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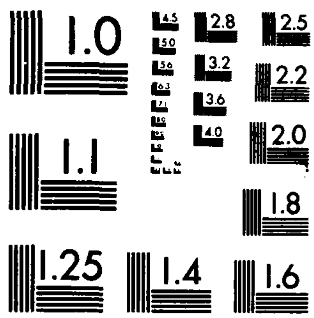
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19. Abstract

Let M be a two-parameter continuous martingale bounded in L^2 and null on the axes. The positive submartingale M^2 has the following Doob-Meyer decomposition

$$M_{st}^2 = 2\int_{R_{st}} M_z dM_z + 2\tilde{M}_{st} + <M_{s\cdot}>_t + <M_{\cdot t}>_s - [M]_{st} \;,$$

(cf. [2], [3], [8]), where R_{st} denotes the rectangle $[0,s]\times[0,t]$; $<M_{s\cdot}>_t$ (resp. $<M_{\cdot t}>_s$) is the quadratic variation of the one-parameter martingale $\{M_{st},\ t\geq 0\}$ (resp. $\{M_{st},\ s\geq 0\}$), and $\tilde{M}$ is a martingale obtained in the following way: For each $z\in \mathbf{R}_+^2$, consider an increasing sequence of grids, $\{\rho^n,\ n\geq 1\}$, of $T=R_z$ whose mesh tends to zero, then $\tilde{M}_z$ is the L^1-limit of the sequence

$$\Sigma_{(n)}[M(s_{i+1},t_j)-M(s_i,t_j)][M(s_i,t_{j+1})-M(s_i,t_j)] \;.$$

The purpose of this paper is to relate the measures induced by the quadratic variations $[M]_{st}$, $[\tilde{M}]_{st}$, $<M_{s\cdot}>_t$ and $<M_{\cdot t}>_s$, in terms of the "absolute continuity" property. Since we are dealing with random measures, different definitions are possible:

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ON THE RELATIONS BETWEEN INCREASING FUNCTIONS ASSOCIATED

WITH TWO-PARAMETER CONTINUOUS MARTINGALES

by

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1. Introduction

Let M be a squared two-parameter continuous martingale bounded in L^2 Squared and null on the axes. The positive submartingale M^2 has a ~~the following~~ Doob-Meyer decomposition,

$$M_{st}^2 = 2\int_{R_{st}} M_z\,dM_z + 2\tilde{M}_{st} + \langle M_{s\cdot}\rangle_t + \langle M_{\cdot t}\rangle_s - [M]_{st} ,$$

(cf. [2], [3], [8]), where R_{st} denotes ~~the rectangle~~ $[0,s]\times[0,t]$; $\langle M_{s\cdot}\rangle_t$ (resp. $\langle M_{\cdot t}\rangle_s$) is the quadratic variation of the one-parameter martingale $\{M_{st},\ t\geq 0\}$ (resp. $\{M_{st},\ s\geq 0\}$), and $\tilde{M}$ is a martingale obtained in the following way: For each $z\in\mathbb{R}_+^2$, consider an increasing sequence of grids, $\{\rho^n,\ n\geq 1\}$, of $T=R_z$ whose mesh tends to zero, then $\tilde{M}_z$ is the L^1-limit of the sequence

$$\Sigma_{(n)}[M(s_{i+1},t_j)-M(s_i,t_j)][M(s_i,t_{j+1})-M(s_i,t_j)] .$$

The purpose of this paper is to relate the measures induced by certain ~~the~~ quadratic variations $[M]_{st}$, $[\tilde{M}]_{st}$, $\langle M_{s\cdot}\rangle_t$ and $\langle M_{\cdot t}\rangle_s$, in terms of the "absolute continuity" property. Since we are dealing with random measures, different definitions are possible: (Keywords; Convergence; [illegible]; [illegible]; Stochastic processes

Definition 1.1. Given two random finite measures μ,ν defined on some measurable space (S,Σ), μ is said to be *absolutely continuous with respect to* ν, *a.s.* (and we will write $\mu\ll\nu$, a.s.) if, for any $w\notin N$, $P(N)=0$, the following property holds:

If $A\in\Sigma$ is such that $\nu(w)(A)=0$, then also $\mu(w)(A)=0$.

Definition 1.2. Let μ and ν be as in the preceding definition. μ is said to be *weakly absolutely continuous with respect to* ν if, for any $A\in\Sigma$, $P\{w,\ \nu(w)(A)=0,\ \mu(w)(A)\neq 0\}=0$.

Comparing these two definitions, we easily realize that the first one requires a stronger property, but it seems to be the more "natural" in view of applications, and will be mainly considered along the paper.

The contents of the paper are roughly as follows. Section 2 is devoted to preliminaries on two-parameter processes. In particular we introduce the measure $[M]*[M]$ which gives an exact meaning to the expression $\int_{R_{st}}[M]_{\sigma,d\tau}[M]_{d\sigma,\tau}$ (this turns out to be a special case of generalized exterior products introduced by Wong and Zakai in [14]). The measure $[\tilde{M}]$ can also be viewed as an exterior product of this type and a probabilistic proof of this fact is given in Proposition 2.3. Section 3 relates the measures induced by M on vertical or horizontal lines. It is shown that in general $[M]_{ds,t} \not\ll \langle M_{\cdot,t} \rangle_{ds}$. We have not been able to prove that in general $[M]_{ds,t} \gg \langle M_{\cdot,t} \rangle_{ds}$; we believe it is true and give partial results in this direction. In section 4 we compare the random measures $[M]$,$[\tilde{M}]$ and $[M] * [M]$. It is pointed out that in general $[M] \not\ll [\tilde{M}]$, $[M] \not\ll [M] * [M]$, $[\tilde{M}] \not\ll [M]$. Absolute continuity results are derived in this section under additional conditions. We do not know if $[\tilde{M}] \ll [M] * [M]$ holds in general, we conjecture that it holds and derive partial results in this direction.

In section 5 we apply the results of the previous section to the problem of local time for two parameter martingales. In fact, one of the motivations of our problem has been the study of local time for two-parameter continuous martingales. In the one-parameter case the local time of martingales, $L(x,t)$, (and more general, for semimartingales X) is defined by an extended version of Ito's formula for convex functions. It is a local time with respect to the quadratic variation $\langle X \rangle$, in the sense that, for every $f : \mathbf{R} \to \mathbf{R}$ bounded and Borel it satisfies the "density of occupation" formula

$$\int_{\mathbf{R}} f(x) L(x,t) dx = \int_o^t f(X_s) d\langle X \rangle_s \ , \quad a.s.$$

In the two-parameter case this method leads to a local time with respect to $[\tilde{M}]$ (see [9]). The result seems to be rather surprising, since in comparison with the one-parameter case we expect that the measure $[M]$ should play a role. It follows from the results of the previous sections that $[M]$ alone or $[\tilde{M}]$ alone are not sufficient to carry the local time and, therefore, a measure like $[M]+[\tilde{M}]$ may be more suitable for this purpose.

2. Preliminaries and Notation.

The parameter space is $T=[0,1]^2$ endowed with the partial ordering $(s_1,t_1) \le (s_2,t_2)$ if and only if $s_1 \le s_2$, $t_1 \le t_2$; $(s_1,t_1) < (s_2,t_2)$ means $s_1 < s_2$ and $t_1 < t_2$. If f is a map from T to $\mathbf{R}$, the increment of f on a rectangle $(z_1,z_2] = \{z \in T,\ z_1 < z \le z_2\}$, $z_1=(s_1,t_1)$, $z_2=(s_2,t_2)$ is $f((z_1,z_2]) = f(z_2) - f(s_1,t_2) - f(s_2,t_1) + f(z_1)$.

Let ρ be a grid of T given by

$\rho = \{(s_i,t_j) \in T,\ i=0,\dots,p,\ j=0,\dots,q,\ 0=s_0<s_1<\cdots<s_p<1,\ 0=t_0<t_1<\cdots<t_q<1\}$.

For any $(s_i,t_j) \in \rho$ we define

$\Delta_{ij}=(s_i,s_{i+1}]\times(t_j,t_{j+1}]$, $\Delta_{ij}^1=(s_i,s_{i+1}]\times(0,t_j]$, $\Delta_{ij}^2=(0,s_i]\times(t_j,t_{j+1}]$, with the convention $s_{p+1}=t_{q+1}=1$.

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, and $(\mathcal{F}_z)_{z\in T}$ an increasing family of sub σ-fields of $\mathcal{F}$ satisfying the usual conditions (F1) to (F4) of [2]. We recall that if $M=\{M_z, z\in T\}$ is a real valued integrable and $\mathcal{F}_z$-adapted process, M is a *martingale* if for any $z\le z'$, $E[M_{z'}\mid\mathcal{F}_z]=M_z$. M is a *strong martingale* if M vanishes on the axes, and $E\{M((z,z'])\mid\mathcal{F}_{s1}\vee\mathcal{F}_{1t}\}=0$, for each $z\le z'$, $z=(s,t)$.

Let m_c^p be the class of two-parameter continuous martingales bounded in L^p and null on the axes. Denote by $M_{\cdot t}$ and $M_{s\cdot}$ the one-parameter martingales $\{M_{st}, \mathcal{F}_{s1}, s\ge0\}$ and $\{M_{st}, \mathcal{F}_{1t}, t\ge0\}$ respectively. If $M\in m_c^2$, M is said to be of *path independent variation* if $<M_{\cdot t}>_s=<M_{s\cdot}>_t=[M]_{st}$ (cf. [2], [13]). Here, and throughout the paper, $<X>$ will denote the quadratic variation of a one-parameter martingale X, while $[y]$ will refer to the quadratic variation of a two-parameter martingale y.

The class of strong martingales on m_c^2 is strictly included in the class of path independent variation martingales (see [7]).

All constants will be denoted by C, although they may change from one expression to another.

In order to state the results of this section we consider an increasing sequence of grids $\{\rho^n, n\ge1\}$ of T whose norm tends to zero, and for any $z\in T$ we define $I_z^n=\{(i,j)\in\mathbb{N}^2 \mid (s_i,t_j)\in\rho^n, (s_i,t_j)<z\}$.

Lemma 2.1.- Let $F:\mathbf{R}_+^2\to\mathbf{R}$ be a continuous distribution function. Then $\lim\limits_{n\to\infty}\sum\limits_{(i,j)\in I_z^n}F(\Delta_{ij}^1)F(\Delta_{ij}^2)$ exists and defines a continuous and increasing function $F*F$ (see Lemma 2.6 of [9]).

Proof. Consider the following decomposition:

$$[F(z)]^2 = \left[\sum_{(i,j)\in I_z^n} F(\Delta_{ij}) \right]^2$$

$$= \sum_{(i,j)\in I_z^n} 2[F(s_i,t_j)F(\Delta_{ij}) + F(\Delta_{ij})F(\Delta_{ij}^1) + F(\Delta_{ij})F(\Delta_{ij}^2) + F(\Delta_{ij}^1)F(\Delta_{ij}^2) + \frac{1}{2}F(\Delta_{ij})^2] \ .$$

Let $n \to \infty$, then by the continuity of F we obtain

$$\sum_{(i,j)\in I_z^n} F(\Delta_{ij})F(\Delta_{ij}^1) \le \sup_i (F(s_{i+1},t) - F(s_i,t))F(z) \underset{n\to\infty}{\longrightarrow} 0 \ ,$$

and analogously $\sum_{(i,j)\in I_z^n} F(\Delta_{ij})F(\Delta_{ij}^2) \underset{n\to\infty}{\longrightarrow} 0.$

On the other hand $\sum_{(i,j)\in I_z^n} F(\Delta_{ij})^2 \le \sup_{i,j} F(\Delta_{ij})F(z) \underset{n\to\infty}{\longrightarrow} 0.$ Therefore,

$$\lim_{n\to\infty} \sum_{(ij)\in I_z^n} F(\Delta_{ij}^1)F(\Delta_{ij}^2) = \frac{1}{2}F(z)^2 - \int_{R_z} F(\alpha)F(d\alpha) \ .$$

This limit defines a continuous function $F * F$. It is obvious that for any rectangle Δ with sides parallel to the axes, $(F * F)(\Delta) \ge 0$.

□

If F and G are two continuous distribution functions on R_+^2, $F * G$ can be defined by polarization.

Given $M \in m_c^2$, there exist a continuous and increasing process $\{[M]_z, \ z \in T\}$ such that

$$\lim_{n\to\infty} \sup_z E\{ | \sum_{(i,j)\in I_z^n} M(\Delta_{ij})^2 - [M]_z | \} = 0 \ .$$

(See [2], [8]). Therefore, by Lemma 2.1 we can associate to M a continuous measure $[M] * [M]$. In Section 4 we will compare this measure with the quadratic variations associated with M.

Lemma 2.2. Let $\{M_i^t ; \ \mathcal{F}^t, \ i=0,\ldots,m, \ t \in [0,1]\}$ be a collection of one parameter martingales, bounded in L^p, for some $p \ge 2$, such that $M_i^o = 0$, for any $i=0,\ldots,m$. Suppose that $t \to M_i^t$ is a.s. continuous for every $i=0,\ldots,m$. Then there exists a constant C_p such tha'

$$E\left[\sum_{i=0}^m \sup_{0\le t\le 1} |M_i^t|^2 \right]^{p/2} \le C_p E \, | \sum_{i=0}^m (M_i^1)^2 |^{p/2} \ .$$

Furthermore if $\sum_{j=0}^i M_j^1$ is a martingale sequence in i then $E \, | \sum_{i=0}^m (M_i^1)^2 |^{p/2} \le C_p E \, | \sum_{i=0}^m M_i^1 |^p$.

Proof. Let $\{A_t,\ t\in[0,1]\}$ be the continuous, increasing and $\mathcal{F}^t$-adapted process defined by

$$A_t=\sum_{i=0}^{m}\sup_{0\le\tau\le t}|M_i^{\tau}|^2 .$$

The potential Z_t associated with A_t is computed as follows:

$$\begin{aligned} Z_t &= E[A_1-A_t \mid \mathcal{F}^t] \\ &= E\left\{\sum_{i=0}^{m}[\sup_{0\le\tau\le 1}|M_i^{\tau}|^2-\sup_{0\le\tau\le t}|M_i^{\tau}|^2]\,/\,\mathcal{F}^t\right\} \\ &\le E[\sum_{i=0}^{m}\sup_{t\le\tau\le 1}|M_i^{\tau}|^2/\mathcal{F}^t] .\end{aligned} \tag{1}$$

By Doob's maximal inequality applied to the martingales $\{(M_i^{\tau})\cdot 1_F,\mathcal{F}^{\tau},\tau\in[t,1]\}$, $F\in\mathcal{F}^t$, $i=0,\dots,m$, (1) is bounded by $m_t=C_p\ E[\sum_{i=0}^{m}|M_i^1|^2/\mathcal{F}^t]$, where $\{m_t\ ;\ \mathcal{F}^t,\ t\in[0,1]\}$ is a martingale. By Garsia-Neveu's inequality

$$E(A_1)^{p/2}\le C_p E(m_1)^{p/2}=C_p E[\sum_{i=0}^{m}|M_i^1|^2]^{p/2} , \tag{2}$$

which is the first result of the lemma. Burkholder's inequality applied to the one-parameter discrete martingale $\sum_{j=1}^{i}M_j^1$, yields the second part of the lemma.

□

Note that the first part of this lemma is a slight generalization of the Doob maximal inequality for one-parameter vector valued martingales.

Proposition 2.3. Let $M\in m_c^4$. The following convergence holds:

$$\lim_{n\to\infty}E\{|\sum_{(i,j)\in I_z^n}[M(\Delta_{ij}^1)^2M(\Delta_{ij}^2)^2-(<M_{s_i\cdot}>_{t_{j+1}}-<M_{s_i\cdot}>_{t_j})(<M_{\cdot t_j}>_{s_{i+1}}-<M_{\cdot t_j}>_{s_i})]|\}$$

$=0$, for any $z\in T$.

Proof. To simplify the notations we take $z=(1,1)$. Let $m\ge n$, $\rho^n=\{(s_i,t_j)\in T,\ i=0,\dots,p_n,\ j=0,\dots,q_n\}$, $\rho^m=\{(\sigma_{i'},\tau_{j'})\in T,\ i'=0,\dots,p_m,\ j'=0,\dots,q_m\}$. For every $i=0,\dots,p_n$ (resp. $j=0,\dots,q_n$) define $I_i=\{i',\sigma_{i'}\in[s_i,s_{i+1})\}$ (resp. $J_j=\{j',\tau_{j'}\in[t_j,t_{j+1})\}$). Given points $u'=(\sigma_{i'},t_j)$, $u''=(s_i,\tau_{j'})$ we denote by $\Delta_{i'j}^1=(\sigma_{i'},\sigma_{i'+1}]\times(0,t_j]$, and $\Delta_{ij'}^2=(0,s_i]\times(\tau_{j'},\tau_{j'+1}]$. Then

$$
\begin{aligned}
&E\{\,|\sum_{i,j}[M(\Delta_{ij}^1)^2 M(\Delta_{ij}^2)^2-(<M_{s_i\cdot}>_{t_{j+1}}-<M_{s_i\cdot}>_{t_j})(<M_{\cdot t_j}>_{s_{i+1}}-<M_{\cdot t_j}>_{s_i})]|\} \\
&=E\{\,|\sum_{i,j}[M(\Delta_{ij}^1)^2 M(\Delta_{ij}^2)^2-\lim_{m\to\infty}(\sum_{j'\in J_j} M(\Delta_{ij'}^2)^2)(\sum_{i'\in I_i} M(\Delta_{i'j}^2)^2)]\,|\,\} \qquad (3)\\
&\le \sup_{m\ge n} E\{\,|\sum_{i,j}[(\sum_{i'\in I_i} M(\Delta_{i'j}^1))^2(\sum_{j'\in J_j} M(\Delta_{ij'}^2))^2-(\sum_{j'\in J_j} M(\Delta_{ij'}^2)^2)(\sum_{i'\in I_i} M(\Delta_{i'j}^1)^2)]\,|\,\}\ .
\end{aligned}
$$

For any $i'\in I_i$, $j'\in J_j$, $\sigma_{i'}>s_i$, $\tau_{j'}>t_j$ we define $\overline{\Delta}_{ii'j}^1=(s_i,\sigma_{i'}]\times(0,t_j]$, and $\overline{\Delta}_{ijj'}^2=(0,s_i]\times(t_j,\tau_{j'}]$. Using this notation we obtain that the last term in (3) is bounded by $C\sup_{m\ge n}(a_{nm}+b_{nm}+c_{nm})$, where

$$
\begin{aligned}
a_{nm}&=E[\,|\sum_{i,j}\sum_{i'\in I_i}\sum_{j'\in J_j} M(\Delta_{i'j}^1)^2 M(\Delta_{ij'}^2) M(\overline{\Delta}_{ijj'}^2)\,|\,]\ ,\\
b_{nm}&=E[\,|\sum_{i,j}\sum_{i'\in I_i}\sum_{j'\in J_j} M(\Delta_{ij'}^2)^2 M(\Delta_{i'j}^2) M(\overline{\Delta}_{ii'j}^1)\,|\,]\ ,\\
c_{nm}&=E[\,|\sum_{i,j}\sum_{i'\in I_i}\sum_{j'\in J_j} M(\Delta_{i'j}^1) M(\Delta_{ij'}^2) M(\overline{\Delta}_{ii'j}^1) M(\overline{\Delta}_{ijj'}^2)\,|\,]\ .
\end{aligned}
$$

We next show that each one of these terms tends to zero when $n\to\infty$, uniformly in m.

(i) By Davis inequality we obtain

$$
\begin{aligned}
a_{nm}&\le C\ E[\sum_{j;j'\in J_j}(\sum_{i;i'\in I_i} M(\Delta_{i'j}^1)^2 M(\Delta_{ij'}^2) M(\overline{\Delta}_{ijj'}^2))^2]^{1/2}\\
&=C\ E[\sum_{j;j'\in J_j}(\sum_i M(\Delta_{ij'}^2) M(\overline{\Delta}_{ijj'}^2)\sum_{i'\in I_i} M(\Delta_{i'j}^1)^2)^2]^{1/2}\\
&\le C\ E[\sum_j(\sum_{i;i'\in I_i} M(\Delta_{i'j}^2)^2)^2\sum_{j'\in J_j}\sup_i M(\Delta_{ij'}^2)^2 \sup_i M(\overline{\Delta}_{ijj'}^2)^2]^{1/2}\\
&\le C\ E[\sup_{|z_1-z_2|\le|\rho^n|}|M_{z_2}-M_{z_1}|^2\sup_j(\sum_{i;i'\in I_i} M(\Delta_{i'j}^1)^2)^2\sum_{j;j'\in J_j}\sup_i M(\Delta_{ij'}^2)^2]^{1/2} \qquad (4)\\
&\le C\{E[\sup_j(\sum_{i;i'\in I_i} M(\Delta_{i'j}^1)^2)^2]\}^{1/2}\\
&\cdot\{E[\sup_{|z_1-z_2|\le|\rho^n|}|M_{z_2}-M_{z_1}|^4]\}^{1/4}\cdot\}E(\sum_{j;j'\in J_j}\sup_i M(\Delta_{ij'}^2)^2)^2\}^{1/4}\ ,
\end{aligned}
$$

where $|\rho^n|$ denotes the norm of ρ^n, i.e. $|\rho^n|=\max\{\,|(s_i,t_j)-(s_{i+1},t_{j+1})|\,,$ $i=0,\dots,p_n,\ j=0,\dots,q_n\}$.

By Doob's inequality applied to the positive submartingale $\{\sum_{i;i'\in I_i} M(\Delta_{i'j}^1)^2,\ \mathcal{F}_{1,t_j},\ j=0,\dots,q_n\}$, the first factor of (4) is bounded by $C\ E(\sum_{i;i'\in I_i}(M(\sigma_{i'+1},1)-M(\sigma_{i'},1))^2)^2$, and by Burkholder's inequality this is majorized by

$C\ E(M_{11}^4)$. The second factor of (4) tends to zero as $n \to \infty$, by the continuity of M. Finally,

$$E[(\sum_{j;j'\in J_j} \sup_i M(\Delta_{ij'}^2)^2)^2] \le E[(\sum_{j;j'\in J_j} \sup_{0\le s\le 1} |M(s,\tau_{j'+1})-M(s,\tau_{j'})|^2)^2]$$
$$\le C\ E(M_{11}^4)\ ,$$

where the last bound is given by Lemma 2.2.

For b_{nm} we can use the same arguments as before; we now analyze the last term.

(ii) By Ledoux' version of Davis inequality for two-parameter discrete martingales (see [6]) we obtain

$$c_{nm} \le C\ E[\sum_{i,j}\sum_{i'\in I_i}\sum_{j'\in J_j} M(\Delta_{i'j}^1)^2 M(\Delta_{ij'}^2)^2 M(\overline{\Delta}_{ii'j}^1)^2 M(\overline{\Delta}_{ijj'}^2)^2]^{1/2}$$
$$= C\ E[\sum_{i,j}(\sum_{i'\in I_i} M(\Delta_{i'j}^1)^2 M(\overline{\Delta}_{ii'j}^1)^2)(\sum_{j'\in J_j} M(\Delta_{ij'}^2)^2 M(\overline{\Delta}_{ijj'}^2)^2)]^{1/2}$$
$$\le C\ E[(\sup_{|z_1-z_2|\le|\rho^n|} |M_{z_1}-M_{z_2}|^2)^2 \sum_{i,j}(\sum_{i'\in I_i} M(\Delta_{i'j}^1)^2)(\sum_{j'\in J_j} M(\Delta_{ij'}^2)^2)]^{1/2}$$
$$\le C\ E[\sup_{|z_1-z_2|\le|\rho^n|} |M_{z_1}-M_{z_2}|^4 (\sum_i \sup_j \sum_{i'\in I_i} M(\Delta_{i'j}^1)^2) \sup_i \sum_{j,j'\in J_j} M(\Delta_{ij'}^2)^2]^{1/2}$$
$$\le C\{E[\sup_{|z_1-z_2|\le|\rho^n|} |M_{z_1}-M_{z_2}|^4]\}^{1/2}$$
$$\cdot\{E[\sum_i \sup_j \sum_{i'\in I_i} M(\Delta_{i'j}^1)^2]^2\}^{1/4} \cdot \{E[\sup_i |\sum_{j,j'\in J_j} M(\Delta_{ij'}^2)^2)^2]\}^{1/4}$$
$$\le C\{E[\sup_{|z_1-z_2|\le|\rho^n|} |M_{z_1}-M_{z_2}|^4]\}^{1/2}\{E(M_{11}^4)\}^{1/2}\ ,$$

where the last bound is obtained, as in (i), by Lemma 2.2, and Doob's and Burkholder inequalities.

$\square$

In [9] it is shown that the quadratic variation of the martingale $\tilde{M}$ can be obtained as the L^1-limit of the sums

$$\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^1)^2 M(\Delta_{ij}^2)^2\ .$$

Our Proposition 2.3 provides another way of looking at the measure $[\tilde{M}]$, and shows that, if M is of path independent variation then $[\tilde{M}] = [M] * [M]$.

Using the notation of Lemma 2.1, $[\tilde{M}]_z = \langle M_{x\cdot}\rangle_y * \langle M_{\cdot y}\rangle_x$. The martingale property and the techniques of martingale theory have played a basic role. Notice that it is not clear from the point of view of real analysis if $\langle M_{x\cdot}\rangle_y * \langle M_{\cdot y}\rangle_x$ exists, and defines a measure.

We end this Section by giving some examples. The aim is to suggest what kind of relations of absolute continuity may be expected for the different quadratic variations associated with a two-parameter continuous martingale $M \in m_c^4$.

Example 2.1. Fix $0<z_1<z_2<(s,t)$, and consider the strong martingale

$$M_{st} = \int_{R_{st}} 1_{[z_1,z_2]^c}(z)dw_z \quad ,$$

where $\{w_z,\ z \in R_+^2\}$ is a Brownian sheet. We obviously have $[M]([z_1,z_2])=0$. On the other hand, for a strong martingale $M_{st} = \int_{R_{st}} \phi(z)dw_z$ we obtain

$$[\tilde{M}]_{st} = \int_{R_{st}} \left(\int_{R_{uv}} \phi(u,\tau)^2 \phi(\sigma,v)^2 d\sigma d\tau \right) du dv \quad .$$

It follows that, if $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)$ and $\phi=1_{[z_1,z_2]^c}$,

$$[\tilde{M}]([z_1,z_2]) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} y_1 x_1 du dv = y_1 x_1 (x_2-x_1)(y_2-y_1) \quad .$$

Consequently $[\tilde{M}]$ is not absolutely continuous with respect to $[M]$.

Example 2.2. Let $\{m_u,\ u \in [0,1]\}$ be a continuous martingale, bounded in L^4, $m_o=0$, with respect to some filtration $\{\mathcal{G}_u,\ u \in [0,1]\}$ that satisfies the usual conditions. For each $(s,t) \in T$, define $M_{st}=m_{s \wedge t}$ and $\mathcal{F}_{st}=\mathcal{G}_{s \wedge t}$. It is easy to check that $\{\mathcal{F}_{st},\ (s,t) \in T\}$ is an increasing family of σ-fields satisfying properties (F1) to (F4) of [2], and $\{M_{st}\ ;\ \mathcal{F}_{st}, (s,t) \in T\}$ is a continuous two-parameter strong martingale.

Moreover, $\tilde{M} \equiv 0$ and consequently $[\tilde{M}] \equiv 0$. However $M \neq 0$ and the quadratic variation $[M]$ is a continuous measure which exists in the diagonal of T. In fact, let $\{\Delta_{ij}^n,\ i,j=0,\ldots,p_n\}$ be the dyadic partition of T, $\Delta_{ij}^n=(s_i^n,s_{i+1}^n]\times(t_j^n,t_{j+1}^n]$

$$[M]_{11} = L^1 - \lim_n \sum_{i,j} M(\Delta_{ij}^n)^2$$
$$= L^1 - \lim_n \sum_{i=j} M(\Delta_{ij}^n)^2 = L^1 - \lim_n \sum_i (m_{s_{i+1}^n} - m_{s_i^n})^2 = <m>_1 \quad .$$

This last example shows that $[M]$ is not absolutely continuous with respect to $[\tilde{M}]$ and neither with respect to $[M] * [M]$. The fact that the support of $[M]$ is a set of planar Lebesgue measure zero seems to be the reason for this unexpected result. However we will see in Section 4 that $[M] * [M]$ dominates $[M]$ if the support of the latter measure is not "highly concentrated".

3. One-Dimensional Results

In this Section we consider the measures induced by $<M_{s\cdot}>_t$ (resp. $<M_{\cdot t}>_s$) and $[M]_{st}$ on vertical (resp. horizontal) lines.

The relation $[M]_{sdt} \ll <M_{s\cdot}>_{dt}$, a.s. does not hold. In fact, consider the process defined by $B_{st}=B_s^2 \cdot B_t^2$, where $\{B_s^1,\ s\in[0,1]\}$ and $\{B_t^2,\ t\in[0,1]\}$ are two independent Brownian motions. Let $S=inf\{s>\frac{1}{2},\ B_s^1=0\}$; S is a stopping time, and for any $s>\frac{1}{2}$ $P\{\frac{1}{2}\le S<s\}>0$. Define a two-parameter continuous martingale by $M_{st}=B^1_{s\wedge S}\cdot B_t^2$. Then:

$$<M_{s\cdot}>_{dt}=(B^1)^2_{s\wedge S}\,dt, \quad [M]_{sdt}=<B^1>_{s\wedge S}\,dt \ .$$

Therefore, for any fixed $s>\frac{1}{2}$ and for any $A\in\mathcal{B}(\mathbb{R}_+)$, $|A|\neq 0$, $\int_A <M_{s\cdot}>_{dt}=0$ and $\int_A [M]_{sdt}\neq 0$, with positive probability.

The following lemma is an extension of a well known result in martingale theory (cf. [5]).

Lemma 3.1. Let $M\in m_c^2$. M and $[M]$ have the same rectangles of constancy, almost surely.

Proof. It is very similar as in the one-parameter case. For the sake of completeness we will give the details.

Since M is continuous it is enough to prove that for any fixed $z_1<z_2$, a.s.,

$$\{w,\ M(w)(\Delta)=0, \text{ for any rectangle } \Delta\subset[z_1,z_2]\}$$
$$=\{w,\ [M](w)(\Delta)=0, \text{ for any rectangle } \Delta\subset[z_1,z_2]\}.$$

One inclusion is trivial. In fact, if $M(\Delta)=0$ for every $\Delta\subset[z_1,z_2]$, then $[M]([z_1,z_2])=\lim_n \Sigma M(\Delta_{ij})^2=0$, where the sum extends over an increasing sequence of grids of $[z_1,z_2]$ whose norm tends to zero, and the limit can be taken a.s.

To obtain the other inclusion, define $D = \{z \geq z_1,\ z \in T,\ [M]_z = [M]_{z_1}\}$. D is a predictable stopping set, and for any $z \geq z_1$ we have

$$E\left(\int_{R_z} 1_D(\alpha)dM_\alpha\right)^2 = E\int_{R_z} 1_D(\alpha)d[M]_\alpha = 0 \ .$$

Hence $M(R_z \cap D) = 0$, a.s., and this finishes the proof. □

An easy consequence of this lemma is the absolute continuity of $<M_s.>_{dt}$ with respect to $[M]_{sdt}$ on the algebra generated by intervals.

Proposition 3.2. For any interval I for which $\int_I [M]_{sdt} = 0$, it holds that $\int_I <M_s.>_{dt} = 0$, a.s.

Proof. Assume that this property does not hold. That means, for a set $F \subset \Omega$, with $P(F) > 0$ it is possible to find intervals I_w, such that $\int_{I_w} [M]_{sdt}(w) = 0$, but $\int_{I_w} <M_s.>_{dt}(w) \neq 0$. Lemma 3.1 shows that this is not possible. Indeed, (we omit the dependence on w).

$$0 = \int_I [M]_{sdt} = [M]([0,s] \times I) \ ,$$

therefore $M(\Delta) = 0$, for any rectangle $\Delta \subset [0,s] \times I$, in particular $M(\Delta) = 0$ for any $\Delta = [0,s] \times [t_j^n, t_{j+1}^n]$, where $\{t_j^n,\ j = 1, \ldots, q_n\}$ is a partition of I. Hence $<M_s.>(I) = 0$. □

We do not know if $<M_s.>_{dt} \ll [M]_{sdt}$, a.s., in general. However we will see at the end of this Section that it is possible to prove this property for some classes of martingales.

The next Proposition shows that $<M_s.>_{dt}$ is weakly absolute continuous with respect to $[M]_{sdt}$ in the sense given by Definition 1.2.

Proposition 3.3. For any $s \in [0,1]$ and $f : [0,1] \to \mathbf{R}_+$ measurable and bounded function

$$P\left\{\int_0^1 f(t) <M_s.>_{dt} \neq 0, \quad \int_0^1 f(t)[M]_{sdt} = 0\right\} = 0 \ . \tag{5}$$

Proof. Fix $t\in[0,1]$; the process $\{N_s^t=<M_{s\cdot}>_t-[M]_{st},\ s\in[0,1]\}$ is a continuous martingale, and its quadratic variation satisfies

$$<N^t>_s \le [M]_{st} \sup_{0\le s'\le s} <M_{s'\cdot}>_1 . \qquad (6)$$

Indeed, N_s^t can be obtained as the L^1-limit of a sequence $\sum_{i,j} M(\Delta_{ij}^2)M(\Delta_{ij})$, where the sum is extended to the points $(s_i,t_j)\in\rho^n$, $(s_i,t_j)<(s,t)$. This gives the martingale property. In order to prove (6) we use Lemma 2.1 of [10]. Let $\rho^n=R_1^n\times R_2^n$, where R_1^n and R_2^n are partitions of $[0,1]$ determined by $0=s_o<s_1<\cdots<s_{p_n}<1$ and $0=t_o<t_1<\cdots<t_{q_n}<1$, respectively, whose norms tend to zero as $n\to\infty$. Then

$$<N^t>_s = P-\lim_{n\to\infty}\sum_i\Big(\sum_j M(\Delta_{ij})M(\Delta_{ij}^2)\Big)^2 ,$$

where the $\sum_i$ (resp. $\sum_j$) extends on indexes i (resp. j) such that $s_i<s$ (resp. $t_j<t$). Therefore

$$<N^t>_s \le P-\lim_{n\to\infty}\sum_i\Big(\sum_j M(\Delta_{ij})^2\Big)\Big(\sum_j M(\Delta_{ij}^2)^2\Big)\le P-\lim_{n\to\infty}\Big(\sum_{i,j}M(\Delta_{ij})^2\Big)\Big(\sup_i\sum_j M(\Delta_{ij}^2)^2\Big)$$
$$\le \sup_{0\le s'\le s} <M_{s'\cdot}>_1 [M]_{st} .$$

From this result it easily follows that, if $f:[0,1]\to\mathbf{R}_+$ is a step function, the process $\{N_s^f=\int_0^1 f(t)<M_{s\cdot}>_{dt}-\int_0^1 f(t)[M]_{sdt},\ s\in[0,1]\}$ is a continuous martingale, and

$$<N^f>_s \le \Big(\int_0^1 f(t)[M]_{sdt}\Big)\sup_{0\le s'\le s} <M_{s'\cdot}>_1 .$$

Since $\{\int_0^1 f(t)<M_{s\cdot}>_{dt}\ne 0,\ \int_0^1 f(t)[M]_{sdt}=0\}$ is included in the set $\{N_s^f\ne 0,\ <N^f>_s=0\}$, the property (5) holds if f is a step function.

Let $f:[0,1]\to\mathbf{R}_+$ be a measurable, bounded function, and consider an increasing sequence of positive step functions (f_n) converging to f. Denote by A the set $\{\int_0^1 f(t)<M_{s\cdot}>_{dt}\ne 0,\ \int_0^1 f(t)[M]_{sdt}=0\}$. On A, $\int_0^1 f_n(t)[M]_{sdt}=0$, for any $n\ge 0$, and consequently $\int_0^1 f_n(t)<M_{s\cdot}>_{dt}=0$, for any $n\ge 0$ a.s. Therefore $P(A)=0$.

$\square$

Let us now consider two-parameter martingales with respect to the following filtrations:

(a) $\{\mathcal{F}_z,\ z\in T\}$ generated by a Brownian sheet $\{w_z,\ z\in T\}$

(b) $\{\mathcal{F}_z, z\in T\}$ a product filtration $\mathcal{F}_z=\mathcal{F}_x^1 \vee \mathcal{F}_y^2$, $z=(x,y)$, where $\{\mathcal{F}_x^1,\ x\in[0,1]\}$ (resp. $\{\mathcal{F}_y^2,\ y\in[0,1]\}$) is generated by an n-dimensional (resp. m-dimensional) Brownian motion $\{B_s=(B_s^1,\ldots,B_s^n),\ s\in[0,1]\}$ (resp. $\{\hat{B}_t=(\hat{B}_t^1,\ldots,\hat{B}_t^m),\ t\in[0,1]\}$), B and $\hat{B}$ independent.

These processes will be called Brownian and bi-Brownian martingales, respectively.

Given two random finite measures μ and ν on $([0,1],\ \mathcal{B}([0,1]))$ we can define on $\mathcal{B}([0,1])\otimes\mathcal{F}$ two measures, $\bar{\mu}$ and $\bar{\nu}$, by

$$\bar{\mu}(F)=E\left(\int_0^1 1_F\,d\mu\right),\quad \bar{\nu}(F)=E\left(\int_0^1 1_F\,d\nu\right),$$

for any $F\in\mathcal{B}([0,1])\otimes\mathcal{F}$. Because of the right-continuity of the distribution functions of μ and ν, the property $\mu\ll\nu$, a.s., is equivalent to $\bar{\mu}\ll\bar{\nu}$ on $\mathcal{B}([0,1])\otimes\mathcal{F}$.

Denote by m_w^2 (resp. $m_{B\hat{B}}^2$) the class of two-parameter Brownian (resp. bi-Brownian) martingales, null on the axes and bounded in L^2.

Proposition 3.4. The property $<M_{s\cdot}>_{dt}\ll[M]_{sdt}$ a.s. holds for martingales in the classes m_w^2 and $m_{B\hat{B}}^2$.

Proof. We first recall the representation theorems.

(i) Wong-Zakai representation ([13]). Every $M\in m_w^2$ can be expressed as

$$M_{st}=\int_{R_{st}}\phi_z\,dw_z+\iint_{R_{st}\times R_{st}}\psi(z;z')dw_z\,dw_{z'}\ ,$$

where ϕ is a measurable and adapted process such that $E\int_{R_{z_0}}\phi_z^2dz<\infty$, for any $z_0\in R_+^2$, and ψ is a measurable and $\mathcal{F}_{z\vee z'}$ - adapted process, null except on the set $D=\{(z,z')\in\mathbf{R}_+^2,\ z=(x,y),\ z'=(x',y'),\ x\le x',\ y\ge y'\}$, such that $E\iint_{R_{z_0}\times R_{z_0}}\psi(z,z')^2dzdz'<\infty$, for any $z_0\in\mathbf{R}_+^2$.

(ii) Every $M\in m_{B\hat{B}}^2$ can be represented as

$$M_{st}=\sum_{i=1}^{n}\sum_{j=1}^{m}h_{ij}(x,y)dB_x^i d\hat{B}_y^j\ ,$$

where h_{ij} are measurable and adapted processes such that $E\left(\int_{R_{z_0}}h_{ij}^2(x,y)dxdy\right)<\infty$, for any

$z_o \in \mathbf{R}_+^2$ (see [3]).

Let $M \in m_w^2$. The quadratic variation of M is $[M]_{st} = \int_{R_{st}} g(u,v)\,du\,dv$, with

$$g(u,v) = \phi^2(u,v) + \int_{R_{uv}} \psi^2(x,v;u,y)\,dx\,dy \quad ,$$

and $<M_{s\cdot}>_t = \int_0^t h(s,v)\,dv$, where

$$h(s,v) = \int_0^s (\phi(u,v) + \int_{R_{sv}} \psi(u,v;z')\,dw_{z'})^2 du \quad .$$

Let F be a set of $\mathcal{B}([0,1]) \otimes \mathcal{F}$ such that $E(\int_0^1 1_F(w,t)[M]_{sdt})=0$; we have, using Fubini's theorem,

$$\int_0^1 (\int_{F_t} (\int_0^s g(u,t)\,du)\,dP)\,dt = 0 \quad ,$$

where F_t denotes the section of F through t. Therefore, for all t, a.e., such that $P(F_t)>0$, w - a.s. on F_t we have $\phi(u,t)=0$ and $\psi(x,t;u,y)=0$, for all $u \in [0,1]$, a.e. and for all $(x,y) \in R_{ut}$, a.e.

The conditions on ψ can be expressed in an alternative way, using again Fubini's theorem: For all t, a.e., such that $P(F_t)>0$, w - a.s. on F_t we have $\psi(x,t;u,y)=0$, for all $x \in [0,s]$ a.e. and for all $(u,y) \in [x,s] \times [0,t]$, a.e. Then,

$$\begin{aligned}\int_F dP <M_{s\cdot}>_{dt} &= \int_0^1 dt \int_{F_t} dP\, (\int_0^s (\phi(u,t) + \int_{R_{st}} \psi(u,t;z')\,dw_{z'})^2 du) \\ &= \int_0^1 dt \int_{F_t} dP\, (\int_0^s (\phi(u,t) + \int_u^s \int_0^t \psi(u,t;z')\,dw_{z'})^2 du \\ &= \int_0^1 dt \int_{F_t} dP\, (\int_0^s (\phi(u,t) + \int_x^s \int_0^t \psi(x,t;z')\,dw_{z'})^2 dx = 0 \quad ,\end{aligned}$$

due to the local property of stochastic integrals.

The proof for $M \in m_{BB}^{2}$ follows the same lines. If we restrict ourselves to the case $n=m=1$ we have

$$[M]_{sdt} = (\int_0^s h(u,t)^2 du)dt ,$$

$$<M_{s\cdot}>_{dt} = (\int_0^s h(u,t) dB_u)^2 dt .$$

The simple form of this measure makes computations easier than in the Brownian case. $\square$

4. Two-Dimensional Results

The main purpose of this Section is to analyze in what cases $[M]$ is dominated by $[\tilde{M}]$. Under some hypotheses on $[M]$ it is proved that the class of path independent variation martingales satisfy the property $[M] \ll [\tilde{M}]$, a.s. The measure $[M] * [M]$ introduced in Section 2 plays an important role. In several cases it dominates $[\tilde{M}]$, but we do not know if they are equivalent.

In Section 2 we have given an example of strong martingale for which $[M]$ is not absolutely continuous with respect to $[\tilde{M}]$ a.s. In this example $[M]$ has a special feature: It lives on a subset of T of zero planar Lebesgue measure. We conjecture that with some non-degeneracy hypothesis on $[M]$, it should be absolutely continuous with respect to $[\tilde{M}]$. The next proposition is a partial result in this direction.

Proposition 4.1. Assume that $[M]$ is absolutely continuous with respect to the product of its marginals, a.s., then $[M] \ll [M] * [M]$, a.s.

Proof. Denote by μ_i, $i=1,2$ the marginals corresponding to the measure induced by $[M]$, and let f be a version of $\dfrac{d[M]}{d\mu_1 \times d\mu_2}$. We have

$$([M] * [M])(z) = \int_{R_z} (\int_0^v f(u,\tau)\mu_2(d\tau))(\int_0^u f(\sigma,v)\mu_1(d\sigma))\mu_1(du)\mu_2(dv) .$$

Assume that $([M] * [M])(A) = 0$, for some $A \in \mathcal{B}(T)$, then $(\int_0^t f(s,v)\mu_2(dv))(\int_0^s f(u,t)\mu_1(du)) = 0$, for every $(s,t) \in A$, $\mu_1 \times \mu_2$ a.e. Define

$A_1 = \{(s,t) \in A, \int_0^s f(u,t)\mu_1(du) = 0\}$, $A_2 = \{(s,t) \in A, \int_0^t f(s,v)\mu_2(dv) = 0\}$,

$N = (A_1 \cup A_2)^c \cap A$.

Notice that $(\mu_1 \times \mu_2)(N)=0$, and if $(s,t) \in A_1$, for any $s' \le s$ such that $(s',t) \in A$ we have $(s',t) \in A_1$ (and also an analogue property for A_2). Using Fubini's theorem we obtain $[M](A_1)=[M](A_2)=0$, and consequently $[M](A)=0$.

□

The hypothesis on $[M]$ in the preceding Proposition is obviously satisfied if for any $A \in \mathcal{B}(T)$ such that $[M](A) \neq 0$, there exists a rectangle $R \subset A$ such that $[M](R) \neq 0$.

Corollary 4.2. Let $M \in m_c^4$ be a path independent martingale satisfying the hypothesis of Proposition 4.1, then $[M] \ll [\tilde{M}]$.

Proof. Use Proposition 2.3.

□

Notice that we cannot expect $[M] * [M] \ll [M]$, a.s. Example 2.1 provides a counterexample.

The second part of this Section gives a partial result on the absolute continuity of $[\tilde{M}]$ with respect to $[M] * [M]$.

Lemma 4.3. Let $f : T \to \mathbf{R}$ be a bounded function, and $F : T \to \mathbf{R}$ a continuous distribution function. We have

$$\lim_{n \to \infty} \Big| \sum_{(i,j) \in I_z^n} f(s_i,t_j)[F(\Delta_{ij}^1)F(\Delta_{ij}^2) - (F * F)(\Delta_{ij})] \Big| = 0 .$$

Proof. In order to simplify the notations we take $z=1$. Let $m \ge n$; given a point $(s_i,t_j) \in \rho^n$, we define $I_{ij}^m = \{(i',j'), (\sigma_{i'},\tau_{j'}) \in \rho^m \cap [(s_i,t_j),(s_{i+1},t_{j+1}))\}$. By Lemma 2.1. we have

$$\begin{aligned}
&\Big| \sum_{i,j} f(s_i,t_j)[F(\Delta_{ij}^1)F(\Delta_{ij}^2) - (F * F)(\Delta_{ij})] \Big| \\
&= \Big| \sum_{i,j} f(s_i,t_j)[F(\Delta_{ij}^1)F(\Delta_{ij}^2) - \lim_{m \to \infty} \sum_{(i',j') \in I_{ij}^m} F(\Delta_{i'j'}^1)F(\Delta_{i'j'}^2)] \Big| \\
&\le \sup_{m \ge n} \Big\{ \Big| \sum_{i,j} f_{ij}[(\sum_{i' \in I_i} F(\Delta_{i'j}^1))(\sum_{j' \in J_j} F(\Delta_{ij'}^2)) - \sum_{(i',j') \in I_{ij}^m} F(\Delta_{i'j'}^1)F(\Delta_{i'j'}^2)] \Big| \Big\} ,
\end{aligned} \tag{7}$$

where $f_{ij}=f(s_i,t_j)$, and we have used the notations of Proposition 2.3.

Define $\tilde{\Delta}_{iji'j'}^1$ and $\tilde{\Delta}_{iji'j'}^2$ by $\Delta_{i'j'}^1 = \Delta_{i'j}^1 \cup \tilde{\Delta}_{iji'j'}^1$ and $\Delta_{i'j'}^2 = \Delta_{ij'}^2 \cup \tilde{\Delta}_{iji'j'}^2$, respectively. Taking account of this decomposition we obtain that (7) is bounded by $\sup_{m \ge n} (\alpha_{mn} + \beta_{mn} + \gamma_{mn})$, where

$$\alpha_{mn} = |\sum_{i,j} f_{ij} \sum_{(i',j')\in I_{ij}^m} F(\tilde{\Delta}^2_{iji'j'}) F(\Delta^1_{i'j})| \ ,$$
$$\beta_{mn} = |\sum_{i,j} f_{ij} \sum_{(i',j')\in I_{ij}^m} F(\tilde{\Delta}^1_{iji'j'}) F(\Delta^2_{ij'})| \ ,$$
$$\gamma_{mn} = |\sum_{i,j} f_{ij} \sum_{(i',j')\in I_{ij}^m} F(\tilde{\Delta}^1_{iji'j'}) F(\tilde{\Delta}^2_{iji'j'})| \ .$$

We next prove that each one of these terms tends to zero when $n \to \infty$, uniformly in m

$$\alpha_{mn} \le (\sum_{i,j} \sup_{i'} \sum_{j'} F(\tilde{\Delta}^2_{iji'j'}) \sum_{i'} F(\Delta^1_{i'j})$$
$$\le C \sum_{ij} F(\Delta_{ij}) F(\Delta^1_{ij}) \le C \sup_{|z_2 - z_1| \le |\rho^n|} |F(z_2) - F(z_1)| F(z) \underset{n\to\infty}{\to} 0 \ .$$

In an analogue way, $\sup_{m\ge n} \beta_{mn} \underset{n\to\infty}{\to} 0$. Finally

$$\gamma_{mn} \le C \sum_{i,j} \sum_{j'} \sup_{i'} F(\tilde{\Delta}^2_{iji'j'}) \sum_{i'} F(\tilde{\Delta}^1_{iji'j'})$$
$$\le C \sum_{i,j} \sup_{|z_2 - z_1| \le |\rho^n|} |F(z_2) - F(z_1)| F(z) \underset{n\to\infty}{\to} 0 \ .$$

$\square$

Proposition 4.4. Assume that $<M_{s\cdot}>_{dt}$ and $<M_{\cdot t}>_{ds}$ are absolutely continuous, a.s., with respect to $[M]_{sdt}$ and $[M]_{dst}$, and that there exist versions of their Radon-Nykodim derivatives $\phi_1(s,t)$, $\phi_2(s,t)$ which are jointly continuous, a.s. Then $[\tilde{M}] \ll [M] * [M]$, a.s., and

$$[\tilde{M}]_z = \int_{R_z} \phi_1(x,y)\phi_2(x,y) d([M] * [M])(x,y) \ \text{,a.s.}$$

Proof. Take $z=(1,1)$. By Proposition 2.3 and the hypothesis on absolute continuity we have

$$[\tilde{M}]_{1,1} = \lim_n \sum_{i,j} (<M_{s_i\cdot}>_{t_{j+1}} - <M_{s_i\cdot}>_{t_j})(<M_{\cdot t_j}>_{s_{i+1}} - <M_{\cdot t_j}>_{s_i})$$
$$= \lim_n \sum_{i,j} (\int_{t_j}^{t_{j+1}} \phi_1(s_i, v)[M]_{s_i dv})(\int_{s_i}^{s_{i+1}} \phi_2(u, t_j)[M]_{du t_j}) \tag{8}$$
$$= \lim_n \sum_{i,j} \phi_1(s_i,t_j)\phi_2(s_i,t_j)[M](\Delta^1_{ij})[M](\Delta^2_{ij}) \ ,$$

where the last equality follows from the continuity of ϕ_1, ϕ_2 and $[M]$ in its two variables. Using Lemma 4.3, we obtain that (8) is equal to

$$\lim_n \sum_{i,j} \phi_1(s_i,t_j)\phi_2(s_i,t_j)([M] * [M])(\Delta_{ij})$$
$$= \int_{R_z} (\phi_1 \cdot \phi_2)(x,y) d([M] * [M])(x,y) \ .$$

$\square$

5. Application to Local Time

We have pointed out in the Introduction that one of the motivations of this work has been the problem of finding out what measure is the most "natural" for the purpose of defining local time for two-parameter continuous martingales. This Section is devoted to giving an answer to this question.

We start introducing some terminology. Let $\{x_t, t \in T\}$ be a real valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{T}$ be a σ-field on T, and τ a finite (possibly random) measure.

Following [4], a map $L : \Omega \times \mathbf{R} \times \mathcal{T} \to \mathbf{R}$ is called a *local time* for x with respect to τ if the following conditions hold:

(i) For each $A \in \mathcal{T}$, the function $(w, x) \to L(w, x, A)$ is $\mathcal{F} \otimes \mathcal{B}(\mathbf{R})$ measurable.

(ii) For every $(w, x) \in \Omega \times \mathbf{R}$, the function $A \to L(w, x, A)$ is a finite measure on $\mathcal{T}$.

(iii) For almost every w, we have

$$\int_R f(x) L(w, x, A) dx = \int_A f(x_s) \tau(ds) \; , \tag{9}$$

for each $f : \mathbf{R} \to \mathbf{R}$ bounded, Borel function, and every $A \in \mathcal{T}$.

In the following we will omit the dependence of L on w, for the sake of simplicity.

Lemma 5.1. For almost every w, we have

$$\int_{\mathbf{R} \times T} \phi(x, u) L(x, du) dx = \int_T \phi(x_u, u) \tau(du) \; , \tag{10}$$

for any ϕ: $\mathbf{R} \times T \to \mathbf{R}$ measurable and bounded function.

Proof. Let $\phi(x, u) = f(x) 1_A(u)$, where $f : \mathbf{R} \to \mathbf{R}$ is measurable and bounded, and $A \in \mathcal{T}$; then

$$\int_T f(x_u) 1_A(u) \tau(du) = \int_{\mathbf{R}} f(x) L(x, A) dx$$
$$= \int_{R \times T} f(x) 1_A(u) L(x, du) dx \; .$$

The general case follows by a monotone class argument. □

Remarks.

(1) The equality (10) also holds for ϕ: $\mathbf{R} \times T \to \mathbf{R}$ measurable and positive.

(2) The property (9) means that the distribution function of $L(x,\cdot)$ is the density, with respect to the Lebesgue measure of the τ-measure of sojourn time of the process x on Borel sets.

The following Proposition is an easy consequence of the definition of local time.

Lemma 5.2. Let τ_1,τ_2 be two finite measures on the parameter space $(T,\mathcal{T})$. Assume that the local time of x with respect to τ_2 exists, and denote it by L_2, then we have:

$\tau_1 \ll \tau_2$ a.s., and $\dfrac{d\tau_1}{d\tau_2}(u) = \phi(u)$ a.e. if and only if, for any x a.e. (with respect to the Lebesgue measure on $\mathbf{R}$) there exist the local time of x with respect to τ_1, L_1, and

$$L_1(x,A) = \int_A \phi(u) L_2(x,du) \quad ,\text{a.s.} \tag{11}$$

Proof. By Lemma 5.1 we have

$$\int_T \phi(u)\tau_2(du) = \int_{\mathbf{R}\times T} \phi(u) L_2(x,du)dx \quad ,\text{a.s.}$$

In order to see that (11) defines the local time of x with respect to τ_1, take $f : \mathbf{R} \rightarrow \mathbf{R}$ measurable and bounded. Then

$$\int_{\mathbf{R}} f(x) L_1(x,A)dx = \int_{\mathbf{R}\times A} f(x)\phi(u) L_2(x,du)dx$$
$$= \int_A f(x_u)\phi(u)\tau_2(du) \quad ,\text{by Lemma 5.1}$$
$$= \int_A f(x_u)\tau_1(du) \quad .\text{a.s.}$$

Reciprocally,

$$\tau_1(A) = \int_R L_1(x,A)dx = \int_{R\times A} \phi(u) L_2(x,du)dx = \int_A \phi(u)\tau_2(du) \quad ,\text{a.s.}$$

and therefore $\tau_1 \ll \tau_2$, a.s.

$\square$

For any martingale $M \in m_c^4$ there are two non-trivial measures, $[M]$ and $[\tilde{M}]$, associated in a natural way (see e.g. [2], [3], [8]). Using an Ito's formula for two-parameter continuous martingales, Nualart has proved the existence of a local time for M with respect to $[\tilde{M}]$. On the other hand there exist several results on the existence of a local time for M with respect to $[M]$ (see e.g. [12], [1] for the Brownian sheet, and [11] for a certain class of martingales).

Example 2.2. shows the existence of a non-zero continuous martingale such that $\tilde{M}\equiv 0$. In view of this example it seems that $[\tilde{M}]$ may not be a "good" measure in order to describe the time expended by M on a certain set. Looking at example 2.1., an analogue conclusion can be obtained for the measure $[M]$.

These considerations lead us to propose $[M]+[\tilde{M}]$ as a natural measure to define the local time of M. We will give the precise definition, and discuss the different role of $[M]$ and $[\tilde{M}]$.

Theorem 2.1 of [9] (see also [3]) establishes the following Ito formula: If $f:\mathbf{R}\to\mathbf{R}$ is a C^4-class function, and $M\in m_c^4$, then for any $(s,t)\in\mathbf{R}_+^2$ we have

$$\begin{aligned} f(M_{st})-f(0) &= \int_{R_{st}} f'(M_z)dM_z + \int_{R_{st}} f''(M_z)d\tilde{M}_z + \frac{1}{2}\int_0^s f''(M_{xt})d<M_{\cdot t}>_x \\ &+ \frac{1}{2}\int_0^t f''(M_{sy})d<M_{s\cdot}>_y - \frac{1}{2}\int_{R_{st}} f''(M_z)d[M]_z - \int_{R_{st}} f'''(M_z)d[M,\tilde{M}]_z \\ &- \frac{1}{4}\int_{R_{st}} f^{IV}(M_z)d[\tilde{M}]_z \ . \end{aligned} \tag{12}$$

By means of this formula the existence of a process $\{L_1(x,s,t),\ x\in\mathbf{R},\ (s,t)\in\mathbf{R}_+^2\}$ can be proved, such that it is jointly continuous in (x,s,t), increasing in the sense of the measure, and for almost every w

$$\int_{R_{st}} f(M_z)d[\tilde{M}]_z = \int_{\mathbf{R}} L_1(x,s,t)\, f(x)dx \ , \tag{13}$$

for all bounded and Borel functions f, and every $(s,t)\in\mathbf{R}_+^2$. (See theorem 3.2 of [9]).

The idea of the proof is the same as in the one-parameter case: Apply (12) to a function $g_{\varepsilon x}$ of class C^4 and compact support, such that it is an approximation of $\frac{1}{\varepsilon}1_{[x,x+\varepsilon]}(\cdot)$. It can be checked that

$$L_1(x,s,t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{R_{st}} 1_{\{x\le M_z\le x+\varepsilon\}}\, d[\tilde{M}]_z$$

exists, in the sense of the convergence in probability, and that (13) is satisfied.

In the terminology introduced at the beginning of the Section L_1 is the distribution function of the local time of M with respect to $[\tilde{M}]$.

Notice that, on the set $\{w,\ [\tilde{M}]_{st}(w)=0\}$, $L_1(x,s,t)$ should be zero. Since there exist martingales such that $[\tilde{M}]\equiv 0$, but $[M]\neq 0$, this shows that the time spent by M on a certain set may be not detected by the measure $[\tilde{M}]$, although it can be detected by $[M]$. Therefore, it is important to have a local time with respect to $[M]$, say L_2, and a "good" combination of L_1 and L_2 will provide a reasonable measure of the sojourn time of the martingale on Borel sets.

Unfortunately, we do not have a general result on the existence of L_2 for any $M \in m_c^4$, however we know that it exists and has favorable properties for several classes of martingales (cf. references given before).

Fix $(s,t) \in T$. On the set $\{w, [\tilde{M}]_{st}(w)=0\}$, $L_2(x,u,v),(u,v) \in R_{st}$ can be obtained from Ito's formula using the same approach of [9]. Indeed, using Lemma 3.1 and the local property of the stochastic integral, we get from (12)

$$[f(M_{st})-f(0)]1_{\{[\tilde{M}]_{st}=0\}} = \{\int_{R_{st}} f'(M_z)dM_z + \frac{1}{2}\int_0^s f''(M_{ut})d<M_{\cdot t}>_u$$
$$+\frac{1}{2}\int_0^t f''(M_{sy})d<M_{s\cdot}>_y - \frac{1}{2}\int_{R_{st}} f''(M_z)d[M]_z\} \cdot 1_{\{[\tilde{M}]_{st}=0\}} .$$

Fix $\varepsilon>0$ and $x \in \mathbf{R}$ and consider a C^2-function with compact support such that $g''_{\varepsilon x}$ is an approximation of $\frac{1}{\varepsilon}\, 1_{[x,x+\varepsilon]}(\cdot)$. Then it can be shown that, in the sense of the convergence in probability,

$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\int_{R_{st}} 1_{\{x\le M_z \le x+\varepsilon\}}d[M]_z)\cdot 1_{\{[\tilde{M}]_{st}=0\}} = L_2'(x,s,t)$ exists and

$$L_2'(x,s,t) = \{-2(M_{st}-x)^+ + 2(-x)^+ + 2\int_{R_{st}} 1_{\{M_z>x\}}dM_z$$
$$+\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\int_0^s 1_{\{x\le M_{ut}\le x+\varepsilon\}}d<M_{\cdot t}>_u + \int_0^t 1_{\{x\in M_{sy}\le x+\varepsilon\}}d<M_{s\cdot}>_y]\} \cdot 1_{\{[\tilde{M}]_{st}=0\}} \tag{14}$$
$$= \{-2(M_{st}-x)^+ + 2(-x)^+ + 2\int_{R_{st}} 1_{\{M_z>x\}}dM_z + L^{(1)}(x,s,t) + L^{(2)}(y,s,t)\}\cdot 1_{\{[\tilde{M}]_{st}=0\}} ,$$

where $L^{(1)}(x,\cdot,t)$, $L^{(2)}(y,s,\cdot)$ are the local times of the one-parameter martingales $M_{\cdot t}=\{M_{st},\ s\ge 0\}$ and $M_{s\cdot}=\{M_{st},\ t\ge 0\}$, respectively.

Moreover, L_2' satisfies the "density of occupation" formula

$$(\int_{R_{st}} f(M_z)d[M]_z)\cdot 1_{\{[\tilde{M}]_{st}=0\}} = (\int_R L_2'(x,s,t)f(x)dx)\cdot 1_{\{[\tilde{M}]_{st}=0\}} \ ,a.s.$$

Definition 5.3. The local time of a martingale $M \in m_c^4$ is the process $\{L(x,s,t),\ x \in \mathbf{R},\ (s,t) \in T\}$ given by

$$L(x,s,t) = [L_1(x,s,t) + L_2(x,s,t)] \cdot 1_{\{[\tilde{M}]_{**}>0\}} + L_2'(x,s,t) \ .$$

Notice that L is a local time with respect to the measure $\tau=[M]+[\tilde{M}]$.

To summarize:

(1) If $[M] \ll [\tilde{M}]$, a.s., there exists the local time L with respect to $[M]+[\tilde{M}]$, and it can be expressed in terms of L_1 (cf. Lemma 5.2).

(2) Assume $[M] \not\ll [\tilde{M}]$, a.s. On the set $\{[\tilde{M}]_{st}=0\}$, L always exists and is given by (14). On the set $\{[\tilde{M}]_{st}>0\}$, we know that L exists for a class of martingales on m_c^4 (see [11]), but we do not know about its existence in general.

We end this Section with an application of local time to an example of two-parameter continuous martingale for which the measures $[M]$, $[\tilde{M}]$ and $[M] * [M]$ are equivalent, a.s.

Example. Let $m=\{m_s,\ s \geq 0\}$, $n=\{n_t,\ t \geq 0\}$ be two independent continuous martingales, bounded in L^2, with respect to some filtrations $\{\mathcal{F}_s^1,\ s \geq 0\}$, $\{\mathcal{F}_t^2,\ t \geq 0\}$ respectively. Consider the martingale $M=\{M_{st}=m_s \cdot n_t,\ s,t \geq 0\}$ with respect to the product filtration $\mathcal{F}_{st}=\mathcal{F}_s^1 \vee \mathcal{F}_t^2$. Denote by $L^{(1)}(x,s)$, $L^{(2)}(y,t)$ the local times of m and n with respect to their respective quadratic variations $\langle m \rangle, \langle n \rangle$. We have $[M] \approx [\tilde{M}]$, a.s.

Indeed, for any $A \in \mathcal{B}(R_+^2)$,

$$\begin{aligned}
[\tilde{M}](A) &= \int_{R_+^2} 1_A(s,t) m_s^2 n_t^2 d\langle m \rangle_s d\langle n \rangle_t \\
&= \int_{R_+} \left(\int_{R_+} 1_A(s,t) m_s^2 d\langle m \rangle_s \right) n_t^2 d\langle n \rangle_t \\
&= \int_{R_+} \left(\int_{R \times R_+} 1_A(s,t) x^2 L^{(1)}(x,ds) dx \right) n_t^2 d\langle n \rangle_t \ , \ by \ \ (10) \\
&= \int_{R^2} \int_{R_+^2} 1_A(s,t) x^2 y^2 L^{(1)}(x,ds) L^{(2)}(y,dt) dx dy \ .
\end{aligned}$$

By analogue computations

$$[M](A) = \int_{R^2} \int_{R^2} 1_A(s,t) L^{(1)}(x,ds) L^{(2)}(y,dt)\,dx\,dy \quad ,$$

and consequently the equivalence between $[M]$ and $[\tilde{M}]$. The equivalence between $[\tilde{M}]$ and $[M] * [M]$ is immediate.

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8