteristics do not show up in first order nonlinear differential equations. This is because the continuity and (local) uniqueness of flow severely constrain the kind of behavior possible on the real line: the flow either tends towards an equilibrium, or goes off to infinity.

In this research, I therefore propose to look at dynamical systems – those typically encountered in Physics – to provide a new source of examples for investigation into the fundamental issues of descriptive language, style of reasoning, and representation techniques in qualitative reasoning about nonlinear dynamical systems. Specifically, I will consider two-dimensional discrete dynamical systems defined by area-preserving maps containing a single control parameter. The study of area-preserving maps – transformations of the plane which preserves area – began with the venerable problem of the stability of the solar system. I choose to investigate this simplest non-trivial type of conservative system because many important problems in physics – the restricted 3-body problem, orbits of particles in accelerators, and two coupled nonlinear oscillators, just to mention a few – can be reduced to the study of area-preserving maps.

To illustrate the vocabularies that a physicist uses to describe the qualitative behavior of nonlinear systems, we can examine the following description of one of the phase portraits obtained from the results of numerical experiments with the *quadratic map* (see next section):

*Figure 1a represents a number of trajectories (sequences of points) for  $\cos \alpha = 0.4$  ... a regular structure in a neighborhood of the elliptic fixed point at the origin, and farther away a chaotic zone. In many places the plotted points are so dense that they give the illusion of a continuous curve. Near the origin, the "curves" are almost circular. As we move outward, the curves become distorted. Just inside the outermost regular curve lies a chain of six closed curves, or "chain of islands". Successive points of a trajectory jump from one island to another by application of the mapping. Finally, as the curves become more distorted, there is a sudden break-up and the set of points no longer lies on a curve, but seems rather to fill a two-dimensional region.*

The description is an edited form of a passage from [8, pages 99–100]. What is striking in this description is that the language is entirely geometric: it contains terms like *fixed point*, *continuous curve*, *chain of islands*, *two-dimensional region* etc. The phase portrait is completely characterized by the spatial relationships among these geometric objects.

A goal of this research is to develop a clear understanding of the geometric language displayed in the above description. In particular, I want to understand how representation and reasoning techniques based on the geometric language can be used

as a basis for a program to perform automatic numerical experiment control and interpretation.

The paper is organized as follows. Section two defines the task. Section three reviews the knowledge – qualitative dynamics and bifurcations – required for the task. Section four describes my approach to the experiment control problem. This part has not been implemented. Section five, the main result of this paper, shows the major pieces of an implemented program which solves the interpretation problem.

## 2 THE TASK

Given an one-parameter area-preserving map defining a discrete dynamical system, I am interested in describing the qualitative long-term, or asymptotic, behavior of the system over its entire operating range. Specifically, if  $U$  is an open subset of the phase space  $X \subseteq \mathcal{R} \times \mathcal{R}$ , and  $J \subset \mathcal{R}$  an interval over which the control parameter varies, I want my program to automatically generate (1) a family of phase portraits describing the main dynamical properties of the map for all initial conditions in  $U$  and parameter values in  $J$ , and (2) a summary graph which shows a cross-section of the map along certain axis of symmetry as a function of the parameter.

To explore the complete dynamics of a nonlinear system over a large region of the phase space and parameter space is a fairly typical problem in the physics literature. A good illustration of this task is provided by Henon's well-known paper, "Numerical Study of Quadratic Area-Preserving Mappings" [9]. The goal of Henon's paper is to provide a description of the main properties of the *quadratic map*:

$$\begin{aligned}x_{n+1} &= x_n \cos \alpha - (y_n - x_n^2) \sin \alpha \\y_{n+1} &= x_n \sin \alpha + (y_n - x_n^2) \cos \alpha\end{aligned}\tag{1}$$

where  $x$  and  $y$  are the state variables, and  $\alpha$  is the control parameter. The main results of Henon's paper are shown in Figures 1(a)-(f), which display the output of many numerical simulations, and Figure 2 which summarizes the dynamics of the map in a two-dimensional plot. Dots on the plot indicate regions of invariant curve, circles indicate islands, and blanks indicate chaotic zones.

The simplest approach to this problem is the *brute force* method: it divides the phase space and parameter space into small grids and tries every possible combinations of initial conditions and parameter values. A simple calculation will show

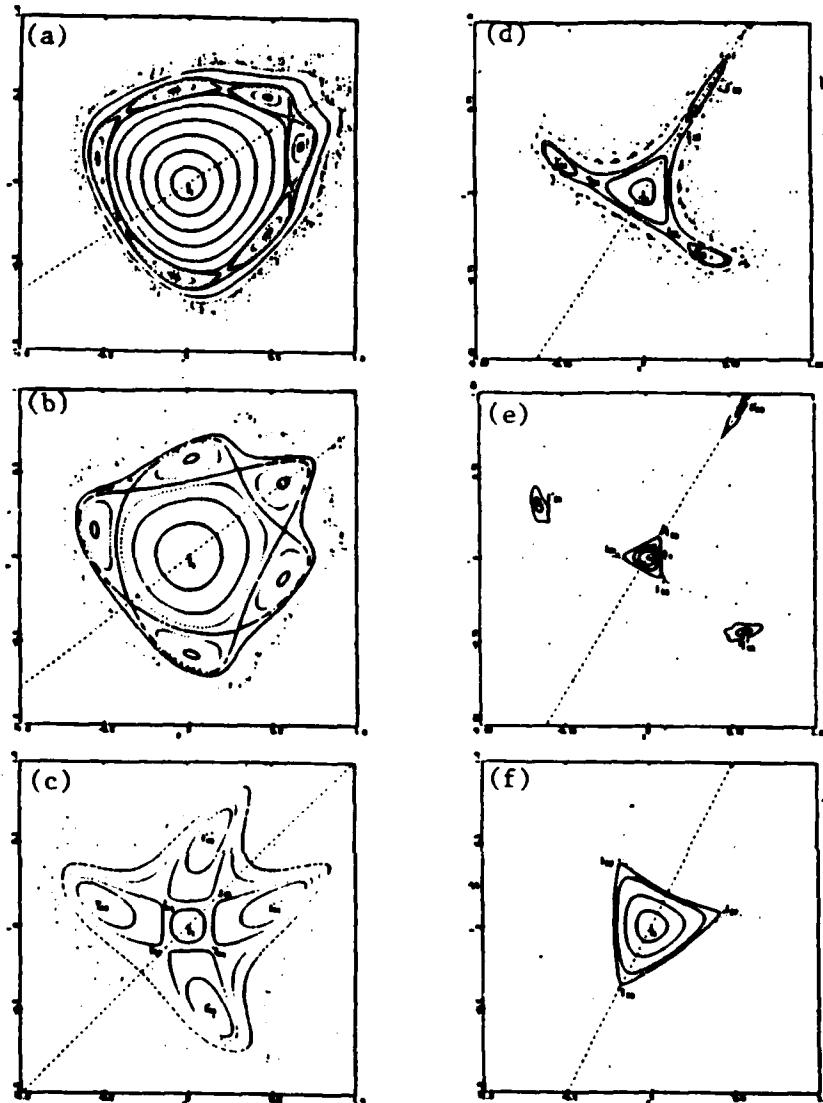


Figure 1: A partial list of phase portraits from numerical experiments. (a)  $\alpha = 1.16$  (b)  $\alpha = 1.33$  (c)  $\alpha = 1.58$  (d)  $\alpha = 2.0$  (e)  $\alpha = 2.04$  (f)  $\alpha = 2.21$ . Dashed line: axis of symmetry.

that this involves an enormous amount of computation. For instance, if we choose a uniform grid size of 0.01, we have to compute approximately  $300 \times 300 \times 600 = 54$  million orbits. Assuming, on the average, 0.02 second is needed to compute a trajectory of 500 points, it will take over 300 hours of computation time to compute all the trajectories.

The brute force method suffers from two serious problems. First, it is grossly inefficient because most of the phase portraits computed will be qualitatively the same. Second, it is not reliable because there is always the danger of missing some important qualitative features when the change occurs at a resolution finer than the grid size.

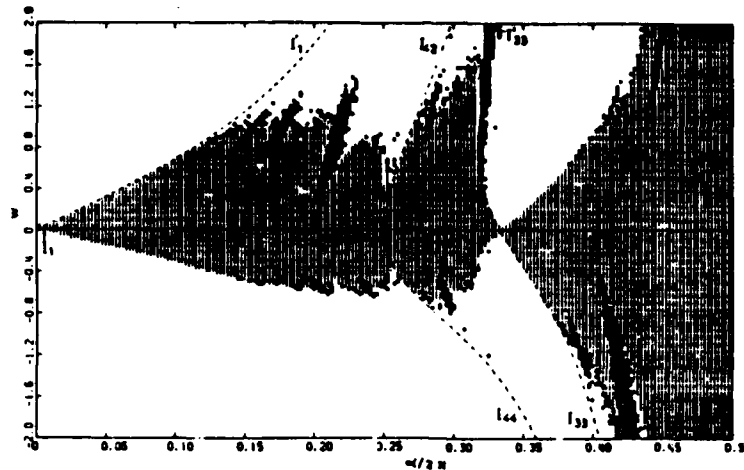


Figure 2: Summary picture for all values of  $\alpha$ , and initial points on the axis of symmetry. Dots: regions of invariant curve. Circles: islands. Blanks: chaotic zones. Dashed curves: unstable invariant points.

A physicist often does much better than this. Figure 3 represents a flow-chart of what an expert physicist does during the numerical experiment. The flow-chart has two nested loops. The outer loop involves deciding when to stop the experiment; the inner loop, when to move on to next parameter value. Controlling what experiment to do next, and interpreting the results of the simulation – these are the two most important decisions the experimenter has to make.

The task of behavior prediction can now be summarized as this: to develop a picture of all possible solutions to the dynamical system from a limited amount of numerical experiments at a limited number of initial conditions and parameter values. The key observation is that knowledge of qualitative dynamics and their geometric manifestations in the phase space provides a strong constraint on the type of behavior possible. As we will see in the next section, this constraint translates into a dramatic reduction of the amount of search required to find those combinations of states and parameter values that lead to “interesting” phase portraits.

### 3 WHAT IS THE KNOWLEDGE?

#### 3.1 Terminology

The purpose of this section is to introduce some concepts and definitions from Dynamical Systems Theory [10]. A dynamical system consists of two parts: (1) the

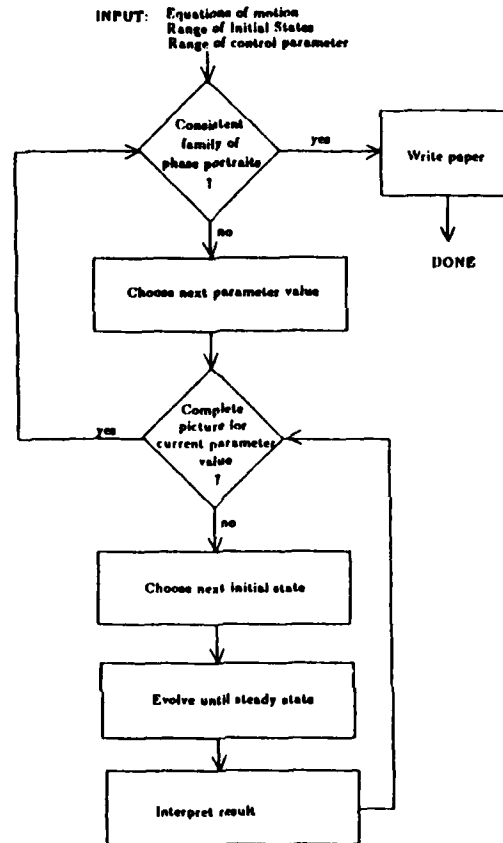


Figure 3: Flow-chart which describes the process of experimenting with a dynamical system.

system state, and (2) the evolution law. The **system state** at any time  $t_0$  is a minimum set of values of variables  $\{x_1, \dots, x_n\}$  which, along with the input to the system for  $t \geq t_0$ , is sufficient to determine the behavior of the system for all time  $t \geq t_0$ . The variables which define the system state are called **state variables**. The conceptual  $n$ -dimensional space with the  $n$  state variables as basis vectors is called the **phase space**. A **state vector** is a set of state variables considered as a vector in the phase space. As the system evolves with time, the state vector traces out a path in the phase space; the path is called an **orbit** or a **trajectory**. Finally, a **phase portrait** is a partition of the phase space into orbits.

The **evolution law** determines how the state vector evolves with time. In a finite dimensional discrete time system, the evolution law is given by difference equations. The difference equation is specified by a function  $f : X \rightarrow X$  where  $X$  is the phase space of the discrete system. The function  $f$  which defines a discrete dynamical system is called a **mapping**, or a **map**, for short. The **multipliers** of the map  $f$  are the eigenvalues of the Jacobian of  $f$ . An **area-preserving map** is a map whose Jacobian has a unit determinant.

The set of iterates of  $f - x, f(x), f^2(x), f^3(x), \dots, f^n(x)$  - as  $n$  becomes large is called the **orbit of  $x$  relative to  $f$** ; it captures the **history** of  $x$  as  $f$  is iterated.

Two types of point have the simplest histories - fixed point, and periodic point. The point  $x$  is a **fixed point** of  $f$  if  $f(x) = x$ . A fixed point  $x$  is called **stable**, or **elliptic**, if all the multipliers of  $f$  at  $x$  lie on the unit circle; it is called **unstable**, or **hyperbolic**, otherwise. The point  $x$  is a **periodic point** of period  $n$  if  $f^n(x) = x$ . The least positive  $n$  for which  $f^n(x) = x$  is called the period of  $x$ . The set of all iterates of a periodic point forms a **periodic orbit**.

### 3.2 Analytical Techniques

The first step in analyzing any dynamical system is to find its equilibria and their stability. For a second order discrete system, this amounts to solving a pair of non-linear algebraic equations. To take an example, the fixed points of quadratic map are the solutions to the following pair of algebraic equations:

$$\begin{aligned} x &= x \cos \alpha - (y - x^2) \sin \alpha \\ y &= x \sin \alpha + (y - x^2) \cos \alpha \end{aligned} \quad (2)$$

There are two solutions:  $x = y = 0$  and  $x = 2 \tan(\frac{\alpha}{2}), y = 2 \tan^2(\frac{\alpha}{2})$ .

Once the fixed points are found, the trace of the Jacobian of the map at each fixed point is computed to determine its linear stability. The stability condition is that the absolute value of the trace of the Jacobian is less than or equal to two [7]. For the quadratic map, the Jacobian is given by:

$$\begin{pmatrix} \cos \alpha + 2x \sin \alpha & -\sin \alpha \\ \sin \alpha - 2x \cos \alpha & \cos \alpha \end{pmatrix}$$

At  $(0,0)$ , the trace of the Jacobian is  $2 \cos \alpha$ . Since  $|2 \cos \alpha| \leq 2$ ,  $(0,0)$  is always stable. Similar stability analysis shows the other fixed point is always unstable.

### 3.3 Qualitative Dynamics and their Geometry

#### 3.3.1 Types of Orbit

In this section, I give a brief outline of the most important types of orbits in area-preserving maps. One important physical situation that gives rise to area-preserving

map is conservative systems of two degrees of freedom. Restricted to constant energy surface, the 4-dimensional phase space becomes 3-dimensional. By the Liouville Theorem on integrable systems [1, Chapter 10], this 3-dimensional subspace is diffeomorphic to a family of concentric 2-dimensional tori lying inside one another (see figure 4). A torus is called **invariant** if an orbit starting at a point on the surface of the torus stays on it forever.

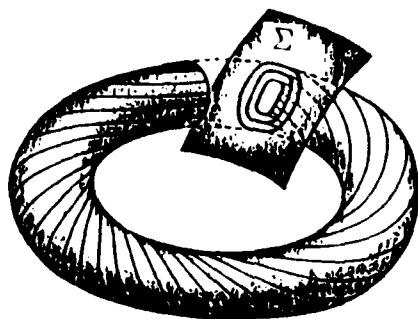


Figure 4: 3-dimensional phase space filled with 2-dimensional concentric tori. A cross-section  $\Sigma$  is drawn.

Qualitatively, three special types of motion are important: (1) **periodic motion** lying on an invariant torus, (2) **quasiperiodic motion** that densely covers an invariant torus, and (3) **chaotic motion** that wanders in a 3-dimensional volume of the phase space.

We construct an area-preserving map by taking a 2-dimensional cross-section  $\Sigma$  of the family of tori. An orbit beginning in  $\Sigma$  returns to it after making a circuit around the torus. The successive intersections of the orbit with  $\Sigma$  defines a map from  $\Sigma$  onto itself. The map is area-preserving because the 3-dimensional flow is conservative [2, Appendix 31].

Now, how do the periodic and quasiperiodic orbits appear on  $\Sigma$ ? When the motion is periodic, the orbit closes after a finite number of intersections with  $\Sigma$ , resulting in a finite number of points on  $\Sigma$ . If the motion is quasiperiodic, it will intersect  $\Sigma$  an infinite number of times, covering a closed curve densely. Finally, if the motion is chaotic, the orbit will intersect  $\Sigma$  erratically at many different places, eventually covering a finite area.

Including the case of unbounded motion for the sake of completeness, we have the following correspondence between special types of orbit in the 3-dimensional phase space and their geometric manifestations on the corresponding 2-dimensional phase space of the map:

|                      |        |                 |
|----------------------|--------|-----------------|
| periodic motion      | $\iff$ | periodic points |
| quasiperiodic motion | $\iff$ | invariant curve |
| chaotic motion       | $\iff$ | chaotic region  |
| unbounded motion     | $\iff$ | escape points   |

### 3.3.2 Bifurcations: Qualitative changes in the phase portrait

Two phase portraits are **qualitatively equivalent** if there exists a homeomorphism between them which preserves fixed points, periodic points, invariant curves, and their stability. **Bifurcation** is said to occur when the dynamical system goes through a qualitative change in its phase portrait as the control parameter is varied. I will focus on one important type of bifurcation: appearance and disappearance of periodic orbits.

I begin with [12] which gives a complete classification of the generic bifurcations of periodic points for one-parameter area-preserving maps. Only bifurcations which are generic are classified because the pathological cases arise rarely, and are much harder to deal with. Let  $\lambda$  and  $\frac{1}{\lambda}$  be the multipliers of a periodic point of an area-preserving map. Meyer has shown that generic bifurcations occur when  $\lambda = n^{\text{th}}$  root of unity where  $n = 1, 2, 3, 4$ , and  $\geq 5$ . Let us consider each of these cases (see figure 5):

#### Case 1. $\lambda = 1$ *Extremal Bifurcation*

The (discrete) flow line forms a cusp at the bifurcation point and as the multiplier passes through  $+1$ , there is a sudden birth of a pair of elliptic and hyperbolic closed orbits. Thus, a pair of fixed points are created for the map. This is also known as *creation, saddle-node bifurcation, or tangent bifurcation*.

#### Case 2. $\lambda = -1$ *Transitional Bifurcation*

As the multiplier passes through  $-1$ , the stable elliptic periodic point loses its stability, i.e., turns hyperbolic, and a stable cycle of twice the period appears. Other names for this phenomenon are: *period doubling, flip bifurcation, subtle division*.

#### Case 3. $\lambda = e^{\frac{2\pi i}{3}}$ *Phantom 3-kiss*

The region of stability of the elliptic fixed point shrinks to zero as the hyperbolic points of an unstable period-3 cycle “kiss” at the origin. After the “kiss”, the fixed point turns elliptic again, and a new unstable period-3 cycle is emitted. Note the change in orientation of the *triangular* region around the elliptic point. The phantom 3-kiss is often preceded by extremal bifurcations in a region a bit further away from the original elliptic fixed point, resulting in the formation of a pair of elliptic and hyperbolic period-3 points.

Case 4.  $\lambda = e^{\frac{2\pi i}{4}}$  Phantom 4-kiss

When the multiplier  $\lambda$  is a 4-th root of unity, it can undergo two types of bifurcation. The first case, phantom 4-kiss, is similar to that of case 3 except that we now have hyperbolic points of a period-4 orbit “kissing”. Tangent bifurcation also occurs at a distance from the center elliptic fixed point, resulting in an unstable period-4 cycle. The second case belongs to the fifth type to be described next.

Case 5.  $\lambda = e^{\frac{2\pi i}{p}}$  where  $p \geq 5$  Emission

As the multiplier passes through a  $p$ -th root of unity where  $p \geq 5$ , there is an emission of elliptic and hyperbolic sub-harmonics of  $\frac{1}{p}$  of the orbital frequency of the original elliptic orbit (shown in the figure for the case  $p = 5$ ). The elliptic points of the stable period- $p$  cycle alternate with the hyperbolic points of the unstable one near the vertices of a regular  $p$ -gon with center at origin. Thus, emission is responsible for the creation of “island chains” in the phase portraits of Figure 1. Meyer called this type of bifurcation the *generic p-bifurcation*.

## 4 APPROACH TO THE CONTROL PROBLEM

### 4.1 How to start the numerical experiment?

Elliptic fixed points are good places to start. We expect that the orbits near an elliptic fixed point, where the linear terms of the map dominate, will be mostly invariant curves. We then search radially outward until we encounter island chains, and eventually chaotic regions.

| Multiplier                       | Discrete Flow Pattern  | Bifurcation Diagram   | Names  |
|----------------------------------|--|---|--|
| $\lambda = 1$                    |  |   | extremal bifurcation<br>creation<br>saddle node bifurcation<br>tangent bifurcation       |
| $\lambda = -1$                   |  |   | transitional bifurcation<br>period doubling<br>flip bifurcation<br>subtle division       |
| $\lambda = e^{\frac{2\pi i}{3}}$ |  |   | phantom 3 line<br><br>(tangent bifurcation<br>further away from<br>elliptic fixed point) |
| $\lambda = e^{\frac{2\pi i}{4}}$ |  |   | phantom 4 line   |
| $\lambda = e^{\frac{2\pi i}{5}}$ |  |   | omission<br>generic p bifurcation  |
|                                  | <ul style="list-style-type: none"> <li>• elliptic periodic point</li> <li>• hyperbolic periodic point</li> </ul> | <ul style="list-style-type: none"> <li>— stable</li> <li>--- unstable</li> <li>nn = period n</li> </ul> |  |

Figure 5: Five Generic types of Bifurcation Geometry. A periodic point bifurcates whenever its multipliers pass through an  $n$ -th root of unity. The bifurcation diagram shows the loci of fixed points as the parameter  $\epsilon$  is varied.

## 4.2 How to decide what experiment to try next?

Knowing the generic bifurcation patterns is valuable for controlling numerical experiments. To begin with, it is difficult to locate the value of the control parameter at which bifurcation occurs: it is of probability almost zero that a randomly chosen point in the  $(x, y, \alpha)$  space will be the bifurcation point. But the pattern of flow near a periodic point just before and after the bifurcation occurs in a finite range of the control parameter; hence the pattern is easier to detect during the experiments.

Once a given flow pattern is found to match some parts in our library of bifurcation geometries, it will give us strong evidence that the corresponding bifurcation exists.

and we should be able to locate the rest of the flow patterns as given by the generic bifurcation. The pre-stored knowledge about these bifurcations gives us the complete information about what geometric objects, and approximately where in the control parameter space to look for.

To take an example, consider the *phantom 3-kiss* seen in figure 1d. The local flow pattern around the fixed point matches that in figure 5. According to the bifurcation pattern, the regular region around the stable fixed point will shrink in size, becoming an unstable fixed point; eventually, a new stable fixed point is born. So, we should expect to see figure 1f at some  $\alpha$  slightly greater than two.

### 4.3 How to decide when to terminate the experiments?

Besides imposing a strong constraint on what can be expected to happen in the phase portrait, the generic bifurcations also provide an answer to the problem of termination: *a simulation experiment is incomplete unless all the major qualitative features in the phase portrait can be explained by this finite list of local generic bifurcations.* An example is the change of stability of a fixed point. Suppose we have numerically located the 3-island chain and the center elliptic point at some  $\alpha = \alpha_0$  as in figure 1d. We know that the family of phase portraits is yet incomplete because we expect a *phantom 3-kiss* bifurcation to occur. In particular, we need to try at least two more experiments to obtain two phase portraits: first, at  $\alpha = \alpha_1$  when the triangular region is flipped, and second, at  $\alpha_2 \in (\alpha_0, \alpha_1)$  when the region becomes vanishingly small, indicating instability of the fixed point.

## 5 SOLVING THE INTERPRETATION PROBLEM

The **Interpretation Problem** consists of the following sub-problems:

1. *Orbit Type.* How can one recognize the orbit type - a 0-dimensional finite point set whose elements are encountered repeatedly, a 1-dimensional smooth curve, or a 2-dimensional region - of a set of iterates?
2. *Clustering.* How can one determine the number of islands in an island chain? This number gives the period of the enclosed periodic point.

3. *Area and Centroid.* How can one estimate the centroid and area enclosed by the curve? The centroid is a good approximation of the location of the enclosed periodic point. The area gives a measure of saliency of the island chain.
4. *Shape.* How can one recognize the shape of a curve? For example, is it a 3-sided figure resembling a triangle?

In the following, I will show how these four problems can be solved by applying techniques from computational geometry and computer vision.

## 5.1 Euclidean Minimal Spanning Tree

Given  $n$  points in the Euclidean plane, a Euclidean minimal spanning tree (EMST) is a tree that interconnects all the points with minimal total edge length. It turns out that EMST provides us with the fundamental data structure to solve the first three sub-problems.

Figure 6 shows the steps of processing. First, the program computes a EMST from the input point set. The current implementation uses the Prim-Dijkstra algorithm which has a run time complexity of  $O(n^2)$  where  $n$  is the number of points [3]. It takes about 100 seconds on the Symbolics to compute a EMST for a set of 512 points. A faster algorithm due to Shamos [13] which makes use of the Voronoi diagram is being implemented. The new algorithm can achieve the optimal lower bound of  $O(n \log n)$  time. Figure 7 shows the EMSTs constructed for three sample point sets.

Second, the program detects clusters in the EMST by looking for edges in the tree that are significantly longer than nearby edges. Such edges are called *inconsistent* [17]. The following criterion suggested by Zahn is used to determine if an edge is inconsistent. An edge is inconsistent if (1) its length is more than two times the average of nearby edges, and (2) its length is more than two standard deviations larger than the average of the lengths of nearby edges (where the statistics are taken from the set of all nearby edges). A point  $P$  is nearby point  $Q$  if  $P$  is connected to  $Q$  in the EMST containing two or fewer edges. Inconsistent edges are then deleted. This breaks up the EMST into connected sub-components which are collected by a depth-first tree walk. Figure 8 shows the result of the clustering procedure on the input point set of figure 7b. The procedure finds three inconsistent edges. For the other two point sets, no inconsistent edge is detected. The output of the clustering

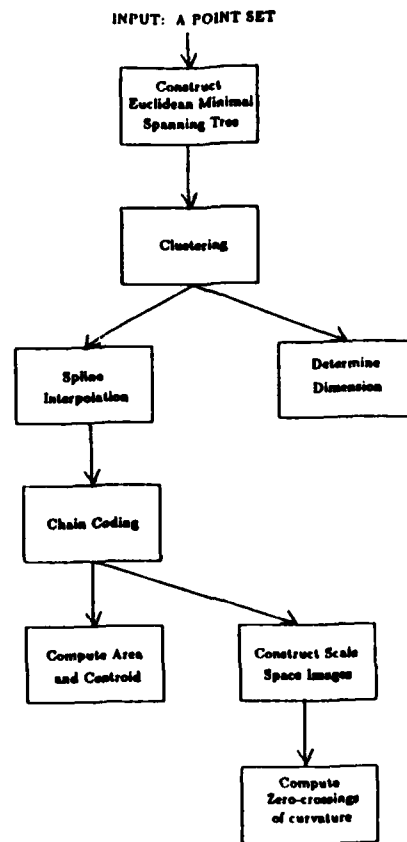


Figure 6: *Main processing steps for solving the interpretation problem*

procedure is a collection of connected sub-components each of which represents a sub-tree of the original EMST.

Third, for each sub-tree of the EMST, the program examines the degree of each of its nodes, where the degree of a node is the number of nodes connected to it in the sub-tree. For a smooth curve, the EMST consists of two terminal nodes of degree one; the rest, degree two. For a point set that fills an area, its corresponding EMST consists of many nodes having degree three or higher.

Finally, to compute the area and centroid of the region bounded by a curve, the program generates an ordered sequence of points from the EMST, and spline-interpolates the sequence to obtain a smooth curve. The smooth curve is encoded using chain coding [6]. Straightforward algorithms are then applied to compute the area and centroid.

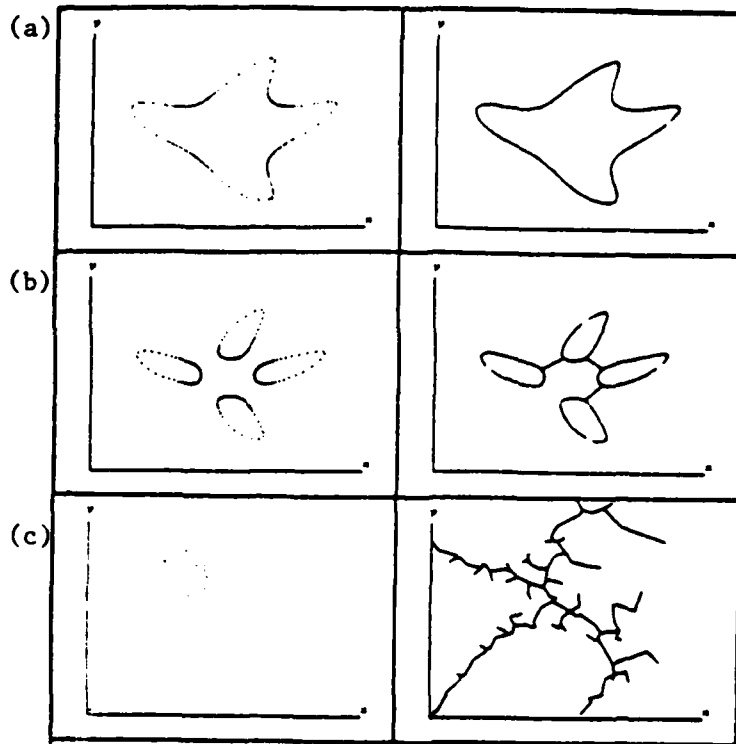


Figure 7: Minimal Spanning Tree for three sample point sets.

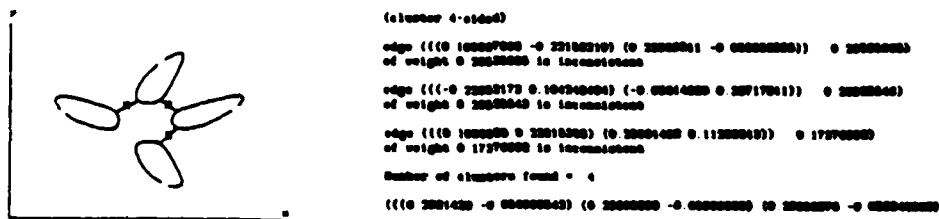


Figure 8: Result of clustering. Three inconsistent edges are detected in the point set of figure 7(b).

## 5.2 Scale Space Image

The shape of a curve, as we have seen in section 4, is an important clue to determine what type of bifurcation is occurring. The number of sides, the locations and types of extremal curvature – these are two of the most important geometric properties to recognize.

The recognition algorithm is based on the method of *scale space image* introduced

by Witkin [16]. First, a curve is parameterized by  $C(s) = (x(s), y(s))$  where  $s$  is the arc length along the curve. The two functions  $x(s)$  and  $y(s)$  are periodic for a closed curve, and they can be computed from the chain code representation. Second,  $x(s)$  and  $y(s)$  are smoothed by the Gaussian and its first two derivatives at multiple spatial scales. The scales are separated by one octave: 4, 5, 7, 11, 15, and 22, which are results of successively multiplying 4 by  $\sqrt{2}$  and rounding. Third, the curvature function  $\kappa(s)$  is computed by the formula:

$$\kappa = \frac{x\ddot{y} - \ddot{y}x}{(x^2 + y^2)^{\frac{3}{2}}}$$

Finally, the zero-crossings of  $\kappa(s)$ , and the signs of  $\kappa(s)$  are computed to determine the locations and type of the extrema.

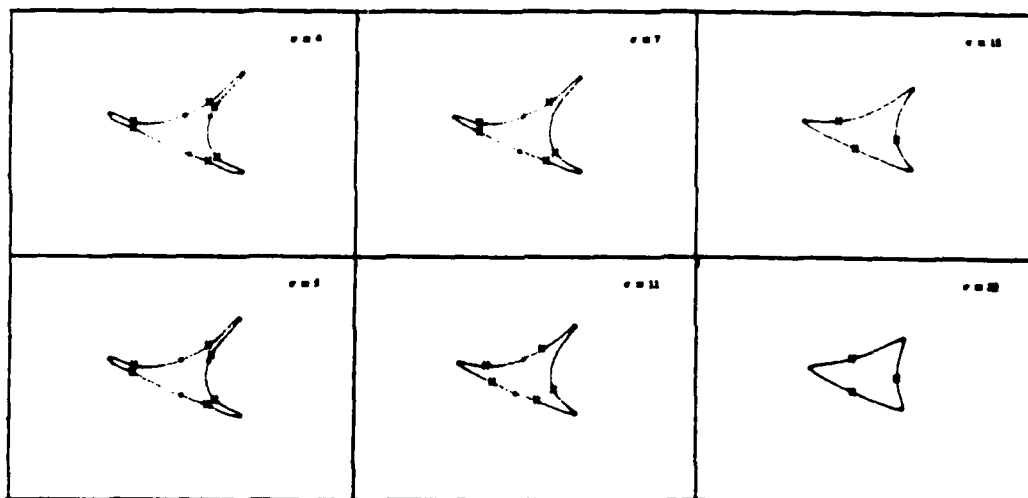


Figure 9: Type and locations of extrema of curvature. Spatial scales: 4, 5, 7, 11, 15, and 22. ● = maximum curvature; × = minimum curvature.

An example of the operation of the recognition algorithm is shown in Figure 9. The input is an orbit segment of 256 points which seems to fill a 3-sided curve. The positions of the extrema of curvature found at various smoothing scales are marked: ● for maximum curvature; × for minimum curvature.

## 6 SUMMARY

In this paper, I have studied the task of qualitative analysis of nonlinear area-preserving map by numerical experiments. I have also described how to approach the two major problems in automating the experimenting process: (1) experiment control, and (2) result interpretation. The basic idea is that knowledge of qualitative dynamics and bifurcations provides a strong constraint on the type of behavior possible. Finally, I have described a program which solves the interpretation problem by using techniques from computational geometry and computer vision.

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