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COMPARING DISPERSION EFFECTS AT VARIOUS LEVELS OF  
FACTORS IN FACTORIAL EXPERIMENTS(U) CALIFORNIA UNIV  
RIVERSIDE DEPT OF STATISTICS S GHOSH EE AL. AUG 87  
TR-159 AFOSR-TR-87-1086 #AFOSR-87-0046

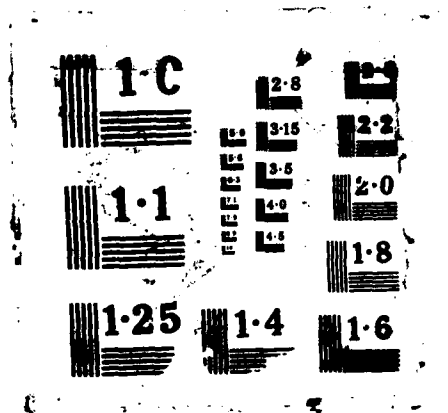
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REPORT DOCUMENTATION PAGE

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1a. REPORT SECURITY CLASSIFICATION Unclassified			1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY NA			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release, distribution unlimited	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE NA			5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR-87-1086	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report No. 159			7a. NAME OF MONITORING ORGANIZATION AFOSR/NM	
6a. NAME OF PERFORMING ORGANIZATION University of California Riverside		3b. OFFICE SYMBOL (If applicable)	7b. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB DC 20332-6448	
6c. ADDRESS (City, State and ZIP Code) Department of Statistics University of California, Riverside Riverside, CA 92521			9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-87-0048	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable)	10. SOURCE OF FUNDING NOS.	
6c. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB DC 20332-6448			PROGRAM ELEMENT NO. 61102F	TASK NO. 2304
11. TITLE (Include Security Classification) Comparing Dispersion effects at various levels of factors in factorial experiment.			WORK UNIT NO. A.5	
12. PERSONAL AUTHOR(S) Subir Ghosh and Eric E. Lagergren				
13a. TYPE OF REPORT Interim journal		13b. TIME COVERED FROM 12/86 TO 08/87	14. DATE OF REPORT (Yr., Mo., Day) August 1987	15. PAGE COUNT 17
16. SUPPLEMENTARY NOTATION Submitted to Technometrics				
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB. GR.	Adjusted residuals, Design, Dispersion effects, Error, Factorial experiments, Linear models, Quality control.	
19. ABSTRACT (Continue on reverse if necessary and identify by block number)				
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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS <input type="checkbox"/>			21. ABSTRACT SECURITY CLASSIFICATION Unclassified	
22a. NAME OF RESPONSIBLE INDIVIDUAL Major Peter M. Woodruff			22b. TELEPHONE NUMBER (Include Area Code) (202) 767-5027	22c. OFFICE SYMBOL AFOSR/NM

87 9 24 214

AFOSR-TR. 87-1086

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Of Factors In Factorial Experiments

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Technical Report No. 159

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DTIC TAB	<input type="checkbox"/>
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August, 1987

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Summary

This paper is an attempt to understand, measure and compare dispersion effects at different levels of factors in factorial experiments. The simplest setting is considered in order to develop better comprehension and insight. The properties of the proposed descriptive measures are examined. A method of "adjusting" residuals and its use in comparing dispersion effects are discussed. Illustrative examples are also given. The problem considered in this paper arises in quality control studies and the methodologies are applicable to industrial experiments.

Key Words: Adjusted residuals, Design, Dispersion effects, Error,  
Factorial experiments, Linear models, Quality control.

\*The work of the first author is sponsored by the Air Force Office Of  
Scientific Research under Grant AFOSR-87-0048.

## 1. Introduction

An important problem in quality control studies is to find an optimum combination of levels of control factors in achieving stability against noise factors (see Taguchi and Wu 1985). Both "location" and "dispersion" effects of levels of factors are pertinent to measure from the data in resolving this problem. This article considers the problem of measuring dispersion effects of levels of factors in both replicated and unreplicated factorial experiments. The concept of dispersion effects in factorial experiments was introduced in Taguchi and Wu (1985) for replicated factorial experiments and in Box and Meyer (1986) for unreplicated factorial experiments. Factorial experiments may be complete or fractional factorial under completely randomized designs. Although for clarity we consider  $2^m$  factorial experiments in this article, the ideas presented can easily be generalized to any symmetric or asymmetric factorial experiments. Kacker (1985), Phadke et. al. (1983) and Nair (1986) made pioneering contributions to this area of research. Ghosh (1986) used the search linear models (see Srivastava 1975) to explain dispersion effects in factor screening experiments.

We first assume that for the fitted model to the data there is no significant lack of fit. We then propose three measures of dispersion effects at levels 0 and 1 of  $m$  factors. All three of them are relevant in replicated factorial experiments and two of them are applicable to unreplicated factorial experiments. We observe that the measures of dispersion effects, based on residuals obtained by the least squares fit

of the model to the data, at levels 0 and 1 of a factor are correlated in most situations. We introduce a method of adjusting residuals and then propose measures based on residuals and adjusted residuals.

## 2. Dispersion Effects

We consider a  $2^m$  factorial experiment under a completely randomized design. Let  $T(n \times m)$  be the design. The rows of  $T$  denote treatments and the columns denote factors. The design  $T$  is called an inner array for  $m$  controlled factors. For various level combinations of noise factors (outer array), we get replicated observations for every treatment in  $T$  (see Taguchi and Wu 1985). In the experiment, we take  $r$  ( $\geq 1$ ) observations for every treatment. The case  $r = 1$  is called the unreplicated experiment and the case  $r > 1$  is called the replicated experiment. Again, for simplicity equal replication is considered for the replicated experiment and the idea is easily extendable to unequal replications. Let  $y_{ij}$  be the  $j$ th observation for the  $i$ th treatment,  $\bar{y}_i$  be the mean of all observations for the treatment  $i$ ,  $i=1, \dots, n$  and  $j=1, \dots, r$ , and  $(N = nr)$  be the total number of observations. The standard linear model for the experiment is

$$E(\underline{y}) = X\underline{\beta}, \quad (1)$$

$$V(\underline{y}) = \sigma^2 I, \quad (2)$$

$$\text{Rank } X = p, \quad (3)$$

where  $\underline{y}(N \times 1)$  is the vector of observations,  $\underline{\beta}(p \times 1)$  is the vector of factorial effects considered in the experiment,  $X(N \times p)$  is a known matrix that depends on the design  $T$  and  $\sigma^2$  is an unknown constant. We denote  $H = X(X'X)^{-1}X'$  and  $R = (I-H)$ . The vectors  $H\underline{y}$  and  $R\underline{y}$  are the vector of

least squares fitted values and the vector of residuals, respectively. The fitted values for all observations corresponding to the  $i$ th treatment are identical and is denoted by  $\hat{y}_i$ ,  $i=1, \dots, n$ . Suppose that for the fitted model to the data there is no significant lack of fit. The sum of squares of error is  $SSE = \sum_{i=1}^n \sum_{j=1}^r (y_{ij} - \hat{y}_i)^2$ , the mean square of error is  $MSE = (SSE/(N-p))$ , the sum of squares of pure error is  $SSPE = \sum_{i=1}^n \sum_{j=1}^r (y_{ij} - \bar{y}_i)^2$  and the mean square of pure error is  $MSPE = (SSPE/n(r-1))$ . Note that both MSE and MSPE are measures of error variance  $\sigma^2$ . We now take MSE and MSPE as descriptive measures of noise. We then express MSPE as the weighted average of  $(MSPE)_1$  and  $(MSPE)_0$ , where  $(MSPE)_u$  is called the contribution of the level  $u$  ( $u = 0, 1$ ) of the factor to MSPE. Formal expressions of  $(MSPE)_u$ ,  $u = 0, 1$  are given in the next section. We do the same for MSE. Different levels of a factor may contribute differently to MSE and MSPE. In general the contributions of levels of a factor to noise (measured by MSPE or MSE) are called the dispersion effects of levels of the factor. The main theme of this paper is to investigate the possible ways of measuring and comparing dispersion effects of levels of factors. We would like to make it clear that we are not presenting a population model for dispersion effects and the proposed measures in this paper are all descriptive.

### 3. Measuring Dispersion Effects

#### 3.1. First and Second Sets of Measures

We take a single factor out of  $m$  factors and develop the methods of measuring dispersion effects at the level 0 and 1 of the chosen factor. We do not introduce any notation for the chosen factor. This is to keep our presentation neat and clean. We define for  $i=1, \dots, n$ ,

$$\delta_i = \begin{cases} 1 & \text{if the level of the factor in the } i\text{th treatment is 1,} \\ 0 & \text{if the level of the factor in the } i\text{th treatment is 0.} \end{cases}$$

Let  $D_1$  ( $N \times N$ ) be a diagonal matrix with  $n$  sets of diagonal elements and the elements in the  $i$ th ( $i=1, \dots, n$ ) set are equal to  $\delta_i$ . We define

$D_0 = I - D_1$ . It can be seen that  $D_1 D_0 = 0$  and both  $D_1$  and  $D_0$  are idempotent matrices. We have

$$\begin{aligned} \text{SSE} &= \mathbf{y}' \mathbf{R} D_1 \mathbf{R} \mathbf{y} + \mathbf{y}' \mathbf{R} D_0 \mathbf{R} \mathbf{y} \\ &= \sum_{i=1}^n \sum_{j=1}^r \delta_i (y_{ij} - \hat{y}_i)^2 + \sum_{i=1}^n \sum_{j=1}^r (1 - \delta_i) (y_{ij} - \hat{y}_i)^2, \\ \text{SSPE} &= \sum_{i=1}^n \sum_{j=1}^r \delta_i (y_{ij} - \bar{y}_i)^2 + \sum_{i=1}^n \sum_{j=1}^r (1 - \delta_i) (y_{ij} - \bar{y}_i)^2. \end{aligned}$$

The first set of measures of dispersion effects of levels of the factor are

$$\begin{aligned} S_1^2(1) &= \frac{\sum_{i=1}^n \sum_{j=1}^r \delta_i (y_{ij} - \bar{y}_i)^2}{\left( \sum_{i=1}^n \delta_i \right) (r-1)}, \\ S_0^2(1) &= \frac{\sum_{i=1}^n \sum_{j=1}^r (1 - \delta_i) (y_{ij} - \bar{y}_i)^2}{\left( \sum_{i=1}^n (1 - \delta_i) \right) (r-1)}, \end{aligned} \tag{4}$$

at the levels 1 and 0, respectively.

We have

$$\text{MSPE} = \left( \frac{\sum_{i=1}^n \delta_i}{n} \right) S_1^2(1) + \left( \frac{\sum_{i=1}^n (1 - \delta_i)}{n} \right) S_0^2(1).$$

Thus  $S_1^2(1)$  and  $S_0^2(1)$  are regarded as  $(MSPE)_1$  and  $(MSPE)_0$  in the notation of the previous section. If  $S_1^2(1) > S_0^2(1)$ , we then say that the level 0 of the factor has less contribution to MSPE and therefore would be preferred to the level 1 in view of stability against noise factors.

We denote Rank  $R D_1 R = V_1$  and Rank  $R D_0 R = V_0$ . We now present the second set of measures of dispersion effects of levels of the factor as

$$\begin{aligned} S_1^2(2) &= (\underline{y}' R D_1 R \underline{y}) / V_1, \\ S_0^2(2) &= (\underline{y}' R D_0 R \underline{y}) / V_0, \end{aligned} \quad (5)$$

at the levels 1 and 0, respectively. We have

$$MSE = \left( \frac{V_1}{(N-p)} \right) S_1^2(2) + \left( \frac{V_0}{(N-p)} \right) S_0^2(2).$$

We now investigate the situation where  $S_u^2(1) = S_u^2(2)$ ,  $u = 0, 1$ . In other words, we like to characterize designs for which  $\hat{y}_i = \bar{y}_i$ . We denote the row of the matrix  $X$  corresponding to the treatment  $i$  by  $\underline{x}_i'(1 \times p)$ . Note that for each  $i$ ,  $i=1, \dots, n$ , the row  $\underline{x}_i'$  is repeated  $r$  times in  $X$ . Let  $X^*(n \times p)$  be a matrix whose  $i$ th row is  $\underline{x}_i'$ . Notice that rows of  $X^*$  are infact distinct rows of  $X$ . We have  $X'X = r(X*'X^*)$ .

Theorem 1. For  $i=1, \dots, n$ ,  $\hat{y}_i = \bar{y}_i$  if and only if  $X^*(X*'X^*)^{-1} X*' = I_n$ .

The proof is in Appendix.

Corollary. For  $n = p$ , we have  $S_u^2(1) = S_u^2(2)$ ,  $u = 0, 1$ .

We thus observe that for designs with  $n = p$ , two sets of measures are identical. The class of designs with  $n = p$  includes the known Plackett and Burman designs (see Plackett and Burman 1947). We however strongly

feel that this class of designs is very weak in view of measuring dispersion effects, particularly for the condition that there is no significant lack of fit.

### 3.2 Adjusted Residuals and Third Set of Measures.

We denote

$$y = \begin{pmatrix} y_1 \\ \dots \\ y_0 \end{pmatrix}, R = \begin{pmatrix} r_1 \\ \dots \\ r_0 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{10} \\ R_{01} & R_{11} \end{pmatrix}, X = \begin{pmatrix} X_1 \\ \dots \\ X_0 \end{pmatrix}, \hat{y} = \begin{pmatrix} \hat{y}_1 \\ \dots \\ \hat{y}_0 \end{pmatrix},$$

where  $y_u$  is the vector of all observations corresponding to treatments with  $\delta_1 = u$ ,  $u = 0, 1$  for the chosen factor. Two vectors of residuals  $r_1 y$  and  $r_0 y$  at the levels 1 and 0 of the factor are generally correlated under the model (1-3). We now present a vector of "adjusted residuals" at the level 0 of the factor, adjusted w.r.t.  $r_1 y$  so that it is uncorrelated with  $r_1 y$ . We denote

$$r_1 = \begin{pmatrix} r_{11} \\ \dots \\ r_{12} \end{pmatrix}, R_{11} = \begin{pmatrix} R_{111} & R_{112} \\ R_{112} & R_{113} \end{pmatrix}, R_{10} = \begin{pmatrix} R_{101} \\ R_{102} \end{pmatrix},$$

where  $R_{111}$  ( $V_1 \times V_1$ ) with its rank  $V_1$ ,  $r_{11}$  ( $V_1 \times N$ ) with its rank  $V_1$ . In fact we have  $R_{111} = r_{11} r_{11}'$ . We now write

$$r_{0a} = r_0 - R_{101}' R_{111}^{-1} r_{11}. \quad (6)$$

It can be seen that  $\text{Rank } r_{0a} = [(N-p) - V_1] = V_{0a}$  (say) and furthermore,  $\text{Cov}(r_{11} y, r_{0a} y) = 0$ . We call  $r_{0a} y$  the vector of "adjusted residuals" at the level 0 of the factor, adjusted w.r.t. the residuals at the level 1 of the factor. We denote

$$r_{0a} = \begin{pmatrix} r_{0a1} \\ r_{0a2} \end{pmatrix},$$

where  $r_{0a1}$  ( $v_{0a} \times N$ ) with its rank  $v_{0a}$ . We now have the sum of squares of the sets of linear functions  $r_{11} \underline{y}$  and  $r_{0a1} \underline{y}$  [see Scheffé 1959] as

$$\begin{aligned} SS(r_{11} \underline{y}) &= \underline{y}' r_{11} [r_{11} r_{11}']^{-1} r_{11} \underline{y}, \\ SS(r_{0a1} \underline{y}) &= \underline{y}' r_{0a1} [r_{0a1} r_{0a1}']^{-1} r_{0a1} \underline{y}, \end{aligned} \quad (7)$$

with d.f.  $v_1$  and  $v_{0a}$ , respectively. We present the measures of dispersion and adjusted dispersion effects of levels of the factor

$$\begin{aligned} s_1^2(3) &= [SS(r_{11} \underline{y})/v_1], \\ s_{0a}^2(3) &= [SS(r_{0a1} \underline{y})/v_{0a}], \end{aligned} \quad (8)$$

at the levels 1 and 0 (adjusted for level 1), respectively. We have

$$MSE = \left( \frac{v_1}{(N-p)} \right) s_1^2(3) + \left( \frac{v_{0a}}{(N-p)} \right) s_{0a}^2(3). \quad (9)$$

Following the above approach we find the vector  $r_{1a} \underline{y}$  of adjusted residuals at the level 1 of the factor adjusted w.r.t.  $r_0 \underline{y}$  so that it is uncorrelated with  $r_0 \underline{y}$ . Let  $r_{1a1}$  ( $v_{1a} \times N$ ) be a submatrix of  $r_{1a}$  such that  $\text{Rank } r_{1a1} = \text{Rank } r_{1a} = v_{1a}$ ,  $r_{01}$  ( $v_0 \times N$ ) be a submatrix of  $r_0$  with  $\text{Rank } r_{01} = \text{Rank } r_0 = v_0$ . We again present the measures of dispersion and adjusted dispersion effects of levels of the factor

$$\begin{aligned} s_{1a}^2(3) &= [SS(r_{1a1} \underline{y})/v_{1a}], \\ s_0^2(3) &= [SS(r_{01} \underline{y})/v_0], \end{aligned} \quad (10)$$

at the levels 1 (adjusted for the level 0) and 0, respectively. We have

$$MSE = \left( \frac{v_{1a}}{(N-p)} \right) s_{1a}^2(3) + \left( \frac{v_0}{(N-p)} \right) s_0^2(3), \quad (N-p) = v_{1a} + v_0 = v_1 + v_{0a}. \quad (11)$$

**Theorem 2.** The following results are true.

$$i. \quad v_{1a} \geq \left( \sum_{i=1}^n \delta_i \right) (r-1), \quad v_{0a} \geq \left( \sum_{i=1}^n (1-\delta_i) \right) (r-1),$$

$$ii. V_{1a} S_{1a}^2(3) \geq \left( \sum_{i=1}^n \delta_i \right) (r-1) S_1^2(1),$$

$$V_{0a} S_{0a}^2(3) \geq \left( \sum_{i=1}^n (1-\delta_i) \right) (r-1) S_0^2(1),$$

$$iii. \text{ If } V_{1a} = \left( \sum_{i=1}^n \delta_i \right) (r-1) \text{ then } S_{1a}^2(3) = S_1^2(1),$$

$$iv. \text{ If } V_{0a} = \left( \sum_{i=1}^n (1-\delta_i) \right) (r-1) \text{ then } S_{0a}^2(3) = S_0^2(1).$$

We now study the measures in two extreme situations: (1)  $r_1 \underline{y}$  and  $r_0 \underline{y}$  are uncorrelated, i.e.,  $R_{10} = 0$ , (ii)  $r_1 \underline{y}$  and  $r_0 \underline{y}$  are completely correlated, i.e.,  $r_{01} = D r_{11}$  for some matrix D.

Theorem 3. Consider the situation  $R_{10} = 0$ . Then  $S_u^2(3) = S_u^2(2) = S_{ua}^2(3)$ ,  $u=0,1$ .

Theorem 4. If  $r_{01} = D r_{11}$  then we have  $r_{0a} = 0$ ,  $V_{0a} = 0$  and  $SS(r_{0a1}\underline{y}) = 0$ .

Theorem 3 tells that in case  $R_{10} = 0$  there is no need for the adjustment of residuals. Theorem 4 tells that in case  $r_0 \underline{y}$  is linearly dependent on dependent on  $r_1 \underline{y}$  then the level 1 of the factor makes all contribution to SSE and the level 0 does not make any additional contribution to SSE. In case  $V_{0a} = V_{1a} = 0$ , we have  $V_0 = V_1 = (N-p)$ ,  $r_{01} = D r_{11}$  and D is nonsingular. This is a situation where the levels 0 and 1 have equal dispersion effects because of the design influence. It follows from Theorem 2 that for  $r > 1$ ,  $V_0$  and  $V_1$  are both nonzero. (We assume naturally that there is at least one  $\delta_i = 1$  and at least one  $(1-\delta_i) = 1$ .) For the case  $r = 1$ , at least one of  $V_{0a}$  and  $V_{1a}$  could be zero or both of them could be nonzero. We now consider the important situation when

both  $V_{0a}$  and  $V_{1a}$  are nonzero. If  $S_1^2(3) > \text{Maximum}(S_0^2(3), S_{0a}^2(3))$ , then the level 0 of the factor has an advantage edge over the level 1. On the other hand if  $S_1^2(3) < \text{Minimum}(S_0^2(3), S_{0a}^2(3))$ , the level 1 of the factor is superior to the level 0 in terms of smaller dispersion effects.

Example 1

We consider the example from Box and Meyer (1986), page 20, and Taguchi and Wu (1985), page 68. Daniel's normal probability plot indicates that, over the ranges studied, only factors B and C affect tensile location by amounts not readily attributed to noise (see Box and Meyer 1986). We now fit the following standard linear model to the data

$$E(y(x_1, x_2)) = \mu + \alpha_1 B + \alpha_2 C,$$

where  $x_i = 0, 1$ ,  $\alpha_i = (2x_i - 1)$ ,  $\mu$  is the general mean, B and C are the main effects of the factors. We can write the above model in the form (1-3). Notice that  $N = 16$ ,  $n = 4$ ,  $p = 3$ ,  $\left( \sum_{i=1}^n \delta_i \right) = \left( \sum_{i=1}^n (1 - \delta_i) \right) = 2$  for both factors B and C. The F value for the lack of fit test under the assumption of normality is .1971 ( $< 1$ ) and we therefore conclude that there is no significant lack of fit. It can be seen that  $R_{10} \neq 0$  for both factors. We observe that  $V_1 = V_0 = 7$ ,  $V_{1a} = V_{0a} = 6$ . It follows from Theorem 2 that  $S_{1a}^2(3) = S_1^2(1)$  and  $S_{0a}^2(3) = S_0^2(1)$ . In Table 1 we display numerical values of various measures of dispersion effects for factors B and C. We find that the level 1 has a lower dispersion effect than the level 0 for both factors. We also observe that the discrepancy between dispersion effects at the levels 0 and 1 w.r.t. all measures is more for the factor C than the factor B.

Table 1

Numerical Values of Measures Of Dispersion Effects For Factors B and C

Factor	$S_1^2(1) = S_{1a}^2(3)$	$S_0^2(1) = S_{0a}^2(3)$	$S_1^2(2)$	$S_0^2(2)$	$S_1^2(3)$	$S_0^2(3)$
B	.2863	.3479	.2498	.3027	.2543	.3071
C	.0279	.6063	.0284	.5241	.0329	.5286

4. Properties

We now state some properties of the descriptive measures proposed in Section 3 under the model (1-3). We first observe that the measures  $S_1^2(1)$  and  $S_0^2(1)$  do not depend on the fitted model and all other measures depend on the fitted model. The measures  $S_1^2(1)$  and  $S_0^2(1)$  are always uncorrelated under the model (1-3). The measures  $S_1^2(2)$  and  $S_0^2(2)$  may however be correlated. Two sets of linear functions of observations  $D_1 R y$  and  $D_0 R y$  are uncorrelated if and only if  $D_1 R D_0 = 0$ . Therefore if  $D R D_1 = 0$  then  $S_1^2(2)$  and  $S_0^2(2)$  are uncorrelated. We have the following results:

Theorem 5. Suppose  $y \sim N(x\beta, \sigma^2 I)$ . A necessary and sufficient condition that

$$(1) \frac{y' R D_1 R y}{\sigma^2} \sim \text{Central } X^2 \text{ with d.f.} = \text{Trace } R_{11},$$

$$(2) \frac{y' R D_0 R y}{\sigma^2} \sim \text{Central } X^2 \text{ with d.f.} = \text{Trace } R_{00},$$

(3) and furthermore, (1) and (2) are statistically independent,

is that  $R_{10} = 0$

Notice that  $D_1 R D_0 = 0$  if and only if  $R_{10} = 0$ . Moreover,  $V_1 + V_0 = (N-p)$  if  $R_{10} = 0$  and  $V_1 + V_0$  could be greater than  $(N-p)$  if  $R_{10} \neq 0$ . We question the use of estimators  $S_1^2(2)$  and  $S_0^2(2)$  for comparison unless  $R_{10} = 0$ . We of course realize that the condition  $R_{10} = 0$  is too stringent to satisfy even for one out of  $m$  factors.

Theorem 6. The following results are true.

a. 
$$\sum_{i=1}^n \sum_{j=1}^r \delta_i (y_{ij} - \hat{y}_i) = \sum_{i=1}^n \sum_{j=1}^r (1 - \delta_i) (y_{ij} - \hat{y}_i) = 0,$$

b. If for the factor  $R_{10} = 0$ , then

b.1.  $R_{11}$  and  $R_{00}$  are idempotent matrices,

b.2.  $(\underline{y}_u - \hat{\underline{y}}_u) = R_{uu} \underline{y}_u, u = 0, 1,$

b.3.  $X'_u R_{uu} = 0, u = 0, 1,$

b.4. 
$$\sum_{i=1}^n \sum_{j=1}^r \delta_i \hat{y}_i (y_{ij} - \hat{y}_i) = \sum_{i=1}^n \sum_{j=1}^r (1 - \delta_i) \hat{y}_i (y_{ij} - \hat{y}_i) = 0.$$

The measures  $S_u^2(3)$  and  $S_{(1-u)a}^2(3), u = 0, 1,$  are always uncorrelated.

The reason for adjusting residuals is to obtain uncorrelated dispersion effects.

Example 2. We consider a  $2^5$  factorial experiment, i.e.,  $m = 5$ . Let the design  $T(8 \times 5)$  be

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

We consider a situation where the model for a main effect plan fits the data adequately i.e., there is no significant lack of fit. Therefore  $n = 8$  and  $p = 6$ . We note that it is sufficient to consider the distinct rows of  $X$ . The first column of the matrix  $X^*$  has all entries unity and the other five columns of  $X^*$  are obtained from  $T$  replacing 0 by  $(-1)$ . We denote

$$A = \begin{pmatrix} 2 & -2 & -2 & 2 \\ -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 \\ 2 & -2 & -2 & 2 \end{pmatrix}$$

It can be easily seen that

$$I_8 - X^*(X^{*'}X^*)^{-1}X^{*'} = \frac{1}{8} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} .$$

It now follows that for factors 1, 2, 4 and 5,  $S_1^2(2) = S_0^2(2)$ ,  $S_{0a}^2(3) = S_{1a}^2(3) = 0$ , and  $S_1^2(3) = S_0^2(3) = \text{MSE}$ . For the factor 3,  $R_{10} = 0$ , i.e.,  $S_1^2(2)$  and  $S_0^2(2)$  are uncorrelated and by Theorem 3,  $S_u^2(3) = S_u^2(2) = S_{ua}^2(3)$ ,  $u = 0, 1$ .

##### 5. Conclusions.

In industrial experiments for quality improvement, dispersion effects at various levels of factors play an important role. They are instrumental in the choice of an optimum combination of levels of control factors. This article presents the descriptive methods of measuring and comparing dispersion effects at the preliminary stage of investigation. The outcome of such comparisons will suggest more appropriate complex models for further investigation. We however believe that the implementation of the methods discussed in this article will result in highly informative conclusions. Although in this paper we find the dispersion

effects of levels of a factor using one factor at a time, the same approach can be used to find the dispersion effects of level combinations of factors using many factors at a time. Unless the number of observations is sufficiently large in every cell, the reliability of the measures using many factors will be questionable.

APPENDIX: Proofs Of The Theorems

Theorem 1.

The condition  $\hat{y}_i = \bar{y}_i$ ,  $i=1, \dots, n$ , holds if

$$\underline{x}_i' (X'X)^{-1} \underline{x}_i = \begin{cases} \frac{1}{r} & \text{for } i = i' \\ 0 & \text{for } i \neq i'; i, i' \in [1, \dots, n]. \end{cases}$$

The above condition may be expressed as  $X^*(X'X)^{-1} X^{*'} = \frac{1}{r} I_n$ , or, equivalently,  $X^*(X^*X^*)^{-1} X^{*'} = I_n$ . This completes the proof.

Theorem 2.

It can be checked that  $\text{Cov}(y_{ij}, \bar{y}_u - \hat{y}_u) = \text{Cov}(\bar{y}_i, \bar{y}_u - \hat{y}_u)$  and therefore  $\text{Cov}(y_{ij} - \bar{y}_i, \bar{y}_u - \hat{y}_u) = 0$ . Moreover,  $\text{Cov}(y_{ij} - \bar{y}_i, y_{uw} - \bar{y}_u) = 0$ ,  $i \neq u$ . It now follows that any contrast of  $(y_{ij} - \bar{y}_i)$ ,  $j = 1, \dots, r$  for a fixed  $i$  with  $\delta_i = 1$ , is orthogonal to any contrast of  $(y_{uw} - \bar{y}_u)$ ,  $w = 1, \dots, r$  for a fixed  $u$  with  $(1 - \delta_u) = 1$ . Furthermore, any contrast of  $(\bar{y}_i - \hat{y}_i)$  for all  $i$  with  $\delta_i = 1$  is orthogonal to any contrast of  $(y_{uw} - \bar{y}_u)$ ,  $w = 1, \dots, r$ , for a fixed  $u$  with  $(1 - \delta_u) = 1$ . The results (i-iv) follow immediately from the above facts, the relationship between the rank and the number of orthogonal contrasts, and the fact that the sum of squares is equal to the sum of sums of squares of orthogonal contrasts.

Theorem 3.

We first show that  $S_1^2(3) = S_1^2(2)$ , or, in other words,  $SS(r_{11} \underline{y}) = \underline{y}' R D_1 R \underline{y}$ . We observe that

$$\underline{y}' r'_{11} [r_{11} r'_{11}]^{-1} r_{11} \underline{y}$$

$$\begin{aligned}
 &= \underline{y}' r_1' \begin{bmatrix} (r_{11} r_{11}')^{-1} & 0 \\ 0 & 0 \end{bmatrix} r_1 y \\
 &= \underline{z}' r_1 \underline{y}, \text{ where } R_{11} \underline{z} = r_1 \underline{y} = D_1 R \underline{y}, \\
 &= \underline{z}' R_{11} \underline{z} = (R_{11} \underline{z})' (R_{11} \underline{z}) \\
 &= (D_1 R \underline{y})' (D_1 R \underline{y}) = \underline{y}' R D_1 R \underline{y}.
 \end{aligned}$$

It follows from (9) and (11) that  $V_{0a} S_{0a}^2(3) = \underline{y}' R D_0 R \underline{y}$ , i.e.,  $S_{0a}^2(3) = S_0^2(3)$ , and consequently  $V_0 = V_{0a}$ . The rest is similar. This completes the proof.

Theorem 4.

We write  $r_0 = D_0 r_{11}$  for a matrix  $D_0$  whose independent rows are rows of  $D$ . This implies that  $R'_{101} = D_0 R_{111}$  and thus  $D_0 = R'_{101} R_{111}^{-1}$ . Hence from (6) we get  $r_{0a} = 0$ . The rest is clear. This completes the proof.

Theorem 5.

It is known (see Rao 1973) that a necessary and sufficient condition of (1), (2) and (3) to be true, is that  $(R D_1 R) (R D_0 R) = R D_1 R D_0 R = 0$ . Now

$$R D_1 R D_0 R = \left( \begin{array}{ccc|ccc} R_{11} & R_{10} & R_{01} & R_{11} & R_{10} & R_{00} \\ R_{01} & R_{10} & R_{01} & R_{01} & R_{10} & R_{00} \end{array} \right)$$

We see that  $R D_1 R D_0 R = 0$  implies  $R_{01} R_{10} R_{01} = 0$  and this in turn implies  $R_{10} = 0$ . Again  $R_{10} = 0$  implies that  $R D_1 R D_0 R = 0$ . This completes the proof.

Theorem 6.

The fact  $X'R = 0$  implies  $X'R \underline{y} = X'(\underline{y} - \hat{\underline{y}}) = 0$ . Considering the columns of  $X$  for the general mean and the factor chosen, the result (a) follows. The results b.1 and b.2 follow directly from the structure of  $R$  and the fact that  $R$  is an idempotent matrix. The result b.3 follows from  $X'R = 0$ . From b.3, we get  $\underline{\beta}' X'_u R_{uu} \underline{y}_u = 0$ , i.e.,  $\hat{\underline{y}}'_u R_{uu} \underline{y}_u = 0$ . The result b.2 implies that  $\hat{\underline{y}}'_u (\underline{y}_u - \hat{\underline{y}}_u) = 0$  and hence the result b.4 is true. This completes the proof.

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