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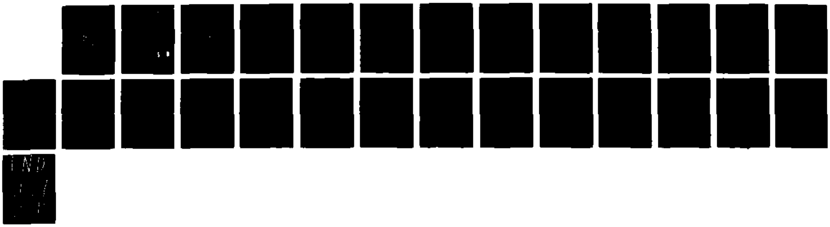

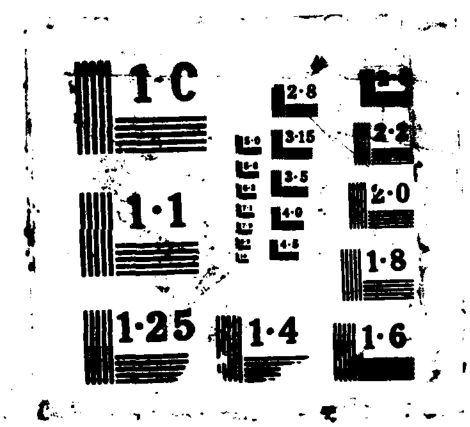
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EIGENVECTORS OF DISTANCE-REGULAR GRAPHS

by David L. Powers

Department of Mathematics and Computer Science

Clarkson University

Potsdam NY 13676

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ABSTRACT

The objective of this work is to find properties of a distance-regular graph G that are expressed in the eigenvectors of its adjacency matrix. The approach is to consider the rows of a matrix of orthogonal eigencolumns as (coordinates of) points in euclidean space, each one corresponding to a vertex of G. For the second eigenvalue, the symmetry group of the points is isomorphic to the automorphism group of G. Adjacency of vertices is related to linear dependence, linear independence and proximity of points. Relative position of points is studied by way of the polytope that is their convex hull. Several families of examples are included.

1. INTRODUCTION

The objective of this work is to find properties of a graph that are reflected in properties of the eigenvectors of the adjacency matrix. As a starting point, we investigate the case of distance-regular graphs. Graphs of this class have convenient algebraic properties, yet need not be either vertex- or edge-transitive. Nevertheless, the most interesting examples have large automorphism groups. In this introduction, we present definitions and a key lemma.

Definition: A partition of the vertex set of a graph G into V_1, $V_2, \ldots, V_k$ is a <u>coloration</u> if, for each i and j, each vertex u in V_i is adjacent to the same number, b_{ij}, of vertices in V_j. The square matrix B = $[b_{ij}]$ is called the <u>coloration matrix</u> of the partition.

A matrix-theoretic definition is this. Let X be the incidence matrix of the partition; that is, X has a 1 in the i,j-position if vertex i is in V_j or a 0 otherwise. Then the partition is a coloration if and only if AX = XB for some B, where A is the adjacency matrix of the graph. If the condition is fulfilled, then B is in fact the coloration matrix. Note that every row of X is a row of the identity, and no column is empty; thus X has independent columns. A general reference for colorations is Cvetkovic, et al. (1980, ch. 4).

Definition: A graph G is <u>distance-regular</u> if, for each vertex i of G, the <u>distance partition starting at</u> i,

$$V_0 = \{ i \},\ V_k = \{ j : \mathrm{dist}(i,j) = k \}$$

is a coloration, and the coloration matrix B is independent of i.

The matrix B has d+1 rows and columns, where d is the diameter of G. The triangle inequality in the graph guarantees that B is tridiagonal, and the implied connectedness makes the entries next to the diagonal nonzero. Furthermore, a distance-regular graph must be regular, of valency ρ, say, and the sum of the entries in each row of B is ρ. We follow the notation of Biggs (1974) for the elements of B (although the matrix called B there is the transpose of this):

$$B = \begin{bmatrix} 0 & b_0 & 0 & 0 \ldots & & & & \\ c_1 & a_1 & b_1 & 0 & 0 \ldots & & & \\ 0 & c_2 & a_2 & b_2 & 0 & 0 & & \\ & & \ldots & & & & & \\ 0 & 0 & \ldots & 0 & 0 & & c_d & a_d \end{bmatrix}$$

A familiar example of a distance-regular graph is the skeleton of a cube, shown in Fig. 1. The coloration matrix of the distance partition is

$$B = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

If G is any graph on n vertices, the <u>distance matrices</u> $A_0, A_1, \ldots, A_d$, where d is the diameter of G, are defined by the requirement that A_k have a 1 in the i,j-position if dist(i,j) = k and a 0 otherwise. Obviously, $A_0 = I$ and $A_1 = A$, the adjacency matrix. The following theorem about distance-regular graphs is well known. (See Biggs, 1974, p. 140.)

Theorem A. If G is a distance-regular graph, then the distance matrices A_0 , A_1 ,..., A_d form a basis for the algebra of polynomials in the adjacency matrix $A = A_1$.

For any symmetric n x n matrix A, if α is an eigenvalue of multiplicity m, there is an n x m matrix Z satisfying $AZ = \alpha Z$, $Z^TZ = I_m$. Such a matrix, composed of orthonormal eigencolumns associated with α , we call a complete eigenmatrix. Then the projector associated with α , $L = ZZ^T$, also satisfies (see, e.g. Lancaster and Tismenetsky, 1985, p. 196)

$$AL = \alpha L, \; L^2 = L, \text{ and } \operatorname{rank}(L) = m.$$

If A is the adjacency matrix of a distance-regular graph, Theorem A guarantees that there are coefficients such that

$$L = y_0 A_0 + y_1 A_1 + \dots + y_d A_d, \qquad (1)$$

because L is a polynomial in A. As the following lemma shows, the coefficients have a further significance in this case. This lemma is implicit in Theorem 21.4 of Biggs (1974, p. 143).

Lemma 1. Let A be the adjacency matrix of a distance-regular graph G, α an eigenvalue with multiplicity m, and let L be the associated projector. Then α is an eigenvalue of the coloration matrix B, and the column matrix $y = [y_0, y_1, \dots, y_d]^T$, containing the coefficients from Eq.(1), satisfies $By = \alpha y$ and $y_0 = m/n$.

Proof. Let X be the incidence matrix of the distance partition starting at vertex i, and consider column i of L (e_i is column i of the identity):

$$
\begin{aligned}
Le_i &= y_0\, e_i + y_1\, Ae_i + \dots + y_d\, A_d e_i \\
&= y_0\, Xe_1 + y_1\, Xe_2 + \dots + y_d\, Xe_{d+1} \\
&= Xy.
\end{aligned}
$$

Now $AXy = \alpha\, Le_i = \alpha\, Xy$

$= XBy.$

Since X has independent columns, we conclude that $By = \alpha\, y$. The second conclusion is proved by taking traces of both sides of Eq.(1).

2. SYMMETRY OF POINT SETS

Definition. Let C be an n x m matrix of coordinates of n not necessarily distinct points in m-dimensional euclidean space such that $C^TC = I$. The <u>symmetry group</u> of C, denoted by orth(C), is the set of m x m matrices R satisfying the condition CR = PC for some permutation matrix P. This definition is based on one of Coxeter (1973, p.253).

Theorem B. The set orth(C) forms a group of orthogonal matrices. Furthermore, the set perm(CC^T), composed of the permutation matrices that commute with CC^T, is also a group and contains precisely those permutations P for which the equation CR = PC has a solution R. Indeed, the mapping P -> C^TPC is a group homomorphism of the second onto the first, whose kernel is the group of permutations that satisfy PC = C.

Proof. Parts of the proof will be found in Godsil (1978). The remaining parts are routine.

We wish to apply the ideas of the theorem above to the case where C is the complete eigenmatrix associated with an eigenvalue of a distance-regular graph. When interpretation in terms of the graph is desired, we speak of the vertex i corresponding to the ith row of Z and/or to the point in euclidean space whose coordinates are found there. We need several technical lemmas.

Lemma 2. Let G be a connected graph, α its second eigenvalue. Then either $\alpha \geq 0$ or else G is a complete graph.

Proof. This lemma is a minor variant on a theorem of Smith (1970), cited by Cvetkovic et al. (1980).

Lemma 3. Let B be the coloration matrix of a distance-regular graph

G, α its second eigenvalue, $y = [y_0, y_1, \ldots, y_d]^T$ a corresponding eigenvector with $y_0 > 0$. Then $y_0 > y_1 > y_2 \geq \ldots \geq y_d$.

Proof. If G is a complete graph, then d = 1 and the result is trivial; from here on, assume that d is at least 2. Since B is tridiagonal, it is easy to show that y_0 cannot be 0. From the first row of the equation By = α y, one easily determines that

$$y_1 = (\alpha/\rho) y_0.$$

Now consider row i of the equation By = αy. Recall that the sum of the nonzero entries in any row is ρ, and use this fact to replace the diagonal entry. Then row i reads

$$c_i y_{i-1} + (\rho - c_i - b_i) y_i + b_i y_{i+1} = \alpha y_i,$$

which is algebraically equivalent to

$$(\rho - \alpha) y_i + c_i (y_{i-1} - y_i) = b_i (y_i - y_{i+1}) \qquad (2).$$

We know that y_0 is greater than y_1, which is nonnegative. Then, using i = 1 in Eq.(2), we see that $y_1 > y_2$. In general, if y_i is nonnegative and not greater than y_{i-1}, then y_{i+1} is not greater than y_i. Thus, the y's decrease until they reach a negative minimum. An argument developed by Fiedler (1975) and continued in Powers (1987) shows that the sequence of y's can change sign only once. Hence, y_d is negative. Analysis of the last row of the equation By = αy shows that $y_{d-1} > y_d$. Thus the sequence decreases as claimed.

Lemma 4. Let G be a distance-regular graph, α the second eigenvalue of its adjacency matrix, A. If Z is a complete eigenmatrix associated with α, then Z has distinct rows.

Proof. Rows i and j of Z are equal if and only if $c^T Z = 0$, where $c = e_i - e_j$. This is true if and only if $c^T Z Z^T = 0$, or $c^T L = 0$ -- that is, if rows i and j of L are equal. However, from Eq.(1) we see that row i of L has y_0 in the ith column, while row j has some y_k, $k > 0$, in the ith column. By Lemma 3, these numbers are different.

Theorem 1. Let G be a distance-regular graph, Z a complete eigenmatrix of the adjacency matrix A associated with the second eigenvalue, α. Then the symmetry group orth(Z) is isomorphic to the automorphism group of G.

Proof. Lemma 4 shows that the rows of Z are distinct, so orth(Z) is isomorphic to $\text{perm}(ZZ^T) = \text{perm}(L)$. Now consider Eq.(1). According to Lemma 3, the coefficient of $A_1 = A$ is greater than any of the subsequent coefficients. Therefore, a permutation matrix that commutes with L must also commute with A. On the other hand, L is a polynomial in A, so any matrix that commutes with A must also commute with L. Thus, perm(L) = perm(A), and the latter is well known to be isomorphic to the automorphism group of G.

This theorem sharpens results of Babai (1978) and Godsil (1978) for the class of distance-regular graphs. A convenient example is supplied by the cube in Fig. 1, whose second eigenvalue is 1, with multiplicity 3. A complete eigenmatrix associated with 1, shown below, is made up of the coordinates of the vertices of a cube. Frucht (1936) showed that the group of the skeleton of the cube is precisely what we have called orth(Z), thus confirming the results of the theorem.

$$Z = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^T$$

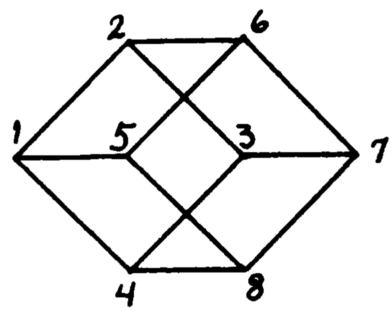

Fig. 1

3. ROWS OF AN EIGENMATRIX

Theorem 1 reveals a significant way in which the rows of an eigenmatrix reflect properties of the distance-regular graph that it belongs to. In this section, we find other properties of the graph that are tied to properties of the rows of an eigenmatrix. In all the theorems of this section, we assume that G is distance-regular, α is its second eigenvalue, the multiplicity of α is m, Z is a complete eigenmatrix associated with α, and $L = ZZ^T$ is the associated projector.

Theorem 2. Let $w_i^T = e_i^T Z$ be the ith row of Z. Then, for any vertex i of G, the minimum of $||w_i - w_j||$, $i \neq j$, is achieved at precisely those vertices j that are adjacent to i in G.

Proof. The square of the quantity to be minimized is

$$||w_i||^2 + ||w_j||^2 - 2 w_i^T w_j.$$

However, we know that $w_i^T w_j$ is the i,j-entry of $ZZ^T = L$. From Eq.(1), we see that this quantity is $2y_0 - 2y_r$, where $r = \text{dist}(i, j)$. By Lemma 3, this is minimized when $r = 1$, which is the conclusion of the theorem.

Theorem 2 tells us that proximity of the rows of Z corresponds exactly to adjacency in G. The next theorem connects linear independence of rows to mutual adjacency.

Theorem 3. Let $G \neq K_n$, and let U be a set of q mutually adjacent vertices of G. Then the corresponding rows of Z, w_i^T, i in U, are linearly independent matrices.

Proof. First note that the set of matrices cited contains q distinct elements, by Lemma 4. Moreover, they are independent if and only if the matrices Le_i, i in U, are independent. Now, the column matrix Le_i has, by Eq.(1), a y_0 in the ith entry and a y_1 in the jth entry, if i and j are adjacent. Thus, the submatrix of L whose row and column indices are in U has y_0 on the diagonal and y_1 elsewhere, forming the q x q matrix (e is the column of 1's)

$$y_0 I + y_1 (ee^T - I) = (y_0 - y_1) I + y_1 ee^T$$

with eigenvalues $y_0 - y_1$ (multiplicity q-1) and $y_0 + (q-1)y_1$ (multiplicity 1). The former is nonzero by Lemma 3. The latter can be zero only if y_1 is negative, which is impossible by Lemma 2, since $G \neq K_n$.

Corollary. Let $G \neq K_n$ be distance-regular with girth 3. If G contains a q-clique, then the multiplicity of the second eigenvalue is at least q.

Note that Terwilliger (1982) gives 2 as a lower bound on the multiplicity of any eigenvalue (other than ρ) of a distance-regular graph of girth 3. While Theorem 3 identifies some independent rows of Z, Theorem 4 will identify some dependent rows. The corollary is a geometric restatement.

Theorem 4. If i is any vertex of G, Z a complete eigenmatrix associated with any eigenvalue α of A. Then the rows of Z corresponding to i and its neighbors are dependent.

Proof. Row i of the adjacency matrix is $e_i^T A = \sum_{j@i} e_j^T$ (where @ means "adjacent to"). Thus row i of the equation $AZ = \alpha Z$ reads

$$\sum_{j@i} w_j^T = \alpha w_i^T,$$

which is precisely the conclusion.

Corollary. For each vertex i let $u_i = \rho^{-1} \sum_{j@i} w_j$ be the center of the set of rows of Z corresponding to the neighbors of i. Then (i) if $\alpha \neq 0$, w_i and u_i are collinear, and on the same side of the origin if $\alpha > 0$ or on opposite sides if $\alpha < 0$;

(ii) if $\alpha = 0$, then $u_i = 0$, and the rows of Z corresponding to neighbors of i surround the origin.

4. CONVEX POLYTOPES

Godsil (1978) suggested studying the m-polytope $P(\alpha)$ that is the convex hull of the points whose coordinates are the rows of a complete $n \times m$ eigenmatrix Z associated with an eigenvalue α of A. In the case of a distance-regular graph, all the rows w_i^T have the same norm, so all the distinct rows will be extreme points of their convex hull. (We use the term "extreme point" instead of the more usual "vertex," reserving the latter for graph usage.) Our reference for convex polytopes is Brondsted (1983).

Each facet of such a polytope P is the intersection of P with a hyperplane

$$H(u;\gamma) = \{x^T : x^T u = \gamma\} \qquad (\gamma \geq 0)$$

having the property that all points x of P satisfy $x^T u \leq \gamma$, with equality for at least m extreme points of P. The extreme points of the facet are the extreme points of P for which equality holds. For any eigenvalue $\alpha \neq \rho$, we have $e^T Z = 0$; hence we are assured that $\gamma > 0$.

In terms of the eigenmatrix Z, a hyperplane $H(u;\gamma)$ defining a facet can be identified by the condition: $Zu \leq \gamma e$ with at least m equalities. In this case we call $f = Zu$ a facet vector of P. Obviously, a facet vector of $P(\alpha)$ is also an eigenvector of A associated with α.

Theorem 5. Let $G \neq K_n$ be a distance-regular graph, Z a complete eigenmatrix associated with α, the second eigenvalue of A. If G contains a q-clique, and q is the multiplicity of α, then the rows of Z corresponding to a q-clique are extreme points of a facet of $P(\alpha)$, and that facet is a simplex.

Proof. Lemma 4 guarantees that the rows of Z are distinct. Let U be a set of vertices that form a q-clique, and let f be the sum of the columns of $L = ZZ^T$ whose indices are in U. Then the ith entry of f is $f_i = y_0 + (q-1) y_1 = \gamma$ if i is in U, but $f_i \leq q y_1 < \gamma$ if i is not in U. Since γ cannot equal 0 (see the proof of Theorem 3), f is a facet vector. Now $P(\alpha)$ is q-dimensional, and its facets are (q-1)-dimensional. Since the facets we have constructed have q extreme points, they must be simplices.

Theorem 6. Let G be a distance-regular graph, Z a complete eigenmatrix associated with α, the second eigenvalue of A. Let U be a set composed of a vertex i and all its neighbors. Then the rows of Z corresponding to U are not the extreme points of a facet of the polytope $P(\alpha)$.

Proof. Let a^T be the ith row of $A - \alpha I$: a has a $-\alpha$ in the ith entry and ρ 1's in the entries corresponding to neighbors of i. Then $a^T Z = 0$, and $a^T f = 0$ for any facet vector f. But if $f_j = 1$ for all j in U, then $a^T f = \rho - \alpha \neq 0$, so no such facet vector can exist.

5. ANTIPODAL GRAPHS

Definition. A distance-regular graph is antipodal if, for each vertex i, there is a unique vertex i' whose distance from i is the diameter d of the graph.

This definition differs from that of Biggs (1974, p. 151). By way of example, note that all the platonic solids except the tetrahedron have antipodal skeletons.

Lemma 5. Let i be a vertex in a distance-regular graph G, and let $\text{dist}(i, i') = d$, the diameter of G. If h is any vertex of G, then $\text{dist}(i, h) + \text{dist}(h, i') = d$.

Proof. By the triangle inequality, $\text{dist}(i, h) + \text{dist}(h, i') \geq \text{dist}(i, i')$, but inequality is impossible because d is the diameter of G.

Lemma 6. Let $V_0, V_1, \ldots, V_d$ be the distance partition starting from vertex i of a distance-regular graph G. Then V_d is a singleton $\{i'\}$ if and only if $V'_k = V_{d-k}$, for $k = 0, 1, \ldots, d$, is the distance partition starting from i'.

Proof. Follows immediately from Lemma 5.

Theorem 7. Let G be distance-regular with distance coloration matrix B. Then B is centrosymmetric if and only if G is antipodal.

Proof. A centrosymmetric matrix is one that commutes with the permutation matrix S that has 1's on the secondary diagonal; in terms of the elements of B, $c_i = b_{d-i}$ and $a_i = a_{d-i}$ for $i = 0, 1, \ldots, d$. Biggs (1974, p.140) gives the formula

$$k_i = b_0 b_1 \ldots b_{i-1} / c_1 c_2 \ldots c_i$$

for the number of vertices in set V_i of a distance partition. It is easy to prove that centrosymmetry implies that $k_i = k_{d-i}$, and in particular, that $k_d = 1$.

If G is antipodal, then the identity $X_i S = X_{i'}$, where i' is the antipode of i, follows immediately from Lemma 6. From the coloration equation $AX_i = X_i B$ we have

$$\begin{aligned} AX_i S &= X_i BS \\ &= AX_{i'} = X_{i'} B = X_i SB \end{aligned}$$

Then, because the columns of X_i are independent, BS = SB follows.

Lemma 7. Let G be distance-regular and antipodal. Then $A_d A_k = A_{d-k}$, for k = 0, 1,..., d.

Proof. The distance matrix A_d has a 1 in the i,j-position if and only if dist(i, j) = d. Thus, A_d is a permutation matrix that interchanges each vertex i with its antipode. Now, row i of the product $A_d A_k$ is row i' of A_d, so it has a 1 in the jth column if and only if dist(j, i') = k, which means that dist(i, j) = d-k, by Lemma 5.

Theorem 8. Let G be distance-regular and antipodal. Then (i) the interchange of each vertex with its antipode is an automorphism of G; (ii) the polytope associated with the second eigenvalue of G admits central inversion as a symmetry; (iii) the facets of this polytope occur in parallel pairs.

Proof. (i) As observed in the proof of Lemma 7, the distance matrix

A_d is a permutation matrix that interchanges each vertex with its antipode. Since A_d is a polynomial in A, it commutes with A and thus represents an automorphism of G.

(ii) Now let $P = A_d$. We need to show that $PZ = -Z$, or equivalently that $PL = -L$, where $L = ZZ^T$. By Eq.(1) and Lemma 7,

$$A_d L = y_0 A_d + y_1 A_{d-1} + \ldots + y_d A_0.$$

Now, the eigenvector y of B is unique when normalized by $y_0 = m/n$. Because S commutes with B, any eigenvector of B is also an eigenvector of S: $Sy = \pm y$. From Lemma 3, we know that $y_d < 0$, so $Sy = -y$, or $y_k = -y_{d-k}$ for $k = 0, 1, \ldots, d$. Thus $PL = -L$, as required.

(iii) If f is a facet vector, then $Pf = -f$ is one also.

6. EXAMPLES

In this section we list some distance-regular graphs and families of graphs for which the polytopes associated with the second eigenvalue can be determined. The results are summarized in a table at the end. For each graph, the information needed is: the coloration matrix B, its eigenvectors in the form of a square matrix Y, the spectrum of A in the style of Biggs (1974) with eigenvalues above their multiplicities, and the number of vertices in each set of the distance partition as a column k.

1. $G = K_n$, the complete graph. The essential information is shown below.

$$B = \begin{bmatrix} 0 & n-1 \\ 1 & n-2 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & n-1 \\ 1 & -1 \end{bmatrix} \quad k = \begin{bmatrix} 1 \\ n-1 \end{bmatrix}$$

$$\text{spec}(A) = \left\{ \begin{matrix} n-1 & -1 \\ 1 & n-1 \end{matrix} \right\}$$

The eigenvector information shows that each column of the projector L has (aside from a normalizing factor of 1/n) an n-1 in the diagonal position and a -1 in all other positions; that is, $L = (nI - ee^T)/n$. A complete eigenmatrix Z is composed of n-1 mutually orthogonal columns, each orthogonal to e. Since all off-diagonal elements of $L = ZZ^T$ are the same, the n rows of Z are equidistant. Thus the polytope P(-1) is a simplex.

2. $G = K_{2m} - mK_2$, $m \geq 2$. Suppose that edges {1, m+1},..., {m, 2m} are deleted from the complete graph on 2m vertices. The resulting graph is distance-regular with diameter 2 and is described by the following information.

$$B = \begin{bmatrix} 0 & 2(m-1) & 0 \\ 1 & 2(m-2) & 1 \\ 0 & 2(m-1) & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 1 & m-1 \\ 1 & 0 & -1 \\ 1 & -1 & m-1 \end{bmatrix} \quad k = \begin{bmatrix} 1 \\ 2(m-1) \\ 1 \end{bmatrix}$$

$$\mathrm{spec}(A) = \left\{ \begin{matrix} 2m-2 & 0 & -2 \\ 1 & m & m-1 \end{matrix} \right\}$$

In G, each vertex is a member of 2^{m-1} different m-cliques and no larger clique. By Theorem 6, the vertices of each clique correspond to the extreme points of a facet of P(0). From the eigenvector information, we can see that a complete eigenmatrix associated with the eigenvalue 0 is

$$Z = \frac{1}{\sqrt{2}} \begin{bmatrix} I \\ -I \end{bmatrix}$$

Since the rows of Z are the coordinates of the m-dimensional cross-polytope (see Coxeter, 1973, p. 122), which has exactly the facets described, we conclude that the description of P(0) is complete.

3. $G = K_{m,m}$, the complete bipartite graph. Require that $m \geq 2$, so that the diameter is 2. The essential information is:

$$B = \begin{bmatrix} 0 & m & 0 \\ 1 & 0 & m-1 \\ 0 & m & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & m-1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad k = \begin{bmatrix} 1 \\ m \\ m-1 \end{bmatrix}$$

$$\text{spec}(A) = \begin{Bmatrix} m & 0 & -m \\ 1 & 2m-2 & 1 \end{Bmatrix}$$

Assume that the two parts contain the vertices 1, 2,..., m and m+1, m+2,..., 2m respectively. Then the adjacency matrix A and the projector associated with 0 are

$$A = \begin{bmatrix} 0 & ee^T \\ ee^T & 0 \end{bmatrix} \quad L = \frac{1}{m} \begin{bmatrix} mI-ee^T & 0 \\ 0 & mI-ee^T \end{bmatrix}$$

Each column of mL contains 0 in m places, m-1 in one and -1 in m-1 places. Thus, the product of mL by the column

$$g_{ij} = - \begin{bmatrix} e_i \\ e_j \end{bmatrix}$$

is an eigenvector with -2(m-1) in two places and 1 in the remaining 2m-2 places. Since the multiplicity of 0 is 2m-2, this product is a facet vector, and the facet is a simplex, which is "opposite" a pair of adjacent vertices, i and m+j. Clearly there are m^2 such facets.

Any facet must contain at least 2m-2 vertices. However, by Theorem 5, no facet can contain a vertex and all its neighbors. Therefore, we have found all the facets of P(0), which is consequently simplicial.

4. $G_m = L(K_m)$, the line graph of K_m. This graph is also known as a

triangular graph because the number of vertices is the triangular number $t_m = m(m-1)/2$. We require that $m \geq 4$, so that the diameter will be 2.

$$B = \begin{bmatrix} 0 & 2m-4 & 0 \\ 1 & m-2 & m-3 \\ 0 & 4 & 2m-8 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 2m-4 & (m-2)(m-3) \\ 1 & m-4 & -(m-3) \\ 1 & -4 & 2 \end{bmatrix}$$

$$\text{spec}(A) = \begin{Bmatrix} 2m-4 & m-4 & -2 \\ 1 & m-1 & m(m-3)/2 \end{Bmatrix} \qquad k = \begin{bmatrix} 1 \\ 2m-4 \\ (m-2)(m-3)/2 \end{bmatrix}$$

It is helpful to think of the vertices as unordered pairs, (a,b), $a \neq b$, $a,b = 1,\dots, m$. Then (a,b) is adjacent to (a,x) for any $x \neq b$. Indeed, we see that the set of all vertices of the form (a,x), $x \neq a$, form an $(m-1)$-clique. By Theorem 5, these are extreme points of a facet that is a simplex, and there are m of these. Closer inspection shows the facet vector to have m-2 in the m-1 entries corresponding to the vertices of a clique and a -2 in the remaing $(m-1)(m-2)/2$ entries. Since $m \geq 4$, the negative of this vector is the facet vector of a different facet, supplying m more facets. On the basis of the number of vertices alone, it is conjectured that these latter facets are polytopes associated with the second eigenvalue of the next smaller traingular graph.

To find further facets of P(m-4), note that the <u>negative</u> of the column of L that corresponds to a vertex (a,b) has (see Y above) the same positive number in the $(m-2)(m-3)/2$ places corresponding to vertices at distance 2 from (a,b) and lesser numbers in the rest. If $(m-2)(m-3)/2 \geq m-1$, that is if $m \geq 6$, such a column is a positive multiple of a facet vector. This construction produces $n = m(m-1)/2$ new facets.

There are two interesting special cases. First, G_4 is the skeleton of the octahedron, which is also the special case m = 3 of item 2. We have found the eight facets, in four parallel pairs. Second, G_5 is the complement of Petersen's graph. The facets, which are 3-dimensional, are five tetrahedra paired with five octahedra. By direct computation, it has been shown that there are no more facets. For G_m, $m \geq 6$, it is not yet known whether we have found all the facets.

5. $G = Q_d$, the d-dimensional cube. The graph is distance-regular with diameter d. The spectrum consists of the numbers d - 2i, with multiplicity d!/i!(d-i)!, for i = 0, 1,..., d. (See Biggs, 1974, p.138, 145) In Licata and Powers (1987), it is shown by induction that the polytope P(d-2) is the d-dimensional cube itself. The information about the coloration matrix of G given by Biggs (1974, p. 138) shows these graphs to be antipodal, and of course they admit central inversion as a symmetry.

Table. Examples of Polytopes

Graph	n	λ_2	mult	(a)	(b)	(c)	facet	Description of polytope
K_n	n	-1	n-1	n	n-1	n-1	simplex	simplex
$K_{2m}-mK_2$	2m	0	m	2^m	m	2^{m-1}	simplex	cross-polytope
$K_{m,m}$	2m	0	2m-2	m^2	2m-2	m	simplex	simplicial
$L(K_m)$	t_m	m-4	m-1	m	m-1	2	simplex	
				m	t_{m-1}	m-2	?	
				t_m	t_{m-2}	t_m	?	
Q_d	2^d	d-2	d	2d	2^{d-1}	d	Q_{d-1}	generalized cube

Notes: (a) number of facets (b) number of extreme points on each facet (c) number of facets each point lies on. $t_m = m(m-1)/2$.

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