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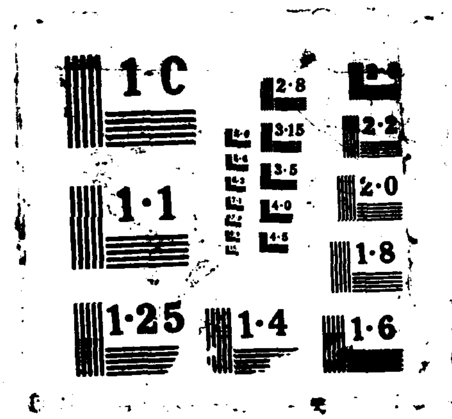
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EXTREMA OF SKEWED STABLE PROCESSES

by

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ABSTRACT

We study extremes of (generally) skewed stable processes. In particular we find the asymptotic behavior of the distribution function of the order statistics from a (dependent) stable sample.

We give necessary conditions for a.s. boundedness of general stable processes. These conditions turn out to be sufficient when $0 < \alpha < 1$. Further, asymptotic lower bounds for the supremum and infimum distribution functions are given. Again, in the case $0 < \alpha < 1$ those bounds are shown to give the exact asymptotic behavior of the supremum and infimum distribution functions.

1. Introduction

Let T be an arbitrary parameter set. A real stochastic process $\{X(t), t \in T\}$ is said to be *stable*, if for any $A > 0$ and $B > 0$, and independent copies $\{X_1(t), t \in T\}$ and $\{X_2(t), t \in T\}$ of $\{X(t), t \in T\}$, there are $C > 0$ and a function $x: T \to R$ such that

$$(1.1) \qquad \{C(AX_1(t) + BX_2(t)) + x(t), t \in T\} \stackrel{d}{=} \{X(t), t \in T\}.$$

It turns out that the constant C has always the form $C = (A^\alpha + B^\alpha)^{-1/\alpha}$ for some $0 < \alpha \le 2$, and the process $\{X(t), t \in T\}$ is called α-*stable* for the corresponding α. If $\alpha = 2$, the process is called Gaussian, and if $\alpha = 1$, the process is called Cauchy.

The process $\{X(t), t \in T\}$ is called *strictly stable* if one can always choose $x(t) \equiv 0$ in (1.1). This process is called *symmetric stable* if $\{X(t), t \in T\} \stackrel{d}{=} \{-X(t), t \in T\}$. Clearly any symmetric stable process is also strictly stable, but the converse is true only for $\alpha = 1$ and $\alpha = 2$.

In this paper we investigate extremes of stable processes. After Section 2, which discusses certain basic properties of stable random variables, random integrals with respect to stable random measures, and integral representation of stable processes, we treat in Section 3 extremes of finite-dimensional stable vectors. We characterize the conditional probability that all the components of a stable vector are big, given one of them is big, and then apply this result to obtain the tail behavior of the distributions of order statistics from a (dependent) stable sample.

In Section 4 we consider stable processes defined on a countable set. Here we consider the tails of the distributions of the suprema and infima of those processes along with conditions for a.s. boundedness. These results generalize those obtained in Samorodnitsky (1987) for the case of symmetric stable processes.

2. Stable variables, stable integrals and integral representation of stable processes

Let $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ be a stable vector in $\mathbb{R}^n$, which means that $\underset{\sim}{Y}$, considered as a stochastic process on $T = \{1,2,\ldots, n\}$, satisfies condition (1.1). It is well known that the characteristic function of $\underset{\sim}{Y}$ is necessarily of the form

$$\text{(2.1)}\quad E[\exp(i\sum_{j=1}^{n}\theta_j Y_j)] = \exp\{i\sum_{j=1}^{n}\theta_j\mu_j - \int_{S_n}|\sum_{j=1}^{n}\theta_j s_j|^{\alpha}\Gamma(ds_1, ds_2, \ldots, ds_n) + C_\alpha(\theta_1, \theta_2, \ldots, \theta_n)\}$$

for any real vector $(\theta_1, \theta_2, \ldots, \theta_n)$, where $0 < \alpha \le 2$, $(\mu_1, \mu_2, \ldots, \mu_n)$ is a real vector, Γ is a finite measure on the unit ball S_n of $\mathbb{R}^n$, and

$$\text{(2.2)}\quad C_\alpha(\theta_1, \theta_2, \ldots, \theta_n) = \begin{cases} \tan(\pi\alpha/2)\int_{S_n}|\sum_{j=1}^{n}\theta_j s_j|^{\alpha}\operatorname{sign}(\sum_{j=1}^{n}\theta_j s_j)\Gamma(ds_1, ds_2, \ldots, ds_n) & \text{if } \alpha \ne 1,2 \\ 2/\pi\int_{S_n}\sum_{j=1}^{n}\theta_j s_j \ln|\sum_{j=1}^{n}\theta_j s_j|\Gamma(ds_1, ds_2, \ldots, ds_n), & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha = 2 \end{cases}$$

see, for example, Kuelbs (1973).

Conversely, any random vector with characteristic function of the form (2.1) is stable (more specifically, α-stable with the α that appears in (2.1)). In particular, a one dimensional random variable Y is α-stable if and only if its characteristic function is of the form

$$\text{(2.3)}\quad \ln\{E[\exp(i\theta Y)]\} = \begin{cases} -|\theta|^{\alpha}\sigma^{\alpha}(1 - i\beta\tan\frac{\pi\alpha}{2}\,\mathrm{sign}(\theta)) + i\mu\theta, & \alpha \neq 1 \\ -|\theta|\sigma(1 + i\beta(2/\pi)\,\mathrm{sign}(\theta)\ln|\theta|) + i\mu\theta, & \alpha = 1 \end{cases}$$

for some $0 < \alpha < 2$, $\sigma > 0$, $|\beta| \le 1$, μ real. A random variable whose characteristic function is given by (2.3) will be said to have $S_\alpha(\sigma,\beta,\mu)$ distribution.

Remark 2.1. In the particular case $|\beta| = 1$, $\mu = 0$ the distribution $S_\alpha(\sigma,\beta,0)$ is said to be totally skewed (to the right if $\beta = 1$ and to the left if $\beta = -1$). If, in addition, $0 < \alpha < 1$, then $S_\alpha(\sigma,1,0)$ and $S_\alpha(\sigma,-1,0)$ are concentrated on positive and negative parts of the real line, correspondingly.

The tail behavior of $S_\alpha(\sigma,\beta,\mu)$ distributions is well known.

Lemma 2.1. Let X *be a random variable with the distribution* $S_\alpha(\sigma,\beta,\mu)$. *Then*

$$\text{(2.4)}\qquad \lim_{\lambda\to\infty}\lambda^{\alpha}P(X > \lambda) = \frac{1+\beta}{2}\sigma^{\alpha}\cdot c_\alpha$$

$$\text{(2.5)}\qquad \lim_{\lambda\to\infty}\lambda^{\alpha}P(X < -\lambda) = \frac{1-\beta}{2}\sigma^{\alpha}c_\alpha,$$

where

$$\text{(2.6)}\qquad c_\alpha = \left[\int_0^\infty \frac{\sin y}{y^\alpha}dy\right]^{-\alpha/2}.$$

Proof: See Feller (1966), Theorem XVII, 5.3 and also Weron (1984).

The following result provides estimates that will be needed later.

Lemma 2.2. Let X *be a random variable with the distribution* $S_\alpha(\sigma,\beta,0)$.

(i) Suppose $1 < \alpha < 2$. *Then there exists a positive constant* a,

independent of β *and* σ, *such that* $P(X \geq 0) \geq a$.

(ii) Suppose $\alpha = 1$ *and* $\sigma \leq K$ *for some finite number* K. *Then there is a finite constant* γ_K *which depends only on* K, *and an absolute positive constant* a *such that* $P(X \geq \gamma_K) \geq a$.

<u>Proof</u>: (i) Let Y_i, $i = 1,2$, be i.i.d. $S_\alpha(1,1,0)$ distributed random variables. Then

(2.7) $$X \stackrel{d}{=} [(1+\beta)/2]^{1/\alpha}\sigma Y_1 - [(1-\beta)/2]^{1/\alpha}\sigma Y_2.$$

Consequently,

$$P(X \geq 0) \geq P(Y_1 \geq 0)\cdot P(Y_2 \leq 0) > 0.$$

(ii) With Y_1, Y_2 as above we have by Lemma 1.2 of Hardin (1987)

(2.8) $$X \stackrel{d}{=} \sigma[(1+\beta)/2]Y_1 - \sigma[(1-\beta)/2]Y_2 + (1/\pi)[\sigma f(\beta) + 2\beta\sigma \ell n \sigma],$$

where

(2.9) $$f(\beta) = (1+\beta)\ell n[(1+\beta)/2] - (1-\beta)\ell n[(1-\beta)/2],\ -1 < \beta < 1,\ f(-1) = f(1) = 0.$$

It is easy to see that f is continuous on [-1,1], thus it is bounded. Let $\max_{-1\leq\beta\leq 1} f(\beta) = -\min_{-1\leq\beta\leq 1} f(\beta) = c < \infty$. We conclude that

$$\gamma_K := (1/\pi) \inf_{0\leq\sigma\leq K, -1\leq\beta\leq 1} [\sigma f(\beta) + 2\beta\sigma\ell n\sigma] \geq (1/\pi)[-cK - 2\sup_{0<\sigma\leq K} \sigma|\ell n\sigma|] > -\infty.$$

The claim of the part (ii) now follows from (2.8) by taking $a = P(Y_1 \geq 0)\cdot P(Y_2 \leq 0) > 0$.

Let (S,Σ,m) be a σ-finite measure space and let $\beta: S \to [-1,1]$ be a Σ-measurable function. Let

$$\Sigma_0 = \{A \in \Sigma : m(A) < \infty\}$$

An independently scattered σ-additive set function $M:\Sigma_0 \to L^0(\Omega)$ is called an α-*stable random measure* if for any $A \in \Sigma_0$, any real θ

(2.10) $$E[\exp(i\theta M(A))] = \exp\{-m(A)|\theta|^\alpha + i\tan\frac{\pi\alpha}{2}|\theta|^\alpha \mathrm{sign}(\theta)\int_A \beta(s)m(ds)\},$$

if $0 < \alpha < 2$, $\alpha \neq 1$ and

(2.11) $$E[\exp(i\theta M(A))] = \exp\{-m(A)|\theta| - i(2/\pi)\theta \ln|\theta| \int_A \beta(s)m(ds)\}$$

if $\alpha = 1$. The meausre m is usually called the *control measure* of the random measure M. To the best of the author's knowledge the function β has not yet acquired a special name, and we will call it the *skewness intensity* of the random measure M. This description of stable random measures in given in Theorem 2.1, Remark (ii) of Hardin (1987). In the same paper stochastic integrals with respect to stable random measures are defined. Although the original definition of Hardin is given for the case $S \subseteq \mathbb{R}$, $\Sigma = \mathcal{B}$, the analogous definition in the general case is obvious. Hardin (1987) shows (Theorem 2.3) that integrals of the type

(2.12) $$I_f = \int_S f(s)M(ds)$$

can be defined for any $f \in L^\alpha(S,\Sigma,m)$ if $\alpha \neq 1$. These integrals are linear in f and their characteristic functions are given by

(2.13) $$\ln\{E[\exp(i\theta I_f)]\} = -|\theta|^\alpha \int_S |f(s)|^\alpha m(ds) + i\tan\frac{\pi\alpha}{2}|\theta|^\alpha \mathrm{sign}(\theta)\int_S |f(s)|^\alpha \mathrm{sign}(f(s))\beta(s)m(ds)$$

Thus, I_f has a $S_\alpha(\sigma_f,\beta_f,0)$ distribution with

(2.14) $$\sigma_f = [\int_S |f(s)|^\alpha m(ds)]^{1/\alpha}$$

(2.15) $$\beta_f = \sigma_f^{-\alpha}\int_S |f(s)|^\alpha \mathrm{sign}(f(s))\beta(s)m(ds).$$

Moreover, the result of Hardin shows that two integrals I_f and I_g are independent if and only if $f \cdot g = 0$ a.e. (m).

In the case $\alpha = 1$ Hardin (1987) defines integrals of the type (2.12) under assumption that the measure m is finite. The integral is defined then for the functions belonging to a certain Orlicz space, $L\log^+L(m)$ which is defined as the collection of all measurable functions $f: S \to \mathbb{R}$ for which $\int_S |f(s)|[\ell n|f(s)|]_+ m(ds) < \infty$, equipped with the norm

$$\|f\|_{L\log^+L(m)} := \inf\{c > 0: \int_S |f(s)/c|[\ell n|f(s)/c|]_+ m(ds) < 1\}.$$

Here

$$(2.16) \quad [a]_+ := \begin{cases} a, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases}$$

The integrals (2.12) in the case $\alpha = 1$ are also linear in f, and the distribution of I_f is $S_1(\sigma_f, \beta_f, \mu_f)$, where σ_f and β_f are given by (2.14) and (2.15) accordingly with $\alpha = 1$ and

$$(2.17) \quad \mu_f = -(2/\pi)\int_S f(s)\ell n|f(s)|\beta(s)m(ds).$$

It seems that the assumption that the measure m is finite is unnecessarily restrictive, and one can also define 1-stable integral for functions which lie outside of $L\log^+L(m)$. In the sequel we outline a possible extension of the definition of this integral.

Let M be a 1-stable random measure on a σ-finite measure space (S,Σ,m) with skewness intensity β. We suppose first that

$$(2.18) \quad \int_S |\beta(s)|m(ds) < \infty.$$

This is the same as to say that the measure m_β on (S,Σ) defined by

$$m_\beta(A) := \int_A |\beta(s)| m(ds), \ A \in \Sigma$$

is finite. A linear space of functions

$$(2.19) \quad I(m,\beta) := L^1(m) \cap L\log^+ L(m_\beta)$$

equipped with the norm

$$(2.20) \quad \|f\|_{I(m,\beta)} := \max(\|f\|_{L^1(m)}, \|f\|_{L\log^+L(m,\beta)})$$

is easily seen to be a Banach space in which simple functions form a dense subset.

Repeating then the steps of the proof of Theorem 2.3 of Hardin (1987) we conclude that the 1-stable integral $I_f = \int_S f(s)M(ds)$ can be defined for all f in $I(m,\beta)$. This integral still possesses all the properties mentioned above, i.e. it is linear in f, its distribution is $S_1(\sigma_f,\beta_f,\mu_f)$ with σ_f, β_f and μ_f given by (2.14), (2.15) and (2.17) respectively, and it is still true that two integrals I_f and I_g are independent if and only if $f\cdot g = 0$ a.e. (m).

Now we can drop the assumption (2.18) while still being able to integrate all f's in $I(m,\beta)$, through the following argument. Let $S_1, S_2, \ldots$ be a partition of S into Σ-sets of finite m-measure. For $n = 1,2,\ldots$ denote by $m^{(n)}$ the restriction of the measure m to S_n, i.e. $m^{(n)}(A) = m(A \cap S_n)$ for each $A \in \Sigma$. Then $m^{(n)}$ is a finite measure for each $n = 1,2,\ldots$. It is clear also that $I(M,\beta) \subset I(m^{(n)},\beta)$ for each $n = 1,2,\ldots$.

For each $n = 1,2,\ldots$ let M_n be the restriction of the random measure M to S_n. That means, M_n is defined by

$$M_n(A) := M(A \cap S_n),\ A \in \Sigma.$$

The obtained random measure M_n is then a 1-stable random measure with a finite control measure $m^{(n)}$ and skewness intensity β. The measures M_n, $n = 1,2,\ldots$ are mutually independent, as assured by independent scatteredness of the measure M.

Let $f \in I(m,\beta)$. Then $f \in I(m^{(n)},\beta)$ for any n. Thus the integrals

$$I_f^{(n)} = \int_S f(s) M_n(ds),\ n = 1,2,\ldots$$

are well defined by the above. The sequence $I_f^{(n)}$, $n = 1,2,\ldots$ is then a sequence of independent 1-stable random variables such that $I_f^{(n)}$ has distribution $S_1(\sigma_f^{(n)},\beta_f^{(n)},\mu_f^{(n)})$, where

$$\sigma_f^{(n)} = \int_{S_n} |f(s)| m(ds),$$

$$\beta_f^{(n)} = \int_{S_n} f(s)\beta(s) m(ds) / \int_{S_n} |f(s)| m(ds),$$

$$\mu_f^{(n)} = -(2/\pi) \int_{S_n} f(s) \ell n |f(s)| \beta(s) m(ds),$$

$n = 1,2,\ldots$. Since $f \in I(m,\beta)$ we conclude that both series $\Sigma_{n=1}^{\infty} \sigma_f^{(n)}$ and $\Sigma_{n=1}^{\infty} \mu_f^{(n)}$ converge. Thus, the series $\Sigma_{n=1}^{\infty} I_f^{(n)}$ converge with probability 1, and we define

$$(2.21) \qquad I_f := \sum_{n=1}^{\infty} I_f^{(n)}.$$

The integral defined in this way preserves all the properties of 1-stable integrals mentioned above. Moreover, it is easily seen that the integral defined in this way does not depend on a particular partition $S_1, S_2, \ldots$ (in spite of the fact that the series (2.21) does not have to converge absolutely!).

The reason why we are talking so much about integrals with respect to random stable measures is that these integrals provide a very convenient way of representing and handling stable processes.

Historically, Schilder (1970) was the first to prove a representation theorem of the kind

$$\text{(2.22)} \qquad \{X(t), t \in T\} \stackrel{d}{=} \{\int_0^1 f(t,s) M(ds), t \in T\}$$

for α-stable processes $\{X(t), t \in T\}$ with $0 < \alpha < 2$. Schilder's result was extended then by Kuelbs (1973) and Hardin (1987) to the form given below.

A process $\{X(t), t \in T\}$ defined on a topological space T is said to satisfy assumption S is there is a countable dense subset T_0 of T such that every X(t) is a limit in probability of a sequence from the set of all finite linear combinations $\Sigma_j a_j X(t_j)$, $t_j \in T_0$.

<u>Theorem 2.1</u>. *Suppose that an* α-*stable process* $\{X(t), t \in T\}$ *satisfies assumption* S.

(i) If $\{X(t), t \in T\}$ *is symmetric, then there is a finite measure* m *on* $([0,1], \mathcal{B})$ *and a family of functions* $f(t,\cdot) \in L^\alpha(m), t \in T$, *such that the representation* (2.22) *holds, where* M *is an* α-*stable random measure with control measure* m *and skewness intensity* $\beta \equiv 0$.

(ii) If $\{X(t), t \in T\}$ *is strict then there is a family of functions* $f(t,\cdot) \in L^\alpha(\lambda)$ *(*λ *is Lebesgue measure on* $([0,1], \mathcal{B}), t \in T)$*), such that the representation* (2.22) *holds, where* M *is an* α-*stable random measure with control measure* λ *and skewness intensity* $\beta \equiv 1$.

Let $\{X(t), t \in T\}$ be an arbitrary α-stable process. It is well known that, unless $\alpha = 1$, there is a deterministic function $\mu: T \to \mathbb{R}$ such that the process $\{X(t) - \mu(t), t \in T\}$ is strictly α-stable.

Suppose that T is a countable set. Then Theorem 2.1(ii) together with the previous remark imply that

$$(2.23) \quad \{X(t), t \in T\} \stackrel{d}{=} \{\int_0^1 f(t,s) M(ds) + \mu(t), t \in T\}$$

for some sequence of functions $f(t,\cdot) \in L^{\alpha}(\lambda)$, $t \in T$ and a sequence of real numbers $\mu(t), t \in T$. Here M is an α-stable random measure on $([0,1], \mathcal{B})$ with control measure λ and skewness intensity $\beta \equiv 1$.

Unfortunately, a 1-stable process cannot, in general, be made strict by shifting. Thus, Theorem 2.1 is insufficient to establish the representation (2.23) when $\alpha = 1$. However, the following result can be obtained directly using a representation of the characteristic functions of stable probability measures on a separable Hilbert space. We omit the proof since it essentially repeats the steps of the proof of Theorem 4.1 of Kuelbs (1973).

Theorem 2.2. Let $\{X(t), t \in T\}$ *be a 1-stable process on a countable set* T. *Then there is a sequence of functions* $f(t,\cdot) \in L\log^{+}L(\lambda), t \in T$ *and a sequence of real numbers* $\mu(t), t \in T$ *such that the representation* (2.23) *holds.*

3. Extremes of stable vectors in $\mathbb{R}^n$

Let M be an α-stable random measure on a σ-finite measure space (S,Σ,m) with skewness intensity β. Let $f_1,f_2,\ldots,f_n$ be a sequence of functions which is assumed to belong either to $L^\alpha(S,\Sigma,m)$ if $\alpha \neq 1$, or to $I(m,\beta)$ if $\alpha = 1$. Let $\mu_1,\mu_2,\ldots,\mu_n$ be arbitrary real numbers. Then

$$(3.1) \qquad X_i = \int_S f_i(s)M(ds) + \mu_i, \quad i=1,2,\ldots,n$$

is a stable vector in $\mathbb{R}^n$, and it follows from Theorems 2.1 and 2.2 that any stable vector in $\mathbb{R}^n$ can be represented in the form (3.1).

Fix $s \in S$. Consider two sequences of non-negative numbers, $[f_i(s)]_+$, $i=1,2,\ldots,n$ and $[-f_i(s)]_+$, $i=1,2,\ldots,n$. Here again

$$[a]_+ = \begin{cases} a, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases}$$

Let $[f(s)]_+^{(1)} \geq [f(s)]_+^{(2)} \geq \ldots \geq [f(s)]_+^{(n)}$ and $[-f(s)]_+^{(1)} \geq [-f(s)]_+^{(2)} \geq \ldots \geq [-f(s)]_+^{(n)}$ denote the above sequence arranged in the nonincreasing order. Define

$$(3.2) \qquad h_+(k;s) := [f(s)]_+^{(k)}, \quad s \in S, \ k=1,2,\ldots,n,$$

$$(3.3) \qquad h_-(k;s) := [-f(s)]_-^{(k)}, \quad s \in S, \ k=1,2,\ldots,n.$$

Because of their special importance in the sequel, $h_+(n;\cdot)$ and $h_-(n;\cdot)$ get shorter names. We put

$$(3.4) \qquad h_+(s) := h_+(n;s), \quad s \in S,$$

$$(3.5) \qquad h_-(s) := h_-(n;s), \quad s \in S.$$

Our first result in this section describes the asymptotic probability that all the components of a stable fector are big given one of them is big.

Theorem 3.1. Suppose that

$$(3.6)\qquad m\{s\in S: f_1(s)\neq 0, \beta(s)\,\mathrm{sign}(f_1(s))\neq -1\}>0.$$

Then

$$(3.7)\qquad \lim_{\lambda\to\infty} P(X_2>\lambda,\dots,X_n>\lambda\,|\,X_1>\lambda) = \frac{\int_S h_+(s)^\alpha(1+\beta(s))m(ds)+\int_S h_-(s)^\alpha(1-\beta(s))m(ds)}{\int_S |f_1(s)|^\alpha m(ds)+\int_S |f_1(s)|^\alpha \mathrm{sign}(f_1(s))\beta(s)m(ds)}.$$

The proof of this result is similar to that of Theorem 4.1 in Samorodnitsky (1986). We defer it to the end of this section. Our next result gives the asymptotic behavior of the distribution of the k^{th} order statistic $X^{(k)}$, $k=1,2,\dots,n$ from a jointly stable sample $(X_1,X_2,\dots,X_n)$ given in the form (3.1).

Theorem 3.2.

$$(3.8)\qquad \lim_{\lambda\to\infty}\lambda^\alpha P(X^{(k)}>\lambda) = \frac{1}{2}c_\alpha\Big[\int_S h_+(k;s)^\alpha(1+\beta(s))m(ds)+\int_S h_-(k;s)^\alpha(1-\beta(s))m(ds)\Big]$$

$$(3.9)\qquad \lim_{\lambda\to\infty}\lambda^\alpha P(X^{(k)}<-\lambda) = \frac{1}{2}c_\alpha\Big[\int_S h_+(n-k;s)^\alpha(1-\beta(s))m(ds)+\int_S h_-(n-k;s)^\alpha(1+\beta(s))m(ds)\Big]$$

for each $k=1,2,\dots,n$. *The constant* c_α *is given in* (2.6).

Proof: Combining Theorem 3.1 and Lemma 2.1 with (2.13)-(2.15) we immediately obtain (3.8) in the particular case $k=n$, or using the notation (3.4), we get

$$(3.10)\qquad \lim_{\lambda\to\infty}\lambda^\alpha P(X^{(n)}>\lambda) = \frac{1}{2}c_\alpha\Big[\int_S h_+(s)^\alpha(1+\beta(s))m(ds)+\int_S h_-(s)^\alpha(1-\beta(s))m(ds)\Big].$$

We use now the following version of the inclusion-exclusion formula. Let $a_1, a_2, \ldots, a_n$ be arbitrary real numbers. Let $a^{(1)} \geq a^{(2)} \geq \ldots \geq a^{(n)}$ denote the same numbers arranged in the nonincreasing order. Then

$$(3.11)\quad a^{(k)} = \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} \sum_{1 \leq i(1) < i(2) < \ldots < i(j) \leq n} \min(a_{i(1)}, a_{i(2)}, \ldots, a_{i(j)}),$$

$k = 1, 2, \ldots, n$. We conclude in particular that

$$(3.12)\quad P(X^{(k)} > \lambda) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} \sum_{1 \leq i(1) < i(2) < \ldots < i(j) \leq n} P(X_{i(1)} > \lambda, X_{i(2)} > \lambda, \ldots, X_{i(j)} > \lambda).$$

For any $1 \leq i(1) < i(2) < \ldots < i(j) \leq n$ denote

$$g_+(i(1), i(2), \ldots, i(j); s) := \min_{m=1,\ldots,j} [f_{i(m)}(s)]_+,$$

$$g_-(i(1), i(2), \ldots, i(j); s) := \min_{m=1,\ldots,j} [-f_{i(m)}(s)]_+.$$

We obtain then by (3.10) and (3.12) that

$$(3.13)\quad \lim_{\lambda\to\infty} \lambda^{\alpha} P(X^{(k)} > \lambda)$$

$$= \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} \sum_{1 \leq i(1) < i(2) < \ldots < i(j) \leq n} \lim_{\lambda\to\infty} \lambda^{\alpha} P(X_{i(1)} > \lambda, X_{i(2)} > \lambda, \ldots, X_{i(j)} > \lambda)$$

$$= \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} \sum_{1 \leq i(1) < i(2) < \ldots < i(j) \leq n} \tfrac{1}{2} c_{\alpha} \Big[\int_S g_+(i(1), i(2), \ldots, i(j); s)^{\alpha} (1+\beta(s))\, m(ds)$$

$$+ \int_S g_-(i(1), i(2), \ldots, i(j); s)^{\alpha} (1-\beta(s))\, m(ds)\Big]$$

$$= \tfrac{1}{2} c_{\alpha} \Big[\int_S \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} \sum_{1 \leq i(1) < i(2) < \ldots < i(j) \leq n} g_+(i(1), i(2), \ldots, i(j); s)^{\alpha} (1+\beta(s))\, m(ds)$$

$$+ \int_S \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} \sum_{1 \leq i(1) < i(2) < \ldots < i(j) \leq n} g_-(i(1), i(2), \ldots, i(j); s)^{\alpha} (1-\beta(s))\, m(ds)$$

and (3.8) follows now from (3.11) and (3.13). Finally (3.9) follows from (3.8) applied to the stable random vector $-X_1, -X_2, \ldots, -X_n$.

We turn now to the proof of Theorem 3.1. The proof is via sequence of lemmas in the spirit similar to that of the proof of Theorem 4.1 of Samorodnitsky (1986).

Lemma 3.1. *Let* $Y_1, Y_2, \ldots$ *be independent* α*-stable random variables,* Y_n *having* $S_\alpha(\sigma_n, \beta_n, \mu_n)$ *distribution. Suppose that*

$$(3.14) \quad \sum_{n=1}^{\infty} \sigma_n^\alpha < \infty,$$

$$(3.15) \quad \sum_{n=1}^{\infty} \mu_n \text{ converges,}$$

and assume that there is a k *such that* $\sigma_k \neq 0$, $\beta_k \neq 1$. *Let*

$$A_n(\lambda) := \{Y_n > \lambda, Y_m \le \lambda \text{ for all } m \neq n\}$$

$$A^*(\lambda) = \bigcap_{n=1}^{\infty} A_n(\lambda)^c.$$

Then

$$(3.16) \quad \lim_{\lambda \to \infty} P\Big(A_n(\lambda) \,\Big|\, \sum_{m=1}^{\infty} Y_m > \lambda\Big) = \frac{(1+\beta_n)\sigma_n^\alpha}{\sum_{m=1}^{\infty} (1+\beta_m)\sigma_m^\alpha}$$

$$(3.17) \quad \lim_{\lambda \to \infty} P\Big(A^*(\lambda) \,\Big|\, \sum_{m=1}^{\infty} Y_m > \lambda\Big) = 0.$$

Remark 3.1. Clearly, conditions (3.14) and (3.15) are necessary and sufficient for a.s. convergence of the series $\Sigma_{n=1}^{\infty} Y_n$. We note that necessary and sufficient conditions for a.s. absolute convergence of this series are

$$(3.18)\qquad \sum_{n=1}^{\infty} |\mu_n| < \infty$$

and

$$(3.19)\qquad \begin{cases} \sum_{n=1}^{\infty} \sigma_n^{\alpha} < \infty \text{ if } 0 < \alpha < 1 \\ \sum_{n=1}^{\infty} \sigma_n |\ell n \sigma_n| < \infty \text{ if } \alpha = 1 \\ \sum_{n=1}^{\infty} \sigma_n < \infty \text{ if } 1 < \alpha < 2. \end{cases}$$

<u>Proof</u>: Clearly, only (3.16) needs to be proved. To simplify notation we take $n = 1$ in (3.16). If either $\sigma_1 = 0$ or $\beta_1 = -1$, (3.16) trivially follows from Lemma 2.1. Suppose therefore that $\sigma_1 > 0$, $\beta_1 > -1$. Fix any $\delta > 0$. Then

$$P(A_1(\lambda) | \sum_{m=1}^{\infty} Y_m > \lambda) \geq P(A_1(\lambda), \sum_{m=2}^{\infty} Y_m > -\lambda\delta | \sum_{m=1}^{\infty} Y_m > \lambda)$$

$$\geq \frac{P(Y_1 > \lambda(1+\delta))P(\sum_{m=2}^{\infty} Y_m > -\lambda\delta, Y_m \leq \lambda, m > 1)}{P(\sum_{m=1}^{\infty} Y_m > \lambda)}.$$

Since the distribution of $\Sigma_{m=1}^{\infty} Y_m$ is

$$S_\alpha((\sum_{m=1}^{\infty} \sigma_m^{\alpha})^{1/\alpha}, \sum_{m=1}^{\infty} \beta_m \sigma_m^{\alpha} / \sum_{m=1}^{\infty} \sigma_m^{\alpha}, \sum_{m=1}^{\infty} \mu_m)$$

we conclude using Lemma 2.1 that

$$(3.20)\qquad \underline{\lim}_{\lambda\to\infty} P(A_1(\lambda) \mid \sum_{m=1}^{\infty} Y_m > \lambda)$$

$$\geq (1+\delta)^{-\alpha} \frac{(1+\beta_1)\sigma_1^\alpha}{\sum_{m=1}^{\infty}(1+\beta_m)\sigma_m^\alpha} \underline{\lim}_{\lambda\to\infty} P(\sum_{m=2}^{\infty} Y_m > -\lambda\delta,\ Y_m \leq \lambda,\ m>1).$$

We have

$$(3.21)\qquad \underline{\lim}_{\lambda\to\infty} P(\sum_{m=2}^{\infty} Y_m > -\lambda\delta,\ Y_m \leq \lambda,\ m>1) \geq 1 - \overline{\lim}_{\lambda\to\infty} \sum_{m=2}^{\infty} P(Y_m > \lambda).$$

By the Three Series Theorem the sum in the right hand side of (3.21) is finite for any λ, thus (3.21), bounded convergence and the fact that (3.20) is true for any $\delta > 0$ imply that

$$\underline{\lim}_{\lambda\to\infty} P(A_1(\lambda) \mid \sum_{m=1}^{\infty} Y_m > \lambda) \leq \overline{\lim}_{\lambda\to\infty} \frac{P(Y_1 > \lambda)}{P(\sum_{m=1}^{\infty} Y_m > \lambda)} = \frac{(1+\beta_1)\sigma_1^\alpha}{\sum_{m=1}^{\infty}(1+\beta_m)\sigma_m^\alpha}.$$

This proves (3.16).

<u>*Lemma 3.2*</u>. *Let* $(Y_1^{(j)}, Y_2^{(j)}, \ldots, Y_n^{(j)})$, $j=1,2,\ldots$ *be independent* α-*stable vectors in* $\mathbb{R}^n$. *Suppose that the series*

$$(3.22)\qquad X_i = \sum_{j=1}^{\infty} Y_i^{(j)},\ i=1,2,\ldots,n$$

converge a.s. for each $i=1,2,\ldots,n$. *Suppose also that* $\lim_{\lambda\to\infty} \lambda^\alpha P(X_1 > \lambda) > 0$. *Then*

$$(3.23)\qquad \overline{\lim}_{\lambda\to\infty} P(X_2 > \lambda, \ldots, X_n > \lambda \mid X_1 > \lambda) \leq \sum_{j=1}^{\infty} \overline{\lim}_{\lambda\to\infty} \frac{P(Y_1^{(j)} > \lambda, Y_2^{(j)} > \lambda, \ldots, Y_n^{(j)} > \lambda)}{P(X_1 > \lambda)}$$

$$(3.24)\quad \underline{\lim}_{\lambda\to\infty} P(X_2>\lambda,\ldots,X_n>\lambda \mid X_1>\lambda) \ge \sum_{j=1}^{\infty} \underline{\lim}_{\lambda\to\infty} \frac{P(Y_1^{(j)}>\lambda, Y_2^{(j)}>\lambda,\ldots, Y_n^{(j)}>\lambda)}{P(X_1>\lambda)}.$$

Proof: For $\lambda > 0$ define

$$B_{j(1),j(2),\ldots,j(n)}(\lambda) := \{Y_i^{(j(i))} > \lambda, i=1,2,\ldots,n, Y_i^{(j)} \le \lambda, j\ne j(i), i=1,2,\ldots,n\},$$

$j(1) = 1,2,\ldots,\ldots,\; j(n) = 1,2,\ldots,$

$$B^*(\lambda) := \bigcap_{j(1)=1}^{\infty}\cdots\bigcap_{j(n)=1}^{\infty} B_{j(1),j(2),\ldots,j(n)}(\lambda)^c.$$

Then by bounded convergence theorem

$$(3.25)\quad \overline{\lim}_{\lambda\to\infty} P(X_2>\lambda,\ldots,X_n>\lambda \mid X_1>\lambda) \le \overline{\lim}_{\lambda\to\infty} P(X_2>\lambda,\ldots,X_n>\lambda, B^*(\lambda) \mid X_1>\lambda)$$

$$+ \sum_{j(1)=1}^{\infty}\cdots\sum_{j(n)=1}^{\infty} \overline{\lim}_{\lambda\to\infty} P(X_2>\lambda,\ldots,X_n>\lambda, B_{j(1),j(2),\ldots,j(n)}(\lambda) \mid X_1>\lambda),$$

$$(3.26)\quad \underline{\lim}_{\lambda\to\infty} P(X_2>\lambda,\ldots,X_n>\lambda \mid X_1>\lambda) \ge \underline{\lim}_{\lambda\to\infty} P(X_2>\lambda,\ldots,X_n>\lambda, B^*(\lambda) \mid X_1>\lambda)$$

$$+ \sum_{j(1)=1}^{\infty}\cdots\sum_{j(n)=1}^{\infty} \underline{\lim} P(X_2>\lambda,\ldots,X_n>\lambda, B_{j(1),j(2),\ldots,j(n)}(\lambda) \mid X_1>\lambda).$$

By Lemma 3.1 we have

$$\lim_{\lambda\to\infty} P(X_2>\lambda,\ldots, X_n>\lambda, B^*(\lambda) \mid X_1>\lambda) = 0.$$

Furthermore, for any choice of the multiple index $(j(1),j(2),\ldots, j(n))$ such that $j(i) \ne j(1)$ for some $i = 2,\ldots, n$ we have

$$\overline{\lim}_{\lambda\to\infty} P(X_2>\lambda,\ldots,X_n>\lambda, B_{j(1),\ldots,j(n)}(\lambda) \mid X_1>\lambda) \le \overline{\lim}_{\lambda\to\infty} P(Y_1^{(j(1))}>\lambda, Y_i^{(j(i))}>\lambda \mid X_1>\lambda)$$

$$\le \lim_{\lambda\to\infty} \frac{P(Y_1^{(j(1))}>\lambda)\cdot P(Y_i^{(j(i))}>\lambda)}{P(X_1>\lambda)} = 0$$

by Lemma 2.1. Consequently, (3.25) and (3.26) are reduced to the following.

$$(3.27)\quad \overline{\lim_{\lambda\to\infty}} P(X_2>\lambda,\dots,X_n>\lambda \mid X_1>\lambda) \le \sum_{j=1}^{\infty} \overline{\lim_{\lambda\to\infty}} P(X_2>\lambda,\dots,X_n>\lambda, B_{j,j,\dots,j}(\lambda) \mid X_1>\lambda)$$

$$(3.28)\quad \underline{\lim_{\lambda\to\infty}} P(X_2>\lambda,\dots,X_n>\lambda \mid X_1>\lambda) \ge \sum_{j=1}^{\infty} \underline{\lim_{\lambda\to\infty}} P(X_2>\lambda,\dots,X_n>\lambda, B_{j,j,\dots,j}(\lambda) \mid X_1>\lambda).$$

For any $j=1,2,\dots$ we have

$$(3.29)\quad \overline{\lim_{\lambda\to\infty}} P(X_2>\lambda,\dots,X_n>\lambda, B_{j,j,\dots,j}(\lambda) \mid X_1>\lambda)$$

$$\le \overline{\lim_{\lambda\to\infty}} P(Y_1^{(j)}>\lambda, Y_2^{(j)}>\lambda,\dots,Y_n^{(j)}>\lambda \mid X_1>\lambda) \le \overline{\lim_{\lambda\to\infty}} \frac{P(Y_1^{(j)}>\lambda, Y_2^{(j)}>\lambda,\dots,Y_n^{(j)}>\lambda)}{P(X_1>\lambda)}$$

Together with (3.27) this proves (3.23). To prove (3.24) note that for any $\delta>0$

$$(3.30)\quad \underline{\lim_{\lambda\to\infty}} P(X_2>\lambda,\dots,X_n>\lambda, B_{j,j,\dots,j}(\lambda) \mid X_1>\lambda)$$

$$\ge \underline{\lim_{\lambda\to\infty}} \frac{P(Y_i^{(j)}>\lambda(1+\delta), i=1,2,\dots,n, \sum_{k\ne j} Y_i^{(k)} \ge -\lambda\delta, i=1,2,\dots,n, Y_i^{(k)} \le \lambda, k\ne j, i=1,2,\dots,n)}{P(X_1>\lambda)}$$

$$= \underline{\lim_{\lambda\to\infty}} \frac{P(Y_1^{(j)}>\lambda(1+\delta), Y_2^{(j)}>\lambda(1+\delta),\dots, Y_n^{(j)}>\lambda(1+\delta))}{P(X_1>\lambda)}$$

$$\times \lim_{\lambda\to\infty} P\Big(\sum_{k\ne j} Y_i^{(k)} \ge -\lambda\delta, i=1,2,\dots,n,\ Y_i^{(k)} \le \lambda,\ k\ne j,\ i=1,2,\dots,n\Big).$$

By monotone convergence theorem we conclude that the second limit in the right hand side of (3.30) is equal to 1. Moreover, by Lemma 2.1

$$\lim_{\lambda\to\infty} \frac{P(Y_1^{(j)} > \lambda(1+\delta),\ Y_2^{(j)} > \lambda(1+\delta),\ldots,\ Y_n^{(j)} > \lambda(1+\delta))}{P(X_1 > \lambda)}$$

$$= (1+\delta)^{-\alpha} \lim_{\lambda\to\infty} \frac{P(Y_1^{(j)} > \lambda,\ Y_2^{(j)} > \lambda,\ldots,\ Y_n^{(j)} > \lambda)}{P(X_1 > \lambda)} .$$

Consequently,

$$\lim_{\lambda\to\infty} P(X_2 > \lambda,\ldots,\ X_n > \lambda,\ B_{j,j,\ldots,j}(\lambda) \mid X_1 > \lambda) \geq (1+\sigma)^{-\alpha} \lim_{\lambda\to\infty} \frac{P(Y_1^{(j)} > \lambda, Y_2^{(j)} > \lambda,\ldots,Y_n^{(j)} > \lambda)}{P(X_1 > \lambda)}$$

and, since it is true for any $\delta > 0$, (3.28) implies (3.24).

<u>*Lemma 3.3*</u>. *In addition to the assumptions of Theorem* 3.1, *assume that* $n = 2$, *that the measure* m *is finite and that there is a* Σ-*set*, Λ, *of* m-*measure zero and a positive number* θ *such that for any* $s \in \Lambda^c$, $f_1(s) \geq \theta$, $f_2(s) \leq -\theta$. *Then*

$$\lim_{\lambda\to\infty} P(X_2 > \lambda \mid X_1 > \lambda) = 0.$$

Proof: Using (2.14), (2.15) and (2.17) we conclude that the distribution of X_i is $S_\alpha(\sigma_i, \beta_i, \mu_i)$, $i = 1,2$, where

(3.31) $\quad \sigma_i^\alpha = \int_S |f_i(s)|^\alpha m(ds), \ i = 1,2,$

(3.32) $\quad \beta_i = \sigma_i^{-\alpha} \int_S |f_i(s)|^\alpha \operatorname{sign}(f_i(s)) \beta(s) m(ds), \ i = 1,2,$

(3.33) $\quad \mu_i = \begin{cases} -(2/\pi) \int_S f_i(s) \ln|f_i(s)| \beta(s) m(ds) & \text{if } \alpha = 1 \\ 0 \text{ if } \alpha \neq 1 \end{cases}, \ i = 1,2.$

The assumption (3.6) implies that $\sigma_1 > 0$, $\beta_1 > -1$. For $N \geq 1$ we define a partition of the set Λ^c as follows. Let

$$\Delta(k_1,k_2) := \{s \in \Lambda^c : k_1\theta/N \le f_1(s) < (k_1+1)\theta/N,\ k_2\theta/N \le -f_2(s) < (k_2+1)\theta/N\},\ k_1 \ge N,\ k_2 \ge N.$$

Let

$$X_i(k_1,k_2) := \int_S f_i(s) I(s \in \Delta(k_1,k_2)) M(ds),\ i = 1,2,\ k_1 \ge N,\ k_2 \ge N.$$

It follows from the properties of stable integrals mentioned in Section 2 and from Remark 3.1 that the double array $(X_1(k_1,k_2), X_2(k_1,k_2))$ consists of independent α-stable vectors in $\mathbb{R}^2$, and that

$$(3.34) \qquad (X_1,X_2) \stackrel{d}{=} \Big(\sum_{k_1=N}^{\infty} \sum_{k_2=N}^{\infty} X_1(k_1,k_2),\ \sum_{k_1=N}^{\infty} \sum_{k_2=N}^{\infty} X_2(k_1,k_2)\Big)$$

in the sense that for each fixed order of summation the double sums in the right hand side of (3.31) converge with probability one, and the joint distributions of both sides of (3.31) coincide. By Lemma 3.2 we have

$$(3.35) \qquad \overline{\lim_{\lambda\to\infty}} P(X_2 > \lambda | X_1 > \lambda) \le \sum_{k_1=N}^{\infty} \sum_{k_2=N}^{\infty} \overline{\lim_{\lambda\to\infty}} \frac{P(X_1(k_1,k_2) > \lambda,\ X_2(k_1,k_2) > \lambda)}{P(X_1 > \lambda)}.$$

Since for any $s \in \Delta(k_1,k_2)$

$$-k_1\theta/N \le k_2 f_1(s) + k_1 f_2(s) \le k_2\theta/N$$

we conclude by (3.31), (3.32) and Lemma 2.1 that

$$(3.36) \qquad \overline{\lim_{\lambda\to\infty}} \frac{P(X_1(k_1,k_2) > \lambda,\ X_2(k_1,k_2) > \lambda)}{P(X_1 > \lambda)}$$

$$\le \overline{\lim_{\lambda\to\infty}} \frac{P((k_1+k_2)^{-1}(k_2 X_1(k_1,k_2) + k_1 X_2(k_1,k_2)) > \lambda)}{P(X_1 > \lambda)}$$

$$= \frac{\int_{\Delta(k_1,k_2)} |k_2 f_1(s) + k_1 f_2(s)|^\alpha [1 + \mathrm{sign}(k_2 f_1(s) + k_1 f_2(s))\beta(s)] m(ds)}{(k_1 + k_2)^\alpha (1+\beta_1) \int_S |f_1(s)|^\alpha m(ds)}$$

$$\le \frac{2\theta^\alpha N^{-\alpha} \max(k_1^\alpha, k_2^\alpha) m(\Delta k_1, k_2))}{(k_1 + k_2)^\alpha (1+\beta_1) \int_S |f_1(s)|^\alpha m(ds)}$$

$$\le \frac{2}{1+\beta_1} \cdot \frac{\theta^\alpha}{N^\alpha} \cdot \frac{m(\Delta(k_1,k_2))}{\int_S |f_1(s)|^\alpha m(ds)}$$

We conclude by (3.35) and (3.36) that

$$\overline{\lim_{\lambda\to\infty}} P(X_2 > \lambda \mid X_1 > \lambda) \le \frac{2}{1+\beta_1} \cdot \frac{\theta^\alpha}{N^\alpha} \cdot \frac{m(s)}{\int_S |f_1(s)|^\alpha m(ds)} .$$

Since N can be taken arbitrarily large, the proof of the lemma is completed.

Lemma 3.4. In Lemma 3.3 assume now that $\theta = 0$, and the measure m does not have to be finite any longer. Then the conclusion of Lemma 3.3 remains in force.

Proof: Let

$$\Delta(n,k) := \{s \in \Lambda^c : 1/n \le f_1(s) < 1/(n-1),\ 1/k \le f_2(s) < 1/(k-1)\},$$

$n \ge 1$, $k \ge 1$. Define

$$X_i(n,k) := \int_S f_i(s) I(s \in \Delta(n,k)) M(ds),\ i = 1,2,\ n \ge 1,\ k \ge 1.$$

By Lemma 3.2 we have

$$(3.37)\quad \overline{\lim_{\lambda\to\infty}} P(X_2 > \lambda \mid X_1 > \lambda) \le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \overline{\lim_{\lambda\to\infty}} \frac{P(X_1(n,k) > \lambda,\ X_2(n,k) > \lambda)}{P(X_1 > \lambda)}.$$

For $n \ge 1$, $k \ge 1$ let $m_{n,k}$ be the restriction of m to $\Delta(n,k)$. Then the joint distribution of the random vector

$$\hat{X}_i(n,k) := \int_S f_i(s) M_{n,k}(ds),\ i = 1,2,.$$

($M_{n,k}$ is an α-stable random measure on (S,Σ) with control measure $m_{n,k}$ and skewness intensity β) coincides with the joint distribution of the random vector $(X_1(n,k), X_2(n,k))$. Then Lemma 3.3 implies that all the limits in the double sum in (3.37) are equal to zero. This proves the lemma.

Lemma 3.5. In addition to the assumptions of Theorem 3.1 *assume that there is a* Σ*-set*, Λ, *of* m*-measure zero, such that for any* $s \in \Lambda^c$ $f_1(s) \ge 0$ *and* $\min(f_2(s),\ldots, f_n(s)) \le 0$. *Then*

$$\lim_{\lambda\to\infty} P(X_2 > \lambda, \ldots, X_n > \lambda \mid X_1 > \lambda) = 0.$$

Proof: Let $C_i := \{s \in \Lambda^c : f_i(s) \le 0,\ f_j(s) > 0,\ j = 2,\ldots, i-1\}$, $i = 2,\ldots,n$. Then the sets $C_2,\ldots, C_n$ partition Λ^c. Letting

$$Y_i^{(j)} := \int_S f_i(s) I(s \in C_j) M(ds),\ i = 1,2,\ldots,n,\ j = 2,\ldots,n$$

we conclude by Lemma 3.2

$$(3.38)\quad \overline{\lim_{\lambda\to\infty}} P(X_2 > \lambda, \ldots, X_n > \lambda \mid X_1 > \lambda) \le \sum_{j=2}^{n} \overline{\lim_{\lambda\to\infty}} \frac{P(Y_1^{(j)} > \lambda, Y_2^{(j)} > \lambda, \ldots, Y_n^{(j)} > \lambda)}{P(X_1 > \lambda)}$$

Lemma 3.4 implies that every limit in the right hand side of (3.38) is equal to zero. This proves the lemma.

Lemma 3.6. In addition to the assumptions of Theorem 3.1 assume that there is a Σ-set, Λ, of m-measure zero, such that for any $s \in \Lambda^c$, for any $i = 1,2,\ldots,n$, $f_i(s) \geq f_1(s) \geq 0$. Then

$$\lim_{\lambda\to\infty} P(X_2 > \lambda, \ldots, X_n > \lambda | X_1 > \lambda) = 1.$$

Proof: Let $X_i^* := X_i - X_1$, $i = 2,\ldots,n$. For any $\delta > 0$ we have by Lemma 2.1

$$(3.39)\quad \underline{\lim}_{\lambda\to\infty} P(X_2 > \lambda,\ldots,X_n > \lambda | X_1 > \lambda) \geq \underline{\lim}_{\lambda\to\infty} P(X_1 > \lambda(1+\delta), X_2^* \geq -\lambda\delta,\ldots,X_n^* \geq -\lambda\delta | X_1 > \lambda)$$
$$\geq (1+\delta)^{-\alpha} - \sum_{i=2}^{n} \overline{\lim}_{\lambda\to\infty} P((-X_i^*) > \lambda\delta | X_1 > \lambda).$$

By Lemma 3.4 each limit in the right hand side of (3.39) is equal to zero. Then

$$\underline{\lim}_{\lambda\to\infty} P(X_2 > \lambda,\ldots,X_n > \lambda | X_1 > \lambda) \geq (1+\delta)^{-\alpha},$$

and, since δ can be taken as small as we please, the claim of the lemma follows.

Proof of Theorem 3.1: Let A be a Σ-set. Note that multiplying simultaneously the functions $\beta(s)$, $f_1(s),\ldots,f_n(s)$, $s \in A$, by -1 does not change the joint distribution of the random vector $(X_1, X_2, \ldots, X_n)$. We may and will therefore assume that $f_1(s) \geq 0$ for each s. The claim of the theorem is reduced then to

$$(3.40)\quad \lim_{\lambda\to\infty} P(X_2 > \lambda,\ldots,X_n > \lambda | X_1 > \lambda) = \frac{\int_S h_+(s)^\alpha (1+\beta(s))m(ds)}{\int_S f_1(s)^\alpha (1+\beta(s))m(ds)}.$$

Define

$$B_0 := \{s \in S : f_1(s) = 0\}$$

$$B_1 := \{s \in S: f_1(s) > 0,\ \min(f_2(s), \ldots, f_n(s)) \le 0\}$$

$$B_2 := \{s \in S: 0 < \min(f_2(s), \ldots, f_n(s)) < f_1(s)\}$$

$$B_3 := \{s \in S: 0 < f_1(s) \le \min(f_2(s), \ldots, f_n(s))\}.$$

These four sets constitute a partition of S. Let

$$Y_i^{(j)} := \int_S f_i(s) I(s \in B_j) M(ds),\ i = 1,2,\ldots,n,\ j = 0,1,2,3.$$

Then by Lemma 3.2

$$(3.41)\qquad \lim_{\lambda\to\infty} P(X_2 > \lambda, \ldots, X_n > \lambda | X_1 > \lambda) = \sum_{j=1}^{3} \lim_{\lambda\to\infty} \frac{P(Y_1^{(j)} > \lambda,\ Y_2^{(j)} > \lambda, \ldots, Y_n^{(j)} > \lambda)}{P(X_1 > \lambda)}$$

provided all three limits in the right hand side of (3.41) exist. We will prove that for any $j = 1,2,3$

$$(3.42)\qquad \lim_{\lambda\to\infty} \frac{P(Y_1^{(j)} > \lambda,\ Y_2^{(j)} > \lambda, \ldots, Y_n^{(j)} > \lambda)}{P(X_1 > \lambda)} = \frac{\int_S h_+^{(j)}(s)^\alpha (1 + \beta(s)) m(ds)}{\int_S f_1(s)^\alpha (1 + \beta(s)) m(ds)},$$

where

$$h_+^{(j)}(s) := \min_{i=1,2,\ldots,n} [f_i(s) I(s \in B_j)]_+,\ j = 1,2,3.$$

In that case (3.41) and (3.42) would imply (3.40). Note that if for some $j = 1,2,3$

$$(3.43)\qquad \int_S f_1(s)^\alpha I(s \in B_j)(1 + \beta(s)) m(ds) = 0$$

then Lemma 2.1 implies that for this particular j both sides of (3.42) are equal to zero. We will assume therefore that the integral

(3.43) is positive for each $j = 1,2,3$.

By the definition $h_+^{(1)}(s) \equiv 0$, so the right hand side of (3.42) is zero when $j = 1$. It is easy to see that in that case the limit in the left hand side of (3.42) exists and is equal to zero as well. This follows from Lemma 3.5. This proves (3.42) in the case $j = 1$. Consider now the case $j = 3$. By the definition $h_+^{(3)}(s) = f_1(s)I(s \in B_3)$. Then the expression in the right hand side of (3.42) takes the form

$$\frac{\int_S f_1(s)^\alpha I(s \in B_3)(1 + \beta(s))m(ds)}{\int_S f_1(s)^\alpha (1 + \beta(s))m(ds)}$$

while the left hand side of (3.42) is equal to the same value by Lemmas 2.1 and 3.6. It remains therefore to prove (3.42) in the case $j = 2$. Define

$$B_2(i) := \{s \in B_2 : f_i(s) = \min_{j=1,2,\dots,n} f_j(s) < \min_{j=1,2,\dots,i-1} f_j(s)\},$$

$i = 2,\dots,n$. Then the sets $B_2(2),\dots, B_2(n)$ are disjoint, cover B_2, and on $B_2(i)$, $0 < f_i(s) \le f_j(s)$ for all $j = 1,2,\dots,n$. Let

$$\hat{Y}_i^{(k)} := \int_S f_i(s) I(s \in B_2(k))M(ds), \quad i = 1,2,\dots,n, \quad k = 2,\dots,n.$$

Arguing as above we get

$$(3.44) \quad \lim_{\lambda\to\infty} \frac{P(\hat{Y}_1^{(k)} > \lambda, \hat{Y}_2^{(k)} > \lambda, \dots, \hat{Y}_n^{(k)} > \lambda)}{P(Y_1^{(2)} > \lambda)} = \frac{\int_S f_k(s)^\alpha I(s \in B_2(k))(1 + \beta(s))m(ds)}{\int_S f_1(s)^\alpha I(s \in B_2)(1 + \beta(s))m(ds)},$$

$k = 2,\dots,n$. Now Lemma 3.2 implies (3.42) in the case $j = 2$. This completes the proof of the theorem.

4. Stable processes on countable sets - conditions for boundedness and high excursions

Stable processes considered in this section are given in the form

$$(4.1)\qquad X_i = \int_S f_i(s)M(ds) + \mu_i,\ i=1,2,\dots,$$

where, as in Section 3, M is an α-stable measure on a σ-finite measure space (S,Σ,m) with skewness intensity β, $\{f_i\}_{i=1}^{\infty}$ is a sequence from $L^{\alpha}(S,\Sigma,m)$ if $\alpha \neq 1$ or from $I(m,\beta)$ if $\alpha = 1$, and $\{\mu_i\}_{i=1}^{\infty}$ is a sequence of reals.

We will find conditions for a.s. boundedness of this process and will study the asymptotic behavior of the distribution functions of $\sup_{i\geq 1} X_i$ and $\inf_{i\geq 1} X_i$. Knowing those properties of stable processes defined on countable sets we would be able to handle processes defined on more general sets. The point is that in order to make $\sup_{t\in T} X(t)$ a well defined random variable, one usually considers a separable version of the process, which reduces, in effect, the parameter set to its certain countable subset. Our first result generalizes Theorem 6.1 of Samorodnitsky (1987) to the nonsymmetric case.

Theorem 4.1. *Suppose the process* $\{X_i, i\geq 1\}$ *is given by* (4.1). *Then*

(i)

$$(4.2)\qquad \underline{\lim}_{\lambda\to\infty} \lambda^{\alpha} P\Big(\sup_{i=1,2,\dots} X_i > \lambda\Big) \geq \frac{1}{2}c_{\alpha}\Big[\int_S g_+(s)^{\alpha}(1+\beta(s))m(ds) + \int_S g_-(s)^{\alpha}(1-\beta(s))m(ds)\Big]$$

$$(4.3)\qquad \underline{\lim}_{\lambda\to\infty} \lambda^{\alpha} P\Big(\inf_{i=1,2,\dots} X_i < -\lambda\Big) \geq \frac{1}{2}c_{\alpha}\Big[\int_S g_+(s)^{\alpha}(1-\beta(s))m(ds) + \int_S g_-(s)^{\alpha}(1+\beta(s))m(ds)\Big],$$

where

$$(4.4)\qquad g_+(s) := \sup_{i=1,2,\dots} [f_i(s)]_+,\ s\in S,$$

$$(4.5)\qquad g_-(s) := \sup_{i=1,2,\dots} [-f_i(s)]_+,\ s\in S.$$

(ii) If $0 < \alpha < 1$ *and* $\sup_{i \ge 1} \mu_i < \infty$ *then*

$$(4.6)\quad \lim_{\lambda\to\infty} \lambda^{\alpha} P(\sup_{i=1,2,\ldots} X_i > \lambda) = \frac{1}{2} c_\alpha [\int_S g_+(s)^{\alpha}(1+\beta(s))m(ds) + \int_S g_-(s)^{\alpha}(1-\beta(s))m(ds)].$$

If $0 < \alpha < 1$ *and* $\inf_{i \ge 1} \mu_i > -\infty$, *then*

$$(4.7)\quad \lim_{\lambda\to\infty} \lambda^{\alpha} P(\inf_{i=1,2,\ldots} X_i < -\lambda) = \frac{1}{2} c_\alpha [\int_S g_+(s)^{\alpha}(1-\beta(s))m(ds) + \int_S g_-(s)^{\alpha}(1+\beta(s))m(ds)].$$

The constant c_α *is given in* (2.6).

<u>Proof</u>: (i) For each $n = 1,2,\ldots$ define

$$g_+^{(n)}(s) := \max_{1 \le i \le n} [f_i(s)]_+, \quad s \in S,$$

$$g_-^{(n)}(s) := \max_{1 \le i \le n} [-f_i(s)]_+, \quad s \in S.$$

Then by Theorem 3.2, for each fixed $n = 1,2,\ldots$ we obtain

$$\underline{\lim}_{\lambda\to\infty} \lambda^{\alpha} P(\sup_{i=1,2,\ldots} X_i > \lambda) \ge \lim_{\lambda\to\infty} \lambda^{\alpha} P(\max_{i=1,2,\ldots,n} X_i > \lambda)$$

$$= \frac{1}{2} c_\alpha [\int_S g_+^{(n)}(s)^{\alpha}(1+\beta(s))m(ds) + \int_S g_-^{(n)}(s)^{\alpha}(1-\beta(s))m(ds)].$$

Since this holds for any $n = 1,2,\ldots$, (4.2) follows by monotone convergence theorem. The assertion (4.3) follows now by (4.2) applied to the α-stable process $\{-X_i, i = 1,2,\ldots\}$.

(ii) We prove (4.6) first. We may and will assume here, without loss of generality, that $\mu_i = 0$ for each $i = 1,2,\ldots$. Moreover, in view of (4.2) it is enough to prove that

$$(4.8)\quad \overline{\lim}_{\lambda\to\infty} \lambda^{\alpha} P(\sup_{i\ge 1} X_i > \lambda) \le \frac{1}{2} c_\alpha [\int_S g_+(s)^{\alpha}(1+\beta(s))m(ds) + \int_S g_-(s)^{\alpha}(1-\beta(s))m(ds)].$$

If either one of the two integrals in the right hand side of (4.8) is infinite, then there is nothing to prove. Assume therefore that

$$\int_S g_+(s)^\alpha (1+\beta(s))m(ds) < \infty \tag{4.9}$$

$$\int_S g_-(s)^\alpha (1-\beta(s))m(ds) < \infty. \tag{4.10}$$

Let K_1 and K_2 be independent α-stable random measures on the same σ-finite measure space (S,Σ,m) with skewness intensities $\beta \equiv 1$. Let

$$W_i := \int_S f_i(s)(1+\beta(s))^{1/\alpha} K_1(ds), \quad i=1,2,\ldots,$$

$$Z_i := \int_S f_i(s)(1-\beta(s))^{1/\alpha} K_2(ds), \quad i=1,2,\ldots .$$

Then $\{W_i, i=1,2,\ldots\}$ and $\{Z_i, i=1,2,\ldots\}$ are two independent α-stable processes. A direct computation of the joint characteristic functions shows that

$$\{2^{-1/\alpha}(W_i - Z_i), i=1,2,\ldots\} \stackrel{d}{=} \{X_i, i=1,2,\ldots\}.$$

Thus

$$\begin{aligned} P(\sup_{i\ge 1} X_i > \lambda) &= P(\sup_{i\ge 1}(W_i - Z_i) > 2^{1/\alpha}\lambda) \\ &\le P(\sup_{i\ge 1} W_i + \sup_{i\ge 1}(-Z_i) > 2^{1/\alpha}\lambda). \end{aligned} \tag{4.11}$$

Let

$$W_* := \int_S g_+(s)(1+\beta(s))^{1/\alpha} K_1(ds)$$

$$Z_* := \int_S g_-(s)(1-\beta(s))^{1/\alpha} K_2(ds).$$

By the assumptions (4.9) and (4.10) we conclude that W_* and Z_* are well defined α-stable random variables. They are also independent and a.s. positive (see Remark 2.1). Note that for each $i = 1,2,\ldots$, each $s \in S$, $g_+(s) - f_i(s) \geq 0$. Consequently, for each $i = 1,2,\ldots$

$$W_* - W_i = \int_S (g_+(s) - f_i(s))(1 + \beta(s))^{1/\alpha} K_1(ds) \geq 0 \text{ a.s.}$$

We conclude that

$$(4.12) \qquad W_* \geq \sup_{i \geq 1} W_i \text{ a.s.}$$

The same argument shows that

$$(4.13) \qquad Z_* \geq \sup_{i \geq 1} (-Z_i) \text{ a.s.}$$

as well. It follows from (4.11)-(4.13) that

$$\overline{\lim_{\lambda \to \infty}} \lambda^\alpha P(\sup_{i \geq 1} X_i > \lambda) \leq \lim_{\lambda \to \infty} \lambda^\alpha P(W_* + Z_* > 2^{1/\alpha}\lambda)$$

$$= \frac{1}{2} c_\alpha [\int_S g_+(s)^\alpha (1 + \beta(s)) m(ds) + \int_S g_-(s)^\alpha (1 - \beta(s)) m(ds)]$$

by independence of W_* and Z_* and by Lemma 2.1. This proves (4.6). The assertion (4.7) follows now by applying (4.6) to the α-stable process $\{-X_i, i = 1,2,\ldots\}$.

We now turn to the problem of a.s. boundedness of the process (4.1).

Theorem 4.2. *(i) The following are necessary for the α-stable process* (4.1) *to be a.s. bounded*

$$(4.14) \qquad \int_S f^*(s)^\alpha m(ds) < \infty,$$

where $f^*(s) := \sup_{i\ge 1}|f_i(s)|$, $s \in S$, *and*

(4.15) $\quad \sup_{i\ge 1}|\mu_i| < \infty$, *if* $\alpha \neq 1$

or

(4.16) $\quad \sup_{i\ge 1}|\mu_i - (2/\pi)\int_S f_i(s)\ln|f_i(s)|\beta(s)m(ds)| < \infty \quad$ *if* $\alpha = 1$.

(ii) If $0 < \alpha < 1$, *then the conditions* (4.14)-(4.15) *are also sufficient for a.s. boundedness of the process* (4.1).

Proof: (i) If the process (4.1) is a.s. bounded, then by the general result of de Acosta (1977) it follows that

$$\lim_{\lambda\to\infty}\lambda^{\alpha}P(\sup_{i\ge 1}|X_i| > \lambda) < \infty.$$

By Theorem 4.1(i) this implies that the expressions in the right hand sides of (4.2) and (4.3) must be finite. Summing those two expressions we conclude that

(4.17) $$\int_S [g_+(s)^{\alpha} + g_-(s)^{\alpha}]m(ds) < \infty.$$

The necessity of (4.14) follows now from (4.17) and the following obvious relation.

(4.18) $$\max(g_+(s)^{\alpha}, g_-(s)^{\alpha}) = f^*(s)^{\alpha} \le g_+(s)^{\alpha} + g_-(s)^{\alpha}.$$

We prove now the necessity of (4.15) in the case $1 < \alpha < 2$. Suppose that (4.15) does not hold. We may assume without loss of generality that $\sup_{i=1,2,\ldots} \mu_i = \infty$. Fix any $\lambda > 0$. By our assumption there is an $i(\lambda)$ such that $\mu_{i(\lambda)} > \lambda$. Then by Lemma 2.2(i)

$$P(\sup_{i=1,2,\ldots} X_i > \lambda) \ge P(X_{i(\lambda)} > \lambda) \ge P(X_{i(\lambda)} - \mu_{i(\lambda)} > 0) \ge a$$

for some positive a that does not depend on λ. Then $P(\sup_{i=1,2,\dots} X_i = \infty) \geq a$ so by the zero-one law (see Dudley and Kanter (1974)), $P(\sup_{i=1,2,\dots} |X_i| = \infty) = 1$ and the process is a.s. unbounded. Next we prove the necessity of the condition (4.16) in the case $\alpha = 1$. Let $K = \int_S f^*(s)m(ds)$. Since we have already proved the necessity of the condition (4.14), we may assume that $K < \infty$. Denote

$$\Delta_i := \mu_i - (2/\pi)\int_S f_i(s)\ln|f_i(s)|\beta(s)m(ds), \quad i = 1,2,\dots$$

and let

(4.19) $$Z_i := X_i - \Delta_i, \quad i = 1,2,\dots .$$

We conclude by (2.14), (2.15) and (2.17) that Z_i has an $S_1(\sigma_i, \beta_i, 0)$ distribution, $i = 1,2,\dots$, where

$$\sigma_i = \int_S |f_i(s)|m(ds), \quad i = 1,2,\dots,$$

$$\beta_i = \sigma_i^{-1}\int_S f_i(s)\beta(s)m(ds), \quad i = 1,2,\dots .$$

Clearly, $\sigma_i \leq K$ for each $i = 1,2,\dots$. Suppose that $\sup_{i=1,2,\dots} |\Delta|_i = \infty$. As before we assume, without loss of generality, that $\sup_{i=1,2,\dots} \Delta_i = \infty$. By Lemma 2.2(ii) there is a finite constant γ_K such that for each $i = 1,2,\dots$

(4.20) $$P(Z_i > \gamma_K) \geq a$$

for a certain positive a that does not depend on i. Fix any $\lambda > 0$. By our assumption there is an $i(\lambda)$ such that $\Delta_{i(\lambda)} > \lambda - \gamma_K$. Then by (4.20)

$$P(\sup_{i=1,2,\dots} X_i > \lambda) \geq P(X_{i(\lambda)} > \lambda) = P(Z_{i(\lambda)} > \lambda - \Delta_{i(\lambda)})$$

$$\geq P(Z_{i(\lambda)} > \gamma_K) \geq a.$$

As before, by the zero-one law we conclude that the process $\{X_i, i=1,2,\ldots\}$ is a.s. unbounded. To complete the proof of the part (i) of the theorem we have to prove the necessity of (4.15) in the case $0<\alpha<1$. We defer this task until after we prove the part (ii) of the theorem.

(ii) By the part (ii) of Theorem 4.1 we conclude that

$$\lim_{\lambda\to\infty}\lambda^{\alpha}P(\sup_{i=1,2,\ldots}|X_i|>\lambda) = c_\alpha\int_S f^*(s)^\alpha m(ds) < \infty \tag{4.21}$$

Thus the process $\{X_i, i=1,2,\ldots\}$ is a.s. bounded. We complete now the proof of the part (i). Suppose that $0<\alpha<1$ and that the condition (4.15) does not hold. Since the necessity of the condition (4.14) has already been proved, we assume that this condition is satisfied. Then by the part (ii) of this theorem the process

$$Y_i := X_i - \mu_i,\ i=1,2,\ldots$$

is a.s. bounded. Consequently $\{X_i, i = 1,2,\ldots\}$ can be represented as a sum of an a.s. bounded process and an unbounded sequence. This proves that the process $\{X_i, i=1,2,\ldots\}$ is itself a.s. unbounded. This completes the proof of the theorem.

It should be mentioned that in general the conditions (4.14), (4.15) if $1<\alpha<2$ and (4.14), (4.16) if $\alpha=1$ are not sufficient for a s. boundedness of the process (4.1), as follows from Example 6.1 of Samorodnitsky (1987). The above example shows also that, in general, the second part of Theorem 4.1 (or its obvious modification if $\alpha=1$) is false when $\alpha\geq 1$. It is a conjecture of the author that the second part of Theorem 4.1 is still true even if $\alpha\geq 1$, if it is known that the process (4.1) is a.s. bounded.

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