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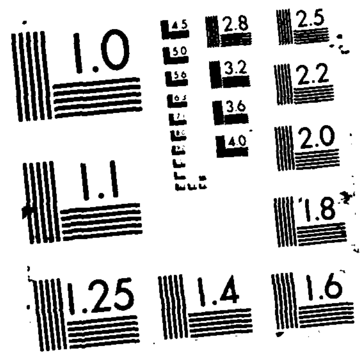
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The equations for the spatial and temporal modulations of the phase shift for slowly varying strongly nonlinear oscillators and dispersive waves have been determined for the first time. The effects of dissipative perturbations have been investigated for nonlinear oscillatory solutions of ordinary and partial differential equations (described by Klein-Gordon and Korteweg-de Vries type equations). The phase shift equations were derived using the method of multiple scales by evaluating the small perturbation to the exact action equation, a somewhat simpler technique than the usual elimination of secular terms at an even higher order in the asymptotic expansion. It has been shown that, for dissipative perturbations, the frequency and action equations are valid to higher order and that their variations are only due to perturbations in the wave number and the averaged amplitude parameters. For second-order ordinary differential equations, the phase shift is determined from initial conditions in straight-forward manner since it was shown that there exists a new adiabatic invariant.

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Variation of Wave Action:
Modulations of the Phase Shift
for Strongly Nonlinear Dispersive Waves
with Weak Dissipation

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Abstract

Strongly nonlinear dispersive waves described by a general Klein-Gordon equation with slowly varying coefficients and a dissipative perturbation are considered. The method of multiple scales shows that the equation for wave action and the dispersion relation (based on the averaged energy) are valid to a higher order than anticipated. Thus, higher order terms are calculated from the first variation of wave action. The spatial and temporal slow modulations of the phase shift are shown to be governed by a new equation, which is universal for small but otherwise arbitrary dissipation. This result extends to nonlinear partial differential equations the quite recent work by the authors on new adiabatic invariants for nonlinear oscillations governed by ordinary differential equations.

1. Introduction.

Oscillatory solutions of strongly nonlinear dispersive waves are quite common, perhaps the most well-known being the cnoidal waves for the Korteweg-deVries equation. In order to understand the effects of a slowly varying medium, Luke [1] in 1966 utilized the method of multiple scales to analyze a model nonlinear problem, the nonlinear Klein-Gordon equation. This extended Kuzmak's work [2] on oscillatory solutions of nonlinear ordinary differential equations to the case of nonlinear partial differential equations. In this way, the amplitude-dependent dispersion relation could be used to show that the slow time and spatial evolution of the amplitude of strongly nonlinear dispersive waves was governed by conservation of wave action, the appropriate generalization of the adiabatic invariant of nonlinear ordinary differential equations. With a small perturbation representing damping, Whitham [3] showed how wave action was dissipated.

Although the modulated phase shift for a nonlinear wave is part of the leading order long time slowly varying solution, it has eluded previous efforts until now. For unperturbed non-dissipative systems, Luke [1] correctly observed that the phase shift satisfied a second-order equation, one solution of which is a constant. However, the equation was not stated, which apparently has led to some common misconceptions concerning the phase shift. The same non-dissipative situation was reexamined by Dobrokhotov and Maslov [4]. They introduced some imaginative ideas that enabled the perturbation method to be carried out to the higher order necessary for the calculation of the phase shift. Unfortunately, their technique is not as easy to implement as ours, which we believe is the reason they incorrectly omit a term in their analysis. For the case of strongly nonlinear ordinary differential equations with small arbitrary damping, we (Bourland and Haberman [5]) quite recently obtained the second-order differential equation for the modulated phase shift. We showed that the phase shift $\phi(T)$ is determined from a new and unexpected adiabatic invariant of universal form, independent of the type of small (but arbitrary) damping; the small frequency modulation $d\phi/dT$ was shown to be a constant multiple of the derivative (with respect to energy) of the amplitude-dependent frequency of the nonlinear oscillator.

In §2, we apply the method of multiple scales to the nonlinear Klein-Gordon equation with slowly varying coefficients and a small arbitrary perturbation. There we briefly rederive the well-known equation for the dissipation of wave action:

$$\frac{\partial}{\partial T}(-\omega J) - c^2(\underset{\sim}{X},T)\nabla\cdot(\underset{\sim}{k}J) = -R \quad , \tag{1.1}$$

where $\underset{\sim}{k}$ is the wave number vector, ω the frequency, $-\omega J$ the wave action, $-c^2 J\underset{\sim}{k}$ the flux of wave action, and R the dissipation of wave action. In §3, we begin to analyze the perturbation expansion to a higher order. If the leading order perturbation is dissipative, then we show that wave action satisfies (1.1) to at least one higher order than would have been anticipated. Thus, we show that the equation needed to describe the modulations of the phase shift $\phi(\underset{\sim}{X},T)$ may be obtained in a simpler way, by just considering the first variations of the well-known equation for the wave action. We conjecture that this is a general principle.

We obtain the phase shift in §4 by eliminating secular terms at a higher order in our perturbation expansion. Here, we show that the leading order perturbation u_1 of the leading order solution u_0 satisfies

$$u_{1_{even}} = \hat{D}u_0 \quad , \tag{1.2}$$

where in §4 we obtain $\hat{D}$ and show it to be a linear first-order differential operator in the slow parameters, energy and wave number vector. We choose $\hat{D}$ in the appropriate way so that it is easy to derive that the partial differential equation for the modulated phase shift $\phi(\underset{\sim}{X},T)$ is

$$\hat{D}_T(-\omega J) - c^2(\underset{\sim}{X},T)\hat{D}_\nabla \cdot (\underset{\sim}{k} J) = 0 \quad . \tag{1.3}$$

Later (in §6) we show that $\hat{D}$ is the Taylor series operator in the parameters E and $\underset{\sim}{k}$, so that

$$\hat{D} = E_1 \frac{\partial}{\partial E} + \underset{\sim}{k}_1 \cdot \nabla_k \quad , \tag{1.4}$$

where $\underset{\sim}{k}_1 = \nabla\phi$ is the perturbation of the wave number and E_1 is the average of the perturbation of the energy. In (1.3) $\hat{D}_T$ and $\hat{D}_\nabla$ are partial derivatives of the operator $\hat{D}$. (In one-spatial dimension $\hat{D}_\nabla = \hat{D}_X$, while in higher spatial dimensions $\hat{D}_\nabla$ is defined in §4.) It is interesting to note that once $\hat{D}$ is determined, (1.3) can be obtained from the wave action equation (1.1) by replacing $\frac{\partial}{\partial T}$ by $\hat{D}_T$, $\nabla\cdot$ by $\hat{D}_\nabla\cdot$, and omitting the dissipation. Equation (1.3) is a partial differential equation since the coefficients for $\hat{D}$ depend on ϕ_T and $\nabla\phi$.

In §5 we obtain equivalent results for the Klein-Gordon equation by elementary (but innovative) considerations of the first variation of wave action (1.1) with respect to perturbations of the wave number and averaged energy. We show that the perturbation of the frequency is accurately determined from a higher-order dispersion relation which includes the averaged perturbed energy. From these ideas, we obtain

$$E_{1_T} \frac{\partial}{\partial E}(-\omega J) + \nabla\phi_T \cdot \nabla_k(-\omega J) - c^2 \nabla E_1 \cdot \frac{\partial}{\partial E}(\underset{\sim}{k} J) - c^2 J \nabla^2 \phi = 0 \quad , \tag{1.5}$$

where $E_1 = (-\phi_T - \nabla_k \omega \cdot \nabla\phi)/\omega_E$ is the average of the perturbation of the energy. Equation (1.5) is shown to be equivalent to (1.3). Its form is universal, since it is independent of the dissipation. We conjecture that (1.5) is a new general principle for the determination of the modulations of the phase shift. In the case of ordinary differential equations, our [5] recently proposed adiabatic invariant follows directly from (1.5), and §5 dramatically simplifies its interpretation. Stability of these nonlinear periodic waves are discussed in §7.

2. Nonlinear Dispersive Waves.

In this section, we will derive in the standard way the equation for wave action for the perturbed nonlinear Klein-Gordon equation with slowly varying coefficients

$$u_{tt} - c^2(\underset{\sim}{X},T)\nabla_X^2 u + V_u(u,\underset{\sim}{X},T) + \varepsilon h(u,u_t,\nabla u,\underset{\sim}{X},T) + \varepsilon^2 h_1(u,u_t,\nabla u,\underset{\sim}{X},T) = 0 \quad , \tag{2.1}$$

where $V(u,\underset{\sim}{X},T)$ is a nonlinear potential depending on the slow-time scale $T = \varepsilon t$ and slow spatial scales $\underset{\sim}{X} = \varepsilon \underset{\sim}{x}$. The notation $\nabla_X^2 \equiv \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is being used. We assume the potential admits oscillatory solutions of (2.1) (for all fixed $\underset{\sim}{X}$, T) on the fast time scale t without the perturbation (i.e., when $h = h_1 = 0$). Unlike Luke [1] and Dobrokhotov and Maslov [4], we allow an arbitrary slowly varying small $O(\varepsilon)$ perturbation, εh. To show the effects of an even smaller perturbation, we include the $\varepsilon^2 h_1$ term. For the early parts of this paper, we make no further assumptions concerning these perturbations. However, the most interesting conclusions (that we will reach in later sections) occur when εh is a dissipative perturbation, so that h is odd in the combined arguments u_t and ∇u [i.e., $h(u, -u_t, -\nabla u, \underset{\sim}{X}, T) = - h(u, u_t, \nabla u, \underset{\sim}{X}, T)$] and $\varepsilon^2 h_1$ is non-dissipative (even in the combined arguments).

We use the method of multiple scales [6] with the fast scale ψ and the slow scales T and $\underset{\sim}{X}$:

$$\psi = \frac{\theta(\underset{\sim}{X},T)}{\varepsilon} + \phi(\underset{\sim}{X},T)$$

$$T = \varepsilon t$$

$$\underset{\sim}{X} = \varepsilon \underset{\sim}{x} \quad ,$$

and follow the procedure we [5] used for ordinary differential equations. This form of the method of multiple scales is particularly suited for the eventual first determination of the phase shift $\phi(\underset{\sim}{X},T)$. For example,

$$u_t = (\theta_T + \varepsilon\phi_T)u_\psi + \varepsilon u_T \quad ,$$

so that (2.1) becomes exactly

$$\begin{aligned}(\theta_T + \varepsilon\phi_T)^2 u_{\psi\psi} &+ \varepsilon[(\theta_{TT} + \varepsilon\phi_{TT})u_\psi + 2(\theta_T + \varepsilon\phi_T)u_{\psi T}] \\ &- c^2(\nabla\theta + \varepsilon\nabla\phi)^2 u_{\psi\psi} - \varepsilon c^2[(\nabla^2\theta + \varepsilon\nabla^2\phi)u_\psi + 2(\nabla\theta + \varepsilon\nabla\phi)\cdot\nabla u_\psi] \\ &+ \varepsilon^2 u_{TT} - \varepsilon^2 c^2 \nabla^2 u + V_u(u,\underset{\sim}{X},T) + \varepsilon h + \varepsilon^2 h_1 = 0 \quad , \qquad (2.2)\end{aligned}$$

where from here on $\nabla \equiv \sum_{i=1}^{N} \hat{e}_i \frac{\partial}{\partial X_i}$ refers to differentiation of a quantity with respect to the slow spatial variables. We now introduce the perturbation expansion

$$u(\psi,\underset{\sim}{X},T) = u_0(\psi,\underset{\sim}{X},T) + \varepsilon u_1(\psi,\underset{\sim}{X},T) + \varepsilon^2 u_2(\psi,\underset{\sim}{X},T) + \ldots$$

It is convenient to note that

$$V_u(u,\underset{\sim}{X},T) = V_u(u_0,\underset{\sim}{X},T) + (\varepsilon u_1 + \varepsilon^2 u_2)\, V_{uu}(u_0,\underset{\sim}{X},T) + \varepsilon^2 \frac{u_1^2}{2} V_{uuu}(u_0,\underset{\sim}{X},T) + O(\varepsilon^3)$$

$$\begin{aligned} h(u,u_t,\nabla_x u,\underset{\sim}{X},T) &= h(u,\theta_T u_\psi + \varepsilon\phi_T u_\psi + \varepsilon u_T\ ,\ \nabla\theta u_\psi + \varepsilon\nabla\phi u_\psi + \varepsilon\nabla u\ ,\ \underset{\sim}{X},T) \\ &= h(u_0,\theta_T u_{0_\psi},\nabla\theta u_{0_\psi},\underset{\sim}{X},T) \\ &\quad + \varepsilon[u_1 h_u + (\theta_T u_{1_\psi} + \phi_T u_{0_\psi} + u_{0_T})h_v + (\nabla\theta u_{1_\psi} + \nabla\phi u_{0_\psi} + \nabla u_0)\cdot \underset{\sim}{h}_\nabla] \\ &\quad + O(\varepsilon^2)\ , \end{aligned}$$

where $h_u \equiv h_u(u_0,\theta_T u_{0_\psi},\ u_{0_\psi}\nabla\theta,\underset{\sim}{X},T)$, $h_v \equiv h_{u_t}(u_0,\theta_T u_{0_\psi},u_{0_\psi}\nabla\theta,\underset{\sim}{X},T)$,

and $\underset{\sim}{h}_\nabla \equiv \sum_{i=1}^{N} \frac{\partial}{\partial(\partial u/\partial x_i)}\, \hat{e}_i\, h(u_0,\theta_T u_{0_\psi},u_{0_\psi}\nabla\theta,\underset{\sim}{X},T)$.

Substituting the perturbation expansion into (2.2) yields, to leading order, the nonlinear oscillator equation:

$$[\theta_T^2 - c^2(\underset{\sim}{X},T)(\nabla\theta)^2]u_{0_{\psi\psi}} + V_u(u_0,\underset{\sim}{X},T) = 0 \quad . \tag{2.3}$$

The higher order terms become

$$L(u_i)=R_i \quad \text{for} \quad i = 1,2, \ldots \quad , \tag{2.4}$$

where $L \equiv [\theta_T^2 - c^2(\underset{\sim}{X},T)(\nabla\theta)^2] \frac{\partial^2}{\partial\psi^2} + V_{uu}(u_0,\underset{\sim}{X},T)$ is the linearization of the nonlinear operator in (2.3). It is not difficult to obtain the right-hand sides. For example,

$$\begin{aligned} R_1 = &- 2[\phi_T\theta_T - c^2(\underset{\sim}{X},T)\nabla\theta\cdot\nabla\phi]u_{0_{\psi\psi}} \\ &- 2[\theta_T \frac{\partial}{\partial T} - c^2(\underset{\sim}{X},T)\nabla\theta\cdot\nabla]u_{0_\psi} - [\theta_{TT} - c^2(\underset{\sim}{X},T)\nabla^2\theta]u_{0_\psi} \\ &- h(u_0,\theta_T u_{0_\psi},u_{0_\psi}\nabla\theta,\underset{\sim}{X},T) \end{aligned} \tag{2.5}$$

$$\begin{aligned} R_2 = &- 2[\phi_T\theta_T - c^2(\underset{\sim}{X},T)\nabla\theta\cdot\nabla\phi]u_{1_{\psi\psi}} - 2[\theta_T \frac{\partial}{\partial T} - c^2(\underset{\sim}{X},T)\nabla\theta\cdot\nabla]u_{1_\psi} \\ &- [\theta_{TT} - c^2(\underset{\sim}{X},T)\nabla^2\theta]u_{1_\psi} - [(\phi_T)^2 - c^2(\underset{\sim}{X},T)(\nabla\phi)^2]u_{0_{\psi\psi}} \\ &- 2[\phi_T \frac{\partial}{\partial T} - c^2(\underset{\sim}{X},T)\nabla\phi\cdot\nabla]u_{0_\psi} - [\phi_{TT} - c^2(\underset{\sim}{X},T)\nabla^2\phi]u_{0_\psi} \\ &\quad - \frac{1}{2} V_{uuu}(u_0,\underset{\sim}{X},T)u_1^2 - [\frac{\partial^2}{\partial T^2} - c^2(\underset{\sim}{X},T)\nabla^2]u_0 \\ &- h_1 - u_1 h_u - [\theta_T u_{1_\psi} + \phi_T u_{0_\psi} + u_{0_T}]h_v - [\nabla\theta u_{1_\psi} + \nabla\phi u_{0_\psi} + \nabla u_0]\cdot \underset{\sim}{h}_\nabla \quad , \end{aligned} \tag{2.6}$$

where h_1 is evaluated at the leading order.

The leading order equation (2.3) is a nonlinear ordinary differential equation in the fast variable ψ. Multiplying by u_{0_ψ} and integrating once yields the conservation of energy equation,

$$\frac{1}{2}\,[\theta_T^2 - c^2(\nabla\theta)^2](u_{0_\psi})^2 + V(u_0,\underset{\sim}{X},T) = E(\underset{\sim}{X},T) \quad , \tag{2.7}$$

where $E(\underset{\sim}{X},T)$ satisfies a well-known equation, which we will rederive. Standard phase plane analysis shows that the solution $u_0(\psi,\underset{\sim}{X},T)$ is periodic in ψ for appropriate potentials $V(u_0,\underset{\sim}{X},T)$, oscillating between $u_{0_{min}}$ and $u_{0_{max}}$ with $V(u_{0_{min}},\underset{\sim}{X},T) = E$ and a similar expression involving $u_{0_{max}}$. As Luke [1] showed and we [5] repeated, $u_0(\psi,\underset{\sim}{X},T)$ can be defined as an even function of ψ with $u_0(0,\underset{\sim}{X},T) = u_{0_{min}}$ or $u_{0_{max}}$ and $\frac{\partial u_0}{\partial\psi}(0,\underset{\sim}{X},T) = 0$. The period of the oscillation in the fast variable is constant [1], and we normalize it to 1 (although 2π is often used) which gives

$$(\theta_T)^2 - c^2(\underset{\sim}{X},T)(\nabla\theta)^2 = \left\{ 2\int_{u_{0_{min}}(\underset{\sim}{X},T)}^{u_{0_{max}}(\underset{\sim}{X},T)} \frac{1}{\sqrt{2}}\,[E(\underset{\sim}{X},T) - V(u_0,\underset{\sim}{X},T)]^{-\frac{1}{2}}\,du_0 \right\}^{-2} . \tag{2.8}$$

In this situation, the notation $\underset{\sim}{k} \equiv \nabla\theta$ for the wave number vector and $\omega \equiv -\theta_T$ for the frequency is appropriate.* Equation (2.8) may be regarded as the dispersion relation, $\omega = \omega(\underset{\sim}{k}(\underset{\sim}{X},T),E(\underset{\sim}{X},T),\underset{\sim}{X},T)$, for the propagation of nonlinear waves in a nonuniform media as described by (2.1).

Some well-known results on the linearized operator are needed to analyze the higher order equations, $L(y_i) = R_i$. Taking the derivative of (2.3) with respect to ψ yields $L(u_{0_\psi}) = 0$, showing that u_{0_ψ} is a homogeneous solution. Since $u_0(\psi,\underset{\sim}{X},T)$ is an even periodic function, u_{0_ψ} is an odd periodic homogeneous solution; usually it is the only periodic solution [1]. In order for the solutions of (2.4) to be periodic, the Fredholm alternative implies that

*In comparing this to the case of ordinary differential equations, note that $\omega_{ode} = -\omega_{pde}$.

$$\int_0^1 u_{0_\psi} R_1 \, d\psi = 0 \quad , \tag{2.9}$$

equivalent to the usual secularity condition.

For $i = 1$, using (2.5), (2.9) becomes

$$\begin{aligned}
&-2[\phi_T\theta_T - c^2(\underset{\sim}{X},T)\nabla\theta\cdot\nabla\phi] \int_0^1 u_{0_{\psi\psi}} u_{0_\psi} \, d\psi \\
&-2[\theta_T \frac{\partial}{\partial T} - c^2(\underset{\sim}{X},T)\nabla\theta\cdot\nabla] \int_0^1 u_{0_\psi} u_{0_\psi} \, d\psi \\
&-[\theta_{TT} - c^2(\underset{\sim}{X},T)\nabla^2\theta] \int_0^1 u_{0_\psi} u_{0_\psi} \, d\psi \\
&\qquad - \int_0^1 h(u_0, \theta_T u_{0_\psi}, u_{0_\psi}\nabla\theta, \underset{\sim}{X}, T) d\psi = 0 \quad .
\end{aligned}$$

The first integral is zero. The next two integrals combine to yield Whitham's [3] well-known partial differential equation describing the possible dissipation of wave action:

$$\begin{aligned}
&\frac{\partial}{\partial T} [\theta_T \int_0^1 (u_{0_\psi})^2 \, d\psi] - c^2(\underset{\sim}{X},T)\nabla\cdot[(\nabla\theta) \int_0^1 (u_{0_\psi})^2 \, d\psi] \\
&\qquad + \int_0^1 h(u_0, \theta_T u_{0_\psi}, u_{0_\psi}\nabla\theta, \underset{\sim}{X}, T) u_{0_\psi} \, d\psi = 0 \quad .
\end{aligned} \tag{2.10}$$

We define the latter integral to be the dissipation R and

$$J \equiv \int_0^1 (u_{0_\psi})^2 \, d\psi \quad , \tag{2.11}$$

so that (2.10) may be written as

$$\frac{\partial}{\partial T} (\theta_T J) - c^2\nabla\cdot(J\nabla\theta) + \int_0^1 h(u_0, \theta_T u_{0_\psi}, u_{0_\psi}\nabla\theta, \underset{\sim}{X}, T) \, d\psi = 0 \quad . \tag{2.12}$$

For ordinary differential equations [2], this simply reduces to the possible dissipation of action (an adiabatic invariant if $h = 0$).

3. Higher Order Wave Action.

In this section, we show that wave action satisfies (2.10) to at least one higher order. We will use this in §4 to determine an equation for the modulations of the phase shift $\phi(\underset{\sim}{X},T)$. To obtain these results, we eliminate the secular terms in the $O(\varepsilon^2)$ equation:

$$\int_0^1 R_2\, u_{0_\psi}\, d\psi = 0 \quad , \tag{3.1}$$

where R_2 is given in (2.6). As with the case of ordinary differential equations ([4],[5]), the following difficult term can be calculated by integration by parts, then by recalling the definition of L, and finally by using $L(u_1) = R_1$:

$$\begin{aligned}
\int_0^1 V_{uuu}(u_0,\underset{\sim}{X},T)u_1^2\, u_{0_\psi}\, d\psi &= -2\int_0^1 V_{uu}(u_0,\underset{\sim}{X},T)u_1 u_{1_\psi}\, d\psi \\
&= -2\int_0^1 [L(u_1) - (\omega^2 - c^2k^2)u_{1_{\psi\psi}}]u_{1_\psi}\, d\psi \\
= 2\int_0^1 \{2(\phi_T\theta_T - c^2\nabla\theta\cdot\nabla\phi)u_{0_{\psi\psi}} &+ 2(\theta_T\tfrac{\partial}{\partial T} - c^2\nabla\theta\cdot\nabla)u_{0_\psi} \\
&+ (\theta_{TT} - c^2\nabla^2\theta)u_{0_\psi} + h(u_0,\theta_T u_{0_\psi}, u_{0_\psi}\nabla\theta,\underset{\sim}{X},T)\}\, u_{1_\psi}\, d\psi \quad ,
\end{aligned}$$

where the periodicity of some expressions has also been used. In this manner, from (3.1), we obtain

$$\frac{\partial}{\partial T}\left[\phi_T \int_0^1 (u_{0_\psi})^2\, d\psi\right] - c^2\nabla\cdot\left[\nabla\phi \int_0^1 (u_{0_\psi})^2\, d\psi\right]$$

$$2\frac{\partial}{\partial T}\left[\theta_T \int_0^1 u_{0_\psi} u_{1_\psi}\, d\psi\right] - c^2\nabla\cdot\left[\nabla\theta \int_0^1 u_{0_\psi} u_{1_\psi}\, d\psi\right] \tag{3.2}$$

$$+ \int_0^1 h u_{1_\psi}\, d\psi + \int_0^1 u_1 u_{0_\psi} h_u\, d\psi + \int_0^1 (\theta_T u_{1_\psi} + \phi_T u_{0_\psi} + u_{0_T})\, u_{0_\psi} h_v\, d\psi$$

$$+ \int_0^1 (\nabla\theta u_{1_\psi} + \nabla\phi u_{0_\psi} + \nabla u_0)\, u_{0_\psi} \cdot \underset{\sim}{h}_\nabla\, d\psi + \int_0^1 h_1 u_{0_\psi}\, d\psi = 0 \quad .$$

A similar calculation was presented by Dobrokhotov and Maslov [4]. They incorrectly calculated one term and thus did not obtain (3.2). Furthermore, their results were for unperturbed systems ($h = h_1 = 0$). Equation (3.2) is not a difficult equation; by comparing it to (2.10), we see that most of the terms are the first variation of the wave action equation (2.10), as we now show. Wave action is a concept valid for the leading order terms in a perturbation expansion. The wave action defined in terms of total frequency, total wave number vector, and total u will not satisfy (2.10). However, we expect there might be an $O(\varepsilon)$ correction:

$$\frac{\partial}{\partial T}\left[(\theta_T + \varepsilon\phi_T) \int_0^1 (u_{0_\psi} + \varepsilon u_{1_\psi} + \ldots)^2\, d\psi\right]$$

$$- c^2\nabla\cdot\left[(\nabla\theta + \varepsilon\nabla\phi) \int_0^1 (u_{0_\psi} + \varepsilon u_{1_\psi} + \ldots)^2\, d\psi\right]$$

$$+ \int_0^1 h(u_0 + \varepsilon u_1 + \ldots, (\theta_T + \varepsilon\phi_T)(u_{0_\psi} + \varepsilon u_{1_\psi} + \ldots),\ (\nabla\theta + \varepsilon\nabla\phi)(u_{0_\psi} + \varepsilon u_{1_\psi} + \ldots), \underset{\sim}{X}, T)\, d\psi$$

$$= \varepsilon P \quad .$$

We compare the above to (3.2) using (2.10) and determine that

$$P = - \int_0^1 (u_{0_T} h_V + \nabla u_0 \cdot \underset{\sim}{h}_\nabla + h_1) u_{0_\psi} \, d\psi + O(\varepsilon) \quad .$$

This gives us a simple method to obtain higher order corrections to wave action. In general, action is not satisfied to higher order (i.e., $P \neq 0$). However, if εh is a dissipative perturbation and $\varepsilon^2 h_1$ is not a dissipative perturbation, then the integrals above vanish (since h_V, $\underset{\sim}{h}_\nabla$ and h_1 are even). In this important case, our perturbation method has proved that wave action is dissipated in the same way as (2.10) to at least one higher order than would have been anticipated.

4. Modulated Phase Shift.

We restrict our attention to the situation in which εh is dissipative (and thus odd in ψ) and $\varepsilon^2 h_1$ is non-dissipative (and thus even in ψ). To obtain an equation for the modulations of the phase shift $\phi(\underset{\sim}{X},T)$, we determine u_1 and substitute it into (3.2). We will obtain a universal form. Because u_0 is even in ψ and h is odd in ψ, we only need the even part of u_1, $u_{1_{even}}$. From (2.4) and (2.5)

$$L(u_{1_{even}}) = - 2(- \omega \phi_T - c^2 \underset{\sim}{k} \cdot \nabla\phi) u_{0_{\psi\psi}} \quad . \tag{4.1}$$

We can easily solve (4.1) by noting the result of differentiating (2.3) with respect to E (keeping $\underset{\sim}{X}$, T, and $\underset{\sim}{k}$ fixed):

$$L(u_{0_E}) = - 2\omega\omega_E \, u_{0_{\psi\psi}} \quad , \tag{4.2}$$

since $\omega(E,\underset{\sim}{k},\underset{\sim}{X},T)$. This idea was used by Luke [1] and Dobrokhotov and Maslov [4]. However, we can also take partial derivatives with respect to each component of the wave number vector $\underset{\sim}{k}$. Since $\omega\nabla_k\omega = c^2\underset{\sim}{k}$ from (2.8), it follows that

$$L(\nabla_k u_0) = -2\,(\omega\nabla_k\omega - c^2\underset{\sim}{k})u_{0_{\psi\psi}} = \underset{\sim}{0} \tag{4.3}$$

Thus, u_0 is independent of $\underset{\sim}{k}$ (as is also clear from (2.7) and (2.8)) since $\nabla_k u_0$ is even and periodic. [The only even homogeneous solution is not periodic (see [1] or [5]).] By comparing (4.1) with (4.2):

$$u_{1_{even}} = \frac{-\phi_T - \nabla_k\omega \cdot \nabla_\phi}{\omega_E}\, u_{0_E} \quad . \tag{4.4}$$

Thus, $u_{1_{even}}$ can be obtained by operating on u_0:

$$u_{1_{even}} = \hat{D}u_0 \tag{4.5}$$

It is convenient and allowable (but not necessary) to include derivatives with respect to $\underset{\sim}{k}$ since $\nabla_k u_0 = \underset{\sim}{0}$. Therefore, from (4.4), we define $\hat{D}$ as the following linear first-order differential operator in the slow parameters E and $\underset{\sim}{k}$:

$$\hat{D} \equiv \frac{-\phi_T - \nabla_k\omega \cdot \nabla\phi}{\omega_E}\frac{\partial}{\partial E} + \underset{\sim}{\mu} \cdot \nabla_k \quad , \tag{4.6}$$

where $\underset{\sim}{\mu}$ is arbitrary since $\nabla_k u_0 = \underset{\sim}{0}$. However, we will discover that our derivation is simplified for a specific choice in which $\underset{\sim}{\mu} \neq \underset{\sim}{0}$.

We note that for any linear first-order differential operator

$$\hat{D}\int_0^1 h(u_0,-\omega u_{0_\psi},\underset{\sim}{k}u_{0_\psi},\underset{\sim}{X},T)u_{0_\psi}d\psi =$$

$$\int_0^1 hu_{1_\psi}d\psi + \int_0^1 u_{0_\psi}[h_u u_1 + h_v(-\omega u_{1_\psi} - \hat{D}(\omega)u_{0_\psi}) + (\underset{\sim}{k}u_{1_\psi} + \hat{D}(\underset{\sim}{k})u_{0_\psi})\cdot \underset{\sim}{h}_\nabla]d\psi \quad , \tag{4.7}$$

since derivatives in the fast variable ψ commute with $\hat{D}$ and since only the even part of u_1 contributes to the integrals. These are precisely the terms that appear in (3.2) if

$$\hat{D}(\omega) = -\phi_T \tag{4.8a}$$

$$\hat{D}(\underset{\sim}{k}) = \nabla\phi \quad , \tag{4.8b}$$

which will greatly simplify our derivation. Thus, we will show only one $\underset{\sim}{\mu}$ can be chosen to satisfy (4.8). From (4.6)

$$\hat{D}(\omega) = -\phi_T - \nabla_k\omega\cdot\nabla\phi + \underset{\sim}{\mu}\cdot\nabla_k\omega$$

$$\hat{D}(\underset{\sim}{k}) = \underset{\sim}{\mu} \quad .$$

Thus, (4.8b) is satisfied only if

$$\underset{\sim}{\mu} = \nabla\phi \quad ,$$

which fortunately does satisfy (4.8a), so that

$$\hat{D} = \frac{-\phi_T - \nabla_k\omega \cdot \nabla\phi}{\omega_E} \frac{\partial}{\partial E} + \nabla\phi \cdot \nabla_k \quad . \tag{4.9}$$

(In §6, we show that $\hat{D}$ is the Taylor series operator in the parameters E and $\underset{\sim}{k}$.) Substituting $u_{1_{\text{even}}} = \hat{D}u_0$ into (3.2) yields relatively easy calculations because $\hat{D}$ is a linear first-order differential operator. For example,

$$\hat{D} \int_0^1 (u_{0_\psi})^2 d\psi = 2 \int_0^1 u_{0_\psi} u_{1_\psi} d\psi \quad .$$

Thus, (3.2) simplifies:

$$\begin{aligned} &\frac{\partial}{\partial T}(\phi_T J) - c^2\nabla \cdot (\nabla\phi J) \\ &\quad + \frac{\partial}{\partial T}(\theta_T \hat{D}J) - c^2\nabla \cdot (\nabla\theta\hat{D}J) + \hat{D}\int_0^1 hu_{0_\psi} d\psi = 0 \quad , \end{aligned} \tag{4.10}$$

where again J is given by (2.11). However, from (4.8)

$$\hat{D}(\theta_T J) = \theta_T\hat{D}J + \hat{D}(\theta_T)J = \theta_T\hat{D}J + \phi_T J$$

$$\hat{D}(\nabla\theta J) = \nabla\theta\hat{D}J + \hat{D}(\nabla\theta)J = \nabla\theta\hat{D}J + \nabla\phi J \quad ,$$

since again $\hat{D}$ is a first-order differential operator. Thus, the conservation form of the equation becomes

$$\frac{\partial}{\partial T}\hat{D}(\theta_T J) - c^2\nabla\cdot\hat{D}(\nabla\theta J) + \hat{D}\int_0^1 hu_{0_\psi} d\psi = 0 \quad . \tag{4.11}$$

This is the partial differential equation for the modulated phase $\phi(\underset{\sim}{X},T)$ since $\hat{D}$ is given by (4.9). This result is particularly pleasing; it can be obtained from the well-known dissipation of wave action, (2.10) or (2.12), by replacing $\frac{\partial}{\partial T}$ by $\frac{\partial}{\partial T}\hat{D}$, $\nabla\cdot$ by $\nabla\cdot\hat{D}$ and the dissipation integral by $\hat{D}$ operating on it. However, (5.11) can be further simplified by just operating $\hat{D}$ on (2.12):

$$\hat{D}\int_0^1 hu_{0_\psi}\,d\psi = -\hat{D}\frac{\partial}{\partial T}(\theta_T J) + c^2\hat{D}\nabla\cdot[\nabla\theta J] \quad .$$

The above can be used to "eliminate" the dissipation term in (4.11):

$$\frac{\partial}{\partial T}\hat{D}(\theta_T J) - \hat{D}\frac{\partial}{\partial T}(\theta_T J) - c^2\nabla\cdot\hat{D}(\nabla\theta J) + c^2\hat{D}\nabla\cdot(\nabla\theta J) = 0 \quad . \tag{4.12a}$$

Therefore, we obtain a particularly simple representation:

$$\hat{D}_T(\theta_T J) - c^2\hat{D}_\nabla\cdot(\nabla\theta J) = 0 \quad , \tag{4.12b}$$

since $\hat{D}_T \equiv \frac{\partial}{\partial T}\hat{D} - \hat{D}\frac{\partial}{\partial T}$ and $\hat{D}_\nabla \equiv \nabla\cdot\hat{D} - \hat{D}\nabla$. Equation (4.12) has been derived for rather arbitrary dissipation; its form is universal. The dependence on dissipation is due to the dissipation of wave action (2.12). This result suggests the following algorithm for obtaining (4.12): Starting from dissipation of wave action (2.12), replace $\frac{\partial}{\partial T}$ by $\hat{D}_T$, ∇ by $\hat{D}_\nabla$, and ignore the dissipation integral. Note that in one spatial dimension, $\hat{D}_\nabla = \hat{D}_X$, which is as simple as $\hat{D}_T$. Similar results are valid for Korteweg-deVries type equations [7]. For clarity, we note that after using (4.9) and $\nabla_k J = 0$ [see (7.6)] the equation for the modulated phase shift $\phi(\underset{\sim}{X},T)$ becomes

$$0 = \left(\frac{-\phi_T - \nabla_k\omega \cdot \nabla\phi}{\omega_E}\right)_T \frac{\partial}{\partial E}(-\omega J) + \nabla\phi_T \cdot \nabla_k(-\omega J)$$

$$- c^2 \nabla\left(\frac{-\phi_T - \nabla_k\omega \cdot \nabla\phi}{\omega_E}\right) \cdot \frac{\partial}{\partial E}(\underset{\sim}{k}J) \tag{4.13}$$

$$- c^2 J\nabla^2\phi \ .$$

Here, for example, $\frac{\partial}{\partial T}$ refers to keeping only $\underset{\sim}{X}$ fixed (not E or $\underset{\sim}{k}$).

In the spatially independent case ($c^2 = 0$), (2.1) is a nonlinear ordinary differential equation and (4.13) becomes

$$\frac{d}{dT}\left[\frac{d\phi/dT}{\omega_E}\right] = 0 \ ,$$

the adiabatic invariant for "arbitrary" dissipative systems that we [5] recently discovered.

5. A Higher-order Accurate Dispersion Relation Depending on Averaged Energy.

In §4, we derived (4.13), the equation for the modulations of the phase shift $\phi(\underset{\sim}{X},T)$. Here, we will show how this result can be obtained from two intuitive physical principles (and justify these ideas mathematically). We have already shown one of these physical principles: the wave action equation

$$(-\omega J)_T - c^2 \nabla \cdot (\underset{\sim}{k} J) + R = 0 \tag{5.1}$$

is valid to at least one higher order in an asymptotic expansion in powers of ε, if εh is a dissipative perturbation and $\varepsilon^2 h_1$ is a non-dissipative one.

We conjecture (and later verify) a second physical principle, one based on the averaged energy. We have defined via (2.7) a leading order slowly varying energy $E_0(\underset{\sim}{X},T)$ in terms of the leading order frequency ω_0, wave number vector $\underset{\sim}{k}_0$, and solution u_0. We will first introduce more accurate expressions for the energy. There are two different methods for doing this. We can define a mathematical expression similar to (2.7):

$$E_{math} \equiv \frac{1}{2}\left[(\theta_T + \varepsilon\phi_T)^2 - c^2(\nabla\theta + \varepsilon\nabla\phi)^2\right]u_\psi^2 + V(u,\underset{\sim}{X},T) \quad , \tag{5.2a}$$

based on the complete frequency, wave number vector, and solution. Alternatively, we can use the physical expression for energy density:

$$E_{physical} \equiv \frac{1}{2}(u_t)^2 - \frac{1}{2}c^2(\nabla_x u)^2 + V(u,\underset{\sim}{X},T) \quad ,$$

which, using the multiple scale approach, can be rewritten exactly as

$$E_{physical} = \frac{1}{2}\left[(\theta_T + \varepsilon\phi_T)u_\psi + \varepsilon u_T\right]^2 - \frac{c^2}{2}\left[(\nabla\theta + \varepsilon\nabla\phi)u_\psi + \varepsilon\nabla u\right]^2 + V(u,\underset{\sim}{X},T) \quad . \tag{5.2b}$$

It is clear that both equal E_0 to leading order [defined by (2.7)]. It is straightforward to show that both $E_{physical}$ and E_{math}, as defined above, oscillate on the fast scale with a small $O(\varepsilon)$ amplitude. Moreover, they differ at $O(\varepsilon)$:

$$E_{physical} - E_{math} = \varepsilon u_{0_\psi}\left(\theta_T u_{0_T} - c^2\nabla\theta\cdot\nabla u_0\right) + O(\varepsilon^2) \quad .$$

Both oscillate on the fast scale, so that we might expect their averages to be important. Since $\int_0^1 u_{0_\psi} u_{0_T} \, d\psi = 0$ and $\int_0^1 u_{0_\psi} \nabla u_0 \, d\psi = 0$ by parity considerations, we note that

$$< E_{physical} > - < E_{math} > = O(\varepsilon^2) \quad ,$$

where $< \, > \equiv \int_0^1 d\psi$ represents averaging over the fast nonlinear oscillation. Thus, we can use either expression for the averaged energy and obtain identical results valid to at least one higher order in powers of ε.

By expanding either energy (5.2a) or (5.2b) to $O(\varepsilon)$ and by integrating, we obtain the averaged perturbation E_1:

$$E_1 = (-\omega_0 \phi_T - c^2 \underset{\sim}{k}_0 \cdot \nabla\phi) \int_0^1 u_{0_\psi}^2 \, d\psi + (\omega_0^2 - c^2 k_0^2) \int_0^1 u_{0_\psi} u_{1_\psi} \, d\psi + \int_0^1 V_{u_0} u_1 \, d\psi \quad . \tag{5.3}$$

This expression can be simplified using our results from §4. Only the even part of u_1 is needed in (5.3), namely from (4.5) $u_{1_{even}} = \hat{D} u_0$. If we operate $\hat{D}$ on (2.7), we obtain

$$(\omega_0^2 - c^2 k_0^2) u_{0_\psi} \hat{D} u_{0_\psi} + u_{0_\psi}^2 [\omega_0 \hat{D}(\omega_0) - c^2 \underset{\sim}{k}_0 \cdot \hat{D}(\underset{\sim}{k}_0)] + V_{u_0} \hat{D} u_0 = \hat{D} E_0 \quad ,$$

which, after averaging (integrating) and using (4.8), yields an expression for E_1 from (5.3):

$$E_1 = \hat{D} E_0 = \frac{-\phi_T - \nabla_k \omega_0 \cdot \nabla\phi}{\omega_{0_E}} \quad . \tag{5.4}$$

We will show that (5.4) is an immediate consequence of a simple physical conjecture. The local amplitude-dependent dispersion relation (2.8), $\omega_0 = \omega[\underset{\sim}{k}_0(\underset{\sim}{X},T),\ E_0(\underset{\sim}{X},T),\ \underset{\sim}{X},\ T]$, is well-known to be valid to leading order. Intuitively, we might expect that the dispersion relation is valid to at least one higher order if both the more accurate wave number vector $\underset{\sim}{k}_0 + \varepsilon\,\underset{\sim}{k}_1$ (with $\underset{\sim}{k}_1 = \nabla\phi$) and the more accurate averaged energy $E_0 + \varepsilon\, E_1$ are used. We would then conjecture on the basis of physical intuition that the perturbed frequency $\omega_1 = -\,\phi_T$ is obtained from the elementary Taylor series of this highly-accurate dispersion relation:

$$\omega_1 = E_1\, \omega_{0_E} + \underset{\sim}{k}_1 \cdot \nabla_k \omega_0 \quad . \tag{5.5}$$

Since this is equivalent to (5.4), we have proved the validity of a second physical principle: the dispersion relation is valid to at least one higher order if an accurate expression for averaged energy is utilized in (2.7). Frequency changes are only due to the perturbed wave number vector and to the averaged perturbed energy. This is valid only if the perturbation εh is dissipative and $\varepsilon^2 h_1$ non-dissipative (for otherwise we would expect further frequency changes due to these).

We use this result to obtain the modulations of the phase shift by analyzing the first variations of the wave action. Since (5.1) is valid to at least one higher order, we will substitute $\underset{\sim}{k} = \underset{\sim}{k}_0 + \varepsilon\,\underset{\sim}{k}_1$ and $\omega = \omega_0 + \varepsilon\,\omega_1$ into (5.1), where $\underset{\sim}{k}_1 = \nabla\phi$ and $\omega_1 = -\,\phi_T$. Furthermore, we must use the more accurate integrals R and J defined in terms of total $\underset{\sim}{k}$, ω, and u. However, it can be shown from §4 that R and J are functions of $\underset{\sim}{k}$ and the average energy:

$$R = R_0 + \varepsilon\,(E_1 R_{0_E} + \underset{\sim}{k}_1 \cdot \nabla_k R_0) + \mathcal{O}(\varepsilon^2)$$

$$J = J_0 + \varepsilon\, E_1\, J_{0_E} + \mathcal{O}(\varepsilon^2) \quad ,$$

since J is independent of $\underset{\sim}{k}$ [see (7.6)]. In this way, (5.1) becomes

$$\begin{aligned} &- [(\omega_0 + \varepsilon\omega_1)(J_0 + \varepsilon\, E_1\, J_{0_E})]_T \\ &- c^2\,\nabla \cdot [(\underset{\sim}{k}_0 + \varepsilon \underset{\sim}{k}_1)(J_0 + \varepsilon\, E_1 J_{0_E})] \qquad (5.6) \\ &+ R_0 + \varepsilon(E_1\, R_{0_E} + \underset{\sim}{k}_1 \cdot \nabla_k\, R_0) + \mathcal{O}(\varepsilon^2) = 0 \quad . \end{aligned}$$

The leading O(1) terms cancel, and many of the $O(\varepsilon)$ terms cancel since the partial derivatives with respect to E and $\underset{\sim}{k}$ of the leading-order wave action equation are also valid. Thus, the remaining $\mathcal{O}(\varepsilon)$ terms must be in balance:

$$\begin{aligned} &E_{1_T}(-\omega J)_E + \nabla\phi_T \cdot \nabla_k(-\omega J) \\ &\qquad - c^2\nabla\, E_1 \cdot \underset{\sim}{k}\, J_E - c^2\, J\, \nabla^2\phi = 0 \quad , \end{aligned} \qquad (5.7)$$

where the zeroth-order subscripts have been dropped. We call (5.7) the equation for the variation of wave action. When the expression for averaged perturbed energy E_1 is used, (5.4), we obtain (4.13), the linear partial differential equation for the modulations of the phase shift. Since (5.7) is the appropriate first variation of the wave action (5.1), it is in essence the linearization of (5.1). Thus, the characteristic velocities of (5.7) will be the same as those for (5.1), as shown in §7.

In summary, the equation for the phase shift can be obtained from the correct higher-order accurate versions of the dispersion relation and the wave action equation.

6. A Linearization Principle for $\hat{D}$.

In §4, we noted that calculations needed for higher-order perturbations were simplified if we observed that $u_{1_{\text{even}}} = \hat{D}\, u_0$, where $\hat{D}$ is given by (4.9). Here, we will show that this operator follows from an elementary understanding of the linearization of the solution. For the Klein-Gordon equation, to leading order, nonlinear oscillatory waves satisfy

$$u_0 = u_0(\psi; E, \underset{\sim}{k}, \underset{\sim}{X}, T) \quad , \tag{6.1}$$

where the usual dispersion relation (2.8) is satisfied

$$\omega = \omega(E, \underset{\sim}{k}, \underset{\sim}{X}, T) \quad . \tag{6.2}$$

If the perturbation is of the restricted class (dissipative) discussed throughout, then higher-order terms do not effect the phase modulations (other than via wave action). In this case and others, some of the higher-order terms may be obtained simply by perturbing the parameters E and $\underset{\sim}{k}$ in (6.1) and (6.2). By this Taylor-series approach,

$$u_{1_{even}} = E_1\, u_{0_E} + \underset{\sim}{k}_1 \cdot \nabla_k\, u_0 \qquad (6.3)$$

$$\omega_1 = E_1\, \omega_E + \underset{\sim}{k}_1 \cdot \nabla_k \omega \quad . \qquad (6.4)$$

Here E_1 represents (as in §5) an averaged perturbed energy. We again introduce $\underset{\sim}{k}_1 = \nabla\phi$ and $\omega_1 = -\phi_T$, in which case (6.3) and (6.4) can be combined to yield

$$u_{1_{even}} = \hat{D}\, u_0 \quad , \qquad (6.5a)$$

where

$$\hat{D} = E_1 \frac{\partial}{\partial E} + \underset{\sim}{k}_1 \cdot \nabla_k = \left(\frac{-\phi_T - \nabla\phi \cdot \nabla_k \omega}{\omega_E}\right) \frac{\partial}{\partial_E} + \nabla\phi \cdot \nabla_k \quad , \qquad (6.5b)$$

agreeing with (4.9). As defined this way, it automatically follows that $\hat{D}(\underset{\sim}{k}) = \underset{\sim}{k}_1 = \nabla\phi$, $\hat{D}(\omega) = \omega_1 = -\phi_T$, and $\hat{D}(E) = E_1$.

The equation for the modulations of the phase shift can be obtained by higher-order perturbations of the wave action equation and dispersion relation. These principles are easily extended to other systems, such as KdV type equations [7], which has two amplitude parameters. Furthermore, the equations for the phase modulations of harder problems, such as nonlinear ones with multiple fast phases, should be analyzed in these ways as well. If perturbations are not necessarily restricted, then the same type of linearized system will occur with nonhomogeneous terms (representing, for example, non-dissipative effects).

7. Characteristic Velocities and Stability.

In this section, we first show that the characteristic velocities for wave action are the same as the characteristic velocities for the modulations of the phase shift. Then, we calculate these characteristic velocities for nonlinear waves periodic in one-spatial dimension. We show that Klein-Gordon waves are stable for hard potentials (i.e., $\omega_E/\omega > 0$), while wildly unstable for soft potentials (i.e., $\omega_E/\omega < 0$). These soft Klein-Gordon waves will be shown to satisfy an elliptic partial differential equation, and hence are not well-posed as a slowly varying wave, in the same manner in which the Benjamin-Feir instability arises for water waves and certain nonlinear Schrödinger equations (see Whitham [3]). For this unstable case, nonlinear waves with nearly one wave number and frequency will not persist. Instead, solitary waves (if they exist) or multi-phased spatially periodic waves (if they exist) might develop (Newell [8]).

The wave number vector, frequency, and amplitude of a nonlinear periodic wave slowly evolves in space and time according to the coupled system composed of the dispersion relation (2.3), conservation of waves

$$\underset{\sim}{k}_T + \nabla\omega = \underset{\sim}{0} \quad , \tag{7.1a}$$

and the principle of wave action

$$(-\omega J)_T - c^2\nabla \cdot (\underset{\sim}{k}J) + R = 0 \quad . \tag{7.1b}$$

By considering the frequency ω as a function of $\underset{\sim}{k}$, E, $\underset{\sim}{X}$, T, via the dispersion relation, the system (7.1) is expressed in terms of the fundamental unknowns E and $\underset{\sim}{k}$:

$$k_{i_T} + E_{X_i}\omega_E + \underset{\sim}{k}_{X_i} \cdot \nabla_k \omega = -\partial\omega/\partial X_i \tag{7.2a}$$

$$E_T(-\omega J)_E + \underset{\sim}{k}_T \cdot \nabla_k(-\omega J) - c^2 J \nabla \cdot \underset{\sim}{k} - c^2(\underset{\sim}{k} \cdot \nabla E) J_E$$

$$= -R - \frac{\partial}{\partial T}(-\omega J) + c^2 \nabla_X \cdot (\underset{\sim}{k} J) \quad , \tag{7.2b}$$

where [only in the rhs of (7.2)] $\frac{\partial}{\partial X_i}$, $\frac{\partial}{\partial T}$, and ∇_X refer to differentiation keeping $\underset{\sim}{k}$ and E fixed. Equation (7.2b) has been simplified somewhat since J does not depend on $\underset{\sim}{k}$ [see (7.6)]. By comparing equations for the wave action (7.2) to the phase shift (4.13), it is apparent that the characteristic velocities are identical.

We determine the characteristic velocities for slowly varying Klein-Gordon waves in one spatial dimension. In this case, (7.2) becomes

$$k_T + \omega_E E_X + \omega_k k_X = \text{l.o.t.} \tag{7.3a}$$

$$(-\omega J)_E E_T + (-\omega J)_k k_T - c^2 J k_X - c^2 k J_E E_X = \text{l.o.t.} \quad , \tag{7.3b}$$

where l.o.t. designates lower order terms, which are not necessary to calculate the characteristic velocities. Equations (7.3) form a system of quasi-linear first-order partial differential equations. By the usual method of diagonalization (described and calculated by Whitham [3]), we obtain the two characteristic velocities

$$\frac{dX}{dT} = \frac{c^2 k J_E \pm c\sqrt{J\omega_E/\omega}}{(\omega J)_E} \quad , \tag{7.4}$$

where $\omega\omega_k = c^2 k$ has been used to derive this result as well as

$$(\omega^2 - c^2k^2)J_E + \omega\omega_E J = 1 \quad . \tag{7.5}$$

The latter equation, (7.5), is derivable from conservation of energy,

$$\frac{1}{2}(\omega^2 - c^2k^2)J = \int_0^1 (E - V)d\psi = (\omega^2 - c^2k^2)^{\frac{1}{2}} \int_{u_{0_{min}}}^{u_{0_{max}}} \sqrt{2}\,(E - V)^{\frac{1}{2}}\, du_0 \quad , \tag{7.6}$$

when the dispersion relation (2.8) is used. From (2.8), $\omega^2 - c^2k^2$ only depends on $\underset{\sim}{X}$, T and E. Thus, J does not depend on $\underset{\sim}{k}$, a result we have found useful occasionally in this paper.

The characteristic velocities, given by (7.4), determine the stability and well-posedness of nonlinear periodic waves for the Klein-Gordon equation. The sign of ω_E/ω is important, since $J > 0$. Note that ω_E is determined from the potential [see the dispersion relation (2.8)]. If $\omega_E/\omega > 0$, which we define as corresponding to hard potentials since the frequency increases with the wave amplitude or energy, then the nonlinear periodic waves are stable. The initial value problem is well-posed (see [5]). However, if $\omega_E/\omega < 0$, corresponding to so-called soft potentials, the velocities have a non-zero imaginary part, implying that the partial differential equations are elliptic. In this case the partial differential equation is not well-posed as an initial value problem; the assumption of a slowly varying periodic wave is not valid.

The periodic wave would immediately be wildly unstable, yielding some other type of solution. Thus, the phase shift analysis in this paper is applicable only for hard potentials ($\omega_E/\omega > 0$).

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References

[1] J. C. Luke, Proc. Roy. Soc. London Ser. A 292 (1966) 403.

[2] G. E. Kuzmak, Prikl. Mat. Meh. 23 (1959), 515. J. Appl. Math. Mech. 23 (1959) 730.

[3] G. B. Whitham, Linear and Nonlinear Waves (John Wiley & Sons, New York, 1974).

[4] S. Yu. Dobrokhotov and V. P. Maslov, Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, 15 (1980) 3.

[5] F. J. Bourland and R. Haberman, submitted.

[6] J. Kevorkian and J. D. Cole, Perturbation Methods in Applied Mathematics (Springer-Verlag, New York, 1981).

[7] R. Haberman, in preparation.

[8] A. C. Newell, Solitons in Mathematics and Physics (S.I.A.M., Philadelphia, 1985).

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A New Adiabatic Invariant Involving the Modulated Phase Shift for Strongly Nonlinear, Slowly Varying, and Weakly Damped Oscillators

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Abstract

The phase shift and corresponding small frequency modulation for weakly dissipated nonlinear oscillators with slowly varying coefficients is calculated for the first time. This extends and corrects earlier work by Kuzmak, Luke, and Dobrokhotov and Maslov. A new adiabatic invariant is derived, the ratio of the perturbation of the frequency to the derivative (with respect to energy) of the amplitude-dependent frequency of the nonlinear oscillator.

1. Introduction.

Kuzmak [4] analyzed the long time effect of weak damping on strongly nonlinear slowly varying oscillators. In his important work the leading order solution was obtained. However, it was unjustifiably assumed that the phase shift of the oscillations was constant, and thus, that the frequency was not perturbed. Since that work, refinements have been suggested ([1],[2],[3],[5]). Luke [5] developed a more systematic approach to the problem using the method of multiple scales that allowed the calculation of higher order terms. His paper, however, only investigated the use of the method on the nonlinear Klein-Gordon equation and did not examine the dissipative perturbations discussed here. He correctly observed that the phase shift satisfied a second-order differential equation, one solution of which is a constant. He did not, however, state the differential equation or the form of the second solution. This has led to some misconceptions among some authors about the phase shift. Dobrokhotov and Maslov [1] examined the same (non-dissipative) equation as Luke [5]. Using some excellent new ideas, they determined a differential equation for the phase shift. Unfortunately, they inadvertantly omitted a term in their dissipationless analysis which caused an error in their result. In this paper, we will derive the correct second-order differential equation for the phase shift and do it in a simpler manner, including arbitrary damping. We also obtain the solutions to this equation and show how it leads to a new and unexpected adiabatic invariant for dissipative equations.

In §2, we analyze second-order nonlinear differential equations of the form

$$\frac{d^2y}{dt^2} + \varepsilon h\left(y, \frac{dy}{dt}, T\right) + V_y(y,T) = 0 \quad , \tag{1.1}$$

with coefficients varying on the slow time $T = \varepsilon t$. Here $V(y,T)$ is a nonlinear potential that admits periodic solutions when $\varepsilon = 0$. The term $h(y, \frac{dy}{dt}, T)$ is assumed to be odd in dy/dt in order to represent an arbitrary small nonlinear damping. This generalizes the approach of Kuzmak [4] who only included quasi-linear damping. We first quickly show well-known results. To leading order, $y = y_0(\psi)$, which is even in ψ, where

$$[\theta'(T)]^2 y_{0_{\psi\psi}} + V_y(y_0,T) = 0 \quad . \tag{1.2}$$

Thus, the energy $E(T)$ is slowly varying

$$\frac{1}{2}[\theta'(T)]^2(y_{0_\psi})^2 + V(y_0,T) = E(T) \quad . \tag{1.3}$$

Here ψ is the fast variable, where $\psi = \frac{\theta(T)}{\varepsilon} + \phi(T)$. We will eventually determine the phase shift $\phi(T)$. Luke [5] showed the period in ψ must be constant and hence the frequency $\omega(E,T)$ satisfies

$$\omega(E,T) \equiv \theta'(T) = \left\{ 2 \int_{y_{0_{min}}}^{y_{0_{max}}} \frac{dy_0}{\sqrt{2}\,[E(T) - V(y_0,T)]^{\frac{1}{2}}} \right\}^{-1} \quad , \tag{1.4}$$

where $y_{0_{min}}(E,T)$ and $y_{0_{max}}(E,T)$ are, respectively, the minimum and maximum of the fast nonlinear oscillator. After performing a perturbation expansion, $y = y_0 + \varepsilon y_1 + \ldots$, we obtain

$$L(y_i) = R_i \quad , \quad i = 1,2,3,\ldots \quad , \tag{1.5}$$

where L is the linearized operator resulting from (1.2) and where R_i is the corresponding right-hand side described in more detail in §2. By the Fredholm alternative (for (1.5) with $i = 1$), equivalent to eliminating secular terms, we obtain Kuzmak's result [4] that the action I is dissipated:

$$\frac{dI}{dT} + D = 0 \quad , \tag{1.6}$$

where the action is twice the average value of the kinetic energy divided by the frequency:

$$I(E,T) \equiv \omega(E,T) \int_0^1 (y_{0_\psi})^2 \, d\psi \quad , \tag{1.7}$$

and where D is energy dissipated over one fast oscillation:

$$D = \int_0^1 h(y_0, \omega y_{0_\psi}, T)\, y_{0_\psi} \, d\psi \quad . \tag{1.8}$$

Equation (1.4) and (1.6) determine $E(T)$ and $\theta'(T)$. If there is no dissipation, then the action I is conserved (to this order) and is the well-known adiabatic invariant.

Next in §2, we determine the second-order differential equation for the phase shift $\phi(T)$ by examining the $O(\varepsilon)$ terms of the exact version, developed by Whitham ([6] and [10]), of the action equation. Our new result is

$$\frac{d}{dT}\left[\frac{d\phi/dT}{\omega_E(E,T)}\right] = 0 \quad . \tag{1.9}$$

Our method only requires a calculation of y_1, and hence is much simpler than the usual elaborate procedure of applying the secularity condition at $O(\varepsilon^2)$. We relegate this to Appendix A, correcting Dobrokhotov and Maslov [1]. In (1.9), ω_E is the rate of change of frequency with respect to energy keeping T fixed via (1.4). Thus, the perturbation of the modulated frequency $d\phi/dT$ divided by ω_E is a new adiabatic invariant. Equation (1.9) easily determines $\phi(T)$ from initial conditions as is discussed in §3.

Enuation (1.9) is unexpected. In general, perturbations effect the functional form of the dependence of the frequency on the amplitude. However, in §4 we show that for dissipative perturbations (limited to the εh term in (1.1) which is odd in dy/dt) the variation of both frequency, $\omega_1 = d\phi/dT$, and action are only due to changes in the averaged perturbed energy E_1. For example, it is shown that $\omega_1 = E_1 \omega_E$, which uses the leading-order amplitude dependence of the frequency. Thus, (1.9) states that the averaged perturbed energy does not vary. Furthermore, in §4 it is shown that (1.9) results from considerations of the linearization of the equations for action and frequency with respect to changes of the averaged energy. We believe the simplicity of (1.9) is due to damping primarily influencing the amplitude, not the frequency. However, the result (1.9) fundamentally depends on the damping through the dissipation (1.6). We reiterate that action is not an adiabatic invariant but $\phi'(T)/\omega_E$ is. Thus, (1.9) completes the description of the leading order problem for long time; it shows that $\phi(T)$ is not necessarily constant. For other types of perturbations, the phase shift will satisfy nonhomogeneous versions of (1.9).

2. Strongly Nonlinear Oscillators.

We consider the second-order nonlinear differential equation which describes strongly nonlinear, slowly varying, and weakly damped oscillators:

$$\frac{d^2y}{dt^2} + \varepsilon h\left(y, \frac{dy}{dt}, T\right) + V_y(y,T) = 0 \quad , \tag{2.1}$$

where $V(y,T)$ is a nonlinear potential depending on the slow-time scale $T = \varepsilon t$. We assume the potential admits periodic solutions to (2.1) (for all fixed T) on the fast time scale t without dissipation (i.e., when $\varepsilon = 0$). Also, in order for the small perturbation εh to be dissipative, we assume $h(y, dy/dt, T)$ is odd in dy/dt. This allows negative dissipation as well as situations such as the van der Pol oscillator in which both positive and negative dissipation may be present simultaneously.

We use the method of multiple scales with the fast time scale ψ and slow time scale T:

$$\psi = \frac{\theta(T)}{\varepsilon} + \phi(T)$$

$$T = \varepsilon t \quad ,$$

so that, for example,

$$\frac{dy}{dt} = [\theta'(T) + \varepsilon\,\phi'(T)]\,\frac{\partial y}{\partial \psi} + \varepsilon\,\frac{\partial y}{\partial T} \quad .$$

In the analysis to follow, we will rederive $\theta(T)$ in the standard way and correctly derive the phase shift $\phi(T)$ for the first time. In this manner, (2.1) becomes exactly

$$\begin{aligned}(\theta' + \varepsilon\phi')^2 y_{\psi\psi} + \varepsilon[(\theta'' + \varepsilon\phi'')y_{\psi} + 2(\theta' + \varepsilon\phi')y_{\psi T}] + \varepsilon h \\ + \varepsilon^2 y_{TT} + V_y(y,T) = 0 \quad .\end{aligned} \tag{2.2}$$

We now introduce the perturbation expansion

$$y(\psi,T) = y_0(\psi,T) + \varepsilon y_1(\psi,T) + \varepsilon^2 y_2(\psi,T) + \ldots$$

It is convenient to note that

$$V_y(y,T) = V_y(y_0,T) + (\varepsilon y_1 + \varepsilon^2 y_2)V_{yy}(y_0,T) + \varepsilon^2\,\frac{y_1^{\,2}}{2}\,V_{yyy}(y_0,T) + O(\varepsilon^3)$$

$$\begin{aligned}h(y,v,T) &= h(y,\theta' y_{\psi} + \varepsilon\phi' y_{\psi} + \varepsilon y_T,T) \\ &= h(y_0,\theta' y_{0_\psi},T) + \varepsilon[h_y y_1 + h_v(\phi' y_{0_\psi} + y_{0_T} + \theta' y_{1_\psi})] + O(\varepsilon^2) \quad ,\end{aligned}$$

where $v \equiv dy/dt$, $h_y \equiv h_y(y_0, \theta' y_{0_\psi}, T)$ and $h_v \equiv h_v(y_0, \theta' y_{0_\psi}, T)$.

The result of substituting the perturbation expansion into (2.2) is the nonlinear oscillator equation

$$[\theta'(T)]^2 y_{0_{\psi\psi}} + V_y(y_0, T) = 0 \tag{2.3}$$

and its higher order perturbations

$$L(y_k) = R_k \quad , \qquad k = 1,2,3, \ldots \quad , \tag{2.4}$$

where $L \equiv (\theta')^2 \frac{\partial^2}{\partial\psi^2} + V_{yy}(y_0, T)$ is the linearization of the nonlinear operator in (2.3) and

$$R_1 = -2\theta'\phi' y_{0_{\psi\psi}} - \theta'' y_{0_\psi} - 2\theta' y_{0_{\psi T}} - h(y_0, \theta' y_{0_\psi}, T)$$

$$\begin{aligned} R_2 = {} & 2\theta'\phi y_{1_{\psi\psi}} - \theta'' y_{1_\psi} - 2\theta' y_{1_{\psi T}} - (\phi')^2 y_{0_{\psi\psi}} \\ & - y_{0_{TT}} - \phi'' y_{0_\psi} - 2\phi' y_{0_{\psi T}} - \frac{1}{2} V_{yyy}(y_0, T) y_1^2 \\ & - h_y y_1 - h_v(\phi' y_{0_\psi} + y_{0_T} + \theta' y_{1_\psi}) \quad . \end{aligned}$$

We now examine the O(1) equation, (2.3). Multiplying by y_{0_ψ} and integrating once yields the conservation of energy equation

$$\frac{1}{2} (\theta')^2 (y_{0_\psi})^2 + V(y_0, T) = E(T) \quad , \tag{2.5}$$

where E(T) is slowly varying in a well-known way that we will rederive. Equation (2.5) defines curves in the $(y_0, \frac{\partial y_0}{\partial \psi})$ plane for fixed T. Because of our assumptions concerning the potential, these curves will be closed and y_0 will be periodic in ψ, oscillating between $y_{0_{min}}(E,T)$ and $y_{0_{max}}(E,T)$ with $V(y_{0_{min}},T) = E$ and $V(y_{0_{max}},T) = E$. Since only even derivatives with respect to ψ occur in (2.3), we can define ψ, without loss of generality, such that $y_0(\psi,T)$ is an even function of ψ, and so that $\psi = 0$ corresponds to $y_0(0,T) = y_{0_{min}}$ and $\frac{\partial y_0}{\partial \psi}(0,T) = 0$ (see Figure 1).

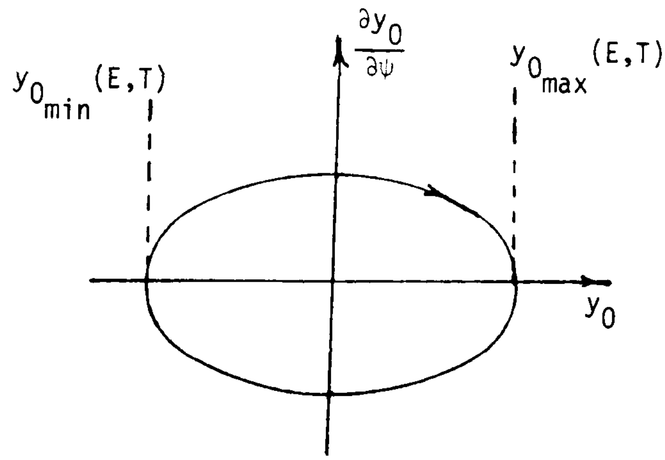

Figure 1: Phase plane solution of y_0 for fixed E,T.

Using quadrature on equation (2.5), we obtain

$$\psi = \theta'(T) \int_{y_{0_{min}}}^{y_0(\psi,T)} \frac{dy_0}{\pm\sqrt{2}\,[E - V(y_0,T)]^{\frac{1}{2}}} \quad .$$

The period of the oscillation in the fast variable is equal to the loop integral around the entire curve, or equivalently:

$$P(T) = 2\theta'(T) \int_{y_{0_{min}}}^{y_{0_{max}}} \frac{dy_0}{\sqrt{2}\,[E(T) - V(y_0,T)]^{\frac{1}{2}}} \quad . \qquad (2.6)$$

If P does depend on T, as is written, then as Luke [5] showed $\frac{\partial}{\partial T}(y_0(\psi + nP(T),T)) = n\frac{dP}{dT}y_{0_\psi}(\psi,T) + y_{0_T}(\psi,T)$, which is unbounded for large n. Thus, $\frac{dP}{dT} = 0$ and the period is constant. The period may be normalized to any convenient constant, and we will choose 1 (although 2π is often used):

$$\omega(E,T) \equiv \theta'(T) = \left\{ 2 \int_{y_{0_{min}}(E,t)}^{y_{0_{max}}(E,t)} \frac{dy_0}{\sqrt{2}\,[E(T) - V(y_0,T)]^{1/2}} \right\}^{-1} . \qquad (2.7)$$

The actual period of the oscillation (to first order) in the real time t is $1/\theta'(T)$ so we are justified in referring to $\theta'(T)$ as the frequency. We have introduced the notation $\omega(E,T)$ because partial derivatives with respect to E (keeping T fixed) will be important. (Note the difference between $\partial\omega/\partial T$ and $d\omega/dT = \partial\omega/\partial T + \partial\omega/\partial E\; dE/dT$.)

In order to examine higher order equations, we first need to obtain some results on the linearized operator. By taking the derivative of equation (2.3) with respect to ψ, we find that y_{0_ψ} is a homogeneous solution of $L(u) = 0$. As was stated before, $y_0(\psi,T)$ is an even function of ψ, which means that y_{0_ψ} is an odd, periodic homogeneous solution. Since y_0 is periodic, $L(u) = 0$ is equivalent to a Hill's equation. From Floquet theory the form of the second homogeneous solution is $u_2 = A(T)\psi y_{0_\psi} + K(\psi,T)$ where K is even and periodic in ψ with period one. Luke [5] showed that $A(T) \neq 0$ if $\frac{\partial P}{\partial E} \neq 0$ in equation (2.6).[1] Thus, in the usual case, y_{0_ψ} is the only periodic, homogeneous solution. In order for solutions to (2.4) to be periodic, the Fredholm alternative states that the right-hand side of (2.4) must be orthogonal to all periodic homogeneous solutions of (2.4). This gives the condition

[1] Note that in the linear case, $\partial P/\partial E = 0$ for all E and there are two periodic solutions: $\cos 2\pi\psi$ and $\sin 2\pi\psi$.

$$\int_0^1 y_{0_\psi} R_k \, d\psi = 0 \quad . \tag{2.8}$$

This is equivalent to eliminating secular terms in the usual way.

For k = 1, (2.8) becomes

$$\int_0^1 2\omega\phi' y_{0_{\psi\psi}} y_{0_\psi} d\psi + 2\omega \int_0^1 y_{0_{\psi T}} y_{0_\psi} d\psi + \frac{d\omega}{dT} \int_0^1 {y_{0_\psi}}^2 d\psi$$
$$+ \int_0^1 h(y_0, \omega y_{0_\psi}, T) y_{0_\psi} d\psi = 0 \quad .$$

The first integral is zero since it is the integral of an odd periodic function over one period. The next two integrals may be combined to give:

$$\frac{d}{dT} [\omega(E,T) \int_0^1 y_{0_\psi}^2 d\psi] + \int_0^1 h(y_0, \omega y_{0_\psi} T) y_{0_\psi} d\psi = 0 \quad . \tag{2.9a}$$

This is a well-known result [6]. We define the action, I(E,T), to be twice the average value of the leading order kinetic energy divided by the leading order frequency:

$$I(E,T) \equiv \omega(E,T) \int_0^1 (y_{0_\psi})^2 d\psi = 2 \int_{y_{min}}^{y_{max}} \sqrt{2} \, [E - V(y_0,T)]^{\frac{1}{2}} dy_0 \quad , \tag{2.10}$$

using (2.5) and a change of variables (i.e., integrating in the phase plane). We also define the dissipation (the energy dissipated over one fast oscillation):

$$D(E,T) \equiv \int_0^1 h(y_0, \omega y_{0_\psi}, T) y_{0_\psi} d\psi \quad . \tag{2.11}$$

Equation (2.9) becomes

$$\frac{dI}{dT} + D = 0 \quad . \tag{2.9b}$$

If there is no dissipation, the action is conserved (to this order) and is the well-known adiabatic invariant. If $h = \hat{h}(T)\,\frac{dy}{dt}$ and thus $h(y_0, \omega y_{0_\psi}, T) = \hat{h}(T)\,\omega y_{0_\psi}$, then (2.9b) simplifies to

$$\frac{dI}{dT} + \hat{h}(T)I = 0 \quad ,$$

which ([2],[4]) has the simple solution $I(T) = I(0)e^{-\int_0^T \hat{h}(s)ds}$. In any case, however, the only unknowns in (2.7) and (2.9b) are $\omega(E,T)$ and $E(T)$, so that these equations provide a closed system for ω and E.

In order to find the equation for $\phi(T)$, it will be necessary to examine higher-order terms and thus to know something about the form of $y_1(\psi,T)$. By taking the partial derivative of equation (2.5) with respect to E, noting that $\omega(E,T)$, we obtain $L(y_{0_E}) = -2\omega\omega_E y_{0_{\psi\psi}}$. By direct substitution, we also find that $L(\psi y_{0_\psi}) = 2\omega^2 y_{0_{\psi\psi}}$. Therefore, $y_H = \omega y_{0_E} + \omega_E \psi y_{0_\psi}$ is a homogeneous solution to (2.4), which is even in ψ. Notice that y_{0_E} is even and periodic, and thus, y_H is in the form predicted by Floquet theory. (Furthermore, we note that there are two periodic solutions if $\omega_E = 0$. For the remainder of this paper we assume $\omega_E(E,T) \neq 0$ since we believe that important cases correspond to the period in real time t being a monotonic function of the energy.) Using the facts that y_0 is even and $h(y,v,T)$ is odd in v (which means h is odd in ψ) allows us to separate $R_1 = -2\omega\omega' y_{0_{\psi\psi}} + R_{1_{odd}}$ where $R_{1_{odd}}$ consists only of terms that are odd in ψ. Since $L(\psi y_{0_\psi}) = 2\omega^2 y_{0_{\psi\psi}}$,

$$y_{1_p} = - \frac{\phi'}{\omega} \psi y_{0_\psi} + y_{1_{p_{odd}}} \quad , \tag{2.12}$$

where a specific odd part of the particular solution may be obtained by reduction of order

$$y_{1_{p_{odd}}} = \frac{y_{0_\psi}}{\omega^2} \int_0^\psi \frac{\int_0^{\psi'} R_{1_{odd}} y_{0_{\bar{\psi}}} \, d\bar{\psi}}{(y_{0_{\psi'}})^2} \, d\psi' \quad .$$

We will not need this expression for $y_{1_{p_{odd}}}$.

To obtain the general solution, we now include the two homogeneous solutions, y_{0_ψ} and y_H given above:

$$y_1 = C_1(T) y_{0_\psi} + B_1(T) \left(\omega y_{0_E} + \omega_E \psi y_{0_\psi}\right) - \frac{\phi'}{\omega} \psi y_{0_\psi} + y_{1_{p_{odd}}} \quad .$$

In order for y_1 to be bounded, the coefficient of ψy_{0_ψ} must be zero; therefore, $B_1(T) = \phi'/\omega\omega_E$, in which case

$$y_1 = C_1(T) \, y_{0_\psi} + \frac{\phi'}{\omega_E} y_{0E} + y_{1_{p_{odd}}} \quad . \tag{2.13}$$

In Appendix A, we show how to determine an equation for $\phi(T)$ by eliminating the secular terms in the $O(\varepsilon^2)$ equation, (2.8) with $k = 2$. Here, we obtain the desired result in a simpler way; we use the periodicity (secular) condition to derive an action equation valid to all orders. Whitham [10] showed that an exact equation for action may be obtained by multiplying (2.2) by y_ψ and integrating over ψ, using the assumption that y is periodic in ψ (with period 1):

$$\frac{d}{dT}\left[(\theta' + \varepsilon\phi') \int_0^1 y_\psi^2 \, d\psi \right] + \int_0^1 h(y,v,T) y_\psi \, d\psi = 0 \quad . \tag{2.14}$$

This is equivalent tothe usual secularity conditions at all orders. In particular, to leading order this yields (2.9). More importantly, the phase shift will be determined by the $O(\varepsilon)$ terms of (2.14):

$$\frac{d}{dT}\left[2\omega \int_0^1 y_{0\psi} y_{1\psi} \, d\psi + \phi' \int_0^1 y_{0\psi}^2 \, d\psi \right]$$
$$+ \int_0^1 \left[h_y y_1 + h_v (\phi' y_{0\psi} + y_{0T} + \theta' y_{1\psi}) \right] y_{0\psi} \, d\psi + \int_0^1 h y_{1\psi} \, d\psi = 0 \quad . \tag{2.15}$$

Since h_v is even in ψ and h_y is odd in ψ, only the even part of y_1 is needed in (2.15). From (2.13)

$$y_{1_{even}} = \frac{\phi'(T)}{\omega_E} y_{0_E} \quad , \tag{2.16}$$

and thus (since $\int_0^1 h_v \, y_{0T} \, y_{0\psi} \, d\psi = 0$ from parity considerations) equation (2.15) becomes

$$2\frac{d}{dT}\left[\frac{\phi'\omega}{\omega_E}\int_0^1 y_{0_\psi} y_{0_{E\psi}}\, d\psi\right] + \frac{d}{dT}\left[\phi' \int_0^1 y_{0_\psi}^2\, d\psi\right]$$

$$+ \frac{\phi'}{\omega_E}\int_0^1 \left\{(\omega\, y_{0_{\psi_E}} + \omega_E\, y_{0_\psi}) y_{0_\psi} h_v \right. \tag{2.17a}$$

$$\left. + y_{0_E} y_{0_\psi} h_y + y_{0_{\psi E}} h\,(y_0, \omega y_{0_\psi}, T)\right\} d\psi = 0$$

$$\text{or } \frac{d}{dT}\left[\frac{\phi'}{\omega_E}\frac{\partial}{\partial E}\left(\omega \int_0^1 y_{0_\psi}^2\, d\psi\right)\right] + \frac{\phi'}{\omega_E}\frac{\partial}{\partial E}\int_0^1 h(y_0, \omega y_{0_\psi}, T) y_{0_\psi}\, d\psi = 0 \ . \tag{2.17b}$$

Equation (2.17a) is equivalent to (A.2), derived from the secularity condition at $O(\varepsilon^2)$. Thus, the elaborate calculations in Appendix A are not necessary to obtain the phase shift. In fact, we do not even need an expression for R_2. Using the definition of action (2.10) and dissipation (2.11), we obtain

$$\frac{d}{dT}\left[\frac{\phi'(T)}{\omega_E(E,T)} I_E\right] + \frac{\phi'(T)}{\omega_E(E,T)} D_E = 0 \ . \tag{2.17c}$$

However, action is dissipated via (2.9b), $D = -dI/dT$, so that (2.17c) becomes

$$\frac{d}{dT}\left[\frac{\phi'(T)}{\omega_E(E,T)} I_E\right] - \frac{\phi'(T)}{\omega_E(E,T)}\frac{\partial}{\partial E}\left(\frac{dI}{dT}\right) = 0 \ ,$$

which is equivalent to

$$I_E \frac{d}{dT}\left[\frac{\phi'(T)}{\omega_E(E,T)}\right] = 0 \ .$$

Since from (2.10) $I_E = 2\int_{y_{min}}^{y_{max}} [E - V(y_0,T)]^{-\frac{1}{2}} \frac{dy_0}{\sqrt{2}} = \frac{1}{\omega} \neq 0$,

it follows that

$$\frac{d}{dT}\left[\frac{\phi'(T)}{\omega_E(E,T)}\right] = 0 \quad . \tag{2.18}$$

Thus, (2.18) defines a new adiabatic invariant. Note that this is valid for nonlinear oscillators with rather arbitrary (small) dissipation. An alternative expression occurs if we observe that, from above, $\omega_E = \frac{-I_{EE}}{I_E^2}$. Here ω_E is determined from (2.7) (do not differentiate under the integral sign), where we note E(T) has been determined earlier from the coupling of (2.7) and (2.9b). We refer to $\phi(T)$ as the phase shift and $\phi'(T)$ as the perturbation of the modulated frequency. In the introduction to this paper we have described the meaning of (2.18). In particular, $\phi(T)$ and $\phi'(T)$ are now easily determined from their initial conditions.

Notice that the only unknown in (2.17) or (2.18) is $\phi(T)$. The arbitrary coefficient of the homogeneous solution in (2.13), $C_1(T)$, has dropped out of the equation as Luke showed [5]. We note that it is possible for $\phi'(T) = 0$, as Kuzmak [4] assumed, but in general (2.18) is a second-order equation. It is important to recognize this for the arbitrary initial value problem (see §3). We can determine C_1 by looking at the $O(\varepsilon^2)$ terms of the action equation (2.14). Proceeding in this manner, we can find C_k by looking at the $O(\varepsilon^{k+1})$ action equation. It is very important that this orthogonality condition not involve C_{k+1}, since P_{k+2} does, in general, involve y_{k+1}. Luke [5] showed this to be true for the non-dissipative case using secularity conditions, and it is easy to show for the general case. As is the case for $\phi(T)$, we would find that $C_k(T)$

satisfies a second-order equation.

3. Initial Value Problem

The importance of the new adiabatic invariant lies in the relevance of $\phi'(T)$ (in particular $\phi'(0)$) to the initial value problem for (2.1):

$$
\begin{aligned}
y(0) &= \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \ldots \\
\frac{dy}{dt}(0) &= \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \ldots
\end{aligned}
\tag{3.1}
$$

with α_i and β_i given. By using the multiple scale assumption and substituting the asymptotic expansion, we obtain

$$y_i(\phi(0),0) = \alpha_i \tag{3.2a}$$

$$\theta'(0)\,\frac{\partial y_0}{\partial\psi}(\phi(0),0) = \beta_0 \tag{3.2b}$$

$$\theta'(0)\,\frac{\partial y_i}{\partial\psi}(\phi(0),0) + \phi'(0)\,\frac{\partial y_{i-1}}{\partial\psi}(\phi(0),0) + \frac{\partial y_{i-1}}{\partial T}(\phi(0),0) = \beta_i \quad , \tag{3.2c}$$

since $\psi(0) = \phi(0)$ because $\theta(0) = 0$. To leading order, (3.2a) for $i = 0$ and (3.2b) form a closed system for $\phi(0)$ and $E(0)$, since from (2.7) $\theta'(T)$ can be written as a function of $E(T)$ and a known function of T ($\theta'(0) = \omega(E(0),0)$). $E(T)$ solves a first order equation (2.9a) or (2.9b), and therefore (3.2a) and (3.2b) determine $E(T)$ uniquely (and $\theta'(T) = \omega(E,T)$). In addition, $\theta(T)$ is uniquely determined by integration since $\theta(0) = 0$.

However, $\phi(T)$ solves a second-order equation and the information for the unique determination of $\phi(T)$ is, as yet, incomplete. Thus, we consider the $O(\varepsilon)$ terms for the initial value problem, (3.2a) and (3.2c), both with $i = 1$. We substitute the form of y_1 from (2.13) and obtain:

$$C_1(0)\, y_{0_\psi}(\phi(0),0) + \phi'(0)\,\frac{y_{0_E}(\phi(0),0)}{\omega_E(E(0),0)} = \alpha_1 - y_{1p_{odd}}(\phi(0),0)$$

$$C_1(0)\theta'(0)y_{0_{\psi\psi}}(\phi(0),0) + \phi'(0)\Big[\frac{\theta'(0)}{\omega_E(E(0),0)}\, y_{0_{E\psi}}(\phi(0),0) + y_{0_\psi}(\phi(0),0)\Big]$$
$$= \beta_1 - \theta'(0)\,\frac{\partial y_{1p_{odd}}}{\partial\psi}(\phi(0),0) - \frac{\partial y_0}{\partial T}(\phi(0),0) \quad . \tag{3.3}$$

In this form the righthand side is given or already determined. This linear system of equations for $C_1(0)$ and $\phi'(0)$ has a unique solution if and only if:

$$\begin{vmatrix} y_{0_\psi}(\phi(0),0) & \dfrac{y_{0_E}(\phi(0),0)}{\omega_E(E(0),0)} \\ \theta'(0)y_{0_{\psi\psi}}(\phi(0),0) & \dfrac{\theta'(0)}{\omega_E(E(0),0)}\, y_{0_{E\psi}}(\phi(0),0) + y_{0_\psi}(\phi(0),0) \end{vmatrix} \neq 0 \quad .$$

This determinant is just $\frac{1}{\omega_E(E(0),0)}$ times the Wronskian of the two homogeneous solutions y_{0_ψ} and y_H of the linearized operator evaluated at $\psi = \phi(0)$ and $T = 0$. By differentiating (2.5) with respect to E, the Wronskian equals $1/\omega$. Since these two solutions are linearly independent, the determinant is non-zero and a unique solution for $\phi'(0)$ and $C_1(0)$ exists. $C_1'(0)$ (necessary since $C_1(T)$ satisfies a second-order equation) and $C_2(0)$ may then be determined in a similar manner from the $O(\varepsilon^2)$ condition and so on. Thus, the method gives the solution for arbitrary initial conditions. Note that the $O(\varepsilon)$ initial conditions are necessary to determine $\phi'(0)$ and hence to determine the phase shift $\phi(T)$ and the small frequency modulation

$\phi'(T)$ corresponding to the leading order long time solution. This is not surprising as perturbed initial conditions change the energy and hence the frequency.

Using the expression for $y_{1_{p_{odd}}}$, we obtain from (3.3) and (2.18)

$$\frac{\phi'(T)}{\omega_E(E,T)} = \frac{\phi'(0)}{\omega_E(E(0),0)} = \left\{ \frac{\partial}{\partial T}\left[\omega \int_0^{\psi} y_{0_\psi}^2 \, d\psi\right] - \omega y_{0_\psi} y_{0_T} + \int_0^{\psi} h y_{0_\psi} \, d\psi \right. \tag{3.4}$$
$$\left. + \, \omega\left(\beta_1 y_{0_\psi} - \alpha_1 y_{0_{\psi\psi}}\right) \right\}_{\substack{T=0 \\ \psi = \phi(0)}} .$$

To simplify this, we differentiate the energy equation (2.5) with respect to T and integrate it from $\psi = 0$ to ψ:

$$\frac{\partial}{\partial T} \int_0^{\psi} \omega y_{0_\psi}^2 \, d\psi - \omega y_{0_\psi} y_{0_T} = \frac{\psi}{\omega}\frac{dE}{dT} - \frac{1}{\omega}\int_0^{\psi} V_T \, d\psi \quad . \tag{3.5}$$

Evaluating (3.5) at $\psi = 1$, yields the following expression for dE/dT, where the action equation (2.9) has been used:

$$\frac{dE}{dT} = \int_0^1 V_T d\psi - \omega \int_0^1 h y_{0_\psi} \, d\psi \quad . \tag{3.6}$$

By substituting (3.6) in (3.5) and by noting that the lhs of (3.5) appears in (3.4), we obtain

$$\frac{\phi'(T)}{\omega_E(E,T)} = \frac{\phi'(0)}{\omega_E(E(0),0)} = \left\{ \phi(0) \int_0^1 \left(\frac{V_T}{\omega} - h y_{0_\psi}\right) d\psi - \int_0^{\phi(0)} \left(\frac{V_T}{\omega} - h y_{0_\psi}\right) d\psi \right. \tag{3.7}$$
$$\left. + \, \omega(\beta_1 y_{0_\psi} - \alpha_1 y_{0_{\psi\psi}}) \right\}_{\substack{T=0 \\ \psi = \phi(0)}} .$$

Equation (3.7) is significant. It determines the perturbed frequency and shows how it depends on the dissipation, $\int h y_{0_\psi} d\psi$, the slow time-dependence of the potential, V_T, and the perturbed initial conditions, α_1 and β_1. For example, a simpler expression results if $\alpha_1 = \beta_1 = 0$ and there is no dissipation ($h = 0$); the frequency changes are only due to a time-dependent potential. If in addition, $V_T = 0$, then $\phi(T)$ is constant, as is clear since in this special case there is no slow variation of the solution.

4. Discussion: Higher-order Frequency and Averaged Perturbed Energy.

The leading-order equations for the slow variation of a nonlinear oscillator are the amplitude-dependent frequency (2.7) (based on the leading-order energy) and the action equation (2.9). Our result concerning the phase shift is best understood by considering higher-order expressions for both. In this section, we note that changes in the frequency and action equations only result from the linearization of them due to perturbing the averaged energy, if the perturbation is dissipative, as we have assumed.

In Appendix B, it is shown that the frequency (2.7) is valid to $O(\varepsilon)$ if the average perturbed energy E is utilized in (2.7). Thus, the Taylor expansion of the frequency $\omega = \omega(E)$ yields

$$\omega_1 = E_1 \omega_E \quad , \tag{4.1}$$

where E_1 is the $O(\varepsilon)$ perturbation of the averaged energy and $\omega_1 = d\phi/dT$ is the $O(\varepsilon)$ perturbation of the frequency (the frequency shift or derivative of the phase shift). Here, <u>frequency changes are only due to the averaged energy changes</u>, and thus, we might reason intuitively that (4.1) should be valid.

Equation (4.1) is quite significant. For example, the leading-order nonlinear oscillator satisfies

$$y_0 = y_0(\theta;\ E(T),\ T) \quad .$$

We expect that some of the perturbed solution y_1 results from changes in E. In fact,

$$y_{1_{even}} = E_1 \, y_{0_E} \tag{4.2}$$

since from (4.1) $E_1 = d\phi/dT/\omega_E$, showing that (4.2) is equivalent to (2.16). Thus, (2.16) or (4.2) corresponds to the response due to perturbed average energy levels.

Furthermore, (4.1) leads to an understanding of the phase shift. Since action is valid to higher-order from (2.14) and both action and the dissipation depend on the averaged energy E [7], it follows that

$$\frac{d}{dT} I(E,T) + D(E,T) = 0 \quad . \tag{4.3}$$

To leading order, this yields (2.9). Higher-order terms may be simply obtained by perturbing the averaged energy:

$$\frac{d}{dT} (E_1 \, I_E) + E_1 \, D_E = 0 \quad . \tag{4.4}$$

However, the partial derivative of (4.3) with respect to E is also valid. Thus, (4.4) becomes

$$I_E \frac{d}{dT} (E_1) = 0 \quad .$$

This shows that the averaged perturbed energy E_1 does not vary. Using (4.1), we obtain the adiabatic invariant, $\frac{1}{\omega_E} \, d\phi/dT$:

$$\frac{d}{dT} \left(\frac{d\phi/dT}{\omega_E}\right) = 0 \quad . \tag{4.5}$$

Our results are only valid for the dissipative perturbation ϵh . For non-dissipative perturbations or $O(\epsilon^2)$ dissipative perturbations, (4.5) becomes a non-homogeneous equation for $d\phi/dT$.

Recently, we have extended the ideas of this paper to obtain the modulated phase shift for oscillatory solutions of nonlinear dispersive waves ([7],[8], and [9]).

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Appendix A: Higher-order secularity condition.

In this appendix, we show that the higher-order action equation (2.15) is equivalent to the result of eliminating the secular terms in the $O(\varepsilon^2)$ equations:

$$\int_0^1 R_2 y_{0_\psi} d\psi = 0 \quad . \tag{A.1}$$

The term $\int_0^1 V_{yyy}(y_0,T)y_1{}^2 y_{0_\psi} d\psi$ in (A.1) appears to be difficult to calculate since y_1 is given by (2.13). However, if we integrate by parts, recall the definition of L, and then use $L(y_1) = R_1$, we obtain

$$\begin{aligned}\int_0^1 V_{yyy}(y_0,T)y_1{}^2 y_{0_\psi} d\psi &= -2\int_0^1 V_{yy}(y_0,T)y_1 y_{1_\psi} d\psi \\ &= -2\int_0^1 [L(y_1) - \omega^2 y_{1_{\psi\psi}}] y_{1_\psi} d\psi \\ &= +2\int_0^1 [2\omega\phi' y_{0_{\psi\psi}} + \frac{d\omega}{dT} y_{0_\psi} + 2\omega y_{0_{\psi T}} + h(y_0,\omega y_{0_\psi},T)] y_{1_\psi} d\psi \quad ,\end{aligned}$$

where the periodicity of many expressions has also been used. In this manner, from (A.1), we obtain

$$\begin{aligned} &2\phi'\omega \int_0^1 \frac{\partial}{\partial\psi}(y_{0_\psi} y_{1_\psi}) d\psi + 2\frac{d}{dT}[\omega \int_0^1 y_{0_\psi} y_{1_\psi} d\psi] \\ &+ \int_0^1 h(y_0,\omega y_{0_\psi},T) y_{1_\psi} d\psi + \phi' \int_0^1 (y_{0_\psi})^2 h_v d\psi + \omega \int_0^1 y_{1_\psi} y_{0_\psi} h_v d\psi \\ &+ \int_0^1 y_{0_\psi} y_1 h_y d\psi + \frac{d}{dT}[\phi' \int_0^1 (y_{0_\psi})^2 d\psi] \\ &+ (\phi')^2 \underline{\int_0^1 y_{0_{\psi\psi}} y_{0_\psi} d\psi} + \int_0^1 y_{0_\psi} y_{0_T} h_v d\psi \\ &+ \underline{\int_0^1 y_{0_{TT}} y_{0_\psi} d\psi} = 0 \quad . \end{aligned} \tag{A.2}$$

This corrects an error in Dobrokhotov and Maslov [1] due to an omitted term. The first integral is exact and vanishes due to periodicity considerations. The underlined integrals vanish from parity considerations. Thus, (A.2) is equivalent to (2.15) derived from the higher-order action equation.

Appendix B: Averaged Perturbed Energy.

It is useful in interpreting the phase shift to show that (4.1) is valid, that is, to evaluate the averaged perturbed energy. The averaged energy is defined to be

$$E \equiv \int_0^1 \left[\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + V(y,T)\right] d\psi \quad . \tag{B.1}$$

When y is expanded and the multiple scale assumption is used, the leading-order satisfies (2.5) with E replaced by E_0. If E_1 is defined as the $O(\varepsilon)$ perturbation of the averaged energy, then from (B.1) we obtain

$$E_1 = \int_0^1 \left[\theta' y_{0\psi}(\theta' y_{1\psi} + \phi' y_{0\psi} + y_{0T} - (\theta')^2 y_1 y_{0\psi\psi}\right] d\psi \quad , \tag{B.2}$$

using (2.3). We note that only $y_{1_{even}}$ is needed in (B.2), and thus, using parity arguments and (2.16), we obtain

$$E_1 = d\phi/dT/\omega_E \quad ,$$

proving that (4.1) is valid. In a similar way it can be shown that (4.3) is valid to one higher order.

References

[1] S. Yu. Dobrokhotov and V. P. Maslov, Finite-zone, almost-periodic solutions in WKB approximations, Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, 15 (1980), pp. 3-94. J. Sov. Math. 15 (1980), 1433-1487.

[2] J. Kevorkian and J. D. Cole, Perturbation Methods in Applied Mathematics, Springer-Verlag, New York, 1981.

[3] S. Kogelman and J. B. Keller, Asymptotic theory of nonlinear wave propagation, SIAM J. Appl. Math. 24 (1973), pp. 352-361.

[4] G. E. Kuzmak, Asymptotic solutions of nonlinear second order differential equations with variable coefficients, Prikl. Mat. Meh. 23 (1959), pp. 515-526. J. Appl. Math. Mech. 23 (1959), pp. 730-744.

[5] J. C. Luke, A perturbation method for nonlinear dispersive wave problems, Proc. Roy. Soc. London Ser. A 292 (1966), pp. 403-412.

[6] G. B. Whitham, Linear and Nonlinear Waves, John Wiley & Sons, New York, 1974.

[7] R. Haberman and F. J. Bourland, Variation of wave action: Modulations of the phase shift for strongly nonlinear dispersive waves with weak dissipation, submitted.

[8] F. J. Bourland and R. Haberman, Dissipatively modulated nonlinear dispersive waves with a Lagrangian formulation, in preparation.

[9] R. Haberman, The modulated phase shift for weakly dissipated nonlinear oscillatory waves of the Korteweg-deVries type, submitted.

[10] G. B. Whitham, Two-timing, variational principles and waves, J. Fluid Mech. 44(1970), pp. 373-395.

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The Modulated Phase Shift for Weakly Dissipated Nonlinear Oscillatory Waves of the Korteweg-de Vries Type

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Abstract

Nonlinear dispersive oscillatory waves are analyzed for Korteweg-deVries type partial differential equations with slowly varying coefficients and arbitrary small perturbations. Spatial and temporal evolution of the amplitude parameters are determined in the usual way by the possible dissipation of the wave actions for both momentum and energy. For dissipative perturbations, both wave actions are shown to be valid to a higher order. Thus, the first variation of the wave action equations is used to derive equations for the slow modulations of the phase shift. It is shown that the phase shift satisfies a universal set of two coupled equations, each independent of the small dissipative perturbation.

Introduction

Kuzmak [4] made fundamental contributions to the method of multiple scales in his analysis of the effects of weak dissipation and variable media on strongly nonlinear oscillators satisfying ordinary differential equations. The slowly varying frequency was shown to equal the local frequency, depending on the amplitude. Kuzmak [4] also obtained the differential equation for the slowly varying amplitude (a nonlinear generalization of the dissipation of action). For strongly nonlinear dispersive waves (but without dissipation), satisfying the Klein-Gordon equation (a partial differential equation) with variable coefficients, Luke [5] obtained the dispersion relation and the partial differential equation for the spatial and temporal slow evolution of the amplitude. Whitham [6] generalized this by introducing the concept of wave action, enabling as an example the calculation of the effects of variable media and weak dissipation on the cnoidal waves for the Korteweg-deVries equation.

Kuzmak [4] assumed the phase shift <u>was</u> constant, while for a dissipationless system, Luke [5] claimed that the phase shift <u>could be</u> constant, an important distinction that has been very understandably overlooked by many. Unfortunately, when Dobrokhotov and Maslov [2] reinvestigated this issue they made a critical error. Although they obtained a second-order equation,

their results were incorrect. Furthermore, they only analyzed unperturbed dissipationless systems.

Quite recently, Bourland and Haberman [1] analyzed nonlinear oscillators with dissipation. Using some of the good ideas of Dobrokhotov and Maslov [2], a simple second-order ordinary differential equation for the phase shift was obtained. Most significantly, it was of universal form, independent of the type of dissipation. Furthermore, there was a simple adiabatic invariant, enabling the phase shift to be determined for arbitrary linear or nonlinear dissipation.

The simplicity and generality of the phase shift for nonlinear oscillatory ordinary differential equations strongly suggested that similar results would be valid for weakly dissipated nonlinear dispersive waves propagating in a slowly varying media. Efforts were simultaneously undertaken for Klein-Gordon and Korteweg-deVries type equations; each guided the development of the other. For the Klein-Gordon equation, the dissipation of wave action is fundamental, in one spatial dimension,

$$\frac{\partial I}{\partial T} + \frac{\partial q}{\partial X} = -R \ , \tag{1.1}$$

where I is the wave action, q is the flux of wave action, and R is the dissipation. Haberman and Bourland [3] showed that the phase shift satisfies the same partial differential equation, independent of the type of dissipation:

$$\hat{D}_T I + \hat{D}_X q = 0 \quad , \tag{1.2}$$

where $\hat{D}$ is the Taylor series operator resulting from changes in the wave number and averaged perturbed energy. In [3] it is shown that $\hat{D}$ depends on the phase shift in a linear way, so that (1.2) is a linear partial differential equation for the phase shift. Once $\hat{D}$ has been determined, (1.2) can be obtained from (1.1) by replacing $\frac{\partial}{\partial T}$ by $\hat{D}_T$ and $\frac{\partial}{\partial X}$ by $\hat{D}_X$ and by neglecting the dissipation.

In this paper, we analyze Korteweg-deVries type equations for a variety of reasons. It is an important type of equation because of its frequent occurrence in physical problems and it can have great mathematical interest in the remarkable cases where it is integrable and known to have exact multi-phase periodic solutions. However, we investigate slow variations because nonlinear periodic waves satisfy a third-order equation (not self-adjoint) which is thus more difficult than the corresponding problem for the Klein-Gordon equation. For Korteweg-deVries type equations, there are two independent amplitude parameters, while for the Klein-Gordon there was only one.

In §2, the method of multiple scales is used to analyze Korteweg-deVries type equations with arbitrary small perturbations. Two equations are derived representing the well-know dissipation of wave action for momentum and energy, both in the form of (1.1), with I, q replaced by I_i, q_i for i = 1,2. In §3, higher order terms are shown to be obtained from the first variation of a form of the wave action equations, an apparently general result since it is also valid for the Klein-Gordon equation [3]. In §4, one operator $\hat{D}$, depending on the phase shift, is determined such that (1.2) is valid for both wave actions. In this way, the phase shift for Korteweg-deVries type equations is shown to satisfy a coupled system of partial differential equations (1.2) of universal form, independent of the type of dissipation. The equations (1.2) are easy to express once the operator $\hat{D}$ is determined. In §5, we show that these results correspond to the linearization of the dispersion relation and the wave action equations with respect to perturbations of the wave number and the averaged values of the amplitude parameters. The operator $\hat{D}$ is now the Taylor series operator in these parameters.

2. Korteweg-deVries Type Equations.

In this paper, we analyze nonlinear partial differential equations with slowly varying coefficients of the Korteweg-deVries (KdV) type:

$$u_t + \frac{1}{2} Q_{uu}(u,X,T)u_x + \alpha^2(X,T)u_{xxx}$$

$$= -\varepsilon h(u,u_x,u_{xx},X,T) - \varepsilon^2 h_1(u,u_x,u_{xx},X,T) \quad , \qquad (2.1a)$$

where $\frac{1}{2} Q_{uu}(u,X,T)$ is the convective velocity. To correspond to nonlinear wave phenomena in a slowly varying media, we allow quantities to depend on the slow time and spatial scales, $T = \varepsilon t$ and $X = \varepsilon x$. As we will show later in this section, it is possible to assume that $\frac{1}{2} Q_{uu}(u,X,T)$ can be chosen such that there are nonlinear oscillatory wave solutions of (2.1a)(for all fixed X and T) on the fast scale t when the perturbations εh and $\varepsilon^2 h_1$ are absent. Typical examples are $Q = \frac{1}{3}\beta(X,T)u^3$, in which case (2.1) becomes the slowly varying and perturbed Korteweg-deVries equation, and $Q = \frac{1}{6}\beta(X,T)u^4$, in which in a similar way (2.1a) relates to the modified Korteweg-deVries equation.

The term $\varepsilon^2 h_1$ is introduced since $O(\varepsilon^2)$ perturbations can influence the leading order solution for long time. At first, we calculate the effects of arbitrary slowly varying perturbations εh and $\varepsilon^2 h_1$. However, our primary interest is in dissipative perturbations. Thus, later in the paper we assume εh is dissipative, but simplifications will only occur if we assume in addition that $\varepsilon^2 h_1$ is non-dissipative.

We briefly review some elementary ideas concerning conservation laws. We interpret u as momentum-density, so that (2.1a)itself expresses the possible dissipation of momentum by the perturbations:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}Q_u + \alpha^2 u_{xx}\right) = \frac{1}{2} Q_{ux} + 2\alpha\alpha_x u_{xx} - \varepsilon h - \varepsilon^2 h_1 , \qquad (2.1b)$$

where $\frac{1}{2}Q_u + \alpha^2 u_{xx}$ is the momentum -flux. Here, momentum also may change due to the variable media, represented by the terms $\frac{1}{2}Q_{ux}$ and $2\alpha\alpha_x u_{xx}$. Similarly, multiplying (2.1a) by u shows how the energy may be dissipated:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}uQ_u - \frac{1}{2}Q\right) + \alpha^2 \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(uu_x) - \frac{3}{2}u_x^2\right] = \frac{1}{2}uQ_{ux} - \frac{1}{2}Q_x$$
$$- \varepsilon hu - \varepsilon^2 h_1 u \quad . \qquad (2.1c)$$

Thus, $\frac{1}{2}uQ_u - \frac{1}{2}Q + \alpha^2[\frac{\partial}{\partial x}(uu_x) - \frac{3}{2}u_x^2]$ is the energy flux.

Typical linear perturbations are $h = h_0 u$, which dissipates both momentum and energy, and $-h = \nu u_{xx}$, a diffusive term which conserves momentum, but dissipates energy. In this latter case, (2.1a) is of the KdV-Burgers type. However, for the moment we will be quite general and make no assumptions concerning εh or $\varepsilon^2 h_1$. Later, when we assume that εh represents dissipation, we will allow any kind including nonlinear types.

To obtain the long time behavior of nonlinear dispersive waves including the ffects of a slowly varying media and the small perturbations, the method of multiple scales will be used. We follow Luke's [5] procedure for the Klein-Gordon equation as re-examined by Haberman and Bourland [3]:

$$\psi = \frac{\theta(X,T)}{\varepsilon} + \phi(X,T)$$

$$T = \varepsilon t$$

$$X = \varepsilon x \quad .$$

This method is particularly suited for determining the modulations of the phase shift $\phi(X,T)$. For example,

$$\frac{\partial u}{\partial X} = (\theta_X + \varepsilon\phi_X)u_\psi + \varepsilon u_X \quad .$$

In this way, (2.1a) becomes "exactly"

$$(\theta_T + \varepsilon\phi_T)u_\psi + \varepsilon u_T + \frac{1}{2}(\theta_X + \varepsilon\phi_X)\frac{\partial}{\partial\psi}Q_u + \frac{\varepsilon}{2}\frac{\partial}{\partial X}Q_u - \frac{\varepsilon}{2}Q_{uX}$$

$$+ \alpha^2(X,T)[(\theta_X + \varepsilon\phi_X)^3 u_{\psi\psi\psi} + 3\varepsilon(\theta_X + \varepsilon\phi_X)^2 u_{\psi\psi X}$$

$$+ 3\varepsilon(\theta_{XX} + \varepsilon\phi_{XX})(\theta_X + \varepsilon\phi_X) u_{\psi\psi} + 3\varepsilon^2(\theta_X + \varepsilon\phi_X) u_{XX\psi}$$

$$+ 3\varepsilon^2(\theta_{XX} + \varepsilon\phi_{XX}) u_{\psi X} + \varepsilon^2(\theta_{XXX} + \varepsilon\phi_{XXX})u_\psi + \varepsilon^3 u_{XXX}]$$

$$= -\varepsilon h[u, \theta_X u_\psi + \varepsilon\phi_X u_\psi + \varepsilon u_X, (\theta_X)^2 u_{\psi\psi} +$$

$$+ 2\varepsilon\theta_X\phi_X u_{\psi\psi} + 2\varepsilon\theta_X u_{\psi X} + \varepsilon\theta_{XX}u_\psi + O(\varepsilon^2), X,T]$$

$$- \varepsilon^2 h_1[u, \theta_X u_\psi + O(\varepsilon), (\theta_X)^2 u_{\psi\psi} + O(\varepsilon), X,T] \quad . \tag{2.2}$$

The substitution of the perturbation expansion,

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \quad ,$$

will be facilitated by noting

$$Q_u(u,X,T) = Q_u(u_0,X,T) + (\varepsilon u_1 + \varepsilon^2 u_2) Q_{uu}(u_0,X,T)$$

$$+ \varepsilon^2 \frac{u_1^2}{2} Q_{uuu}(u_0,X,T) + O(\varepsilon^3)$$

$$h[\text{as in (2.2)}] = h(u_0, \theta_X u_{0_\psi}, (\theta_X)^2 u_{0_{\psi\psi}}, X, T)$$

$$+ \varepsilon[h_u u_1 + h_{u_X}(\theta_X u_{1_\psi} + \phi_X u_{0_\psi} + u_{0_X})$$

$$+ h_{u_{XX}}(\theta_X^2 u_{1_{\psi\psi}} + 2\theta_X \phi_X u_{0_{\psi\psi}} + 2\theta_X u_{0_{\psi X}} + \theta_{XX} u_{0_\psi})] + O(\varepsilon^2) \quad ,$$

where h_u, h_{u_X}, and $h_{u_{XX}}$ are evaluated at u_0, $\theta_X u_{0_\psi}$, $\theta_X^2 u_{0_{\psi\psi}}$, X,T.
We obtain in this way, to leading order, the nonlinear ordinary differential equation which represents travelling wave solutions of the KdV type equations:

$$\theta_T u_{0_\psi} + \frac{1}{2}\theta_X \frac{\partial}{\partial\psi} Q_u(u_0, X, T) + \beta^2(\theta_X)^3 u_{0_{\psi\psi\psi}} = 0 \quad , \qquad (2.3)$$

where $\beta^2 \equiv \beta^2(X,T)$. Equation (2.3) is solved in the usual way. Integration yields

$$\theta_T u_0 + \frac{1}{2}\theta_X Q_u(u_0, X, T) + \beta^2(\theta_X)^3 u_{0_{\psi\psi}} = -B \quad , \qquad (2.4)$$

where $B \equiv B(X,T)$. By multiplying by $2u_{0_\psi}$ and integrating, we obtain an equation which can be interpreted as conservation of energy for the system (2.4):

$$\theta_T u_0^2 + \theta_X Q(u_0, X, T) + \beta^2(\theta_X)^3 (u_{0_\psi})^2 = -2Bu_0 + 2A \quad , \qquad (2.5)$$

where $A = A(X,T)$. The parameters $A(X,T)$ and $B(X,T)$ are similar in spirit to those introduced by Whitham [6] for the slowly varying KdV equation. Our analysis is valid for those $Q(u_0,X,T)$ for which there are periodic solutions, see the "potential" $\theta_T u_0^2 + \theta_X Q(u_0,X,T)$ of (2.5). From (2.5), we can define $u_0(\psi,X,T)$ to be an even function of ψ. We introduce the wave number $k \equiv \theta_X$ and frequency $\omega = -\theta_T$. Luke [5] showed that the period in the fast variable ψ is constant. If we normalize that period to 1, we obtain the amplitude dependent dispersion relation, $\omega = \omega(k(X,T), A(X,T), B(X,T), X,T)$, for the nonlinear dispersive wave in a nonhomogeneous media:

$$\alpha k^{3/2} \int_{u_{0_{min}}}^{u_{0_{max}}} \frac{du_0}{\sqrt{2A - 2Bu_0 + \omega u_0^2 - kQ(u_0,X,T)}} = \frac{1}{2} \quad , \tag{2.6}$$

where the wave oscillates from $u_{0_{min}}$ to $u_{0_{max}}$, two successive zeroes of the denominator of (2.6). Here, there are two amplitude parameters $A(X,T)$ and $B(X,T)$ unlike the case of the Klein-Gordon equation [3] where there is only one $E(X,T)$.

The higher order equations in the perturbation expansion are

$$L(u_i) = R_i \qquad i = 1,2,\ldots \quad , \tag{2.7}$$

where L is the linearization of the nonlinear operator in (2.3):

$$L \equiv \theta_T \frac{\partial}{\partial\psi} + \frac{1}{2}\theta_X \frac{\partial}{\partial\psi} Q_{uu} + \alpha^2\theta_X^3 \frac{\partial^3}{\partial\psi^3} \quad . \tag{2.8}$$

In a straight-forward manner, we obtain R_1, the term we will need to determine the amplitude parameters $A(X,T)$ and $B(X,T)$ and the phase shift $\phi(X,T)$. We do not need R_2, though it will be used in Appendix A to show that our procedure is equivalent to that used by Dobrokhotov and Maslov [2]:

$$
\begin{aligned}
R_1 = {} & -\phi_T u_{0_\psi} - u_{0_T} - \frac{1}{2}\phi_X \frac{\partial}{\partial\psi} Q_u - \frac{1}{2}\frac{\partial}{\partial X} Q_u + \frac{1}{2} Q_{uX} \\
& - \alpha^2 (3\theta_X^2 \phi_X u_{0_{\psi\psi\psi}} + 3\theta_X^2 u_{0_{\psi\psi X}} + 3\theta_{XX}\theta_X u_{0_{\psi\psi}}) \\
& - h(u_0, \theta_X u_{0_\psi}, \theta_X^2 u_{0_{\psi\psi}}, X, T)
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
R_2 = {} & -\phi_T u_{1_\psi} - u_{1_T} - \frac{1}{2}\theta_X \frac{\partial}{\partial\psi}(u_1^2 Q_{uuu}) \\
& - \frac{1}{2}\phi_X \frac{\partial}{\partial\psi}(u_1 Q_{uu}) - \frac{1}{2}\frac{\partial}{\partial X}(u_1 Q_{uu}) + \frac{1}{2} u_1 Q_{uuX} \\
& - \alpha^2 (3\theta_X^2 \phi_X u_{1_{\psi\psi\psi}} + 3\theta_X \phi_X^2 u_{0_{\psi\psi\psi}} + 3\theta_X^2 u_{1_{\psi\psi X}} \\
& + 6\theta_X \phi_X u_{0_{\psi\psi X}} + 3\theta_{XX}\theta_X u_{1_{\psi\psi}} + 3\phi_{XX}\theta_X u_{0_{\psi\psi}} \\
& + 3\theta_{XX}\phi_X u_{0_{\psi\psi}} + 3\theta_X u_{0_{XX\psi}} + 3\theta_{XX} u_{0_{\psi X}} + \theta_{XXX} u_{0_\psi}) \\
& - h_1(u_0, \theta_X u_{0_\psi}, \theta_X^2 u_{0_{\psi\psi}}, X, T) - u_1 h_u - (\theta_X u_{1_\psi} + \phi_X u_{0_\psi} + u_{0_X}) h_{u_X} \\
& - (\theta_X^2 u_{1_{\psi\psi}} + 2\theta_X \phi_X u_{0_{\psi\psi}} + 2\theta_X u_{0_{\psi X}} + \theta_{XX} u_{0_\psi}) h_{u_{XX}} \, .
\end{aligned} \tag{2.10}
$$

Equation (2.7) is a third-order equation. In order for u_i to be periodic (in ψ), the Fredholm alternative implies that R_i must be orthogonal to all periodic solutions, 1 and u_0, of the homogeneous adjoint equation:

$$\int_0^1 R_i \, d\psi = 0 \qquad i = 1,2, \ldots \tag{2.11}$$

$$\int_0^1 R_i \, u_0 \, d\psi = 0 \qquad i = 1,2, \ldots \; . \tag{2.12}$$

At $O(\varepsilon)$, these yield the well-known equations to determine $A(X,T)$ and $B(X,T)$. Using (2.9) and (2.11), directly yields

$$\frac{\partial}{\partial T} \int_0^1 u_0 d\psi + \frac{1}{2} \frac{\partial}{\partial X} \int_0^1 Q_u d\psi = \frac{1}{2} \int_0^1 Q_{uX} d\psi - \int_0^1 h(u_0, \theta_X u_{0_\psi}, \theta_X^2 u_{0_{\psi\psi}}, X, T) d\psi \tag{2.13}$$

since u_0 is periodic. This shows that wave-momentum action may not be conserved due to the *rhs* of (2.13). Using (2.9) and (2.12), after some elementary algebra, we derive a similar result for the possible variation of wave-energy action:

$$\frac{\partial}{\partial T} \int_0^1 \frac{1}{2} u_0^2 d\psi + \frac{1}{2} \frac{\partial}{\partial X} \int_0^1 (u_0 Q_u - Q) d\psi + \frac{1}{2} \int_0^1 Q_X d\psi - \frac{1}{2} \int_0^1 u Q_{uX} d\psi$$

$$- \alpha^2 \frac{\partial}{\partial X} \left[\frac{3}{2}(\theta_X)^2 \int_0^1 (u_{0_\psi})^2 d\psi\right]$$

$$= - \int_0^1 h(u_0, \theta_X u_{0_\psi}, \theta_X^2 u_{0_{\psi\psi}}, X, T) u_0 d\psi \; . \tag{2.14}$$

Here, Q_X is the partial derivative of Q with respect to X keeping u fixed. In many problems, the convective velocity Q_{uu} only depends on u, and thus in this case, $Q_X = 0$, preserving the conservation form of (2.13) and (2.14) if $h = 0$. The integrals in (2.13) and (2.14) depend on $A(X,T)$, $B(X,T)$, $k(X,T)$, and $\omega(A,B,k,X,T)$. Thus the dispersion relation (2.6) and conservation of waves,

$$\frac{\partial}{\partial T} k + \frac{\partial}{\partial X} \omega = 0 \quad , \tag{2.15}$$

combined with (2.13) and (2.14) determine the amplitude parameters $A(X,T)$ and $B(X,T)$ and the dominant phase parameters $k(X,T)$ and $\omega(A,B,k,X,T)$, but not the phase shift $\phi(X,T)$. Equations (2.13) and (2.14) agree with Whitham's [6] result for the KdV. However, for the KdV, $Q = \frac{1}{3}\beta u^3$ and thus the momentum flux Q_u is proportional to the energy density u^2, reducing by one the number of independent integrals in (2.13) and (2.14). These results also show that even though the KDV has an infinite number of conservaiton laws, momentum and energy are the only appropriate ones to consider when perturbations are applied.

3. Higher-Order Wave Actions.

In Appendix A, we determine the equations for the modulations of the phase shift $\phi(X,T)$ by eliminating secular terms at $O(\varepsilon^2)$:

$$\int_0^1 R_2 \, d\psi = 0 \quad \text{and} \quad \int_0^1 R_2 u_0 d\psi = 0 \quad . \tag{3.1a,b}$$

This approach is somewhat involved since it requires R_2 given by (2.10). Here, we use a simpler method which yields equivalent results. We show that the phase shift is determined from the $O(\varepsilon)$ wave action equations.

As Whitham [6] has shown for Klein-Gordon equations, exact equations for the wave actions may be derived. For KdV type equations, an exact expression for wave-momentum action is obtained by integrating (2.2) over ψ from 0 to 1 and applying the periodicity of u:

$$\frac{\partial}{\partial T}\int_0^1 u\,d\psi + \frac{1}{2}\frac{\partial}{\partial X}\int_0^1 Q_u\,d\psi - \frac{1}{2}\int_0^1 Q_{uX}\,d\psi + \varepsilon^2\alpha^2\frac{\partial^3}{\partial X^3}\int_0^1 u\,d\psi$$

$$= -\int_0^1 h\,d\psi - \varepsilon\int_0^1 h_1\,d\psi \quad . \tag{3.2}$$

By first multiplying (2.2) by u and then integrating, we obtain (with more effort) an exact expression for wave-energy action:

$$\frac{\partial}{\partial T}\int_0^1 \frac{1}{2}u^2\,d\psi + \frac{1}{2}\frac{\partial}{\partial X}\int_0^1 (uQ_u - Q)\,d\psi + \frac{1}{2}\int_0^1 Q_X\,d\psi - \frac{1}{2}\int_0^1 uQ_{uX}\,d\psi$$

$$-\frac{3}{2}\alpha^2\frac{\partial}{\partial X}\left[(\theta_X + \varepsilon\phi_X)^2\int_0^1 u_\psi^2\,d\psi\right] - 3\alpha^2\varepsilon\frac{\partial}{\partial X}\left[(\theta_X + \varepsilon\phi_X)\int_0^1 u_\psi u_X\,d\psi\right]$$

$$+\varepsilon^2\alpha^2\int_0^1 uu_{XXX}\,d\psi = -\int_0^1 hu\,d\psi - \varepsilon\int_0^1 h_1 u\,d\psi \quad . \tag{3.3}$$

It is perhaps even simpler to obtain (3.2) and (3.3) by directly introducing the multiple-scale assumption into (2.1b) and (2.1c), and then integrating. Wave-momentum action (3.2) is "identical" to the leading-order result (2.13) if the entire u is used as well as the total wave number, $\theta_X + \varepsilon\phi_X$, except for the effects of the perturbations and $O(\varepsilon^3)$ dispersive term. However, wave-energy action (3.3) is alterred slightly by the additional term $-3\alpha^2\varepsilon\frac{\partial}{\partial X}[(\theta_X + \varepsilon\phi_X)\int_0^1 u_\psi u_X d\psi]$. This is an $O(\varepsilon)$ correction, but since u_0 is even in ψ, it is in fact smaller being at least $O(\varepsilon^2)$. Thus, we have shown that both wave actions are valid to at least one higher order if the effects of the perturbations, εh and $\varepsilon^2 h_1$, are properly introduced.

These exact wave action equations, (3.2) and (3.3), are equivalent to eliminating secular terms at all orders. For example, (3.2) is the same to leading order as (2.13), and (3.3) relates similarly to (2.14). Moreover, evaluating the exact action equations to $\mathcal{O}(\varepsilon)$ is equivalent to eliminating secular terms at $\mathcal{O}(\varepsilon^2)$ (see Appendix A). In particular, the $\mathcal{O}(\varepsilon)$ wave-momentum action terms from (3.2) yield

$$\frac{\partial}{\partial T}\int_0^1 u_1 d\psi + \frac{1}{2}\frac{\partial}{\partial X}\int_0^1 Q_{uu} u_1 d\psi - \frac{1}{2}\int_0^1 Q_{uuX}\, u_1\, d\psi = -\int_0^1 \delta_h d\psi - \int_0^1 \delta_s\, d\psi \quad , \tag{3.4}$$

and similarly the wave-energy action terms from (3.3) become

$$\begin{aligned}\frac{\partial}{\partial T}\int_0^1 u_0 u_1 d\psi &+ \frac{1}{2}\frac{\partial}{\partial X}\int_0^1 u_0 u_1 Q_{uu}\, d\psi - \frac{1}{2}\int_0^1 u_0 u_1 Q_{uuX}\, d\psi \\ &- 3\alpha^2 \frac{\partial}{\partial X}\left[\theta_X^2 \int_0^1 u_{1_\psi}\, u_{0_\psi}\, d\psi\right] - 3\alpha^2 \frac{\partial}{\partial X}\left[\theta_X \phi_X \int_0^1 (u_{0_\psi})^2\, d\psi\right] \\ &= -\int_0^1 \delta_h\, u_0\, d\psi - \int_0^1 \delta_s\, u_0\, d\psi \quad ,\end{aligned} \tag{3.5}$$

where we have separated the terms δ_h resulting from the variation of h and the terms δ_s resulting from the $\mathcal{O}(\varepsilon^2)$ perturbation and the slow derivatives of Q, u_0, and k:

$$\delta_h = h_u u_1 + h_{u_x}(\theta_X u_{1_\psi} + \phi_X u_{0_\psi}) + h_{u_{xx}}(\theta_X^2 u_{1\psi\psi} + 2\theta_X \phi_X u_{0\psi\psi})$$

$$\delta_s = h_{u_x}\, u_{0_X} + h_{u_{xx}}(2\theta_X u_{0_{\psi X}} + \theta_{XX}\, u_{0\psi}) + h_1 \quad .$$

For εh to be a dissipative perturbation,* $h(u,u_x,u_{xx},X,T)$ must be even in the u_x argument [i.e. $h(u,-u_x,u_{xx},X,T) = h(u,u_x,u_{xx},X,T)$]. Since u_0 is even in ψ, in

*Generalizations to other kinds of dissipation (for example, dependent on third or higher derivatives) are easily made.

this case $h(u_{0\psi}, \theta_X u_{0\psi}, \theta_X^2 u_{0\psi\psi}, X, T)$ is even in ψ, h_{u_X} is odd, and $h_{u_{XX}}$ is even. We thus obtain that the slowly varying terms δ_s have no effect, $\int_0^1 \delta_s \, d\psi = \int_0^1 \delta_s u_{0\psi} \, d\psi = 0$, if εh is dissipative and $\varepsilon^2 h_1$ is non-dissipative (odd in the u_x argument). In this case (3.2) and (3.3) are the first variation of the wave action equations. Therefore, the wave action equations (2.13) and (2.14) are valid to at least one higher order if εh is dissipative and $\varepsilon^2 h_1$ is non-dissipative. This means that (2.13) and (2.14) are valid to $O(\varepsilon^2)$ when $u_0 \to u_0 + \varepsilon u_1$ and $\theta_X \to k + \varepsilon \phi_X$, if the contribution from δ_s vanishes. In general, this is not valid, but (3.2) and (3.3) are valid.

4. Modulated Phase Shift for KdV Type Equations.

The equations for the modulations of the phase shift $\phi(X,T)$ will be determined in this section by evaluating the appropriate higher order action equations, (3.4) and (3.5). We will assume εh is dissipative and $\varepsilon^2 h_1$ non-dissipative $\int_0^1 \delta_s(1, u_0) d\psi = 0$. These equations require knowing properties of u_1, which satisfies $L(u_1) = R_1$. By differentiating (2.3) with respect to ψ, we immediately obtain the well-known result that

$$L(u_{0\psi}) = 0 \quad , \qquad (4.1)$$

that is, $u_{0\psi}$ is a homogeneous periodic solution, which is odd in ψ.

It is less well-known that the other homogeneous solutions may be obtained by differentiation with respect to a parameter. Dobrokhotov and Maslov [2] used this idea for both ordinary and partial differential equations of the Klein-Gordon type, but an alternate representation introduced by Haberman and Bourland [3] yields a better result for $\phi(X,T)$, independent of the type of dissipation. We should recall that the parameter ω in (2.3) satisfies the dispersion relation (2.6) and thus is a function of A, B, and k. By directly taking the first partial derivatives of (2.3) with respect to these, we obtain

$$L(u_{0_A}) = \omega_A u_{0_\psi} \tag{4.2}$$

$$L(u_{0_B}) = \omega_B u_{0_\psi} \tag{4.3}$$

$$L(u_{0_k}) = \omega_k u_{0_\psi} - \frac{1}{2} Q_{uu} u_{0_\psi} - 3\alpha^2 k^2 u_{0_{\psi\psi\psi}} \quad .$$

An equivalent form of the latter expression is obtained using (2.3)

$$L(u_{0_k}) = (\omega_k - \frac{3\omega}{k})\, u_{0_\psi} + Q_{uu} u_{0_\psi} \quad . \tag{4.4}$$

This has yielded a second homogeneous solution, $\omega_B u_{0_A} - \omega_A u_{0_B}$, which is periodic and even (and hence independent of u_{0_ψ}). To obtain a third homogeneous solution, our experience with the Klein-Gordon equation suggests we should analyze the non-periodic even function ψy_{0_ψ}. By direct calculation

$$L(\psi u_{0_\psi}) = -\omega u_{0_\psi} + \frac{1}{2} k\, Q_{uu} u_{0_\psi} + 3\alpha^2 k^3 u_{0_{\psi\psi\psi}} \quad .$$

Using (2.3) again, we obtain

$$L(\psi u_{0_\psi}) = 2\omega u_{0_\psi} - k Q_{uu} u_{0_\psi} \quad .$$

Thus from (4.2) - (4.4), a proper combination of u_{0_A}, u_{0_k}, and ψu_{0_ψ} will be a third homogeneous solution, namely $\omega_A(u_{0_k} + \frac{1}{k}\psi u_{0_\psi}) - (\omega_k - \frac{\omega}{k})\, u_{0_A}$, which is even and non-periodic. [u_{0_B} could be used here with $\omega_A \to \omega_B$ instead of u_{0_A}.]

We are now able to solve $L(u_1) = R_1$. We could use variation of parameters. However, it is easier for us to note that from (2.9)

$$L(u_{1_{even}}) = R_{1_{odd}} = -\phi_T u_{0_\psi} - \frac{1}{2}\phi_X \frac{\partial}{\partial\psi} Q_u - 3\alpha^2\theta_X^2 \phi_X u_{0_{\psi\psi\psi}}$$

$$L(u_{1_{odd}}) = R_{1_{even}} = -u_{0_T} - \frac{1}{2}\frac{\partial}{\partial X} Q_u + \frac{1}{2} Q_{uX} - \alpha^2(3\theta_X^2 u_{0_{\psi\psi X}} + 3\theta_{XX}\theta_X u_{0_{\psi\psi}}) + h \quad ,$$

since h is even for dissipative perturbations. The integrals in (3.4) and (3.5) only require the even part of u_1, $u_{1_{even}}$, significantly simplifying the needed calculations. Equation (2.3) is used once more (to eliminate $u_{0_{\psi\psi\psi}}$) so that

$$L(u_{1_{even}}) = u_{0_\psi}\left(-\phi_T - 3\frac{\omega}{k}\phi_X\right) + \phi_X Q_{uu} u_{0_\psi} \quad .$$

By appropriate use of (4.2) and (4.4), we have that

$$L\left[u_{1_{even}} - \phi_X u_{0_k} + \frac{u_{0_A}}{\omega_A}(\phi_T + \omega_k \phi_X)\right] = 0 \quad .$$

Thus, the expression in the bracket is a homogeneous solution, which is periodic and even. The only periodic even homogeneous solution is $\omega_B u_{0_A} - \omega_A u_{0_B}$; this must be proportional to it, so that

$$u_{1_{even}} = \phi_X u_{0_k} - \frac{u_{0_A}}{\omega_A}(\phi_T + \omega_k \phi_X) + C_2(\omega_B u_{0_A} - \omega_A u_{0_B}) \quad , \qquad (4.5a)$$

where $C_2(X,T)$ is an arbitrary multiple of the periodic and even homogeneous solution. We have thus shown that the arbitrary multiple of the other homogeneous solution, u_0, vanishes when the secularity conditions equivalent to (3.4) and (3.6) are applied. Luke [5] had shown a similar result for the Klein-Gordon equation.

The substitution of (4.5a) into (3.4) and (3.5) appears formidable. We can make the calculation elementary by using an observation by Haberman and Bourland [3] for the Klein-Gordon equation. Equation (4.5a) is equivalent to

$$u_{1_{even}} = \hat{D} u_0 , \tag{4.5b}$$

where $\hat{D}$ is a linear first-order (in the slow parameters) differential operator with variable coefficients:

$$\hat{D} = \theta_X \frac{\partial}{\partial k} - \frac{\theta_T + \omega_k \theta_X}{\omega_A} \frac{\partial}{\partial A} + C_2(X,T)\left(\omega_B \frac{\partial}{\partial A} - \omega_A \frac{\partial}{\partial B}\right) . \tag{4.5c}$$

In §5, we show that $\hat{D}$ is the Taylor series operator in the parameters A, B, and k. This operator has the property that

$$\hat{D}(k) = \theta_X \text{ and } \hat{D}(\omega) = - \theta_T , \tag{4.6}$$

which will be quite useful.

The rhs of the higher-order action equations (3.4) and (3.5) are simplified by noting that

$$\hat{D}\int_0^1 h(u_0, \theta_X u_{0_\theta}, \theta_X^2 u_{0_{\theta\theta}}, X, T)\, d\theta =$$

$$\int_0^1 \left\{ h_u \hat{D}u_0 + h_{u_x}[\theta_X \hat{D}u_{0_\theta} + \hat{D}(k)u_{0_\theta}] + h_{u_{xx}}[\theta_X^2 \hat{D}u_{0_{\theta\theta}} + 2\theta_X \hat{D}(k)u_{0_{\theta\theta}}]\right\} d\theta$$

$$= \int_0^1 \delta_h \, d\theta$$

and similarly

$$\hat{D}\int_0^1 h(u_0, \theta_X u_{0_\theta}, \theta_X^2 u_{0_{\theta\theta}}, X, T)u_0 d\theta = \int_0^1 \delta_h u_0 \, d\theta \quad ,$$

since $\hat{D}$ is a first-order partial differential operator in the slow parameters that commutes with the fast derivatives $\partial/\partial\theta$. We also used $u_{1_{even}} = \hat{D}u_0$ and $\hat{D}(k) = \phi_X$. There are corresponding simplifications for the lhs of (3.4) and (3.5). First of all,

$$\hat{D}\int_0^1 Q_u \, d\theta = \int_0^1 Q_{uu}\hat{D}u_0 \, d\theta \quad , \quad \hat{D}\int_0^1 Q_{uX}\, d\theta = \int_0^1 Q_{uuX}\hat{D}u_0 \, d\theta \quad ,$$

$$\hat{D}\int_0^1 (u_0 Q_u - Q)d\theta = \int_0^1 u_0 Q_{uu}\hat{D}u_0 \, d\theta \quad , \quad \text{and}$$

$$\hat{D}\int_0^1 (u_0 Q_{uX} - Q_X)d\theta = \int_0^1 u_0 Q_{uuX}\hat{D}u_0 \, d\theta \quad ,$$

since Q depends on A, B, and k only through u_0. In addition, for (3.5), we note that

$$\hat{D}\int_0^1 \frac{1}{2} u_0^2 \, d\theta = \int_0^1 u_0 \hat{D} u_0 \, d\theta$$

$$\hat{D}\left(\theta_X^2 \int_0^1 u_{0\theta}^2 \, d\theta\right) = \theta_X^2 \int_0^1 2 u_{0\theta} \hat{D}(u_{0\theta}) \, d\theta + 2k \hat{D}(k) \int_0^1 (u_{0\theta})^2 \, d\theta \;,$$

where again $\hat{D}(k) = \theta_X$. In this manner, we derive the remarkable simple result that the higher order wave action equations (respectively momentum and energy) have the following nearly conservation form:

$$\frac{\partial}{\partial T} \hat{D}\int_0^1 u_0 \, d\theta + \frac{1}{2}\frac{\partial}{\partial X}\hat{D}\int_0^1 Q_u \, d\theta = -\hat{D}\int_0^1 h \, d\theta - \frac{1}{2}\hat{D}\int_0^1 Q_{uX} \, d\theta \qquad (4.7)$$

$$\frac{\partial}{\partial T}\hat{D}\int_0^1 \frac{1}{2} u_0^2 \, d\theta + \frac{1}{2}\frac{\partial}{\partial X}\hat{D}\int_0^1 (u_0 Q_u - Q) \, d\theta - \frac{1}{2}\hat{D}\int_0^1 u_0 Q_{uX} \, d\theta$$
$$+ \frac{1}{2}\hat{D}\int_0^1 Q_X \, d\theta - \frac{3}{2}\epsilon^2 \frac{\partial}{\partial X}\hat{D}\left(\theta_X^2 \int_0^1 u_{0\theta}^2 \, d\theta\right) = -\hat{D}\int_0^1 h u_0 \, d\theta \;,$$
$$(4.8)$$

in the case where the perturbation ϵh is dissipative and $\epsilon^2 h_1$ is non-dissipative. It is interesting to note that (4.7) and (4.8) may be obtained from the leading order action equations, (2.13) and (2.14), by replacing the derivative terms of the conservation law by the respective derivative acting on $\hat{D}$ (i.e., $\frac{\partial}{\partial T}$ by $\frac{\partial}{\partial T}\hat{D}$ and $\frac{\partial}{\partial X}$ by $\frac{\partial}{\partial X}\hat{D}$) and the non-derivative terms in the action equations by $\hat{D}$ operating on them (i.e. $\int_0^1 Q_X d\theta$ by $\hat{D}\int_0^1 Q_X d\theta$, $\int_0^1 h\,d\theta$ by $\hat{D}\int_0^1 h\,d\theta$, etc.). We [3] have also shown this phenomena for the Klein-Gordon equation, and thus we believe it to be a general principle for appropriately perturbed systems. Perhaps, a general proof of its validity could be given using Whitham's [6] averaged Lagrangian principle with the necessary inclusion of weak dissipation.

Although (4.7) and (4.8) are remarkably simple, expressing the validity of action to a higher order, they can be reduced to an even more fundamental form. The dissipation terms can be eliminated by operating [illegible] on the leading order action equations, (2.13) and (2.14). Since all non-derivative terms in (4.7) and (4.8) are already [illegible] operating on the terms in (2.13) and (2.14), they will also be eliminated this way. What will remain involves [illegible] and [illegible]. These operators do not commute (fortunately) as they have variable coefficients. However,

$$\hat{D}_T = [illegible] \hat{D} - \hat{D} [illegible] \qquad (4.9a)$$

$$\hat{D}_X = [illegible] \hat{D} - \hat{D} [illegible] \qquad (4.9b)$$

Thus, there is a non-conservation form of the higher order wave action equations, equivalent to (4.7) and (4.8):

$$[illegible] \qquad (4.[illegible])$$

$$[illegible] \qquad (4.[illegible])$$

These equations are <u>universal</u> in the sense that the form of the equations are independent of the particular dissipative perturbation. The effect of the dissipation is that the quantities in (4.1[illegible]) and (4.1[illegible]) must still satisfy the dissipation of action equations, (2.13) and (2.14). It is interesting to note that (4.[illegible]) and (4.[illegible]) may be obtained by making simple changes in (2.13) and (2.14), namely, replacing [illegible] and [illegible] and dropping all the dissipative terms (including the term representing the possible dissipation of energy from a spatial varying potential [illegible]). Again we believe this to be an extremely fundamental general principle, since we [illegible] also obtained this result for all Klein-Gordon equations.

We claim that (4.10) and (4.11) are used to describe the phase shift [illegible]. To understand this it is best to recall the definition of [illegible] (4.5c)

[illegible]

which [illegible] depends on the phase shift [illegible] but not the form of dissipation. Thus, the [illegible] that appear in (4.[illegible]) and (4.[illegible]) are dependent on the phase shift

[illegible]

[illegible]

[illegible]

[illegible] we evaluate the integrals in (4.[illegible]) and (4.[illegible]) [illegible] we show that the two coupled equations (4.[illegible]) and (4.[illegible]) [illegible].

$$[illegible] + \partial_x \left(\frac{\beta}{k} \int_0^1 u_0 \, d\sigma \right) = \partial_x \left(\frac{B}{k} \right) \qquad 4.13$$

$$[illegible] + \partial_x \left(\frac{\beta}{k} \int_0^1 u_0^2 \, d\sigma \right) = \partial_x \left(\frac{A}{k} \right) \quad . \qquad 4.14$$

We have not succeeded yet in obtaining a simple equation for [illegible] alone via the elimination of [illegible] from (4.13) and (4.14).

5. A Higher-Order Accurate Dispersion Relation.

In this section, we will show that the equation for the modulations of the [illegible] (4.[illegible]) and (4.[illegible]), are equivalent to an appropriate higher-order [illegible], when [illegible] is a dissipative perturbation and [illegible]. We have already shown that both wave [illegible] [illegible] (4.[illegible]), [illegible] higher-order than [illegible] [illegible].

[illegible]

are utilized, not just the [illegible] The wave [illegible] but we conjecture [illegible] that [illegible] where [illegible] the average value of perturbation [illegible] we now use the [illegible] assuming that [illegible] the perturbation [illegible] the frequency.

[illegible]

[illegible] and the perturbation [illegible] in this section we [illegible]

[illegible] we find the wave [illegible]

[illegible]

the first variation of each wave action:

$$A_{1_T} I_A + B_{1_T} I_B + k_{1_T} I_k \tag{5.4}$$
$$+ A_{1_X} q_A + B_{1_X} q_B + k_{1_X} q_k = 0 \quad ,$$

where we note that all the dissipation terms have been balanced, leaving a result of universal form.

We now use the conjectured frequency perturbation (5.2) to eliminate A_1 ... B_1 ...

$$A_1 = \frac{\quad_T - \quad_X \quad_k}{\quad_A} - B_1 \frac{\quad_B}{\quad_A} \quad , \tag{5.5}$$

... and $k_1 = \quad_X$. It can be shown [non-trivial, but straightforward using (4.5) - (4.6)] that

$$B_1 \quad - \quad_A \; C_2 \quad , \tag{5.6}$$

... that when (5.6) is substituted into (5.4), we obtain

$$\frac{\quad - \quad_k}{\quad_A}\Big|_T I_A - (\omega_A C_2)_T I_B + (\omega_B C_2)_T I_A + \phi_{XT} I_k$$
$$+ \frac{\quad_T - \quad_k}{\quad_A}\Big|_X q_A - (\omega_A C_2)_X q_B + (\omega_B C_2)_X q_A + \phi_{XX} q_k = 0 \quad ,$$

which is equivalent to (4.10) - (4.12). In this way, we have verified the conjecture that a highly accurate dispersion relation may be obtained using averaged perturbations of the amplitude parameters. Furthermore, the modulations ... phase shift are determined from equations equivalent to the highly accurate wave action equations, when highly accurate dispersion is calculated.

We now explain our earlier remark that $\hat{D}$ is the Taylor series operator in the parameters A, B, and k. The leading-order oscillatory wave solution for KdV type equations is in the form $u_0 = u_0(\psi;A,B,k)$, when the dispersion relation (2.6) is satisfied. Some (but not all) perturbations of this can be obtained by perturbing the parameters A, B, and k (a Taylor series) in which case $u_{1_{even}} = \hat{D}u_0$, where

$$\hat{D} = A_1 \frac{\partial}{\partial A} + B_1 \frac{\partial}{\partial B} + k_1 \frac{\partial}{\partial k} \quad . \tag{5.7}$$

Through the use of (5.5) and (5.6), it is seen that (5.7) is identical to (4.5c).

Appendix A: Secularity Condition at $\mathcal{O}(\varepsilon^2)$.

In this appendix, we show that eliminating the secular terms at $\mathcal{O}(\varepsilon^2)$ is equivalent to analyzing the $\mathcal{O}(\varepsilon)$ wave action equations. From (3.1a) using (2.10), we immediately obtain (3.2) using the periodicity of u_0, u_1, and functions of u_0. Similar results are obtained for wave-energy action, but with much greater effort. One term that is present in $\int_0^1 R_2 u_0 d\psi = 0$ can be evaluated using some ideas that worked on the Klein-Gordon equation ([2],[3]):

$$\theta_X \int_0^1 u_0 \frac{\partial}{\partial\psi}\left(\frac{1}{2} u_1^2 Q_{uuu}\right) d\psi = -\theta_X \int_0^1 u_{0_\psi} \frac{u_1^2}{2} Q_{uuu}\, d\psi$$

$$= \theta_X \int_0^1 Q_{uu} u_1 u_{1_\psi}\, d\psi = -\theta_X \int_0^1 u_1 \frac{\partial}{\partial\psi}(u_1 Q_{uu})\, d\psi$$

$$= -\int_0^1 [L(u_1) - \theta_T u_{1_\psi} - \alpha^2 \theta_X^2 u_{1_{\psi\psi\psi}}] u_1\, d\psi = -\int_0^1 u_1 R_1 d\psi \quad .$$

[illegible], we have integrated by parts three times (equivalent to using the adjoint [illegible], used the definition of L, and noted $L(u_1) = R_1$, where R_1 is given [illegible] Then, after a lengthy but elementary calculation, $\int_0^1 R_2 u_0 d\psi = 0$ becomes (3.3).

Appendix B: Some Integral Identities

The averaged densities and fluxes in (4.10) and (4.11) can be related. By integrating (2.4) using the periodicity of u_0, we obtain

$$- \omega \int_0^1 u_0 \, d\psi + \frac{1}{2} k \int_0^1 Q_u \, d\psi = - B \quad . \tag{B.1}$$

By simply using (B.1), (4.13) is obtained from (4.10). It is not much harder to derive (4.14). Multiplying (2.4) by u_0 and integrating yields

$$- \omega \int_0^1 u_0^2 \, d\psi + \frac{1}{2} k \int_0^1 u_0 Q_u \, d\psi - \alpha^2 k^3 \int_0^1 u_{0_\psi}^2 \, d\psi = - B \int_0^1 u_0 \, d\psi \; . \tag{B.2}$$

However, by integrating (2.5) we obtain

$$- \omega \int_0^1 u_0^2 \, d\psi + k \int_0^1 Q \, d\psi + \alpha^2 k^3 \int_0^1 u_{0_\psi}^2 \, d\psi = - 2B \int_0^1 u_0 \, d\psi + 2A \quad . \tag{B.3}$$

Equations (B.2) and (B.3) may be combined to yield

$$3\alpha^2 k^2 \int_0^1 u_{0_\psi}^2 \, d\psi = \frac{2A}{k} - \frac{\omega}{k} \int_0^1 u_0^2 \, d\psi - \int_0^1 (Q - uQ_u) \, d\psi \quad ,$$

from which (4.11) directly becomes (4.14) since terms involving $\int_0^1 (Q - uQ_u) \, d\psi$ cancel.

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References

[1] F. J. Bourland and R. Haberman, A new adiabatic invariant involving the modulated phase shift for strongly nonlinear, slowly varying, and weakly damped oscillators, accepted SIAM J. Appl. Math.

[2] S. Yu. Dobrokhotov and V. P. Maslov, Finite-zone, almost-periodic solutions in WKB approximations, Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Mathematiki, 15 (1980), pp. 3-94. J. Sov. Math. 15(1980), pp. 1433-1487.

[3] R. Haberman and F. J. Bourland, Variation of wave action: Modulations of the phase shift for strongly nonlinear dispersive waves with weak dissipation, submitted.

[4] G. E. Kuzmak, Asymptotic solutions of nonlinear second order differential equations with variable coefficients, Prikl. Mat. Meh. 23 (1959), pp. 515-526. J. Appl. Math. Mech. 23 (1959), pp. 730-744.

[5] J. C. Luke, A perturbation method for nonlinear dispersive wave problems, Proc. Roy. Soc. London Ser. A 292 (1966), pp. 403-412.

[6] G. B. Whitham, Linear and Nonlinear Waves, John Wiley & Sons, New York, 1974.

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