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A Critical Review of the Development of Several Viscoplastic Constitutive Theories

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The need to model the rate dependence that is inherent in non-recoverable deformation of structural materials and the continued progress in computational capability have served as stimuli for the rapid development of viscoplastic constitutive theory. The structure and development of these theories can be separated into two categories: those with or without a yield condition. All theories reviewed here have common underlying features and limitations: phenomenological, homogeneous material continuum, nonrecoverable deformation is a constant volume process, deformation may be decomposed into linearly additive elastic and inelastic parts. The theories of Bingham, Hohenemser-Prager, Perzyna, Chaboche, Bodner-Partom, Walker, Kremple-Liu, Miller and Valanis are reviewed. Comparisons are made among predictions of a modified Chaboche and Bodner theories, and experiments of Inconel at 1200°F for various loading conditions.

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A CRITICAL REVIEW OF THE DEVELOPMENT OF
SEVERAL VISCOPLASTIC CONSTITUTIVE THEORIES

INTRODUCTION

(i) Background Considerations

The need to model the rate dependence that is inherent in non-recoverable deformation of structural materials, particularly at elevated temperatures, and the continued progress in computational speed and capability, have served as stimuli for rapid development of viscoplastic constitutive theory. Since the time when Bingham and Green, in 1919, first introduced a theory for viscoplastic simple shear, more than a dozen general theories for viscoplastic material response have appeared.

The structure and development of these theories can be separated into two categories. In one it is assumed that for strong coherent solids (e.g. structural metals) at low levels of stress or small magnitudes of strain, the inelastic part of any deformation is negligibly small. The deformation is essentially recoverable and rate independent, i.e., elastic. Inelastic deformation, here taken to mean deformation that is non-recoverable (plastic) and rate dependent, begins only after the stress state reaches a level sufficient to cause the onset of yield. An idealized yield condition is postulated, with the yield function appearing as an integral part of the statement of the constitutive relation.

In the other category, inelastic deformation is assumed to accompany the elastic deformation at any level of stress, however small. There is no yield state of stress that separates the occurrance of the inelastic part of the total deformation from the part that is elastic. A yield condition and a yield function, therefore, are not required in the statement of the constitutive relation, affording thereby a seeming measure of simplicity, comparatively speaking. However, having dispensed with the need for a yield function, and with it the useful geometrical construct of a yield surface in stress space, this category of theories must resort to different means for describing the hardening that accompanies inelastic deformation. Internal variables are accordingly introduced for this purpose, which are loosely identified with microstructural mechanisms associated with non-recoverable deformation in polycrystalline materials.

All of the theories here considered have other common underlying features and limitations that may be enumerated as follows:

(a) All are phenomenological theories, that is, the observed mechanical behavior of a material is assumed to be describable in terms of macroscopically defined measurable variables. Although several theories from the non-yield condition category do introduce terminology borrowed from microstructural models of polycrystalline plastic deformation, the intention nevertheless is to describe macroscopically observable features of inelastic material behavior.

(b) All assume a homogeneous material continuum. Materials with existing microstructural inhomogeneities are viewed, macroscopically speaking, as being statistically homogeneous.

(c) Non-recoverable (plastic) deformation is considered not to be significantly affected by superposed pressure of ordinary engineering magnitudes (less than 10^5 p.s.i.). Inelastic deformation is thus taken to be volume perserving.

(d) The theories discussed are, at this stage of development, presented in a form appropriate for small order deformation. The recoverable and non-recoverable components of any deformation is assumed to be decomposable into linearly additive parts.

(ii) Index of Frequently Used Symbols and Definitions

Since the theories to be discussed in their present form are confined to small small order deformation, the current configuration and the reference configuration of a deforming body may be treated as one, i.e., there is no need for distinction between the two. Also, all time derivatives reduce to the ordinary time derivative, denoted by a superposed dot.

Symbols for the variables appearing most frequently in the discussion to follow are listed below. Any others that subsequently appear will be defined as they appear.

Symbol	Name
$\mathbf{T}, T_{jk}$	Cauchy stress tensor.
$\overset{o}{\mathbf{T}}, \overset{o}{T}_{jk}$	Stress deviator tensor.
$\mathbf{\Sigma}, \Sigma_{jk}$	Infinitesimal strain tensor.
$\overset{o}{\mathbf{\Sigma}}, \overset{o}{\Sigma}_{jk}$	Strain deviator tensor.
$\mathbf{\Sigma}'$	Elastic or recoverable part of the total strain.
$\mathbf{\Sigma}''$	Inelastic or non-recoverable part of the total strain.
$\mathbf{D}, D_{jk}$	Rate of deformation tensor.
$\dot{\mathbf{\Sigma}}$	Infinitesimal strain rate tensor $= \partial \Sigma / \partial t \cong D$.
$\mathbf{Y}, Y_{jk}$	Kinematic hardening tensor variable (stress).
$\overset{o}{\mathbf{Y}}, \overset{o}{Y}_{jk}$	Kinematic hardening deviator tensor.
$\mathbf{u}, u_k$	Displacement vector.
$\mathbf{v} \cong \dot{\mathbf{u}}, v_k \cong \dot{u}_k$	Velocity vector.
$I_{\mathbf{T}}, II_{\mathbf{T}}, III_{\mathbf{T}}$	Principal invariants of $\mathbf{T}$.
R	Isotropic hardening scalar variable (stress).
P, p	Accumulated inelastic strain.
W''	Inelastic work.
E, μ, ν	Young's elastic modulus, elastic shear modulus, and Poisson's ratio, respectively.
Θ	Temperature.

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Referring to a rectangular coordinate system, the trace of a second-order tensor, the inner product of tensors with tensors and vectors are defined as follows:

$$tr\, \mathbf{T} = T_{kk} = T_{11} + T_{22} + T_{33} \tag{ii.1}$$

$$\mathbf{T} \cdot \mathbf{v} = \mathbf{a}: \quad T_{jk} v_k = a_j \tag{ii.2}$$

$$\mathbf{T} \cdot \mathbf{D} = \mathbf{C}: \quad T_{jk} D_{kl} = C_{jl} \tag{ii.3}$$

$$tr\,(\mathbf{T} \cdot \mathbf{D}): \quad T_{jk} D_{kj} = C_{jj}. \tag{ii.4}$$

The identity tensor 1 has the property.

$$\mathbf{T} \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{T} = \mathbf{T}, \qquad \mathbf{v} \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{v} = \mathbf{v}. \tag{ii.5}$$

The transpose and inverse of the second-order tensor $\mathbf{T}$ are denoted by $\mathbf{T}^T$ and $\mathbf{T}^{-1}$, while the deviator tensor $\overset{o}{\mathbf{T}}$ associated with $\mathbf{T}$ is defined by the relation

$$\overset{o}{\mathbf{T}} = \mathbf{T} - \frac{1}{3}(tr\,\mathbf{T})\,\mathbf{1}. \tag{ii.6}$$

The magnitude of a vector and a second-order tensor are specified by

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_k v_k} \ , \quad |\mathbf{T}| = \sqrt{tr\,(\mathbf{T} \cdot \mathbf{T}^T)} = \sqrt{T_{jk} T_{jk}}. \tag{ii.7}$$

The kinematic tensors Σ and $\mathbf{D}$ are given by

$$\Sigma = \frac{1}{2}[\,\text{grad}\,\mathbf{u} + (\text{grad}\,\mathbf{u})^T\,], \qquad \Sigma_{jk} = \frac{1}{2}[u_{j,k} + u_{k,j}] \tag{ii.8}$$

$$\mathbf{D} = \frac{1}{2}[\,\text{grad}\,\mathbf{v} + (\text{grad}\,\mathbf{v})^T], \quad D_{jk} = \frac{1}{2}[v_{j,k} + v_{k,j}]. \tag{ii.9}$$

The invariants of the tensor $\mathbf{T}$ and of its associated deviator tensor $\overset{o}{\mathbf{T}}$ are determined by

$$I_{\mathbf{T}} = tr\,\mathbf{T}$$

$$II_{\mathbf{T}} = \frac{1}{2}\{(tr\,\mathbf{T})^2 - tr\,\mathbf{T}^2\}, \qquad tr\,\mathbf{T}^2 = tr\,(\mathbf{T} \cdot \mathbf{T})$$

$$III_{\mathbf{T}} = \det\mathbf{T} = \frac{1}{6}\{2tr\,\mathbf{T}^3 - 3tr\,\mathbf{T}^2\,tr\,\mathbf{T} + (tr\,\mathbf{T})^3\} \tag{ii.10}$$

and

$$I_{\overset{o}{\mathbf{T}}} = 0$$

$$II_{\overset{o}{\mathbf{T}}} = -\frac{1}{2}tr\,\overset{o}{\mathbf{T}}{}^2, \quad \text{or} \qquad II'_{\overset{o}{\mathbf{T}}} = -2II_{\overset{o}{\mathbf{T}}} = tr\,\overset{o}{\mathbf{T}}{}^2 = tr\,(\overset{o}{\mathbf{T}} \cdot \overset{o}{\mathbf{T}}) \tag{ii.11}$$

$$III_{\overset{o}{\mathbf{T}}} = \frac{1}{3}tr\,\overset{o}{\mathbf{T}}{}^3, \quad \text{or} \qquad III'_{\overset{o}{\mathbf{T}}} = 3III_{\overset{o}{\mathbf{T}}} = tr\,\overset{o}{\mathbf{T}}{}^3 = tr\,(\overset{o}{\mathbf{T}} \cdot \overset{o}{\mathbf{T}} \cdot \overset{o}{\mathbf{T}}).$$

The following derivative formula appears frequently in plastic theory

$$\frac{\partial II'_{\overset{o}{\mathbf{T}}}}{\partial \mathbf{T}} = \frac{\partial II'_{\overset{o}{\mathbf{T}}}}{\partial \overset{o}{\mathbf{T}}} = 2\overset{o}{\mathbf{T}}. \tag{ii.12}$$

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PART I
RIGID/VISCOPLASTIC AND ELASTIC/ELASTIC-VISCOPLASTIC THEORIES

1. Definition of Yield

On the basis of what is always observed from the stress-strain data obtained from uniaxial tension-compression tests of the structural metals, at some level of stress (or possibly for some small range of stress values) the deformational behavior of the material changes from essentially recoverable behavior to combined recoverable and non-recoverable behavior. This fact, observed for uniaxial stress, can be generalized to general conditions of stress. Thus, the state of stress T at any point of a deformed body at which recoverable (elastic) material response passes into partially non-recoverable (elastic-plastic) response defines a yield condition at that point, which can be expressed mathematically by a relation of the form

$$f(\mathbf{T}) = k. \tag{1.1}$$

where k is a material parameter. Equation (1.1) represents an initial yield condition. The locus of all such possible states of yield stress at any point of the deformed body forms, or defines, an initial yield surface in stress space

$$F(\mathbf{T},k) = 0, \tag{1.2}$$

that encloses the origin (zero stress).

For all strong coherent solids (e.g., crystalline and polycrystalline solids) it is an observed fact that during loading, increments of strain are accompanied by non-negative increments of stress (presuming no material microstructural disintegration over this stage of the loading). This characteristic is referred to as stable material behavior. For such behavior it is possible to prove analytically that the yield surface must always be a convex surface [1,2]. This characteristic of a yield surface has been demonstrated experimentally many times over the past thirty years.

When the material response is isotropic up to the point of initial yield the yield function f must be an isotropic scalar valued function of the invariants of the stress tensor,

$$f(I_{\mathbf{T}}, II_{\mathbf{T}}, III_{\mathbf{T}}) = k. \tag{1.3}$$

For low to moderate levels of pressure (e.g., less than 10^5 psi) structural metals exhibit very little pressure sensitivity with respect to yield. Consequently, as a first approximation, it is assumed in plasticity theory that the mean pressure

$$\bar{p} = \frac{1}{3} I_{\mathbf{T}} = -\frac{1}{3} tr\mathbf{T} \tag{1.4}$$

has no influence upon inelastic deformation. Equivalently, inelastic deformation is taken to be incompressible or volume preserving. [cf. comments at the end of this section]. Since

$$\mathbf{T} = \overset{o}{\mathbf{T}} + \frac{1}{3}(tr\mathbf{T})\mathbf{1} = \overset{o}{\mathbf{T}} - \bar{p}\mathbf{1} \tag{1.5}$$

neglect of the mean pressure allows the invariants of the stress tensor to be replaced by the invariants of its deviator tensor. Furthermore since $I_o = 0$, plastic incompressibility reduces the initial yield condition to the form

$$f(II_o, III_o) = k .$$

(1.6)

Experimental observations of initial yield obtained from combined tension-torsion tests have shown that the following von Mises form for the yield function appears to offer the best correlation of the test data.

$$f(II'_o, k) = \sqrt{II'_o} - \sqrt{2}k$$

(1.7)

where $k = (1/\sqrt{3}) Y_t$ and Y_t is the yield stress in uniaxial tension, or where $k = Y_s$ which is the yield stress in simple shear. Note that the third invariant of the deviator stress is assumed to have a negligible effect on the initial yield. The von Mises yield condition states that initial yield begins when the elastic distortional strain energy density reaches a particular value.

The above expression (1.6), or the particular form (1.7), for yield are appropriate for the commencement of yield only. Thereafter yield is accompanied by hardening which, in general, is anisotropic. The simplest reasonably effective description of hardening which includes the Bauchinger effect with load reversal, is a combined isotropic and kinematic hardening. For the isotropic part the constant k is replaced by a scalar function R of the inelastic work, W'', or of the accumulated inelastic strain, P, [3,4]. For the anisotropic component a kinematic form of hardening is most often employed which introduces the tensor $\overset{o}{Y}$ as an argument for the yield function F in the form $\overset{o}{T} - \overset{o}{Y}$ [5,6]. The tensor $\overset{o}{Y}$ represents the center of the migrating yield surface, while the scalar variable R is a measure of its size. Both hardening variables have the dimension of stress.

Thus far in the discussion of yield the known effects of the strain rate and of the temperature at which yield takes place have not been considered. Both effects are non-directional and can therefore be introduced through the isotropic hardening variable [7-11]. Keeping in mind the assumed plastic incompressibility assumption, the form of the yield function appearing in Eq. (1.2), if it is to account for combined isotropic and kinematic hardening, strain rate and temperature, must have the form

$$F(\overset{o}{T}, \overset{o}{Y}, R) = f(\overset{o}{T} - \overset{o}{Y}) - R ,$$

(1.8)

where

$$R = R(W'', \dot{\Sigma}, \theta), \qquad W'' = \int_0^{\Sigma''} tr(T \cdot d\Sigma'')$$

(1.9)

or

$$R = R(P, \dot{\Sigma}, \theta) , \qquad P = \int_0^{\Sigma''} d |\Sigma''| .$$

(1.10)

The assumption of the incompressibility of plastic deformation is, as indicated at the outset, an approximation. The degree of validity of the approximation varies with the structure of the material and the level of the mean stress imposed. Experimental data can be found in references [12-15], while discussions of the theoritical implications are given in [16-18].

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2. Bingham Theory for Rigid-Viscoplastic Simple Shear

Bingham introduced the first theory of what can be interpreted as viscoplastic behavior for a solid in 1919, [19,20], when he proposed a model for the simple shear of a material that differs from a viscous fluid in that it can sustain a shear stress when not deforming, i.e., is at rest.

The theory assumes that the material is rigid and begins to deform in simple shear non-recoverably, when an imposed shear stress exceeds a critical value, Y_s such that the rate of shear deformation is proportional to the excess of the shear stress over the yield stress. Thus

$$D_{12} = C[\,|T_{12}| - Y_s] \quad \text{for} \quad |T_{12}| > Y_s\,,$$

$$D_{12} = 0 \quad\quad\quad \text{for} \quad |T_{12}| \leq Y_s\,. \tag{2.1}$$

The material remains rigid until the imposed shear stress exceeds the material's inherent shear yield strength Y_s. Alternatively, introducing the yield function

$$F = 1 - \frac{Y_s}{|T_{12}|}\,, \tag{2.2}$$

the constitutive relation (2.1) for simple shear can be re-expressed as

$$D_{12} = \frac{1}{2\eta} <F> |T_{12}| \tag{2.3}$$

$$<F> = \begin{cases} F \text{ for } F>0, & \text{or } |T_{12}| > Y_s \\ 0 \text{ for } F \leq 0, & \text{or } |T_{12}| \leq Y_s\,, \end{cases} \tag{2.4}$$

with η interperted as a viscosity coefficient. The material parameters Y_s and η are temperature dependent. Rewriting the above relations in the form

$$|T_{12}| = Y_s + 2\eta D_{12}, \quad F>0 \text{ or } |T_{12}| > Y_s\,, \tag{2.5}$$

we see that in the absence of a shear yield stress (the material is no longer a solid) Eq. (2.5) becomes the constitutive equation for a Newtonian viscous fluid, whereas if the viscosity coefficient is zero, it resembles the Saint Venant-Levy constitutive equation for a rigid perfectly-plastic solid subject to simple shear.

3. Generalization by Hohenemser and Prager

The Bingham theory for rigid viscoplastic simple shear was generalized to arbitrary states of stress by Hohenemser and Prager in 1932 by modifying the yield function and constitutive expressions (2.2) and (2.3) to the forms

$$F = 1 - \frac{\sqrt{2}k}{\sqrt{II'_{\overset{o}{T}}}}\,, \tag{3.1}$$

where k has the same interpretation as given in Eq. (1.7), and

$$\mathbf{D}'' = \frac{1}{2\eta} <F> \overset{o}{\mathbf{T}} \tag{3.2}$$

$$<F> = \begin{cases} F \text{ for } F > 0 \\ 0 \text{ for } F \leq 0. \end{cases} \tag{3.3}$$

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where k and η are temperature dependent material parameters. The "flow" equation (3.2) implies plastic incompressibility since

$$tr(2\eta \, \mathbf{D}'') = tr(F \, \overset{o}{\mathbf{T}}) = F \, tr \, \overset{o}{\mathbf{T}} = 0 \rightarrow tr \, \mathbf{D}'' = \operatorname{div} \mathbf{v}'' = 0$$

which is the condition for constant volume deformation. Eq. (3.2) together with (3.1) can be used to obtain the generalized equivalent of Eq. (2.5).

$$\overset{o}{\mathbf{T}} = \frac{2k}{\sqrt{II'_{\underset{\mathbf{T}}{o}}}} \, \mathbf{D}'' + 2\eta \, \mathbf{D}'', \quad F > 0$$

$$\overset{o}{\mathbf{T}} = 0 \qquad\qquad , \quad F \le 0,$$

(3.4)

Comments:

1. The form of the yield function (3.1) implies no hardening. The theory, therefore, is for rigid perfectly-viscoplastic flow which commences when the value of F becomes positive.

2. When $\eta = 0$ Eq. (3.4), duplicates the Saint Venant-Levy equation for rigid perfectly-plastic flow, with the exception that in the perfectly plastic theory flow commences when $F = 0$, i.e., for $\sqrt{II'_{\underset{\mathbf{T}}{o}}} = \sqrt{2}k$, which is the von Mises yield condition (1.7). For the viscoplastic theory the onset of flow requires that $F > 0$ or $\sqrt{II'_{\underset{\mathbf{T}}{o}}} > \sqrt{2}k$. In the absence of a yield stress, i.e., $k = 0$, Eq. (3.4) reduces to the constitutive equation for a Newtonian viscous fluid.

3. Since the kinematic variable appearing in the theory is the rate of deformation $\mathbf{D}$, equations (3.2) and (3.4) are appropriate for deformations of any magnitude.

4. Perzyna Theory

The theory proposed by Perzyna in 1963 [23-26] generalizes the rigid/perfectly-viscoplastic theory of Hohenemser-Prager into an elastic/elastic-viscoplastic theory that includes hardening. The theory in its initial form is for small deformation. Prior to yield the deformation is considered to be rate-independent and recoverable, i.e., elastic. Whatever inelastic deformation that may actually accompany the elastic deformation is considered to be insignificant.

An initial 'static' yield condition is assumed, defined by the yield function

$$F(\mathbf{T}, \Sigma'', R) = \frac{f(\mathbf{T}, \Sigma'')}{R(W'')} - 1, \tag{4.1}$$

in which the possibility for isotropic and anisotropic hardening are included by virtue of the presence of Σ'' and R as variables. The existence of a static yield condition is an idealization of the observation that below strain rates that are very small, a strain rate effect upon yield cannot be ordinarily observed.

A cornerstone of rate-independent plastic constitutive theory is Drucker's postulate of positive plastic work [1,2], which leads to the conclusions that: (i) The yield surface is a convex surface in stress space, and (ii) the plastic strain increment $d\Sigma''$, or the plastic strain rate $\dot{\Sigma}''$, must be normal to the yield surface $F(\mathbf{T}) = 0$ at all points. Thus $d\Sigma''$ or $\dot{\Sigma}''$ must also be parallel to the gradient of the yield function,

$$d\Sigma'' = \Lambda \frac{\partial F}{\partial \mathbf{T}}, \tag{4.2}$$

where Λ is an arbitrary positive scalar function having the same arguments as the yield function.

Considering rate-independent plasticity as the lower limit of a rate-dependent plasticity, or viscoplasticity, the inelastic strain tensor is postulated to be normal at each point to the 'static' yield surface, and to all subsequent rate dependent yield surface. This implies the relation

$$\dot{\Sigma}'' = \Lambda \frac{\partial F}{\partial \mathbf{T}}. \tag{4.3}$$

The non-negative scalar valued function Λ, other than possibly being a function of the same variables as F, is arbitrary otherwise. Perzyna takes Λ to be an unspecified functional of the yield function F such that

$$\Lambda = \gamma_0 <\Phi(F)> = \begin{cases} \Phi(F) & F > 0 \\ 0 & F \leq 0 \end{cases} \tag{4.4}$$

where γ_0 is a constant. It is presumed that the functional $\Phi(F)$ may be chosen to represent the results of tests on the behavior of metals under dynamic loading. With the choice of Λ in this form

$$\dot{\Sigma}'' = \gamma_0 <\Phi(F)> \frac{\partial F}{\partial \mathbf{T}} = \gamma <\Phi(F)> \frac{\partial f}{\partial \mathbf{T}}. \tag{4.5}$$

$$\gamma = \gamma_0/R$$

The strain rate is assumed to be decomposable into a linear fat SIGMA of elastic and inelastic components

$$\dot{\Sigma} = \dot{\Sigma}' + \dot{\Sigma}'', \tag{4.6}$$

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with the elastic strain rate specified by

$$\dot{\Sigma}' = \frac{1}{2\mu}\frac{\partial}{\partial t}(\overset{o}{T}) + \frac{1-2\nu}{3E}(tr\,\dot{T})\,1 \tag{4.7}$$

The rate dependence of the yield function is not obvious from Eq. (4.1). However from (4.5)

$$tr(\dot{\Sigma}'' \cdot \dot{\Sigma}'') = II'_{\dot{\Sigma}''} = \gamma^2\Phi^2(F)tr(\frac{\partial f}{\partial T} \cdot \frac{\partial f}{\partial T})$$

from which

$$\Phi(F) = \frac{1}{\gamma}\sqrt{II'_{\dot{\Sigma}''}}\left\{\frac{\partial f}{\partial T} \cdot \frac{\partial f}{\partial T}\right\}^{-1/2}.$$

Taking the inverse of Φ

$$F = \frac{f}{R} - 1 = \Phi^{-1}\left[\frac{\sqrt{II'_{\dot{\Sigma}''}}}{\gamma}\left\{tr\left[\frac{\partial f}{\partial T} \cdot \frac{\partial f}{\partial T}\right]\right\}^{-1/2}\right]$$

so that

$$f(T,\Sigma'') = R(W'')\left[1 + \Phi^{-1}\left[\frac{\sqrt{II'_{\dot{\Sigma}''}}}{\gamma}\left\{tr\left[\frac{\partial f}{\partial T} \cdot \frac{\partial f}{\partial T}\right]\right\}^{-1/2}\right]\right]. \tag{4.8}$$

The dependence of the yield function f on the inelastic strain rate appears thru the dependence of Φ^{-1} on the second invariant of Σ''.

In summary:

$$\dot{\Sigma} = \dot{\Sigma}' + \dot{\Sigma}''$$

$$\dot{\Sigma}' = \frac{1}{2\mu}\frac{\partial}{\partial t}(\overset{o}{T}) + \frac{1-2\nu}{3E}(tr\,\dot{T})1$$

$$\dot{\Sigma}'' = \gamma < \Phi(F) > \frac{\partial f}{\partial T} \text{ for } F > 0 \tag{4.9}$$

$$\dot{\Sigma}'' = 0 \text{ for } F \leq 0$$

$$F = \frac{f(T,\Sigma'')}{R(W'')} - 1, \quad \gamma = \gamma_0/R \qquad \mu,\nu,E,\gamma_0, \text{ are constants.}$$

Comments:

1. Up to this stage of development of the theory, the assumption of inelastic incompressibility has not been introduced, since the stress tensor rather than its deviator appears as an argument for the yield function. Also, the specific form of the hardening is left open.

2. As with the Bingham and the Hohenemser-Prager theories, inelastic deformation occurs only for $F > 0$ rather than when $F \geq 0$, as in the rate-independent theories of plasticity. In the Bingham and the Hohenemser-Prager constitutive theories the functional $\Phi(F)$ have the specific forms

$$\Phi(F) = F = 1 - \frac{|T_{12}|}{Y_s},$$

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$$\Phi(F) = F = 1 - \frac{\sqrt{2}k}{\sqrt{II'_{o}}} ,$$

respectively.

3. The dependence of yield upon the strain rate, as indicated by Eq. (4.8), applies for subsequent yield only. The initial yield, however, in this theory is rate-independent. To see this, for onset of yield $F = 0$, and as $\Phi(0) = 0 \rightarrow \dot{\Sigma}'' = 0$ and $W'' = 0$. Thus $\frac{f(\mathbf{T},0)}{R(0)} - 1 = 0$, so that $f(\mathbf{T},0) = R(0) =$ constant, which represents the size of the yield surface at onset of yield, independent of the strain rate at which the yield initiates. For uniaxial stress this constant represents the yield stress Y_t or Y_s.

4a. Inelastic Incompressibility and Isotropic Hardening

More specialized forms of the constitutive equations can be written as additional assumptions are introduced. If inelastic incompressibility is assumed and hardening is limited to isotropic hardening only, the yield function (4.1) appropriate to these conditions has the form

$$F(\overset{o}{\mathbf{T}},R) = \frac{f(II'_{o},III'_{o})}{R(W'')} - 1. \tag{4.10}$$

If, furthermore, for the function f a von Mises form is adopted, that is, $f = (II'_{o})^{1/2}$

$$F(\overset{o}{\mathbf{T}},R) = \frac{\sqrt{II'_{o}}}{R(W'')} - 1, \tag{4.11}$$

and Equations (4.5) and (4.8) accordingly become

$$\dot{\Sigma}'' = \gamma \left\langle \Phi \left[\frac{\sqrt{II'_{o}}}{R} - 1 \right] \right\rangle \frac{\overset{o}{\mathbf{T}}}{\sqrt{II'_{o}}} , \tag{4.12}$$

and

$$\sqrt{II'_{o}} = R(W'') \left[1 + \Phi^{-1} \left[\frac{\sqrt{II'_{\dot{\Sigma}''}}}{\gamma} \right] \right] . \tag{4.13}$$

4b. Uniaxial Constitutive Equations

For axial loadings, or uniaxial stress, Eqs. (4.9) reduce to uniaxial form (cf. Appendix for the stress and strain tensor reductions). Assuming inelastic incompressibility, isotropic hardening only and a von Mises form for the function f, i.e., Eq. (4.11) for the yield function, a reduction to uniaxial stress gives, in summary form,

$$\dot{\epsilon} \quad = \dot{\epsilon}' + \dot{\epsilon}''$$

$$\dot{\epsilon}' \quad = \frac{1}{E}\dot{\sigma}$$

$$\dot{\epsilon}'' \quad = \sqrt{\frac{2}{3}}\,\gamma <\Phi(F)> \quad \text{for } F>0$$

$$\dot{\epsilon}'' \quad = o \quad \text{for } F \le 0 \tag{4.14}$$

$$F \quad = \frac{\sqrt{\frac{2}{3}}\sigma}{R(W'')} - 1, \quad \gamma = \sigma_0/R$$

$$W'' = \int_0^{\epsilon''} \sigma d\epsilon'', E \text{ and } \gamma_o \text{ are constants.}$$

Eq. (4.13) correspondingly reduces to

$$\sqrt{\frac{2}{3}}\sigma = R(W'') \left[1 + \Phi^{-1} \left[\frac{\sqrt{\frac{3}{2}}\dot{\epsilon}''}{\gamma} \right] \right]. \tag{4.15}$$

The following forms for the functional $\Phi(F)$ are suggested by Perzyna as forms that may possibly be used to correlate uniaxial stress-strain experimental data for structural metals:

$$\Phi(F) = F$$

$$\Phi(F) = F^n$$

$$\Phi(F) = \sum_{n=1}^n A_n F^n \tag{4.16}$$

$$\Phi(F) = e^F - 1$$

$$\Phi(F) = \sum_{n=1}^n A_n (e^{F^*} - 1).$$

5. Chaboche Theory

The viscoplastic theory proposed by Chaboche [27-30] is fundamentally similar in its underlying structure to Perzyna's theory, in that a normality assumption of the infinitesimal strain rate to the yield surface defines the flow rule. In this theory specific forms are chosen for the yield function F and for the flow rule functional $\Phi(F)$. The theory also introduces specific rate equations that govern the evolution of the hardening, which is taken as a combination of isotropic and kinematic types of hardening.

The deformation is assumed to be of small order and isothermal, with the inelastic part considered as being volume preserving. The yield function is taken to have the von Mises form, modified for the inclusion of hardening. Thus

$$F(\overset{o}{\mathbf{T}}, \overset{o}{\mathbf{Y}}, R) = F(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}}) - R = \sqrt{II'_{(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})}} - R , \tag{5.1}$$

where

$$II'_{(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})} = tr\left[(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})\cdot(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})\right] , \tag{5.2}$$

$\overset{o}{\mathbf{Y}}$ is the deviator of the kinematic hardening tensor and R, the isotropic hardening variable, is assumed to be a function of the inelastic strain accumulation p, defined by Eqs. (1.10) and (ii.7)$_2$.

The normality assumption [cf. Eq. (4.2)], here written for the inelastic strain rate rather than as its increment, requires, in view of Eqs. (5.2) and (ii.12) that in the space of the stress deviator

$$\dot{\Sigma}'' = \wedge\frac{\partial F}{\partial \overset{o}{\mathbf{T}}} = \wedge\frac{\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}}}{\sqrt{II'_{(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})}}} . \tag{5.3}$$

The scalar proportionality function $\wedge$ which is positive, and which can depend possibly on the same variables as F, is arbitrary otherwise. As in the Perzyna theory, [cf. Eq. (4.4)], it is chosen as a functional of the yield function having the particular power form:

$$\wedge = <\Phi(F)> = \begin{cases} \Phi(F) \text{ for } F > 0 \\ 0 \quad \text{ for } F \leq 0 \end{cases} \tag{5.4}$$

where

$$\Phi(F) = (\frac{F}{K'})^n = \left[\frac{\sqrt{II'_{(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})}} - R}{K'}\right]^n , \tag{5.5}$$

and K', n are temperature dependent material parameters. The so-called flow rule, comparable to the flow rule (4.5) of the Perzyna theory, in this theory has the explicit form

$$\dot{\Sigma}'' = \left[\frac{\sqrt{II'_{(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})}} - R}{K'}\right]^n \frac{(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})}{\sqrt{II'_{(\overset{o}{\mathbf{T}} - \overset{o}{\mathbf{Y}})}}} \quad \text{ for } F > 0. \tag{5.6}$$

The hardening that accompanies the inelastic deformation is described by a pair of rate

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equations. For the isotropic hardening variable $R(P)$

$$\frac{dR}{dt} = B(Q - R)\,\frac{dP}{dt} = B(Q - R)\,|\,\dot{\Sigma}''\,| \tag{5.7}$$

where

$$P = \int_0^{\Sigma''} d\,|\,\Sigma''\,| = \int_0^t \frac{d}{d\tau}\left(\sqrt{II'_{\Sigma''}}\right) d\tau \tag{5.8}$$

since

$$d\,|\,\Sigma''\,| = d\left(tr\sqrt{(\dot{\Sigma}'' \cdot \Sigma'')}\right) = d(\sqrt{II'_{\Sigma''}}). \tag{5.9}$$

It follows therefore that

$$\dot{P} = |\,\dot{\Sigma}''\,| = \sqrt{II'_{\dot{\Sigma}''}}. \tag{5.10}$$

For the anisotropic part of the hardening, the kinematic hardening that is employed is described by the rate equation

$$\frac{d\overset{o}{Y}}{dt} = C\left[A\dot{\Sigma}'' - \sqrt{\frac{2}{3}}\,\overset{o}{Y}\,|\,\dot{\Sigma}''\,|\right] - \Gamma(II'_{\overset{}{Y}})^{\frac{m-1}{2}}\,\overset{o}{Y}. \tag{5.11}$$

In these hardening rate equations, B, Q, C, A, Γ, and m are material parameters that are temperature dependent. The first contribution to the kinematic hardening rate equation is Prager's form [5] which is appropriate for monotonic loading. The second term generalizes this hardening rule so as to allow it to apply for change of load direction, or for cyclic loading [32]. The third models the hardening recovery, or softening, that is often observed at high temperatures. The isotropic hardening rate first-order differential equation (5.7) can be solved directly, giving

$$R(P) = Q + (R_0 - Q)e^{-BP}. \tag{5.12}$$

As the accumulated inelastic strain $P \to 0$, $R(0) \to R_o$, which represents the initial yield stress. As the accumulated inelastic strain becomes large, $R \to Q$, which represents the 'saturation' value for R. In the absence of elevated temperature the hardening recovery (or thermal softening) term in Eq. (5.11) is dropped, i.e., $\Gamma = 0$.

In summary:

$$\dot{\Sigma} = \dot{\Sigma}' + \dot{\Sigma}''.$$

$$\dot{\Sigma}' = \frac{1}{2\mu}\frac{\partial}{\partial t}(\dot{T}) + \frac{1-2\nu}{E}(tr\dot{T})1.$$

$$\dot{\Sigma}'' = \left(\frac{\sqrt{II'_{(\overset{o}{T}-\overset{o}{Y})}} - R}{K'}\right)^n \frac{(\overset{o}{T} - \overset{o}{Y})}{\sqrt{II'_{(\overset{o}{T}-\overset{o}{Y})}}} \qquad \text{for } F > 0.$$

$$\dot{\Sigma}'' = 0 \qquad \text{for } F \leq 0.$$

$$F = \sqrt{II'_{(\overset{o}{T}-\overset{o}{Y})}} - R(P).$$

$$\frac{dR}{dt} = B(Q - R)\frac{dP}{dt} \quad , \quad \dot{p} = |\,\dot{\Sigma}''\,|. \tag{5.13}$$

$$R = Q + (R_o - Q)\,e^{-BP}.$$

$$\frac{d\overset{o}{Y}}{dt} = C(A\dot{\Sigma}'' - \sqrt{\frac{2}{3}}\,\overset{o}{Y}\,|\,\dot{\Sigma}''\,|) - \Gamma(II_o)^{\frac{m-1}{2}}\,\overset{o}{Y}.$$

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$\{\mu, \nu, E\}$ are elastic constants , $\{K', n, B, Q, A, C, \Gamma, m, R_o\}$ are viscoplastic material parameters.

5a. Uniaxial Constitutive Equations

Referring to the Appendix for the reductions and notational changes adopted for axial loading, Equations (5.13) may thereby be reduced to uniaxial form. In so doing, the following transformations are also utilized

$$\sqrt{\frac{2}{3}} P \to p = \int_0^{\epsilon''} |d\epsilon''| = \int_0^t |\dot{\epsilon}''| d\tau, \quad \dot{p} = |\dot{\epsilon}''|, \quad \text{where } \epsilon'' = \dot{\Sigma}_{11}'' \text{ and}$$

$$\sqrt{\frac{3}{2}} R(P) \to R(p), \quad \sqrt{\frac{3}{2}} B \to b, \quad (\frac{2}{3})^{\frac{m-1}{2}} \Gamma \to \gamma$$

$$(\frac{3}{2})^{\frac{n+1}{2}} K' \to K, \quad \sqrt{\frac{3}{2}} A \to a, \quad T_{11} \to \sigma$$

$$\sqrt{\frac{3}{2}} Q \to q, \quad \sqrt{\frac{2}{3}} C \to c, \quad Y_{11} \to Y.$$

For unaxial loads the constitutive equations corresponding to Eqs. (5.13) can be expressed as follows:

$$\dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}''$$

$$\dot{\epsilon}' = \frac{1}{E} \dot{\sigma}$$

$$\dot{\epsilon}'' = \left[\frac{|\sigma - Y| - R}{K} \right]^n Sg(\sigma - Y) \quad \text{for } F > 0.$$

$$\dot{\epsilon}'' = 0 \quad \text{for } F \le 0.$$

$$F = |\sigma - Y| - R(p)$$

$$\frac{dR}{dt} = b(q - R) \frac{dp}{dt}, \quad \dot{p} = |\dot{\epsilon}''| \tag{5.14}$$

$$R = q + (R_0 - q) e^{-bp.}$$

$$\frac{dY}{dt} = c(a\dot{\epsilon}'' - Y|\dot{\epsilon}''|) - \gamma |Y|^m Sg(Y), \quad Sg(Y) = \frac{Y}{|Y|} = \pm 1$$

$\{K, n, b, q, c, a, \gamma, m, R_o\}$ are viscoplastic material parameters.

The non-linear first-order differential equation for the kinematic hardening, Eq. $(5.14)_8$, must be solved numerically. In circumstances for which the hardening recovery term can be neglected, i.e. $\gamma \approx 0$, the remaining differential equation, which is homogeneous in time, can be brought to a separated form and integrated directly, giving

$$Y = sa + (Y_0 - sa) e^{-sc(\epsilon'' - \epsilon''_0)}, \tag{5.15}$$

where $s = +1$ for $d\epsilon'' > 0$, $s = -1$ for $d\epsilon'' < 0$, and Y_0 and ϵ''_0 represent the fixed values of Y and ϵ'' at the previous reversal of direction of the strain (or load) for non-monotonic and cyclic loading. For strictly monotonic loading

$$Y = a(1 - e^{-c\epsilon''}). \tag{5.16}$$

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Inversion of Eq. (5.14) for the axial stress yields

$$\sigma = Y + R + K(\dot{\epsilon}'')^{\frac{1}{n}}. \tag{5.17}$$

For monotonic loading from zero stress

$$\sigma = a(1 - e^{-c\epsilon''}) + q + (R_0 - q)e^{-b\int_0^{\epsilon''} |d\epsilon''|} + K(\dot{\epsilon}'')^{\frac{1}{n}}. \tag{5.18}$$

5b. Modification for the Strain Rate Dependence of Initial Yield

As in Perzyna's theory, the Chaboche theory also fails to account for the fact that the onset of yield, or the initial yield, is strain rate dependent. When yield commences $\epsilon'' = \dot{\epsilon}'' = p = 0$, and from Eq. (5.14), $\lim_{p \to 0} R(p) = R_0$, whereas from Eq. (5.18) $\lim_{(\epsilon'', \dot{\epsilon}'') \to 0} \sigma = R_0$. Thus R_0 represents the initial yield stress for either uniaxial tension or compression, the value of which changes with the rate of strain $\dot{\epsilon}$ at which it takes place. To incorporate this dependency into the constitutive equation, the isotropic hardening variable can be taken to be a function of both the accumulated inelastic strain p and the strain rate $\dot{\epsilon}$, where the dependence on $\dot{\epsilon}$ comes through its appearance as an argument for the initial yield stress R_0. The Chaboche theory can be modified accordingly by assuming that the isotropic hardening variable has the form [32-34]

$$R(p, \dot{\epsilon}) = q + [R_0(\dot{\epsilon}) - q]e^{-bp} \tag{5.19}$$

where

$$\lim_{(p, \dot{\epsilon}) \to 0} R(p, \dot{\epsilon}) = R_0(0) \equiv R_{00} \tag{5.20}$$

is the initial yield stress at quasi-static strain rate, that is, the strain rate below which no significant observable effect upon the yield stress can be detected by the usual means of measurement.

The form of the initial yield function $R_0(\dot{\epsilon})$ can be determined by or suggested from initial yield stress versus strain rate test data covering the strain rate range of interest. As an illustration, test data gathered for the alloy Inconel at 1200°F indicates that the dependence of the initial yield stress on the strain rate is significant only for the low strain rate range $\dot{\epsilon} < 10^{-5}$ sec^{-1} [35-38]. From the experimental information provided by these sources it was possible to correlate the initial yield dependence upon strain rate over the range $10^{-9} < \dot{\epsilon} < 10^{-5}$ sec^{-1} by means of the relation [32,33,63]

$$R_0(\dot{\epsilon}) = R_{oo} \left\{ 1 + \ln\langle 1 + \exp\left[\sum_{i=0}^{5} \alpha_i (\ln \dot{\epsilon})^i \right] \rangle \right\}. \tag{5.21}$$

In this expression the α_i are the correlation coefficients that are obtained from non-linear regression analysis of the data. This form can be generalized to multi-dimensional stress by replacing $\dot{\epsilon}$ by $|\Sigma| = [tr(\Sigma \cdot \Sigma)]^{1/2}$.

PART II
VISCOPLASTIC THEORIES WITHOUT A YIELD CONDITION

6. Bodner-Partom Theory

The initial formulation of the Bodner-Partom theory for viscoplastic behavior [39-41] incorporates only the isotropic contribution to the inelastic hardening. The theory is constructed within the framework of small isothermal deformation with the infinitesimal strain divided into elastic and inelastic parts. Thus

$$\dot{\Sigma} = \dot{\Sigma}' + \dot{\Sigma}'', \tag{6.1}$$

with $\dot{\Sigma}'$ the linear elastic strain rate given by Eq. (4.7). The inelastic component of the deformation is assumed to be incompressible so that $\overset{o}{\Sigma}'' = \Sigma''$. With the absence of a yield condition, inelastic strain always takes place at all levels of stress, however small.

The deviator of the inelastic strain rate is assumed to be proportional to the stress deviator

$$\frac{\partial}{\partial t}(\overset{o}{\Sigma}'') = \frac{\partial}{\partial t}(\Sigma'') = \dot{\Sigma}'' = \Phi \overset{o}{T} \tag{6.2}$$

where

$$\Phi = \Phi(II'_{\overset{}{T}}, Z) \tag{6.3}$$

and Z is an unspecified scalar internal variable that is determined by a rate equation having the form

$$\frac{dZ}{dt} = f(\overset{o}{T}, \Sigma'', Z) . \tag{6.3a}$$

It follows from Eq. (6.2) that

$$tr(\dot{\Sigma}'' \cdot \dot{\Sigma}'') = II_{\dot{\Sigma}''} = \Phi^2 tr(\overset{o}{T} - \overset{o}{T}) = II'_{\overset{}{T}} \Phi^2 \tag{6.3b}$$

from which

$$\Phi = \sqrt{II'_{\dot{\Sigma}''}} \frac{1}{\sqrt{II'_{\overset{}{T}}}} \tag{6.4}$$

and

$$\dot{\Sigma}'' = \sqrt{II'_{\dot{\Sigma}''}} \frac{\overset{o}{T}}{\sqrt{II'_{\overset{}{T}}}} . \tag{6.5}$$

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The development leading from Eqs. (6.1) to (6.5) essentially parallels the development of the Saint Venant-Levy theory for perfectly plastic flow, except that the scalar internal variable Z has been introduced to serve as the isotropic hardening variable. To obtain a specific relation between the hardening variable and the inelastic strain rate, recourse is made to the Gilman-Johnston microstructural model for the motion of dislocations in crystalline solids. According to this model the speed, V, of a moving dislocation is related to the resolved shear stress, τ, by the relation

$$ v = v^* \, e^{-(\frac{d}{\tau})^{a}} . \tag{6.6}$$

In this equation v^* represents the limit value of v as $\tau \to \infty$, and thus represents the limit value of the dislocation velocity. The term d is called the 'drag stress' that impedes the motion of the dislocation, i.e., in macroscopic terms, the source of the hardening. Bodner-Partom adopt this model, generalizing it to multidimensional stress and setting it to a macroscopic description by means of the following correspondences:

$$ v \to II'_{\dot{\Sigma}''} \, , \quad \tau \to II'_{\underset{T}{\circ}} \, , \quad d \to Z \, , \quad v^* \to D_0^2 \, , $$

and the postulated form

$$ II'_{\dot{\Sigma}''} = D_0^2 \, \exp \left[- \left[\frac{Z^2}{3 II'_{\underset{T}{\circ}}} \right]^n \left[\frac{n+1}{n} \right] \right] . \tag{6.7}$$

The internal variable Z is thus identified with the 'drag stress' as a measure of the isotropic hardening associated with the inelastic deformation. The parameter D_0 is interpreted as the limit value of the shear strain rate. Combining Eqs. (6.5) and (6.7) gives

$$ \dot{\Sigma}'' = D_0 \, \exp \left\{ - \left[\frac{Z^2}{3 II'_{\underset{T}{\circ}}} \right]^n \left[\frac{n+1}{2n} \right] \right\} \frac{\overset{o}{T}}{\sqrt{II'_{\underset{T}{\circ}}}} \tag{6.8}$$

as the flow rule, which includes Z explicitly as the isotropic hardening variable. The constants D_0 and n are treated as material parameters that are temperature dependent.

The evolution of Z, that is, its growth with inelastic deformation, is described by the postulated rate equation

$$ \frac{dZ}{dt} = m(z_1 - Z) \frac{dW''}{dt} , \tag{6.9}$$

which has the solution

$$ Z = z_1 + (z_0 - z_1) e^{-mW''} . \tag{6.10}$$

The quantities m, z_0 and z_1 are material parameters, where z_0 is the value of Z when the inelastic strain is zero. Note the similarity in the form of the isotropic hardening Eq. (6.10) to that of Eq. (5.12) of the Chaboche theory. In Eq. (6.10) the hardening is expressed in terms of the inelastic work, whereas in Eq. (5.12) it is determined by the accumulated inelastic strain. The parameter z_1 represents the 'saturation' value of Z as $W'' \to \infty$, and corresponds in this sense to the parameter Q in Eq. (5.12). Also note that while R_0 in Eq. (5.12) can be identified as the initial yield stress, no such interpretation can be given to the parameter Z_0. In circumstances where hardening recovery

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takes place (e.g. thermal softening at elevated temperature) the isotropic hardening rate equation (6.9) is modified by the addition of hardening recovery term.

$$\frac{dZ}{dt} = m\,(z_1 - Z)\,\frac{dW''}{dt} - Az_1 \left[\frac{Z - z_2}{z_1}\right]^r \tag{6.11}$$

in which A, z_2 and r are additional material parameters.

In summary:

$$\dot{\Sigma} = \dot{\Sigma}' + \dot{\Sigma}'' \cdot$$

$$\dot{\Sigma}' = \frac{1}{2\mu}\frac{\partial}{\partial t}\,(\overset{o}{T}) + \frac{1 - 2\nu}{E}\,(tr\,\dot{T})1 \cdot$$

$$\dot{\Sigma}'' = D_0 \exp\left[-\left[\frac{Z^2}{3II'_{\overset{o}{T}}}\right]^n \left[\frac{n+1}{2n}\right]\right] \frac{\overset{o}{T}}{\sqrt{II'_{\overset{o}{T}}}}\cdot$$

$$\frac{dZ}{dt} = m\,(z_1 - Z)\frac{dW''}{dt} - Az_1 \left[\frac{Z - z_2}{z_1}\right]^r \tag{6.12}$$

$$Z = z_1 + (z_0 - z_1)e^{-mW''}$$

$\{m, \nu, E\}$ are elastic constants , $\{D_0, n, z_0, z_1, z_2, m, A, r\}$ are viscoplastic material parameters.

6a. Modification for Directional Hardening: Stouffer-Bodner Theory

A modification of the Bodner-Partom theory was proposed by Stouffer and Bodner [42,43] that allows it to incorporate the Bauschinger effect and the generally anisotropic nature of plastic hardening associated with load reversals. In the absence of a yield surface to provide a geometrical representation of hardening, the anisotropic hardening that is introduced is referred to as a 'directional hardening' that appears to be similar to the more familiar kinematic hardening.

Equation (6.2) is replaced by the relation

$$\frac{\partial}{\partial t}\,(\overset{o}{\Sigma}'') = \frac{\partial}{\partial t}\,(\Sigma'') = \Lambda \cdot \overset{o}{T}, \tag{6.13}$$

in which Λ is a fourth-order tensor which governs the anisotropy of the inelastic behavior. This equation implies inelastic incompressibility since $\overset{o}{\Sigma}'' = \Sigma$. Because of the symmetry of the tensors $\dot{\Sigma}''$ and $\overset{o}{T}$, Λ must have the symmetries

$$\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} , \tag{6.14}$$

which reduces the number of independent non-zero components of the tensor from 81 to 36. This symmetry allows introduction of the following notational change, where the components of the tensors $\overset{o}{T}$ and $\dot{\Sigma}''$ are re-labeled according to the scheme

$$\begin{array}{ll} \overset{o}{T}_{11} - \overset{o}{T}_1 , & 2\overset{o}{T}_{12} - \overset{o}{T}_4 \\[4pt] \overset{o}{T}_{22} - \overset{o}{T}_2 , & 2\overset{o}{T}_{23} - \overset{o}{T}_5 \cdot \\[4pt] \overset{o}{T}_{33} - \overset{o}{T}_3 , & 2\overset{o}{T}_{13} - \overset{o}{T}_6 \end{array} \tag{6.15}$$

Expressions (6.14) and (6.15) enable Eq. (6.13) to be expressed in the form

$$\dot{\Sigma}_\alpha'' = \sum_{\beta=1}^{6} \Lambda_{\alpha\beta} \overset{o}{T}_\beta, \qquad \alpha = 1, 2, \ldots, 6. \tag{6.16}$$

The independent non-zero components of the fourth-order tensor Λ are thus represented by the 6×6 matrix $[\Lambda_{\alpha\beta}]$. This matrix is next assumed to be symmetric, although no explanation is given as to what this means or implies in terms of reduction of the degree of anisotropy. (Analytically speaking the assumed symmetry reduces the number of independent components from thirty-six to twenty-one). With the symmetry, the principle values (eigenvalues), $A_{\alpha\alpha}$, $\alpha = 1, \ldots, 6$ of $[\Lambda_{\alpha\beta}]$ are real and the principal directions (eigenvectors), e_α, are orthogonal. Knowledge of the eigenvectors permits construction of an orthogonal matrix (tensor), $Q^{-1} = Q^T$, such that by means of the similarity transformation $Q \cdot \Lambda \cdot Q^T$, reduces Λ to its diagonal form

$$\left[\Lambda_{\alpha\beta} \delta_{\sigma\beta} \right]_{\{e_a\}} = \begin{bmatrix} A_{11} & & & & \\ & A_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \; A_{66} \end{bmatrix}. \tag{6.17}$$

Relative to the principal direction basis e_α, $\alpha = 1, \ldots, 6$, the constitutive equations (6.16) take the form

$$\dot{\Sigma}_\alpha'' = \sum_{\beta=1}^{6} A_{\alpha\beta} \overset{o}{T}_\beta \tag{6.18}$$

$$A_{\alpha\beta} = \begin{cases} A_{\alpha\alpha}, \alpha = \beta & \\ & \cdot \quad \alpha = 1, \ldots, 6. \\ 0, \quad \alpha \neq \beta \end{cases}$$

From Eq. (6.17) it follows that

$$I_{\dot{\Sigma}''} = \dot{\Sigma}''_1 + \dot{\Sigma}''_2 + \dot{\Sigma}''_3 = A_{11}\overset{o}{T}_1 + A_{22}\overset{o}{T}_2 + A_{33}\overset{o}{T}_3. \tag{6.18}$$

For isotropic hardening, the hardening tensor can have only two independent non-zero components. Thus $A_{11} = A_{22} = A_{33} = \lambda$ can be one of them and, consequently

$$I_{\dot{\Sigma}''} = \lambda (\overset{o}{T}_1 + \overset{o}{T}_2 + \overset{o}{T}_3) = \lambda \, I_{\underset{T}{o}} = 0$$

for all values of λ. This implies plastic incompressibility. For anisotropic hardening in general, $A_{11} \neq A_{22} \neq A_{33}$ and $I_{\dot{\Sigma}''} \neq 0$, which implies plastic compressibility in contradiction with the inelastic incompressibility implied in Eq. (6.13).

Paralleling the assumption (6.3) for isotropic hardening, the principle values of the anisotropic material response tensor Λ are assumed to be functions of the second invariant of the strain deviator tensor and of a tensorial internal variable Z, where the components $Z_{\alpha\beta}$ of Z, referred to principal directions of the matrix $[\Lambda_{\alpha\beta}]$, are such that $Z_{\alpha\beta} = 0$, $\alpha \neq \beta$. Thus

$$A_{\alpha\alpha} = F_\alpha(II'_{\underset{T}{o}}, Z_{\alpha\alpha}), \quad \alpha = 1, \ldots, 6 \; (\text{ no sum on } \alpha). \tag{6.19}$$

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The internal variable **Z** is interpreted as a hardness variable that represents the resistance to anisotropic inelastic deformation. The form of Eqs. (6.19) is taken to be similar to the form of Eq. (6.8). That is, Stouffer and Bodner ascribe to anisotropic hardening the same microstructural 'drag stress' mechanism modeled by Eq. (6.6), that they used to introduce isotropic hardening into the Bodner-Partom theory when they assume that

$$A_{\alpha\alpha} = D_0 \exp \left[- \left[\frac{Z_{\alpha\alpha}^2}{3 II'_{\underset{T}{o}}} \right]^n \left[\frac{n+1}{2n} \right] \right] \frac{1}{\sqrt{II'_{\underset{T}{o}}}} \quad \alpha = 1, \ldots, 6 \; (\text{ no sum on } \alpha). \quad (6.20)$$

The hardness internal variable **Z** is assumed to be governed by a rate equation similar to the rate equation (6.11) for isotropic hardening. To introduce directionality and the Bauschinger effect, a material parameter q is introduced that serves to describe the relative amount of the isotropic part to the directional part of the hardening in the following manner:

$$\dot{Z}_{\alpha\alpha} = q\dot{Z} + (1-q)\dot{Z}u_\alpha \qquad \text{for } \dot{\Sigma}''_\alpha > 0$$

$$\alpha = 1, \ldots, 6, (\text{ no sum on } \alpha) , \qquad (6.21)$$

$$\dot{Z}_{\alpha\alpha} = q\dot{Z} - (1-q)\dot{Z}U_\alpha \qquad \text{for } \dot{\Sigma}''_\alpha > 0$$

where

$$U_\alpha = \frac{\dot{\Sigma}''_\alpha}{|\dot{\Sigma}''|} \quad \text{are the 'direction cosines' of the rate of strain} \qquad (6.22)$$

and

$$\dot{Z} = m(z_1 - Z)\dot{W}'' - Az_1 \left[\frac{Z - z_2}{z_1} \right]^r . \qquad (6.23)$$

Integration of Eqs. (6.21) gives for each $\alpha = 1, \ldots, 6$,

$$Z_{\alpha\alpha} = z_0 + q \int_0^t \dot{Z}(r)d\tau + (1-q)Sg(\dot{\Sigma}_\alpha'') \int_0^t \dot{Z}(\tau)U_\alpha(\tau)d\tau , \qquad (6.24)$$

$$Sg(\dot{\Sigma}_\alpha'') = \frac{\dot{\Sigma}_\alpha''}{|\dot{\Sigma}_\alpha''|} = \pm 1 .$$

A value of $q = 1$ indicates completely isotropic hardening, whereas $q = 0$ signifies that the hardening is entirely directional.

For each value of $\alpha = 1, \ldots, 6$, expressions (6.18) and (6.20) give the six components for the strain rate Σ_α'' as

$$\dot{\Sigma}_\alpha'' = D_0 \exp \left[- \left[\frac{Z_{\alpha\alpha}^2}{3 II'_{\underset{T}{o}}} \right] \left[\frac{n+1}{2n} \right] \right] \frac{T_\alpha^o}{\sqrt{II'_{\underset{T}{o}}}} , \qquad (6.25)$$

in which the six directional hardness variables $Z_{\alpha\alpha}$ are obtained by means of Eqs. (6.22) and (6.23).

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In summary:

$$\dot{\Sigma} = \dot{\Sigma}' + \dot{\Sigma}''$$

$$\dot{\Sigma}' = \frac{1}{2\mu} \frac{\partial}{\partial t} (\overset{o}{T}) + \frac{1-2\nu}{E} (tr\dot{T})1$$

$$\dot{\Sigma}'' = D_0 \exp\left[-\left[\frac{Z_{\alpha\alpha}^2}{3II'_T}\right]^n \left[\frac{n+1}{2n}\right]\right] \frac{\overset{o}{T_\alpha}}{\sqrt{II'_{\overset{o}{T}}}}, \quad \alpha = 1,\ldots,6 \quad (\text{ no sum on } \alpha)$$

$$Z_{\alpha\alpha} = z_0 + q \int_0^t \dot{Z}(\tau) d\tau + (1-q) Sg(\dot{\Sigma}_\alpha'') \int_0^t \dot{Z}(\tau) u_\alpha(\tau) d\tau \qquad (6.26)$$

$$u_\alpha = \dot{\Sigma}_\alpha''/|\dot{\Sigma}''| \quad , \quad Sg(\dot{\Sigma}_\alpha'') = \dot{\Sigma}_\alpha''/|\dot{\Sigma}_\alpha''| = \pm 1$$

$$\dot{Z} = m(z_1 - Z)\dot{W}'' - Az_1 \left[\frac{Z-z_2}{z_1}\right]^r$$

$\{\mu, \nu, E\}$ are elastic constants, $\{D_0, n, z_0, z_1, z_2, m, A, r, q\}$ are viscoplastic material parameters.

Comments:

1. The anisotropic hardening that is described by the Stouffer-Bodner theory is an attempt to simulate the kinematic hardening appearing in yield-based theories without the benefit of the geometrical description of hardening afforded by a yield surface. The anisotropic hardening, described by a 9×9 matrix in the most general sense, is reduced by symmetry considerations and assumptions so as to allow the main diagonal components of the 6×6 reduced hardening matrix to be taken as functions of the principal values of a hardening tensor Z, which is considered to be an internal variable. The components of the hardening tensor are further assumed to be parallel to the components of the inelastic strain rate tensor. In this way a directional hardening analog to kinematic hardening is constructed.

2. The functional form used to relate the components $A_{\alpha\alpha}$ of the material tensor Λ to the components $Z_{\alpha\alpha}$ of the internal variable hardening tensor Z is identical to the one used for the isotropic hardening theory. In other words, the same microstructural model is used to suggest macroscopic relations that describe two different kinds of hardening, although isotropic and anisotropic hardening are generally believed to be caused by different microstructural mechanisms.

3. The parameter q which proportions the hardening into isotropic and directional parts is arbitrarily assigned a value. It is not measured. For cyclic loading, where during the course of the cycling the character of the hardening changes, the value chosen for q would have to be changed repeatedly as the cycling progresses.

4. The parameter D_0, representing the limit value of the shear strain rate, is also a parameter that is not measured. A value of 10^4 sec^{-1} was initially suggested by Bodner. Subsequently values of 10^6 and 10^8 have been assigned for the value of D_0 by others using this viscoplastic constitutive theory.

6b. Uniaxial Constitutive Equations

The reduction of Eqs. (6.26) for uniaxial stress yields as follows:

$$\dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}''$$

$$\dot{\epsilon}' = \frac{1}{E}\dot{\sigma}$$

$$\dot{\epsilon}'' = \frac{2}{\sqrt{3}}D_0 \exp\left[-\left[\frac{Z^2}{\sigma^2}\right]^n\left[\frac{n+1}{2n}\right]\right]Sg(\sigma) \tag{6.27}$$

$$Z = z_0 + q\int_0^t \dot{Z}(\tau)d\tau + (1-q)\,Sg\,(\dot{\epsilon}'')\int_0^t \dot{Z}(\tau)\frac{\dot{\epsilon}''}{|\dot{\Sigma}''|}d\tau$$

$$\dot{Z} = m(z_1 - Z)\dot{W}'' - Az_1\left[\frac{Z - z_2}{z_1}\right]^r$$

$\{D_0, n, z_0, z_1, z_2, m, A, r, q\}$ are viscoplastic material parameters

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7. Walker Theory

The viscoplastic constitutive theory proposed by Walker that is shown here is the theory as of 1981 [44,45]. Walker advances two theories, an integral version and a rate version. Both theories are appropriate for small isothermal deformations that are assumed to be decomposable into elastic and inelastic parts.

7a. Integral Theory

The integral version is obtained by modification of the standard 3-parameter spring-dashpot model for a viscoelastic solid. For a uniaxial stress it may be shown to give the rate equation

$$\dot{\sigma} + \lambda \, \sigma = E_f \dot{\epsilon} + \lambda \, E_s \epsilon \, , \tag{7.1}$$

where E_f, E_s and λ are the three material parameters of the model.

The parameter E_f, representing the elastic modulus of the spring in series with the elastic spring and dashpot that are arranged in parallel to each other, gives the instantaneous response of the system

$$\dot{\sigma} = E_f \dot{\epsilon} \tag{7.2}$$

which is the first term of the right side of Eq. (7.1). For very slow rates of loading or strain, the response of the model is given essentially by the second term

$$E_s \epsilon = Y \, , \tag{7.3}$$

which is referred to as the 'equilibrium stress'. E_s is the elastic modulus of the spring in parallel to the dashpot. The parameter λ, representing the action of the dashpot, is interpreted as a viscosity coefficient.

The Boltzman integral form of the standard 3-parameter model for a viscoelastic solid subject of uniaxial stress is given by

$$\sigma(t) = \int_0^t G(t - \tau) \frac{d\epsilon}{d\tau} d\tau \, , \tag{7.4}$$

where

$$G(t) = E_f e^{-\lambda t} + E_s (1 - e^{-\lambda t}) \tag{7.5}$$

is the relaxation function. Substitution of expressions (7.5) and (7.3) into (7.4) leaves

$$\sigma(t) = \int_0^t \left[f \frac{d\epsilon}{d\tau} - \frac{dY}{d\tau} \right] e^{-\lambda(t-\tau)} d\tau + \int_0^t \frac{dY}{d\tau} d\tau \, . \tag{7.6}$$

Since

$$e^{-\lambda(t-\tau)} = e^{-\lambda \int_\tau^t d\varsigma} = e^{-\int_\tau^t \lambda d\varsigma} \, ,$$

and

$$\int_0^t \frac{dY}{d\tau} d\tau = Y(t) - Y(0) = Y(t) - E_s \epsilon(0) = Y(t) \, ,$$

it follows that

$$\sigma(t) = Y(t) + \int_0^t (E_f \frac{d\epsilon}{d\tau} - \frac{dY}{d\tau}) e^{-\int_\tau^t \lambda d\zeta} d\tau .$$ (7.7)

The modification of the above viscoelastic constitutive theory to one representing viscoplasticity is proposed in the following manner. Inelastic strain is defined as the difference between the total strain and the elastic strain

$$\epsilon'' = \epsilon - \frac{1}{E}\sigma , \qquad \text{where } E = E_f .$$ (7.8)

The linear elastic 'equilibrium stress' relation (7.3) is replaced by a nonlinear relation that depends on the inelastic strain and its accumulation

$$Y = Y(\epsilon'',p).$$ (7.9)

The viscosity coefficient λ is assumed to be nonlinearly proportional to the inelastic strain accumulation rate

$$\lambda = \frac{E}{R} \dot{p}^{(1-\frac{1}{n})} ,$$ (7.10)

with R and n as constants.

In view of Eq. (7.10),

$$e^{-\int_\tau^t \lambda d\zeta} - \frac{E}{R}\int_\tau^t \frac{\partial p}{\partial \zeta}^{(1-\frac{1}{n})} d\zeta = \frac{E}{R}\int_0^t \frac{\partial p}{\partial \zeta}^{(1-\frac{1}{n})} d\zeta - \frac{E}{R}\int_0^\tau \frac{\partial p}{\partial \zeta}^{(1-\frac{1}{n})} d\zeta$$

$$= Q(t) - Q(\tau)$$ (7.11)

where

$$Q(t) = \frac{E}{R}\int_0^t \frac{\partial p}{\partial \zeta}^{(1-\frac{1}{n})} d\zeta.$$ (7.12)

Substitution of expressions (7.9) and (7.11) into Eq. (7.7) yields

$$\sigma(t) = Y(t) + \int_0^t \left[E\frac{d\epsilon}{d\tau} - \frac{dY}{d\tau} \right] e^{-\left[Q(t)-Q(\tau) \right]} d\tau.$$ (7.13)

Expression (7.9) for Y is postulated to have the form

$$Y(t) = Y_0 + n_1 \epsilon''(t) + \int_0^t n_2 e^{-[Z(t)-Z(\tau)]} \left| \frac{d\epsilon''}{d\tau} \right| d\tau ,$$ (7.14)

in which $Z(t)$ is specified by

$$Z(t) = [n_3 + n_4 p(t)] \ln \left[\frac{n_5 p(t)}{1+n_6 p(t)} + 1 \right] + \int_0^t n_7 |Y - Y_0|^{m-1} d\tau .$$ (7.15)

Equations (7.8) and (7.12) - (7.15) constitute Walker's proposed viscoplastic constitutive theory.

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The 'equilibrium stress' Y is interpreted as an internal variable representing the microstructural 'back stress' which — "is a measure of the globally averaged dislocation arrangement and produces directional hardening". The $n_1 \epsilon''$ contribution to Eq. (7.14) is meant to model the Bauschinger effect. Expression (7.15) is intended to model cyclic hardening and softening, where the logarithmic term allows for cyclic hardening and the integral term for the decrease of the equilibrium stress due to thermal recovery (softening) at elevated temperatures. The constant R in Eq. (7.12) is identified as the microstructural 'drag stress' usually associated with isotropic hardening. [cf. Eq. (6.7) of the Bodner-Partom theory]. It is here taken to be constant.

In summary:

$$\epsilon'' = \epsilon - \epsilon' = \epsilon - \frac{1}{E}\sigma$$

$$\sigma(t) = Y(t) + \int_0^t (E\frac{d\epsilon}{d\tau} - \frac{dY}{d\tau}) e^{-[Q(t)-Q(\tau)]} d\tau$$

$$Q(t) = \frac{E}{R}\int_0^t \frac{\partial p}{\partial \zeta}^{(1-\frac{1}{n})} d\zeta$$

$$p(t) = \int_0^{\epsilon''} |d\epsilon''| = \int_0^t |\dot{\epsilon}''| d\tau \qquad (7.16)$$

$$Y(t) = Y_0 + n_1\epsilon''(t) + \int_0^t n_2 e^{-[Z(t)-Z(\tau)]} |\frac{d\epsilon''}{d\tau}| d\tau$$

$$Z(t) = [n_3 + n_4 p] \ln \left[\frac{n_5 p}{1 + n_6 p} + 1 \right] + \int_0^t n_7 |Y - Y_0|^{m-1} d\tau$$

$\{n, R, Y_0, m, n_1, \ldots, n_7\}$ are temperature dependent viscoplastic material parameters.

7b. Rate Theory

A viscoplastic constitutive theory having a rate form is also constructed from the 3-parameter spring-dashpot model for viscoelastic material response. After combining Eqs. (7.1) and (7.3) and relabeling E_f as E, it follows that

$$\dot{\epsilon} = \frac{1}{E}[\dot{\sigma} + \lambda(\sigma - Y)]. \qquad (7.17)$$

The conversion to a theory for viscoplasticity begins by differentiating Eq. (7.8) for the definition of the inelastic strain

$$\dot{\epsilon}'' = \dot{\epsilon} - \frac{1}{E}\dot{\sigma} . \qquad (7.18)$$

This is substituted into Eq. (7.17), along with Eq. (7.10) which replaces the spring-dashpot viscosity parameter λ by the proposed power relation of the inelastic strain rate. The result is

$$\dot{\epsilon}'' = \left[\frac{\sigma - Y}{R}\right] \dot{p}^{(1-\frac{1}{n})} . \qquad (7.19)$$

Since $\dot{p} = |\dot{\epsilon}''|$, Eq. (7.19) may also be expressed by the equivalent form

$$\dot{\epsilon}'' = \left[\frac{|\sigma - Y|}{R}\right]^n Sg(\sigma - Y) . \qquad (7.20)$$

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In the integral theory the 'equilibrium' stress Y is interpreted as an internal variable that represents the microstructural 'back stress' that is associated with directional hardening. It is given by the proposed relations (7.14) and (7.15). For the rate theory, Y is now governed by the proposed rate or evolution equation

$$\frac{dY}{dt} = (n_1 + n_2)\,\dot{\epsilon}'' - (Y - n_1\epsilon'')\dot{Z}\ , \tag{7.21}$$

where

$$\frac{dZ}{dt} = (n_3 + n_4 e^{-n_5 P})\dot{p} + n_6\,|\,Y\,|^{m-1}\ . \tag{7.22}$$

In the integral theory the constant R appearing in Eq. (7.12) was interpreted as representing the microstructural 'drag stress'. Here it is assumed to be a variable (an internal variable) that accounts for isotropic hardening according to the proposed relation

$$R = R_1 - R_2 e^{-n_7 P}\ , \tag{7.23}$$

In summary

$$\epsilon'' = \epsilon - \epsilon' = \epsilon - \frac{1}{E}\sigma$$

$$\dot{\epsilon}'' = \left[\frac{|\,\sigma - Y\,|}{K}\right]^n Sg(\sigma - Y)$$

$$R = R_1 - R_2 e^{-n_7 P} \tag{7.24}$$

$$\frac{dY}{dt} = (n_1 + n_2)\dot{\epsilon}'' - (Y - n_1\epsilon'')\dot{Z}$$

$$\frac{dZ}{dt} = (n_3 + n_4 e^{-n_5 P})\dot{p} + n_6\,|\,Y\,|^{m-1}$$

$\{n, m, n_1, n_2, n_3, n_4, n_5, n_6, n_7, R_1, R_2\}$ are temperature dependent viscoplastic material parameters

Comments:

1. Generalization of Eqs. (7.16) and (7.24) to multiaxial states of stress is straightforward.

2. Although Walker claims that the two theories are equivalent, i.e., that the rate theory may be obtained from the integral theory by differentiation of the latter, it is difficult to see how this is in fact possible.

3. The development of both of these theories have the earmarks of an ad-hoc piecemeal construction.

4. The similarity of several of Walker's rate equations with those of the Chaboche theory should be noted. In particular Eq. (7.24) with Eq. (5.14)$_3$ for the inelastic strain rate; Eq. (7.24)$_3$ with Eq. (5.14)$_7$ for the modeling of the isotropic hardening; and Eqs. (7.24)$_{4,5}$ with Eq. (5.14)$_8$ for the directional, i.e., kinematic hardening.

8. Kremple-Liu Theory

The Kremple-Liu theory [46,47] although lacking a yield condition, is nevertheless constructed such that for loading, or unloading from a plastic state, the predicted model response will be elastic. This theory, for small isothermal deformation, is also developed by modification of a 3-parameter spring-dashpot model for a solid uniaxially loaded. Starting with the rate equation derived from the spring-dashpot model,

$$\frac{\lambda}{E_s}(E_s + E_\infty)\dot{\epsilon} + E_\infty \epsilon = \frac{\lambda}{E_s}\dot{\sigma} + \sigma \tag{8.1}$$

in which E_s, E_∞ are the elastic coefficients and λ the viscous coefficient of the model, a viscoplastic theory is proposed by changing the material coefficients from constants to functions of the stress and strain.

An equilibrium (zero strain rate) stress-strain relation is assumed

$$\sigma = \Phi(\epsilon). \tag{8.2}$$

An overstress $\sigma - \Phi(\epsilon)$ is defined (similar to what appears in the Perzyna and Chaboche theories). The material parameters appearing in the rate equation (8.1) are replaced by unspecified material functional of the overstress:

$$\frac{\lambda}{E_s}(E_s + E_\infty) \rightarrow \Psi(\sigma - \Phi(\epsilon))$$

$$\frac{\lambda}{E_s} \rightarrow X(\sigma - \Phi(\epsilon)),$$

while the stress $E_\infty \epsilon$ is replaced by the equilibrium stress $\Phi(\epsilon)$. The proposed viscoplastic constitutive equation for uniaxial stress is thus

$$\Psi(\sigma - \Phi(\epsilon))\dot{\epsilon} + \Phi(\epsilon) = X(\sigma - \Phi(\epsilon))\dot{\sigma} + \sigma, \tag{8.3}$$

in which the material functional Ψ and X, as well as the function Φ, are to be determined by experiment. These functions are required to be bounded continuous functions that are also positive and even. They are also required to be such that for all values of their argument the ratio of Ψ over X is constant, equal to the value of the elastic modulus.

$$\frac{\Psi(\sigma - \Phi(\epsilon))}{X(\sigma - \Phi(\epsilon))} = E. \tag{8.4}$$

The form of Eq. (8.3) is such that for quasi-static stress or strain rates, $\dot{\epsilon}$ and $\dot{\sigma}$ are negligibly small leaving the equilibrium stress-strain relation (8.2), while for very high stress or strain rates condition (8.4) ensures a linear elastic response.

Equations (8.3) and (8.4) together may be shown to produce the following additional qualitative features of the proposed theory.

(i) There is initial linear elastic behavior for all rates of loading.

(ii) An elastic slope is obtained upon large instantaneous changes in strain (or stress) rates, e.g., at load reversals from the plastic range.

(iii) Stress-strain curves obtained at different rates of loading are non-linearly spaced.

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(iv) In predicting a creep test with tensile creep stress, the model yields $\dot{\epsilon} \geq 0$ and $\ddot{\epsilon} \leq 0$ for the total strain rate, provided that $(d\Phi/d\epsilon) \geq 0$. Primary and secondary creep can be reproduced.

(v) In predicting a stress relaxation test for positive hold strain, the model calculates $\dot{\sigma} \leq 0$ and $\ddot{\sigma} \geq 0$, where relaxation terminates at the equilibrium stress $\sigma = \Phi(\epsilon)$.

Equations (8.3) and (8.4), when combined, give

$$\dot{\epsilon} = \frac{1}{E}\dot{\sigma} + \frac{\sigma - \Phi(\epsilon)}{EX(\Gamma - \Phi(\epsilon))} \ . \tag{8.5}$$

Since the first term is the elastic stress rate, the second term is interpreted as the inelastic strain rate. Thus

$$\dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}'' \tag{8.6}$$

where

$$\dot{\epsilon}'' = \frac{\sigma - \Phi(\epsilon)}{EX(\sigma - \Phi(\epsilon))} \tag{8.7}$$

is defined as the viscoplastic flow rule of the theory. The equilibrium stress function Φ and the material response functional X for a given material, are determined from uniaxial test data at the temperature of interest.

An example of particular representations for these functions, taken from references [48,49], are as follows:

$$X = Ae^{-B\{\sigma - \Phi(\epsilon)\}^{1/4}} , \tag{8.8}$$

$$\Phi(\epsilon) = E_t(\epsilon - \epsilon_0) + \frac{E - E_t}{2C\tanh(CX - 3)} \log\left[\frac{\cosh U}{\cosh V}\right] + \sigma_0 , \tag{8.9}$$

where

$$U = C[X + (\epsilon - \epsilon_0)] - 3$$

$$V = C[X - (\epsilon - \epsilon_0)] - 3 \tag{8.10}$$

$$C = 3.63/X .$$

E_t is a tangent modulus and A,B,C, are parameters determined from uniaxial tests. The stress and strain values σ_0 and ϵ_0 are values at points where a load reversal occurs.

Comments:

1. The above theory has been used to model the room temperature uniaxial stress behavior of AISI 304 stainless steel, featuring changes in strain and stress, instantaneous changes in strain and stress rates, creep and stress relaxation.

2. A generalization to a multi-dimensional form of Eqs. (8.3) - (8.5) will give:

$$\Psi(T - \Phi(\Sigma))\dot{\Sigma} + \Phi(\Sigma) = X(T - \Phi(\Sigma))\dot{T} + T , \tag{8.11}$$

$$\frac{\Psi}{X} = E . \tag{8.12}$$

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and

$$\dot{\Sigma} = \frac{1}{\partial \mu}\frac{\partial}{\partial t}(\overset{o}{\mathbf{T}}) + \frac{1-2\nu}{E}(tr\,\dot{\mathbf{T}})\,\mathbf{1} + \frac{\mathbf{T}-\Phi(\Sigma)}{EX(\mathbf{T}-\Phi(\Sigma))}.\qquad(8.13)$$

The equilibrium stress function Φ and the material functionals Ψ and X have to be determined for general states of stress. There appears to be no reported progress with the development of the theory for multiaxial stress applications thus far.

9. Miller Theory

This constitutive theory is developed for small isothermal deformation and uniaxial stress. It is constructed to model monotonic and cyclic hardening and creep behavior. The inelastic component of the total strain rate is assumed initially to have the form

$$\dot{\epsilon}'' = \Psi \left[\frac{\sigma - Y}{R} \right] , \qquad (9.1)$$

where Ψ is an unspecified functional of the argument shown. The variables Y and R are referred to, respectively, as the 'rest stress' associated with kinematic hardening, and the 'drag stress' associated with isotropic hardening. To give specific form to the functional Ψ, Miller considers the following empirical relation which relates steady state creep strain rate $\dot{\epsilon}_{ss}''$ to the hold stress σ_h for stainless steels, aluminum alloys and ferritic materials

$$\dot{\epsilon}_{ss}'' = B' \left[\sinh (A \sigma_h) \right]^n , \qquad (9.2)$$

in which A, B' and n are material parameters. Miller argues that during a creep test the hardening variables Y and R assume the values Y_{ss} and R_{ss}, at the onset of steady state creep, and remain constant thereafter. He further assumes the existence of a function Ψ_1, such that during steady state creep

$$\dot{\epsilon}_{ss}'' = \Psi_1 \left[\frac{\sigma_h - Y_{ss}}{R_{ss}} \right] = \sigma_h . \qquad (9.3)$$

Substituting expression (9.3) into (9.2)

$$\dot{\epsilon}_{ss}'' = B' \left\{ \sinh \left[A \Psi_1 \left[\frac{\sigma_h - Y_{ss}}{R_{ss}} \right] \right] \right\}^n . \qquad (9.4)$$

It is next supposed that relation (9.4), which holds during steady state creep, is also appropriate for non-steady situations as well, i.e.,

$$\dot{\epsilon}'' = B' \left\{ \sinh \left[A \Psi_1 \left[\frac{\sigma - Y}{R} \right] \right] \right\}^n . \qquad (9.5)$$

Further recourse to empirical argument is made to propose that

$$A \Psi_1 \left[\frac{\sigma - Y}{R} \right] = \left[\frac{\sigma - Y}{R} \right]^{3/2} . \qquad (9.6)$$

Eq. (9.1) thus acquires the specific form

$$\dot{\epsilon}'' = B' \left\{ \sinh \left[\left[\frac{\sigma - Y}{R} \right]^{3/2} \right] \right\}^n . \qquad (9.7)$$

for the inelastic strain rate.

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A temperature dependence is introduced through the material parameter B', based upon the observation that the activation energy for plastic flow varies with temperature. It is expressed by the relations

$$B' = B\omega, \tag{9.8}$$

$$\omega = \exp\left\{\left[-\frac{Q}{0.6\Theta_m}\right]\left[\ln\left[\frac{0.6\Theta_m}{\Theta}+1\right]\right]\right\}, \qquad \Theta \leq 0.6\Theta_m \tag{9.9}$$

$$\omega = \exp\left[-\frac{Q}{K\Theta}\right]. \qquad \theta > 0.6\theta m \tag{9.10}$$

B and k are material constants, Θ_m is the melting temperature and Q is the activation energy necessary for the plastic deformation.

The hardening variables are assumed to be governed by the following rate equations

$$\dot{Y} = H_1\dot{\epsilon}'' - \Psi_2(Y,\Theta) \tag{9.11}$$
$$\dot{R} = H_2|\dot{\epsilon}''| - \Psi_3(R^3,\theta)$$

which include thermally activated softening (or hardening recovery) by means of the recovery functions Ψ_2 and Ψ_3. These functions are taken to have the same form as the steady state creep correlation expression (9.2). Adding the elastic strain rate, and summarizing:

$$\dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}''$$

$$\dot{\epsilon}' = \frac{1}{E}\dot{\sigma}$$

$$\dot{\epsilon}'' = B\omega(\theta)\left\{\sinh\left[\left[\frac{|\sigma-Y|}{R}\right]^{3/2}\right]\right\}^n Sg(T-Y)$$

$$\frac{dY}{dt} = H_1\dot{\epsilon}'' - H_1 B\omega(\theta)\left[\sinh\left[A_1|Y|\right]\right]^n Sg(Y) \tag{9.12}$$

$$\frac{dR}{dt} = H_2|\dot{\epsilon}''| - H_2 B\omega(\theta)\left[\sinh(A_2 R^3)\right]^n$$

$$\omega(\theta) = \begin{cases} \exp\left[-Q/k\theta\right] & \theta \geq 0.6\theta m \\ \exp\{[-Q/0.6\theta m][\ln(\frac{0.6\theta m}{\theta}+1)]\} & \theta \leq 0.6\theta m \end{cases}$$

$\{B,n,H_1,H_2,A_1,A_2,Q,k,\theta_m\}$ are viscoplastic and thermal material parameters.

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10. Valanis Theory

10a. Intrinsic Time Scale

The derivation of the constitutive equations of Valanis' 'endochronic' theory of viscoplasticity [53-61] is lengthy. It is necessary, however, to go through the derivation in order to undersand the theory's strengths and weaknesses. Central to the theory is the notion of 'intrinsic time' and its measure, which is developed as follows. In first considering rate independent plastic deformation, Valanis argues "that the state of stress in the neighborhood of a point in a plastic material depends on the set of all previous states of deformation of that neighborhood, but it does not depend on the rapidity at which such deformation states have succeeded one another. The independence of stress on the rapidity of succession of deformation states is achieved by introducing a time scale ζ, called the 'intrinsic time' which is independent of time t, the external time measured by a clock, but which is intrinsically related to the deformation of the material." This relation is postulated to have the form

$$d\zeta^2 = d\mathbf{C} \cdot \mathbf{P} \cdot d\mathbf{C} \tag{10.1}$$

where $\mathbf{C}$ is the right Green-Cauchy deformation tensor and $\mathbf{P}$ is a fourth-order material tensor which could depend on $\mathbf{C}$. The 'intrinsic time' ζ describes the length of the path traveled by the material in the six-dimensional deformation space and, thus, is representative of the deformation history of the material.

To account for the rate dependence exhibited by structural materials when experiencing non-recoverable deformation, a second time scale, L, the 'intrinsic time measure', coupling the intrinsic time ζ with the natural time t is introduced.

$$dL^2 = d\zeta^2 + g^2 dt^2 \tag{10.2}$$

The scalar g is a material parameter. Finally an 'intrinsic time scale', $z(L)$, is defined such that

$$dz = \frac{1}{f(L)} dL, \quad \frac{dz}{dL} > 0, \quad 0 \le L < \infty \tag{10.3}$$

where $f(L)$ is another material function. Since $\mathbf{P}$ and f are material functions, the intrinsic time scale z is viewed as representing the particular inner properties of the material. Hence the descriptive name of 'endochronic' given to the theory.

10b. General Theory

The general form of the theory, which allows for finite deformation, is developed from continuum thermodynamic considerations in which internal variables are introduced and coupled with the Onsager-DeGroot-Mansur theory of non-equilibrium thermodynamics. Designating ϵ, η and $\Psi = \epsilon - \theta\eta$ as the internal energy, entropy and free energy densities per unit mass, respectively,* q as the heat conduction vector and h as the heat source density, then the first and second laws of thermodynamics expressed in local form require that at each point of the body undergoing a thermomechanical process

$$\rho\dot{\epsilon} = tr(\mathbf{T} \cdot \mathbf{D}) - div\,\mathbf{q} + \rho h \tag{10.4}$$

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*In this section the symbol ϵ is used to designate internal energy and not uniaxial strain as in other parts of this report.

and

$$tr(\mathbf{T} \cdot \mathbf{D}) - \rho \dot{\Psi} - \rho \, \eta \, \dot{\theta} - \frac{1}{\theta} \mathbf{q} \cdot \text{grad } \theta \geq 0. \tag{10.5}$$

In these expressions ρ is the mass density at a point of the current configuration of the body.

For elastic material response it is assumed that the material behavior can be specified by constitutive equations for the free energy Ψ, the stress $\mathbf{T}$, the entropy η and the heat conduction $\mathbf{q}$, when the deformation $\mathbf{C}$ the temperature θ and its gradient $\mathbf{g} = \text{grad } \theta$ are presumed to be known. To describe inelastic response, Valanis argues that associated with non-recoverable deformation are numerous processes that occur at the atomic scale that are responsible for internal entropy generation. It is necessary, therefore, to introduce N additional state variables, which Valanis refers to as 'internal hidden variables' (i.e., not necessarily observable) in order to completely define the thermodynamic state of the material. Thus, according to this view, the complete thermodynamic state of the material requires knowledge of $\mathbf{C}$, θ, $\mathbf{g}$ and N-many unspecified internal variables, $\beta_1, \beta_2, \ldots, \beta_N$, at each point of the material. The internal variables are further assumed to be governed by N differential equations of evolution.

The set of postulated constitutive equations for any thermomechanical process is thus enlarged to include the internal variables as follows:

$$\Psi = \Psi(\mathbf{C}, \theta, \mathbf{g}, \beta_1, \ldots, \beta_N)$$

$$\mathbf{T} = \mathbf{T}(\mathbf{C}, \theta, \mathbf{g}, \beta_1, \ldots, \beta_N) \tag{10.6}$$

$$\eta = \eta(\mathbf{C}, \theta, \mathbf{g}, \beta_1, \ldots, \beta_N)$$

$$\mathbf{q} = \mathbf{q}(\mathbf{C}, \theta, \mathbf{g}, \beta_1, \ldots, \beta_N)$$

and

$$\frac{d\beta_i}{dt} = f_i(\mathbf{C}, \theta, \beta_1, \ldots, \beta_N) \quad i = 1, 2, \ldots, N.$$

Comment:

1. It is noted that Valanis fails to include the temperature gradient $\mathbf{g}$ as a variable for the functions f_i, even though it is included in all of the other constitutive equations. No reasons nor justification is given as to why this should be so.

Following the analysis of Coleman and Guritn [62] substitution of Eqs. (10.6) into the first and second law requirements (10.4) and (10.5) lead to the following conclusions: The constitutive functions for Ψ, $\mathbf{T}$ and η must be independent of the temperature gradient $\mathbf{g}$ as an explicit variable. In addition the free energy determines the stress and the entropy through the relations

$$\mathbf{T} = 2 \frac{\rho}{\rho_0} \frac{\partial \Psi}{\partial \mathbf{C}}, \tag{10.7}$$

$$\eta = -\frac{\partial \Psi}{\partial \theta},$$

where

$$\Psi = \Psi(\mathbf{C}, \theta, \beta_1, \ldots, \beta_N), \tag{10.8}$$

and

$$-\frac{\partial \Psi}{\partial \beta_i} f_i = -\frac{\partial \Psi}{\partial \beta_i} \dot{\beta}_i \geq 0 \qquad i = 1, \ldots, N(\text{ no sum on } i). \tag{10.9}$$

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As mentioned above, in obtaining expressions (10.7)-(10.9) Valanis employs the arguments of Coleman and Gurtin, who follow the Truesdell-Noll-Coleman theory of non-equilibrium thermodynamics. However in developing his theory further, Valanis next turns to the Onsager-DeGroot-Mansur theory when he makes use of a result central to that theory of non-equilibrium thermodynamics. According to the inequality (10.9) the product of $(- \partial \Psi / \partial \beta_i)$ with $(d\beta_i / dt)$ results in an increase of the entropy. They are then interpreted as 'generalized forces' and 'generalized fluxes' when Valanis next assumes the Onsager relations which relate the generalized forces linearly to the generalized fluxes, that is,

$$-\frac{\partial \Psi}{\partial \beta_i} = b_{ij}^* \frac{d\beta_i}{dt}, \quad i = 1, \ldots, N \quad (\text{no sum on } i). \tag{10.10}$$

In the equations the b_{ij}^* can, in general, be functions of the state variables $C, \theta, \beta_1, \ldots, \beta_N$. When viewed as a set of differential equations for the unknown internal variables $\beta_1, \ldots, \beta_N$, Eqs. (10.10) are non-linear differential equations. A linearization is introduced by further assuming the b_{ij}^* are such that

$$b_{ij}^* = k b_{ij}, \tag{10.11}$$

where b_{ij} are constants and $k > 0$ is a parameter that relates natural time t to the intrinsic time scale z whereby

$$dz = \frac{1}{k} dt. \tag{10.12}$$

Insertion of Eqs. (10.11) and (10.12) into Eq. (10.10) then produces the following set of linear differential equations for the internal variables β_j:

$$\frac{\partial \Psi}{\partial \beta_i} + b_{ij} \frac{d\beta_j}{dz} = 0, \quad i = 1, 2, \ldots, N \ (\text{no sum on } j). \tag{10.13}$$

The system of linear differential Eqs. (10.13) for determination of the β_j is set in terms of the intrinsic time scale z, and replaces the system of generally non-linear differential equations (10.6)$_5$ set in terms of the natural time t.

With equations (10.13), the constitutive description of the entropy producing (dissipative) material behavior can now, in principle, be considered to be complete. One needs only to expand the free energy Ψ as a Taylor series, which, in principle, allows solution of the system of equations (10.13) for the internal variables $\beta_1, \ldots, \beta_N$. These values when substituted into Eqs. (10.6)$_2$ give the desired stress-deformation relations.

In summary:

$$\Psi = \Psi(C, \theta, \beta_1, \ldots, \beta_N)$$

$$T = 2 \frac{\rho}{\rho_o} \frac{\partial \Psi}{\partial C} \tag{10.14}$$

$$\eta = -\frac{\partial \Psi}{\partial \theta}$$

$$\frac{\partial \Psi}{\partial \beta_i} + b_{ij} \frac{d\beta_j}{dz} = 0, \quad j = 1, 2, \ldots, N \ (\text{no sum on } j).$$

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Comments:

2. If in the constitutive equations $(10.6)_5$ the temperature gradient $\mathbf{g} = \text{grad } \theta$ had been included as an independent variable as it was for the other constitutive equations, then in place of the dissipative inequality (10.9) there would be

$$-\frac{\partial \Psi}{\partial \beta_i} f_i = -\frac{\partial \Psi}{\partial \beta_i} \dot{\beta}_i + \frac{1}{p\theta} \mathbf{q} \cdot \mathbf{q} \geq 0$$

instead. It appears in light of the Onsager type relations (10.10), which are suggested by the simpler inequality (10.9), and which make possible the expressions (10.13), that herein lies the rationale for Valanis' omission of $\mathbf{g} = \text{grad } \theta$ as a possible variable for the rate functions f_i at the outset. The above inequality would preclude introduction of the Onsager like relations (10.10).

3. A unique solution to the set of N first order differential equations $(10.6)_5$ or (10.13) for the N internal variables $\beta_1, \ldots, \beta_N$, requires stipulation of a set of N initial conditions for these variables. This presents some difficulty in light of the interpretation of these variables as "hidden" unspecified variables. How can physically relevant initial conditions be assigned on to set of variables if they are not defined?

10c. Explicit Constitutive Equations for Small Deformations

The hidden variables heretofore undefined as to generic nature, are now assumed to be N second-order symmetric tensors $\beta_{ij}^\alpha = \beta_{ji}^\alpha$, $i, j = 1, 2, 3$, $\alpha = 1, \ldots, N$ (we hereafter refer to a rectangular coordinate system and resort to a tensor component notation). For small deformation and uniform temperature, Eqs. (10.14), after appropriate modification*, simplify to

$$\Psi = \Psi(\Sigma_{ij}, \beta'_{ij}, \ldots, \beta_{ij}^N) \tag{10.15}$$

$$T_{ij} = \frac{\partial \Psi}{\partial \Sigma_{ij}} \tag{10.16}$$

$$\frac{\partial \Psi}{\partial \beta_{ij}^\alpha} + b_{ijkl}^\alpha \frac{d\beta_{kl}^\alpha}{dz} = 0 \qquad \alpha = 1, \ldots, N(\alpha \text{ not summed }). \tag{10.17}$$

The components b_{ijkl}^α are the components of the N-many fourth-order tensors $\mathbf{b}^\alpha$, $\alpha = 1, \ldots, N$, which are considered to be material tensors.

The free energy is expanded as a Taylor series up to terms of the second order. (The terms of the first-order vanish to satisfy the zero deformation of the initial undeformed state).

$$\Psi = \frac{1}{2} A_{ijkl} \Sigma_{ij} \Sigma_{kl} + \sum_{\alpha=1}^{N} B_{ijkl}^\alpha \Sigma_{ij} \beta_{kl}^\alpha + \frac{1}{2} \sum_{\alpha=1}^{N} C_{ijkl}^\alpha \beta_{ij}^\alpha \beta_{kl}^\alpha. \tag{10.18}$$

Here A, B^α, C^α are material tensors. Assuming material isotropy, initially and throughout any subsequent deformation, the fourth-order material tensors are, therefore, isotropic tensors and can therefore be given the representation

$$A_{ijkl} = A_1 \delta_{ij} \delta_{kl} + A_2 \delta_{ik} \delta_{jl}$$

$$B_{ijkl}^\alpha = B_1^\alpha \delta_{ij} \delta_{kl} + B_2^\alpha \delta_{ik} \delta_{jl} \tag{10.19}$$

$$C_{ijkl}^\alpha = C_1^\alpha \delta_{ij} \delta_{kl} + C_2^\alpha \delta_{ik} \delta_{jl}$$

$$b_{ijkl}^\alpha = b_1^\alpha \delta_{ij} \delta_{kl} + b_2^\alpha \delta_{ik} \delta_{jl}$$

---

*Having the internal variables as second-order tensor quantities, it is correspondingly necessary the the N many constant coefficients in Eqs. (10.13) be replaced by components B_{ijkl}^α of the N-many fourth-order tensors $\mathbf{B}^\alpha$.

where $A_1, A_2, B_1^\alpha, B_2^\alpha, C_1^\alpha, C_2^\alpha, b_1^\alpha, b_2^\alpha$, $\alpha = 1, \ldots, N$ are material constants and δ_{jk} are the Kronecker delta symbols. Thus

$$\Psi = \frac{1}{2}(A_1\delta_{ij}\delta_{kl} + A_2\delta_{ij}\delta_{jl})\Sigma_{ij}\Sigma_{kl} + \sum_{\alpha=1}^{N}(B_1^\alpha\delta_{ij}\delta_{kl} + B_2^\alpha\delta_{ik}\delta_{jl})\Sigma_{ij}\beta_{kl}^\alpha$$

$$+ \frac{1}{2}\sum_{\alpha=1}^{N}(C_1^\alpha\delta_{ij}\delta_{kl} + C_2^\alpha\delta_{ij}\delta_{jl})\beta_{ij}^\alpha\beta_{kl}^\alpha , \tag{10.20}$$

which gives

$$\frac{\partial\Psi}{\partial\beta_{ij}^\alpha} = B_1^\alpha\Sigma_{kk}\delta_{ij} + B_2^\alpha\Sigma_{ij} + C_1^\alpha\delta_{ij}\beta_{kk}^\alpha + C_2^\alpha\beta_{ij}^\alpha. \qquad (\text{no sum on } \alpha). \tag{10.21}$$

Also, from (10.18) and (10.19)$_4$

$$\mathbf{b}_{ijkl}^\alpha\frac{d\beta_{kl}^\alpha}{dz} = b_1^\alpha\frac{d\beta_{kk}^\alpha}{dz}\delta_{ij} + b_2^\alpha\frac{d\beta_{ij}^\alpha}{dz} \qquad (\text{no sum on } \alpha). \tag{10.22}$$

With these results, Eq. (10.14)$_4$ now takes the form

$$\left[B_1^\alpha\Sigma_{kk} + C_1^\alpha\beta_{kk}^\alpha + b_1^\alpha\frac{d\beta_{kk}^\alpha}{dz} \right]\delta_{ij} + B_2^\alpha\Sigma_{ij} + C_2^\alpha\beta_{ij}^\alpha + b_2^\alpha\frac{d\beta_{ij}^\alpha}{dz} = 0 , \tag{10.23}$$

for each value of $\alpha = 1, 2, \ldots, N$. Solution of this set of first-order differential equations is facilitated by a decomposition into spherical and deviatoric parts as follows:

$$B_0^\alpha\Sigma_{kk} + C_o^\alpha\beta_{kk}^\alpha + b_0^\alpha\frac{d\beta_{kk}^\alpha}{dz} = 0, \tag{10.24}$$

$$B_2^\alpha\overset{o}{\Sigma}_{ij} + C_2^\alpha\overset{o}{B}_{ij}^\alpha + b_2^\alpha\frac{d\overset{o}{\beta_{ij}^\alpha}}{dz} = 0, \tag{10.25}$$

where

$$\overset{o}{\beta_{ih}^\alpha} = \beta_{ij}^\alpha - \frac{1}{3}\beta_{kk}^\alpha\delta_{ij}, \tag{10.26}$$

and

$$B_0^\alpha = 3B_1^\alpha + B_2^\alpha, \quad C_0^\alpha = 3C_1^\alpha + C_2^\alpha, \quad b_0^\alpha = 3b_1^\alpha + b_2^\alpha, \tag{10.27}$$

The general solutions of Eqs. (10.24) and (10.25) are given by

$$\beta_{kk}^\alpha = -\frac{B_0^\alpha}{b_o^\alpha}\int_0^z e^{-\lambda_o(z-z')}\Sigma_{kk}(z')dz' + D_\alpha e^{-\lambda_o z} \tag{10.28}$$

and

$$\overset{o}{\beta_{ij}^\alpha} = -\frac{B_2^\alpha}{b_2^\alpha}\int_{0}e^{-\rho_o(z-z')}\overset{o}{\Sigma}_{ij}(z')dz' + F_\alpha e^{-\rho_o z} \tag{10.29}$$

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in which D_α and E_α, $\alpha = 1, \ldots, N$, are arbitrary constants and

$$\lambda_\alpha = C_0^\alpha / b_0^\alpha , \quad \rho_\alpha = C_2^\alpha / b_2^\alpha . \tag{10.30}$$

Comment:

4.   Since no initial conditions for the set of differential equations (10.24) and (10.25) are given (most probably because the internal tensor variables $\beta^1, \beta^2, \ldots, \beta^N$ are not defined, thus there is no physical motivation available to allow assumption of an appropriate set) the set of arbitrary constants D_α and E_α cannot be determined.   In as much as the equations that follow do not include the $D_\alpha e^{-\lambda_\alpha z}$ and $F_\alpha e^{-\rho_\alpha z}$ parts of the solutions (10.28) and (10.29) one can only conclude that the arbitrary constants D_α and F_α were all set to zero, although no argument or explanation as to why this should be so is given.

Eqs. (10.28) and (10.29) (with the terms including D_α and F_α hereafter omitted) determine the undefined internal variables in terms of the strain and the intrinsic time scale.   Upon substitution into the free energy form (10.18), calculation of stress-strain relations (10.16) may then follow.   From (10.18) and (10.16)

$$T_{ij} = A_1 \Sigma_{kk} \delta_{ij} + A_2 \Sigma_{ij} + \sum_{\alpha=1}^{N} \left[B_1^\alpha \beta_{kk}^\alpha \delta_{ij} + B_2^\alpha \beta_{ij}^\alpha \right] , \tag{10.31}$$

which can be decomposed into the deviatoric and spherical parts

$$\overset{o}{T_{ij}} = A_2 \overset{o}{\Sigma_{ij}} + B_2^\alpha \overset{o}{\beta_{ij}^\alpha} \tag{10.32}$$

and

$$T_{kk} = A_0 \Sigma_{kk} + B_0^\alpha \beta_{kk}^\alpha , \tag{10.33}$$

$$A_0 = 3A_1 + A_2, \quad B_0^\alpha = 3B_1^\alpha + B_2^\alpha . \tag{10.34}$$

Substitution of (10.29) into (10.32), followed by integration yields

$$\overset{o}{T_{ij}} = (A_2 - \frac{B_2^\alpha B_2^\alpha}{C_2^\alpha}) \overset{o}{\Sigma_{ij}} + \int_0^z \frac{B_2^\alpha B_2^\alpha}{C_2^\alpha} e^{-\rho_\alpha (z - z')} \frac{d \overset{o}{\Sigma_{ij}}}{dz'} dz' . \tag{10.35}$$

Designating by $H(z)$ the unit step function and defining

$$2\mu(z) = \left[A_2 - \frac{B_2^\alpha B_2^\alpha}{C_2^\alpha} \right] H(z) + \sum_{\alpha=1}^{N} \frac{B_2^\alpha B_2^\alpha}{C_2^\alpha} e^{-\rho_\alpha z} \tag{10.36}$$

Eq. (10.35) takes the form

$$\overset{o}{T_{ij}} = 2\int_0^z \mu(z - z') \frac{d \Sigma_{ij}^0}{dz'} dz' \tag{10.37}$$

in which $\mu(z)$ is interpreted as a shear modulus.   By a similar calculation in which Eq. (10.28) is substituted into Eq. (10.33), integrated, and where the definition

$$K(z) = (A_0 - \sum_{\alpha=1}^{N} \frac{B_0^\alpha B_0^\alpha}{C_0^\alpha} H(z) + \sum_{\alpha=1}^{N} \frac{B_0^\alpha B_0^\alpha}{C_0^\alpha} e^{-\lambda_\alpha z} \tag{10.38}$$

is introduced, the spherical part of the stress tensor takes the form

$$\frac{1}{3} T_{kk} = \int_0^z K(z-z')\frac{d}{dz'}\Sigma_{kk}dz' \, , \tag{10.39}$$

in which $K(z)$ is considered to be a bulk modulus.

After a re-labeling of the expressions for the shear and bulk moduli, in summary:

$$T_{ij} = 2\int_0^z \mu(z-z')\frac{d\overset{o}{\Sigma}_{ij}}{dz'}dz' + \delta_{ij}\int_0^z K(z-z')\frac{d\Sigma_{kk}}{dz'}dz'$$

$$\mu(z) = \mu_0 + \sum_{\alpha=1}^N \mu_\alpha e^{-\rho_\alpha z}$$

$$K(z) = K_0 + \sum_{\alpha=1}^N K_\alpha e^{-\lambda_\alpha z} \tag{10.40}$$

$\{\mu_0, K_0, \mu_\alpha, K_\alpha, \rho_\alpha, \lambda_\alpha, \ \alpha=1,\dots,N\}$ are material parameters

$z =$ intrinsic time scale determined by Eqs. (10.1)-(10.3).

The theory is carried further for rate-independent deformation by assuming the parameter g in Eq. (10.2) is zero. The intrinsic time measure is then (for small deformation)

$$dz^2 = d\Sigma \cdot \mathbf{P} \cdot d\Sigma = P_{ijkl}d\Sigma_{ij}d\Sigma_{kl} \, . \tag{10.41}$$

The material tensor $\mathbf{P}$ is isotropic [cf. Eqs. (10.19)]

$$P_{ijkl} = k_1\delta_{ij}\delta_{kl} + k_2\delta_{ik}\delta_{jl}$$

so that

$$L = \int_0^{\Sigma_{ij}} \left\{ k_1(d\Sigma_{kk})^2 + k_2 d\Sigma_{ij}d\Sigma_{ij} \right\}^{1/2} \, . \tag{10.42}$$

The material function $f(z)$ of Eq. (10.3) is taken as

$$f(z) = 1 + \beta \, L \qquad \beta > 0 \tag{10.43}$$

from which it follows that

$$z = \frac{1}{\beta}\ln(1+\beta L) \, . \tag{10.44}$$

The constants k_1, k_2 and β are additional material parameters.

The strain input into Eq. (10.42) determines the intrinsic time measure L, which then, through Eq. (10.44), specifies the intrinsic time scale z.

Comments:

5. For inelastic and plastic deformations that exhibit anisotropic hardening e.g., kinematic or directional hardening, the theory given by Eqs. (10.40) is clearly inappropriate.

6. The undefined internal variables β_{ij}^{α}, $\alpha = 1, 2, \ldots, N$ are solved for and expressed in terms of six sets of N-many material parameters $A_1^{\alpha}, A_2^{\alpha}$, $B_1^{\alpha}, B_2^{\alpha}, C_1^{\alpha}, C_2^{\alpha}$, $\alpha = 1, \ldots, N$ and the intrinsic time scale z. These are eventually lumped together as the material functions $(10.40)_{2,3}$ involving $4N+2$ undefined parameters, where the number N is left open. Since the internal variables themselves are not identified, the question of what should be the value of N is only vaguely alluded to. In recent applications of the theory the number has ranged from $N = 1$ to $N = \infty$. What does it mean to say that in one situation one internal variable is necessary to describe the material behavior where as in another an infinite number of them are necessary?

PART III
COMPARISONS BETWEEN THE MODIFIED CHABOCHE AND THE
BODNER THEORIES FOR A HIGH TEMPERATURE APPLICATION

11. Introduction

The modified Chaboche and the Bodner-Partom-Stouffer theories were selected to model the behavior of the alloy Inconel at a temperature of 1200°F. Each theory appears to be the best representative from the category of theories in which they were grouped. Specifically, the behavioral characteristics of Inconel at 1200°F, as predicted by both theories, are examined for uniaxial mono-tonic tensile loading at varying strain rates, for sudden increase and decrease of strain rate in the ine-lastic range, for creep and stress relaxation behavior, for load-unload-reload at varying strain rates, and for cyclic loading [34].

12. Viscoplastic Material Parameters for Inconel at 1200°F

For uniaxial loading the modified Chaboche equations, which consist of Eqs (5.14) and (5.19), are repeated here in combined form.

$$\dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}''$$

$$\dot{\epsilon}' = \frac{1}{E}\dot{\sigma}$$

$$\dot{\epsilon}'' = \left[\frac{|\sigma - Y| - R}{K}\right]^n Sg(\sigma - Y) \quad F > 0$$

$$\dot{\epsilon}'' = 0 \qquad\qquad F = \leq 0$$

$$F = |\sigma - Y| - R(p\,\dot{\epsilon}) \qquad Sg\,(\sigma - Y) = \frac{\sigma - Y}{|\sigma - Y|} = \pm 1 \tag{12.1}$$

$$R(p,\dot{\epsilon}) = q + \left[R_0(\dot{\epsilon}) - q\right]e^{-bp} \qquad p = \int_0^{\epsilon''}|d\epsilon''|$$

$$\frac{dY}{dt} = c(a\dot{\epsilon}'' - Y|\dot{\epsilon}''|\,) - \alpha|Y|^m Sg(Y)$$

$\{K, n, b, q, c, a, \gamma, m\}$ are viscoplastic material parameters.

For high temperature application in which hardening recovery (thermal softening) is expected, all eight of the viscoplastic material parameters require determination. In addition, the form of the dependence of the initial yield stress on the strain rate, $R_0(\dot{\epsilon})$, must also be ascertained.

At 1200°F, the available experimental data for Inconel [35-38] shows that the alloy exhibits strain rate sensitivity only over the small strain rate range $\dot{\epsilon} < 10^{-4}$ sec^{-1}. The material also shows typical creep response (primary, secondary, tertiary creep stages), and cyclic softening for approximately 10 to 20 percent of fatigue life before cyclic stabilization. The stress-strain response tends to a constant stress value at 1.0 to 1.5 % of strain.

The manner by which values for the parameters were evaluated from the data of monotonic loading tests, cyclic loading and creep tests, is described in References [33,63], with the results tabulated below.

Parameter	Description	Value
E	Young's modulus	24.73×10^3 ksi
ν	Poisson's ratio	0.336
a	Saturation value of kinematic hardening variable	30.00 ksi
c	Kinematic hardening exponent	350.0
γ	Coefficient of hardening recovery	0.4×10^{-10}
m	Hardening recovery exponent	7.00
K	Overstress parameter	155.0 ksi sec$^{1/n}$
n	Strain rate sensitivity parameter	5.10
q	Saturation value of isotropic hardening variable	50.00 ksi
b	Isotropic hardening exponent	3.75

At the elevated 1200°F temperature, the experimentally observed strain rate sensitivity of the initial yield stress was primarily over the low strain rate range $10^{-9} < \dot{\epsilon} < 10^{-4}$ sec $^{-1}$, where the data is well correlated by Eq. (15.21), [32,63]

$$R_0(\dot{\epsilon}) = R_{00}\left\{ 1 + (1 + \exp\left[\sum_{i=0}^{5} \alpha_i (\ln\dot{\epsilon})^i \right] \rangle \right\} .$$

where

$$\lim_{(p,\dot{\epsilon}) \to 0} R(p,\dot{\epsilon}) = R_0(0) = R_{00} \qquad (12.2)$$

represents the yield stress at quasi-static strain rate and the α_i are non-linear regression coefficients.

The procedures for determination of the material parameters appearing in the Bodner-Partom-Stouffer viscoplastic constitutive equations:

$$\dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}''$$

$$\dot{\epsilon}' = \frac{1}{E}\dot{\sigma}$$

$$\dot{\epsilon}'' = \frac{2}{\sqrt{3}}D_0 \exp\left[-\left[\frac{Z^2}{\sigma^2}\right]^n \left[\frac{n+1}{2n}\right] \right] Sg(\sigma)$$

$$Z = z_0 + q \int_0^t \dot{Z}(\tau)d\tau + (1-q)Sg(\dot{\epsilon}'')\int_0^t \dot{Z}(\tau)\frac{\dot{\epsilon}''}{|\Sigma''|}d\tau \qquad (12.3)$$

$$\dot{Z} = m(z_1 - Z)\dot{W}'' - Az_1\left[\frac{Z-z_2}{z_1}\right]^r ,$$

$$W = \int_0^t \sigma\dot{\epsilon}''dt = \int_0^{\epsilon''} \sigma d\epsilon''$$

$$\left\{ D_0, n, z_0, z_1, z_2, m, A, r, q \right\} \text{ are viscoplastic material parameters}$$

are detailed in Ref. [38] and the values are shown below.

Parameter	Description	Value
E	Young's modulus	23.55×10^3 ksi
ν	Poisson's ratio	0.336
m	Rate of work hardening	2.9
z_0	Initial value of hardness	235.1 ksi
z_1	Saturation value of hardness	260.2 ksi
z_2	Minimum recovery value of hardness	104.1 ksi
A	Hardening recovery parameter	1.5×10^{-3}
r	Hardening recovery exponent	7.0
n	Strain rate sensitivity parameter	3.0
D_0	Limit value of the inelastic strain rate (assigned value)	10^6 sec^{-1}
q	Hardening ratio (assigned value)	-0.05 to -0.10

13. Experimental and Qualitative Comparisons

13a. Experimental Comparisons

Several comparisons of the predictive capability of the two theories with experimental data (to the extent permitted by the availability of the data) are shown in Figures 13.1 thru 13.3 for monotonic tension, for a fully reversed load cycle and for minimum or secondary creep rate. The agreements shown between both sets of predictions and the experimental results are reasonably good.

It is of interest to observe that the Bodner theory which does not employ a yield condition, and which allows for inelastic deformation at all levels of stress, nevertheless shows a stress-strain curve with a rather sharply defined transition from linear (and presumably reversible elastic) behavior to non-linear behavior, at a stress level close to what would be considered as a yield stress in those theories that assume the existence of a yield condition.

13b. Strain Rate Effects

The modification introduced into the Chaboche theory allows the theory to be receptive to the effect of strain rate on the initial yield stress. Figures 13.4 and 13.5 illustrate the effect of strain rate on the stress-strain response of Inconel at the elevated temperature as predicted by the two theories for three different constant strain rates: 5×10^{-5}, 5×10^{-7}, and 5×10^{-9} sec^{-1}. Both theories predict a pronounced strain rate effect in the strain range considered. In both cases the stress saturates at relatively low strains, which is consistent with what is observed experimentally for superalloys at high temperature. Also, the inelastic parts of the stress-strain curves tend to approach one another as the strain rates increase in constant ratio, a result that is consistent with experimental observation in general.

A qualitatively similar stress-rate effect can also be simulated by both theories. This is shown by the stress-strain responses obtained under stress controlled conditions. The stress rates used were E (elastic modulus) times the corresponding strain rates. The results show the stress-controlled predicted curves to lie entirely above the corresponding strain-controlled curves throughout the non-linear range. This difference in the predicted responses is attributed to the non-linearity of the stress-strain rate constitutive relations.

The relative strength of the strain rate effect, that is the change in the stress-strain response as the strain rate is increased, appears to be greater for the Chaboche theory compared to the Bodner theory as evidenced by the relative stress difference between the stress-strain curves for the $\dot{\epsilon}_3$ and $\dot{\epsilon}_1$ strain rates.

13c. Strain Rate History Effect

The response of viscoplastic materials can be influenced by the strain rate history of the material. This can be shown by strain rate jump tests. Experiments of this kind on polycrystalline materials generally show the following characteristics: As the strain rate from a plastic state is suddenly increased the initial response of the material is predominately elastic, followed by yield and subsequent hardening during which the jump response approached asymptotically the monotonic response at the higher strain rate. Whether the jump response approaches the monotonic response from the low side or the high side, has been shown to depend upon the ratio of the strain rate before and after the jump, and on the crystalline structure of the material. For example for FCC and HCP materials the jump response is lower than the monotonic response, while for BCC materials the reverse takes place.
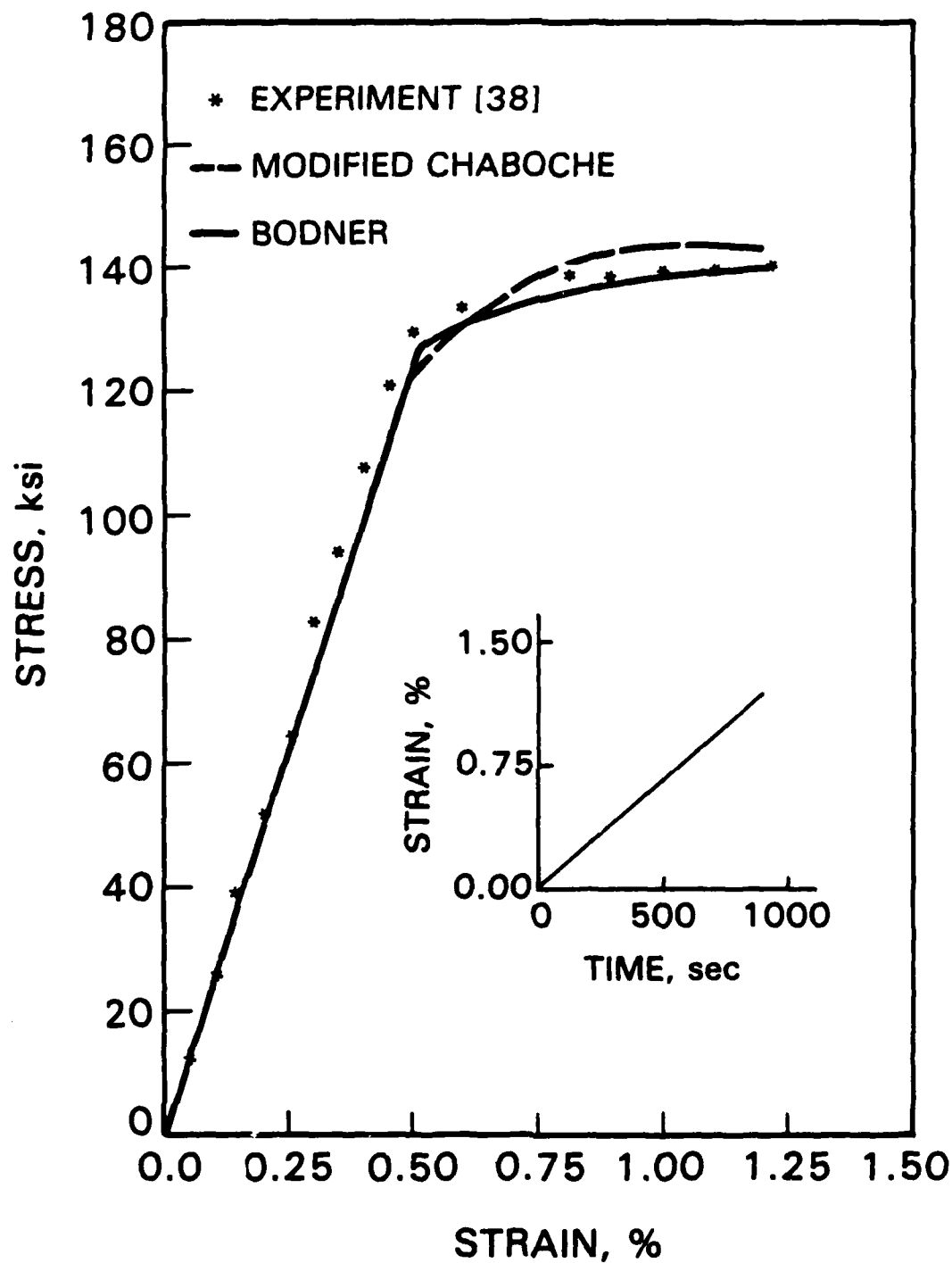
Fig. 13.1 — Experimental and predicted tensile stress-strain curves at $\dot{\epsilon} = 1.333 \times 10^{-5}$ sec^{-1}
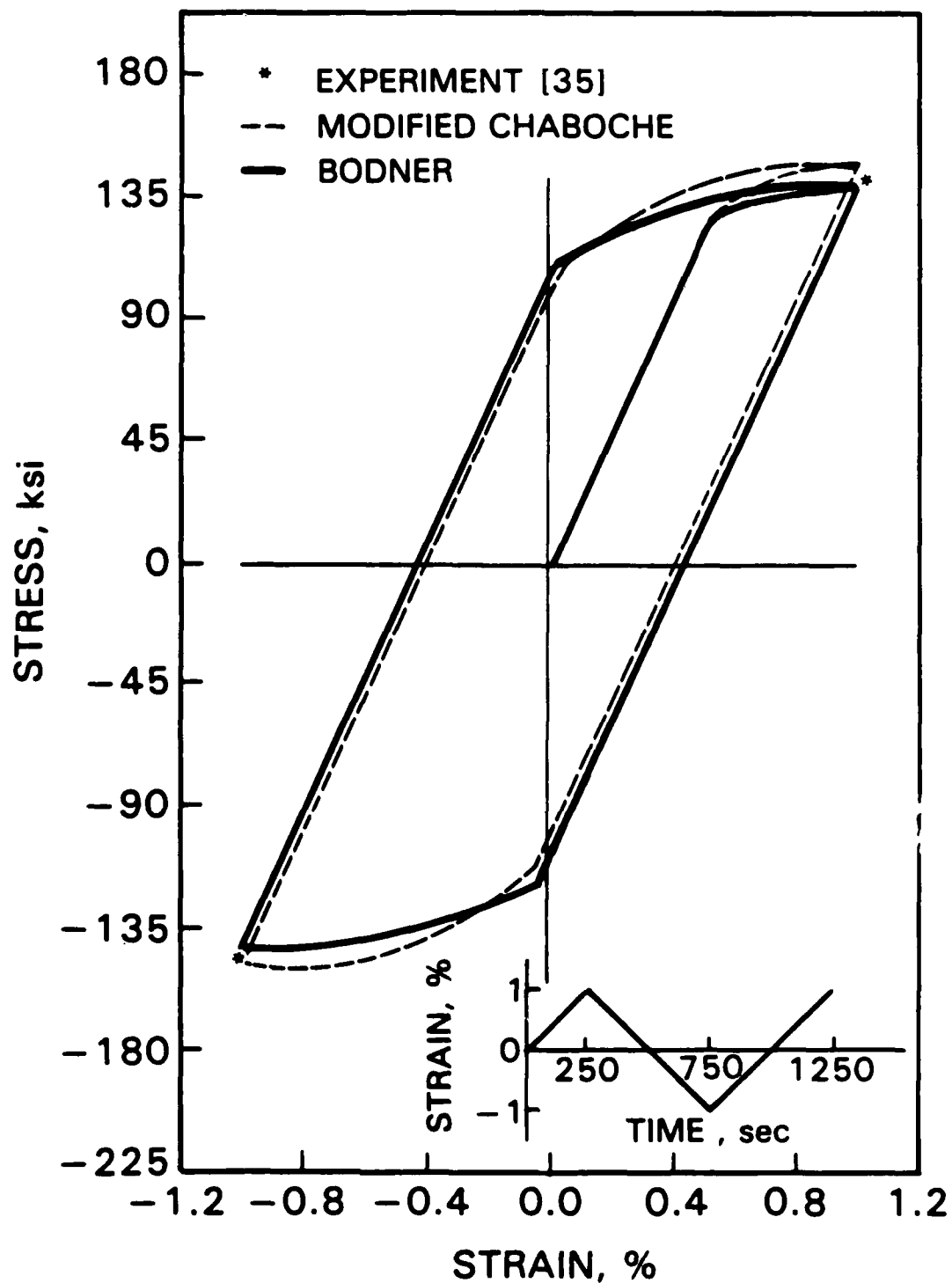using the modified Chaboche and Bodner theories.

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Fig. 13.2 — Experimental and predicted Strain-controlled fully-reversed cyclic behavior at $\epsilon = 4 \times 10^{-5}$ sec^{-1} and $\Delta\epsilon = 2\%$ using the modified Chaboche theory; experimental peak stress only is available.
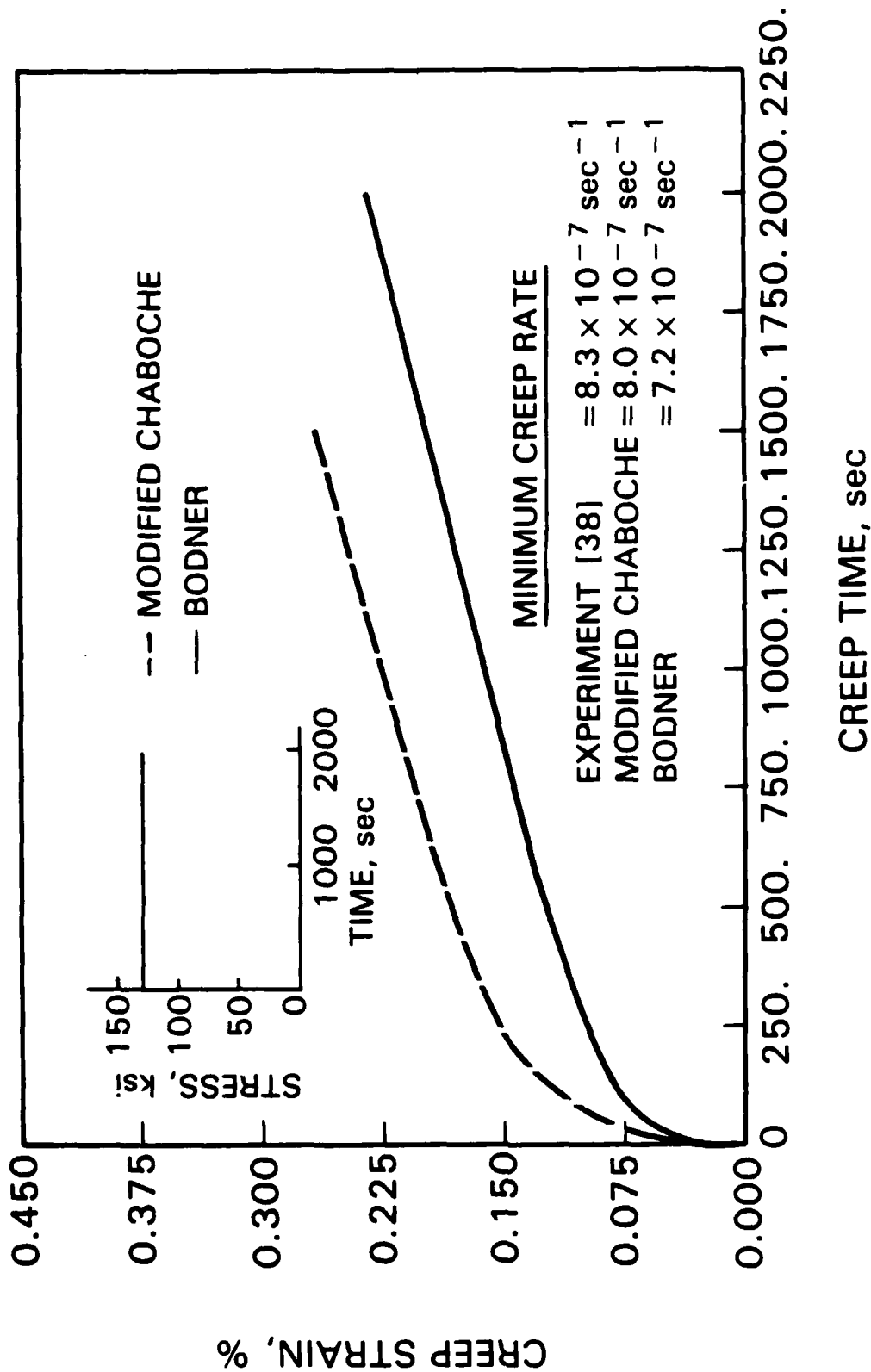
Fig. 13.3 — Predicted primary and secondary creep at 130 ksi using the modified Chaboche and Bodner theories; experimental steady state creep rate only is available.

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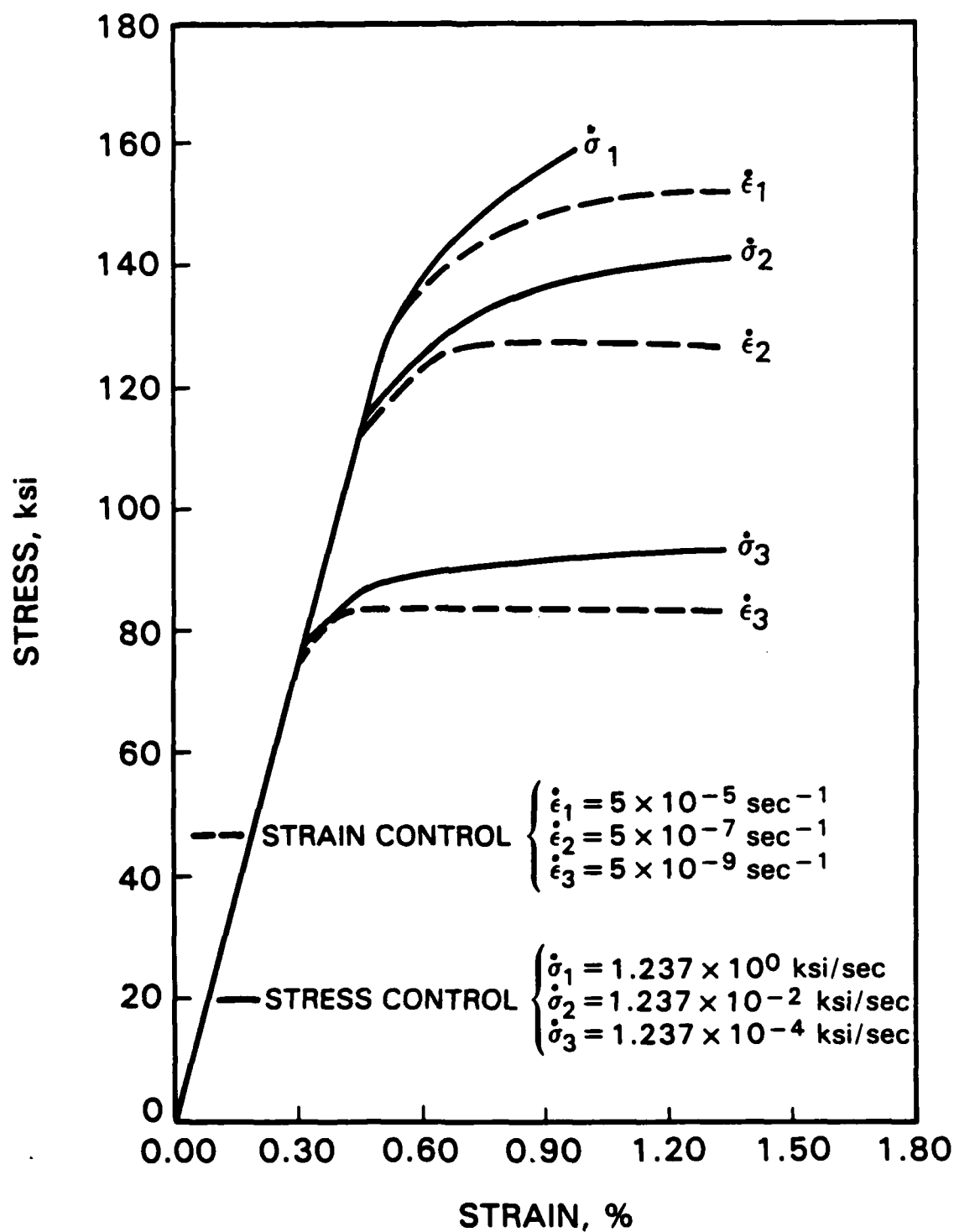
Fig. 13.4 — Predicted tensile response at different rates employing the
modified Chaboche theory.
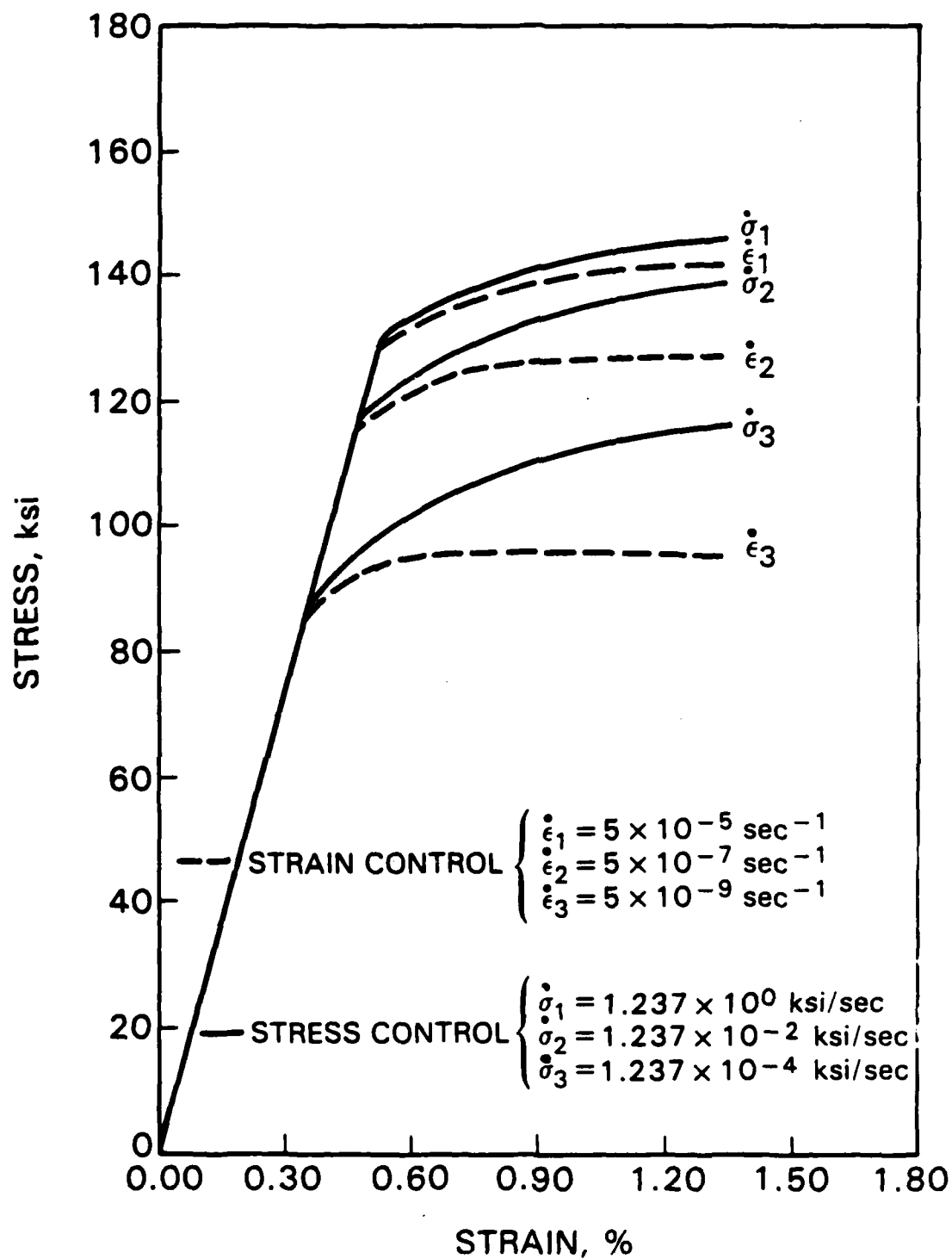
Fig. 13.5 — Predicted tensile response at different rates employing
Bodner's theory.

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Figures 13.6 and 13.7 illustrate the strain rate jump response predicted by the two theories as the strain rates are increased (decreased) by one order of magnitude, from $10^{-6}\,\text{sec}^{-1}$ to $10^{-5}\,\text{sec}^{-1}$. Both theories indicate an initial elastic response immediately following the jump in the strain rate, followed by hardening and a gradual approach to the monotonic stress-strain curve. Both demonstrate the strain rate history effect in the sense that the jump response does not immediately approach the monotonic response because of its prior inelastic pre-strain history at the lower (upper) strain rate. The approach of the jump response to the monotonic response occurs more rapidly according to the Bodner theory.

13d. Creep Behavior

There is experimental data that indicates that the creep behavior of structural materials can be significantly influenced by the rate of stress that is employed in reaching the hold stress σ_h in a uniaxial creep test [64]. Figures 13.8 and 13.9 demonstrate that both viscoplastic theories can model this observed effect, where the higher stress rate ($\dot\sigma_1 = 20$ ksi/sec) to the hold stress shows considerably greater primary creep strain than the lower stress rate ($\dot\sigma_2 = 20$ ksi/sec), prior to the attainment of the minimum creep rate (secondary creep). The minimum creep rate appears to be unaffected by the stress rate to the hold stress.

This effect may be explained in the following manner. The yield stress for the smaller stress rate is lower than for the higher stress rate, i.e., $R_0\,(\dot\epsilon_2) < R_0\,(\dot\epsilon_1)$ for $\dot\sigma_2 < \dot\sigma_1$. The hold stress at $\sigma_h = 140$ ksi is well above both yield stress values. Yield therefore initiates well before the hold stress level is attained, and by the time the creep hold stress is reached the material has hardened. For the smaller stress rate with the correspondingly lower yield stress, the material will experience a greater total strain in getting to the hold stress than for the higher stress rate. However it will also have experienced greater hardening. Therefore when creep begins the material with the greater prior hardening will undergo creep at a smaller rate and thus, for a given time interval, have a smaller creep strain, even though the total strain is greater.

Figure 13.10 shows the predicted creep response after 5 1/2 hours at a hold stress of 125 ksi. The Bodner creep curve shows primary and secondary creep stages although the primary creep predicted is too small to appear on the scale of this plot. The Chaboche creep curve shows the conventional primary and secondary stages of creep. However it also shows gradually increasing creep rates prior to the tertiary creep stage.

Numerical analysis [63] indicates that with the initiation of creep, at 1200°F Inconel starts to harden kinematically and to soften isotropically, where the rate of hardening is considerably greater than the rate of softening, $\dot Y > |-\dot R|$. This is so because R depends upon the inelastic strain accumulation which, during the initial stages of the creep, is very small. Thus with overall hardening there will be decreasing creep rate, which is the primary stage of creep. During the secondary creep stage, where the creep strain rate is constant, it can be easily seen from Eq. (5.17) that $\dot Y = |-\dot R|$. The rate of kinematic hardening is equal to the rate of isotropic softening while the creep strain rate is constant. Thus, when the creep strain rate starts increasing gradually before the beginning of tertiary creep, this must apparently be caused by material softening, that is, $\dot Y < |-\dot R|$, the rate of isotropic softening now exceeds the rate of kinematic hardening.

Instances of gradually increasing creep rates following a period of steady state creep, prior to the tertiary stage of rapidly increasing creep rates associated with material damage, have been observed experimentally for structural materials at elevated temperatures [65,66].
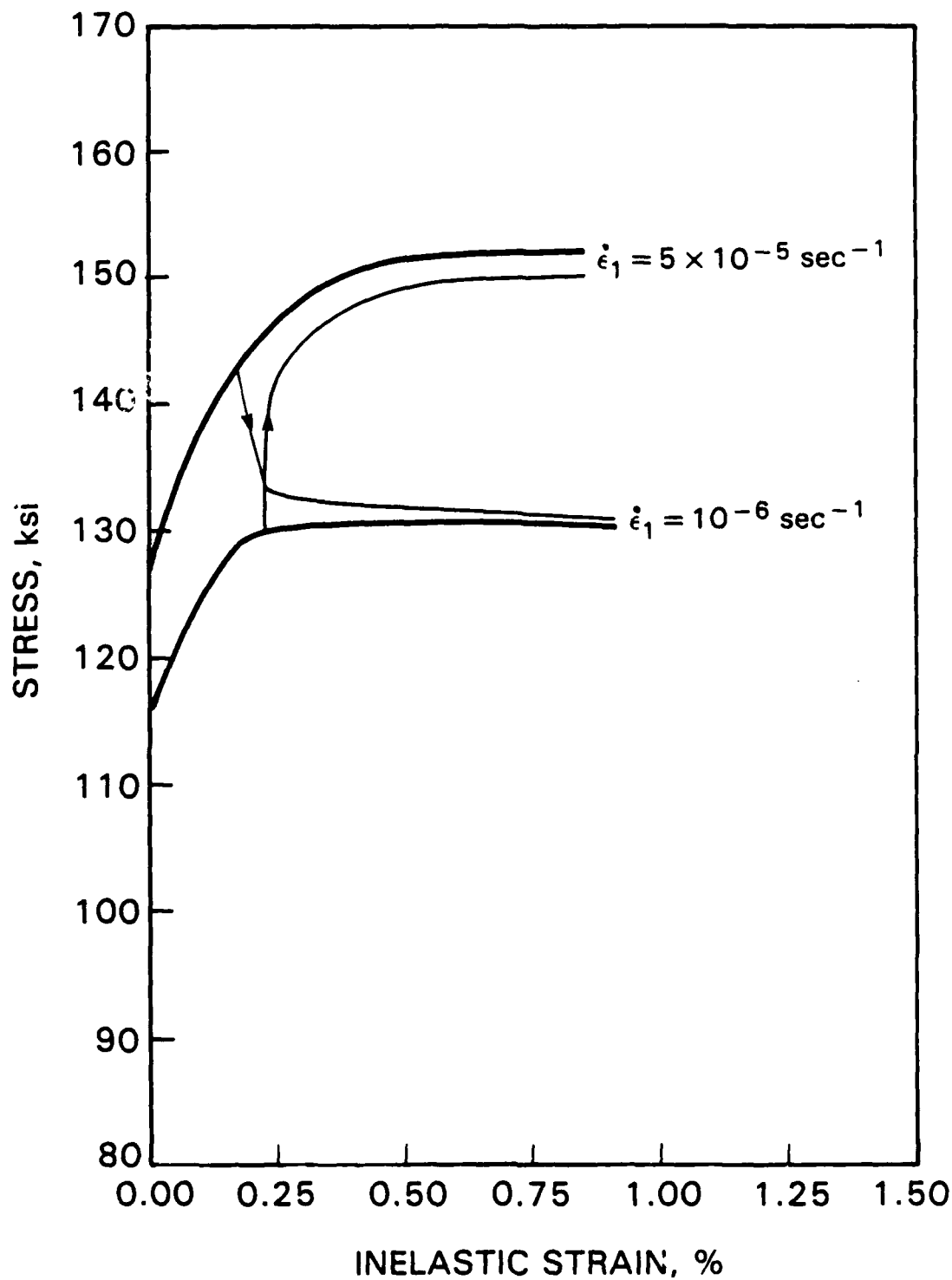
51

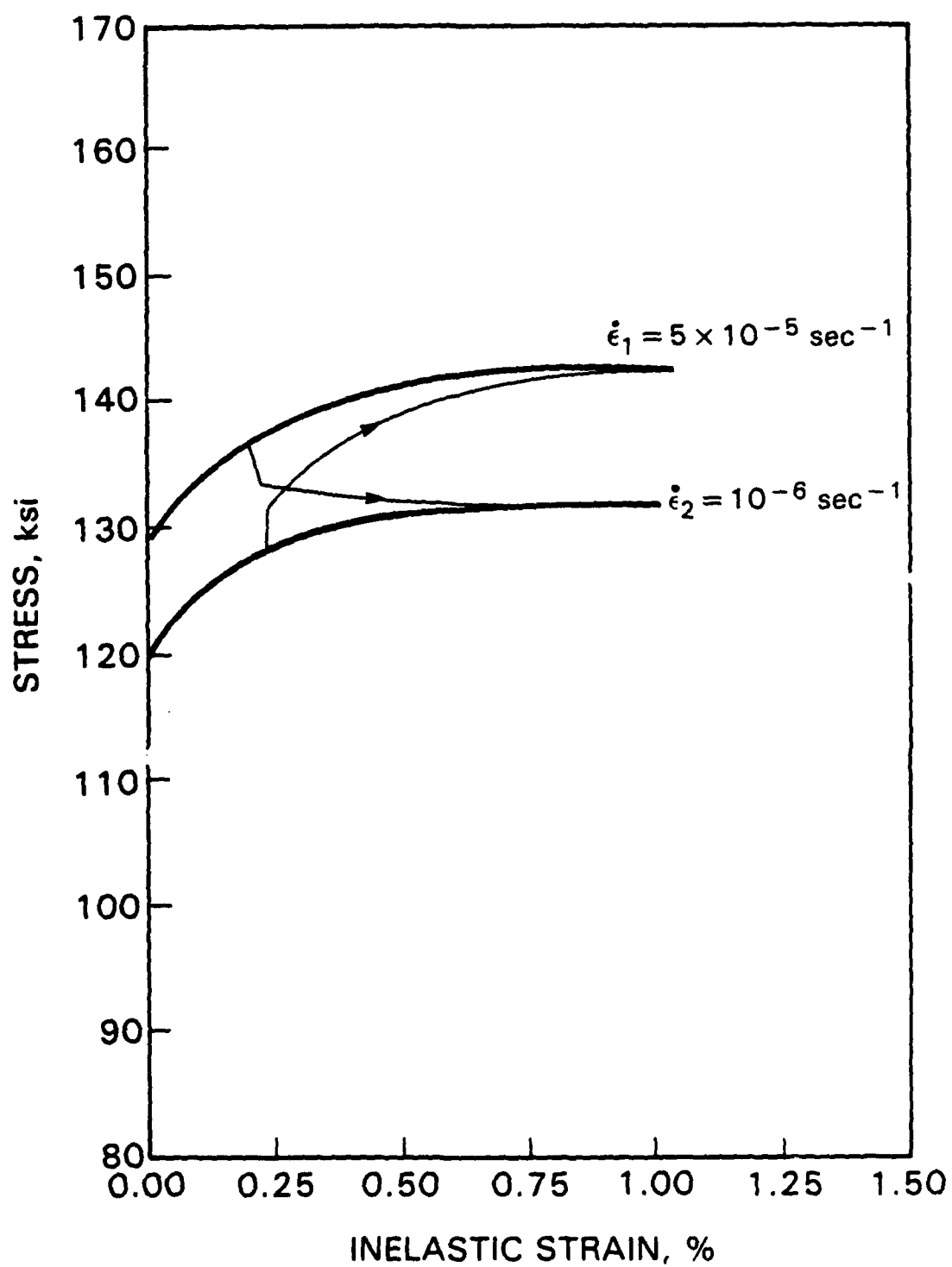Fig. 13.6 — Predicted monotonic and jump responses by the
modified Chaboche theory.

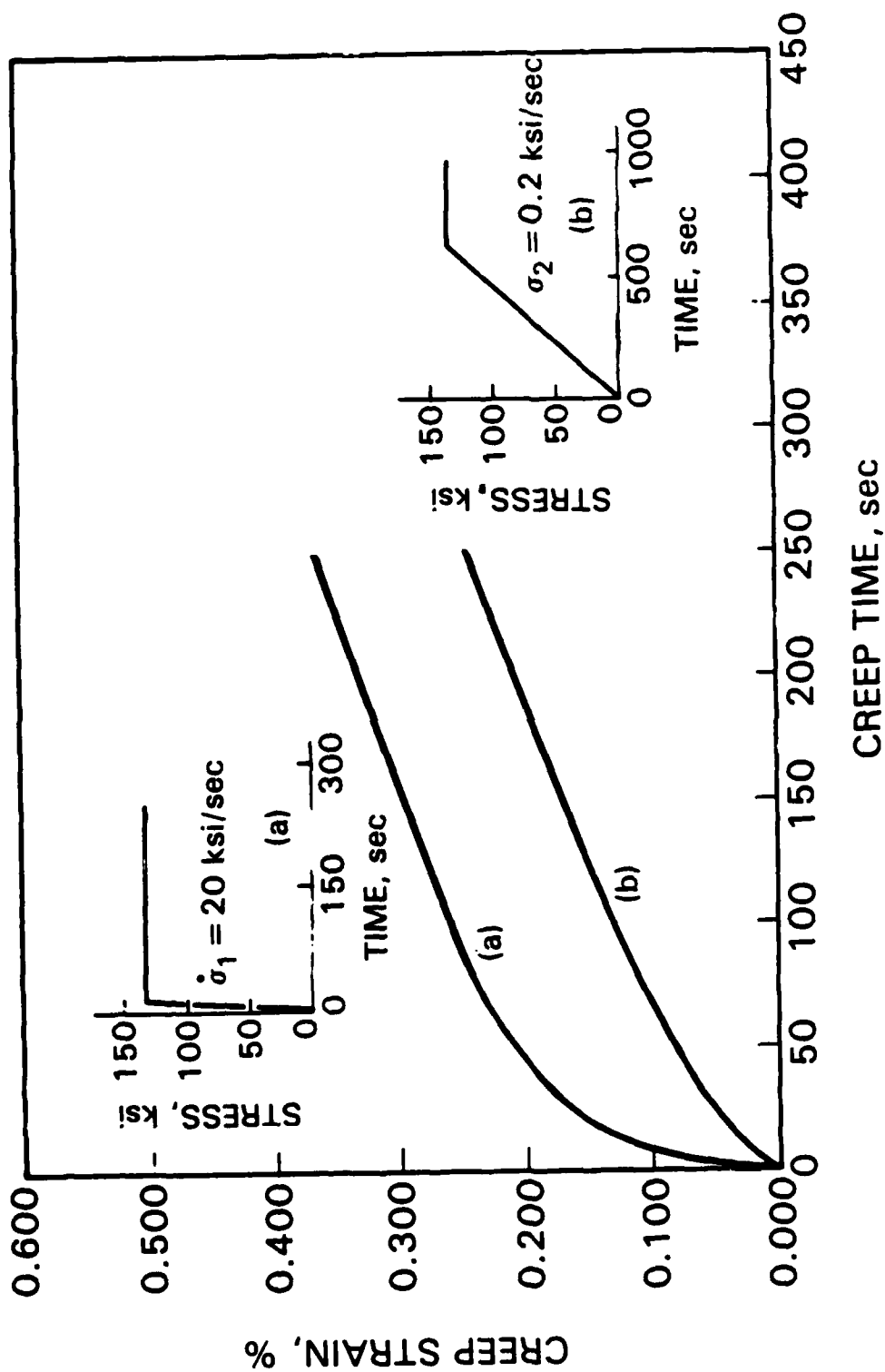Fig. 13.7 — Predicted monotonic and jump responses by Bodner's theory.

Fig. 13.8 — Predicted creep responses at 140 ksi and different stress rates by the modified Chaboche theory.
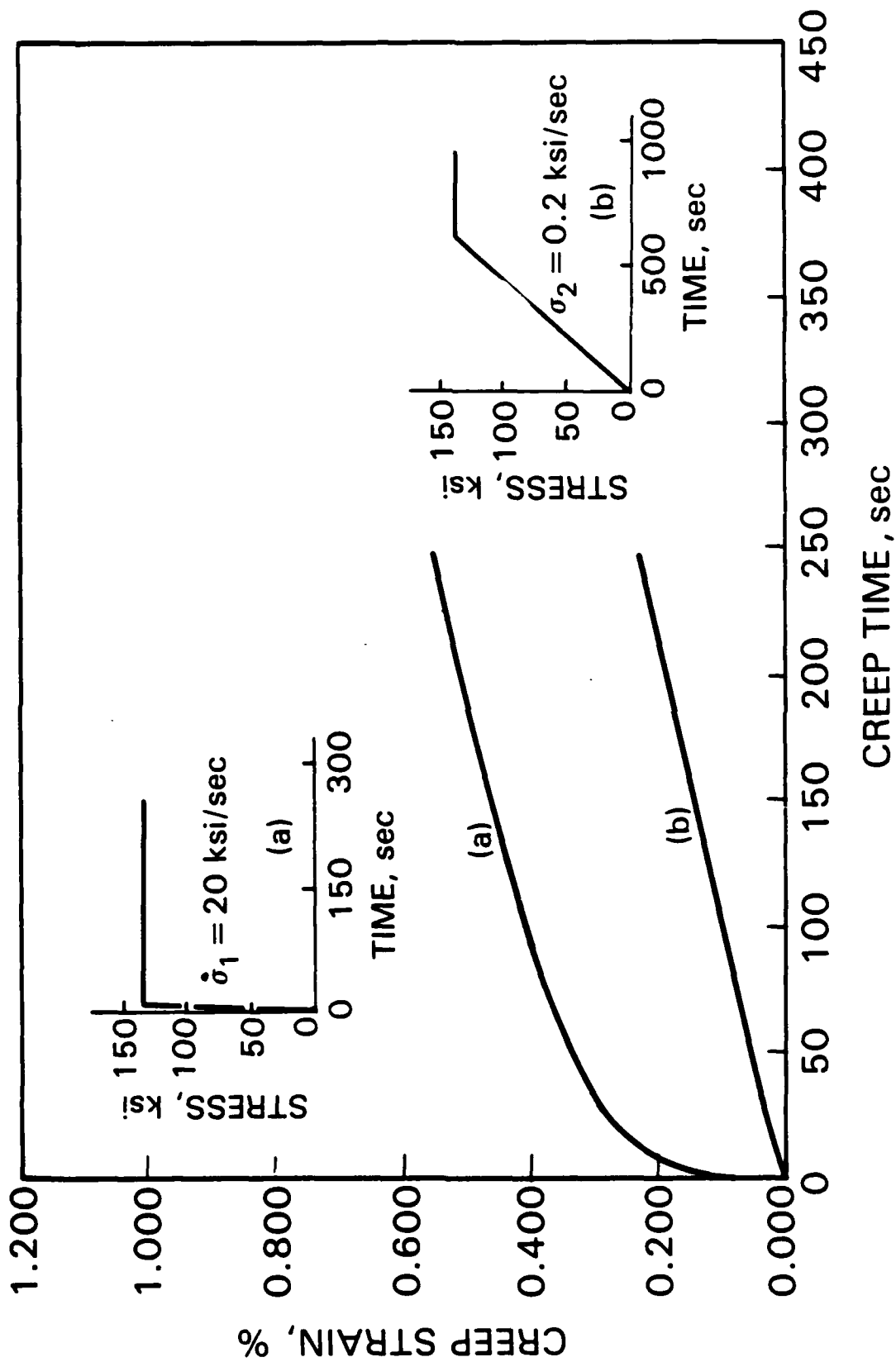
Fig. 13.9 — Predicted creep responses at 140 ksi and different stress rates by Bodner's theory.
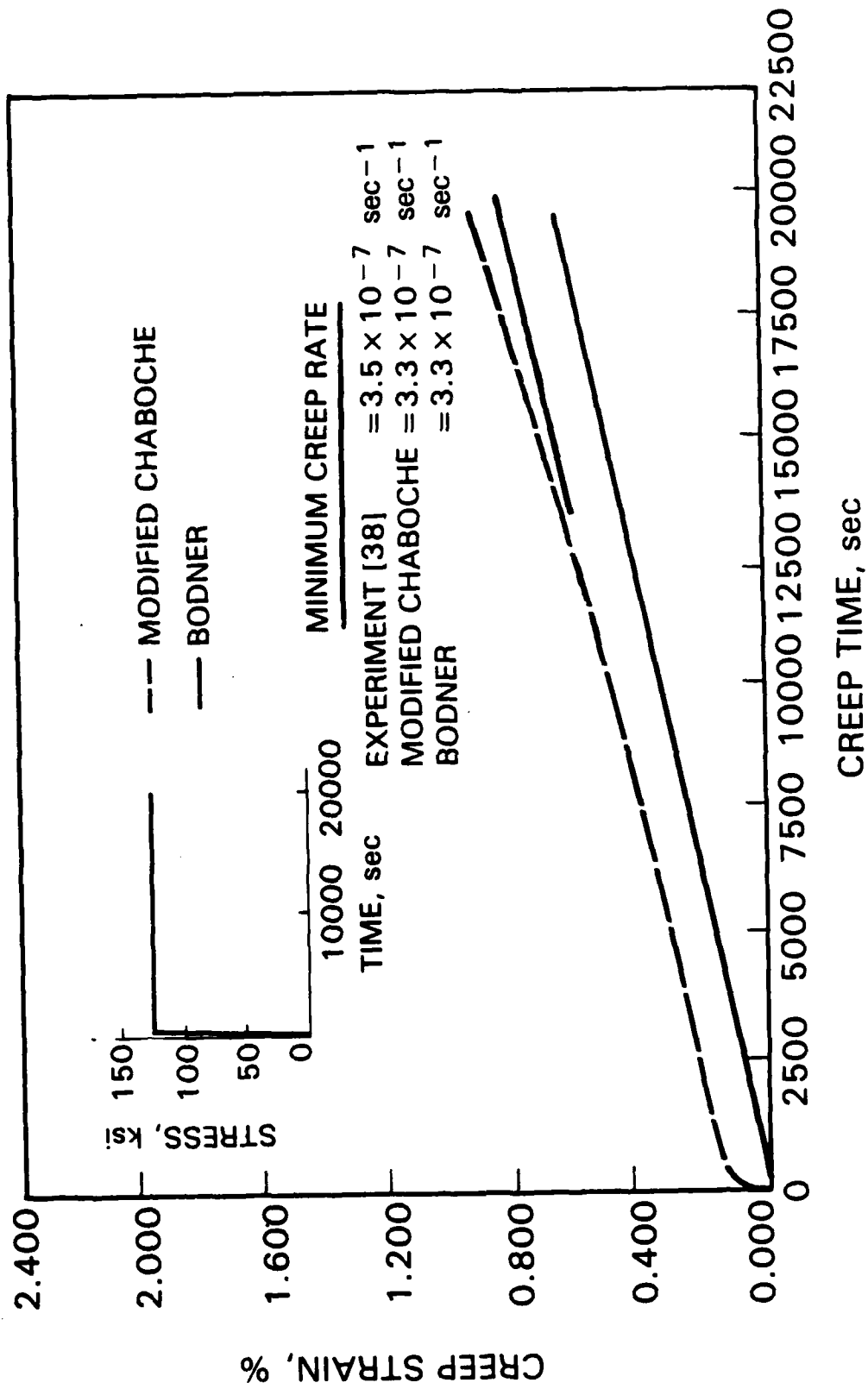
Fig. 13.10 — Predicted creep response at 125 ksi by the modified Chaboche and Bodner theories.

13e. Relaxation Behavior

Structural alloys at high temperature show some degree of stress relaxation at fixed strain, with the stress normally decreasing at a decreasing rate until a stabilized value is reached. At a constant value of the total strain, $\epsilon = \epsilon' + \epsilon''$, it follows that

$$\dot{\epsilon}' = -\frac{1}{E}\dot{\sigma} = -\dot{\epsilon}'' , \tag{13.1}$$

so that

$$\dot{\sigma} = -E\dot{\epsilon}'' \tag{13.2}$$

which indicates that the stress is decreasing, that is, relaxing. At stabilization $\dot{\sigma} = 0$.

For the Chaboche theory, by means of Eqs. $(12.1)_3$ and (13.2) the condition for stabilization requires that

$$-E\left\{\frac{(\sigma - Y) - R}{K}\right\}^n = 0 , \tag{13.3}$$

from which

$$\sigma = Y + R \tag{13.4}$$

as the value for the stress when the stress relaxation has ceased. For Bodner's theory the condition $\dot{\sigma} = 0$ leads to

$$-E\frac{2}{\sqrt{3}}D_0 \; \exp\left[-\left[\frac{Z^2}{\sigma^2}\right]^n \left[\frac{n+1}{2n}\right]\right] = 0 , \tag{13.5}$$

which further requires that

$$e^{-\left[\frac{n+1}{2n}\right]\left[\frac{Z}{\sigma}\right]^{2n}} = 0 \tag{13.6}$$

This condition cannot be satisfied at fixed strain unless $\sigma \to 0$. In other words, the stress continues to *relax with time until it vanishes according to Bodner's theory*.

Figure 13.11 shows the stress relaxation prediction for each theory for a loading history which strains the material at $5 \times 10^{-5} \; \text{sec}^{-1}$ until 1.0% strain is attained and thereafter held fixed for a period of 4 hours. Both theories predict a plausible early response. However, the Chaboche stress relaxation curve shows a maximum stress reduction of approximately 15%, with stabilization occurring at about 2 hours after the strain was held constant, whereas the Bodner stress relaxation curve, as indicated above, continually decreases to zero.
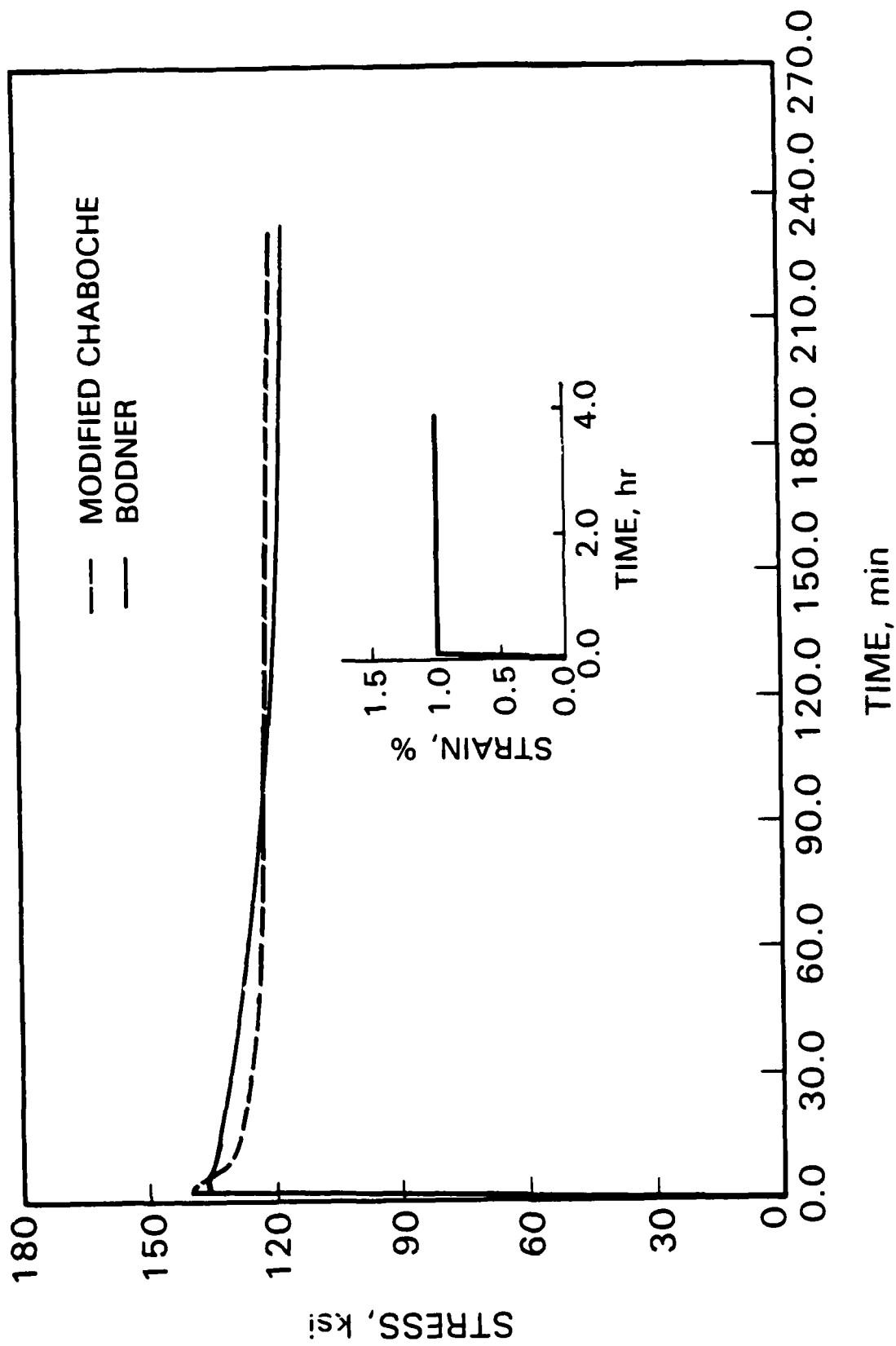
Fig. 13.11 — Predicted stress relaxation behavior after a prestrain of 1% by the modified Chaboche and Bodner theories.

13f. Load-Unload Reload Behavior

For rate-independent plasticity, unloading from a plastic state is invariably elastic. This is not necessarily the situation with viscoplasticity. The initial part of unloading may be inelastic, depending upon the ratio of the loading strain (or stress) rates to the unloading strain (or stress) rates. Figures 13.12 and 13.13 illustrate the predicted response of the two theories at different unloading strain rates after a strain of 0.75% was reached. The symbol δ signifies the ratio of the loading rate to the unloading rate. Thus for a loading strain rate of $5 \times 10^{-7}\ sec^{-1}$, the indicated δ values correspond to the following unloading strain rates:

$$\delta_1 = -10^2 \rightarrow \dot{\epsilon}_1 = -5 \times 10^{-5} sec^{-1}$$

$$\delta_2 = -1 \rightarrow \dot{\epsilon}_2 = -5 \times 10^{-7} sec^{-1}$$

$$\delta_3 = -10^{-2} \rightarrow \dot{\epsilon}_3 = -5 \times 10^{-9} sec^{-1}.$$

Qualitatively speaking, both theories predict the same behavior. They show that the initial unloading behavior is rate dependent. For $\delta_3 = -10^{-2}$, the unload strain rate is two orders of magnitude smaller than the load strain rate, and the initial portion of the unload deformation is inelastic. Along the vertical portion of the unload curve the total strain is constant, so that the inelastic strain is increasing in magnitude equal to the decrease in the elastic strain, $\epsilon'' = -\epsilon'$. This also means that the material is hardening during this part of its unloading history, before passing into elastic unloading.

Figures 13.14 and 13.15 give the predicted responses for loading and unloading at different rates of stress. The stress rates used were determined by multiplying the strain rates that produced Figs. 13.12 and 13.13 by the elastic modulus. The loading stress rate is $\dot{\sigma} = 1.24 \times 10^{-2}$ ksi/sec. Thus the indicated α values correspond to the following unloading stress rates:

$$\gamma_1 = 10^2 \rightarrow \dot{\sigma}_1 = -1.24\ \text{ksi/sec}.$$

$$\gamma_2 = -1 \rightarrow \dot{\sigma}_2 = -1.24 \times 10^{-2}\ \text{ksi/sec}.$$

$$\gamma_3 = -10^{-2} \rightarrow \dot{\sigma}_3 = -1.24 \times 10^{-4}\ \text{ksi/sec}.$$

Both theories show that under stress control the initial unloading behavior is, as to be expected, also rate dependent. At $\gamma_3 = -10^{-2}$ the initial unload response is very nearly akin to creep behavior. A large inelastic strain increase accompanies a small reduction of stress before the elastic unloading begins to take place.
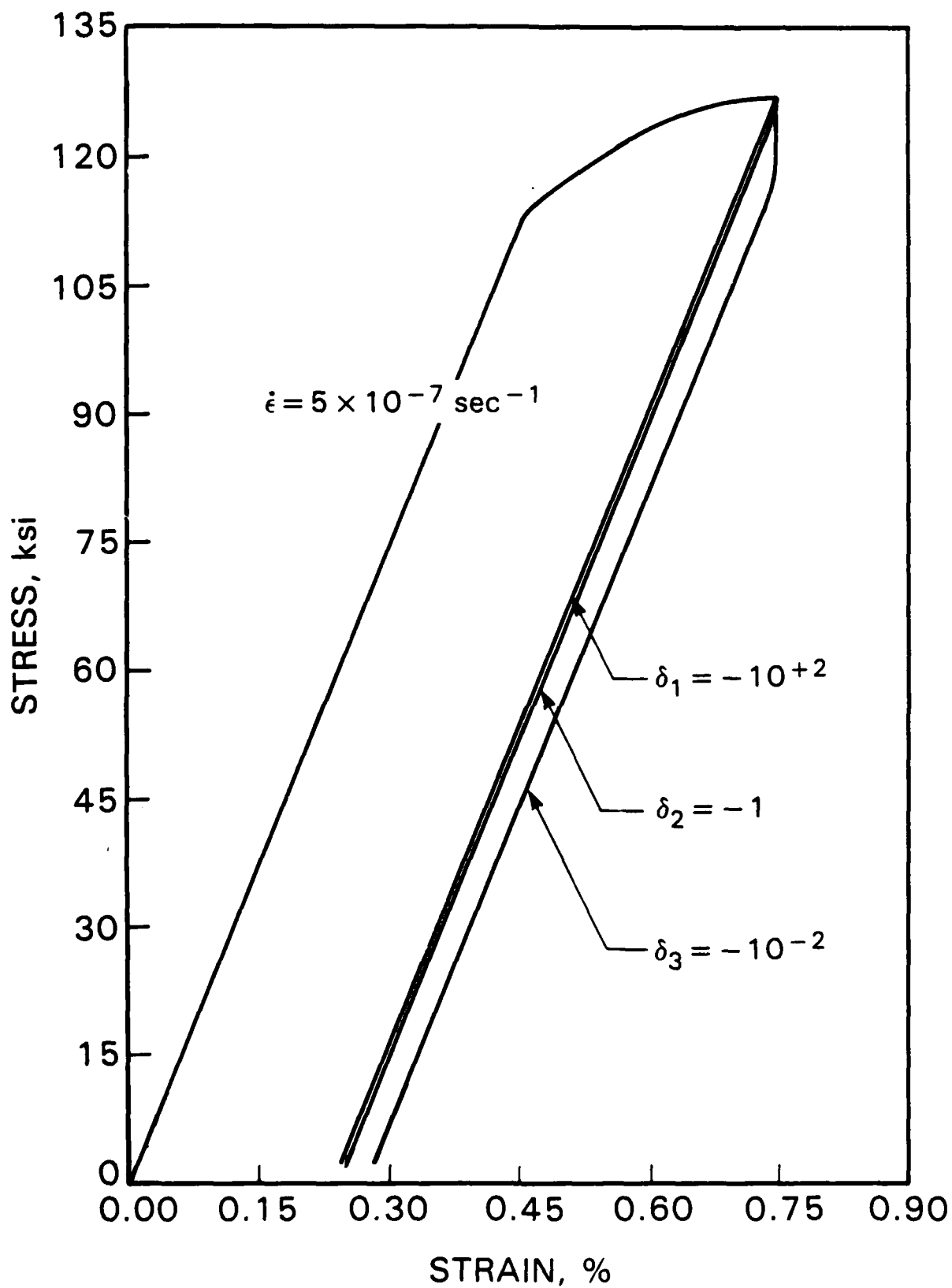
Fig. 13.12 — Predicted loading-unloading response at different unloading strain rates by the modified Chaboche theory.
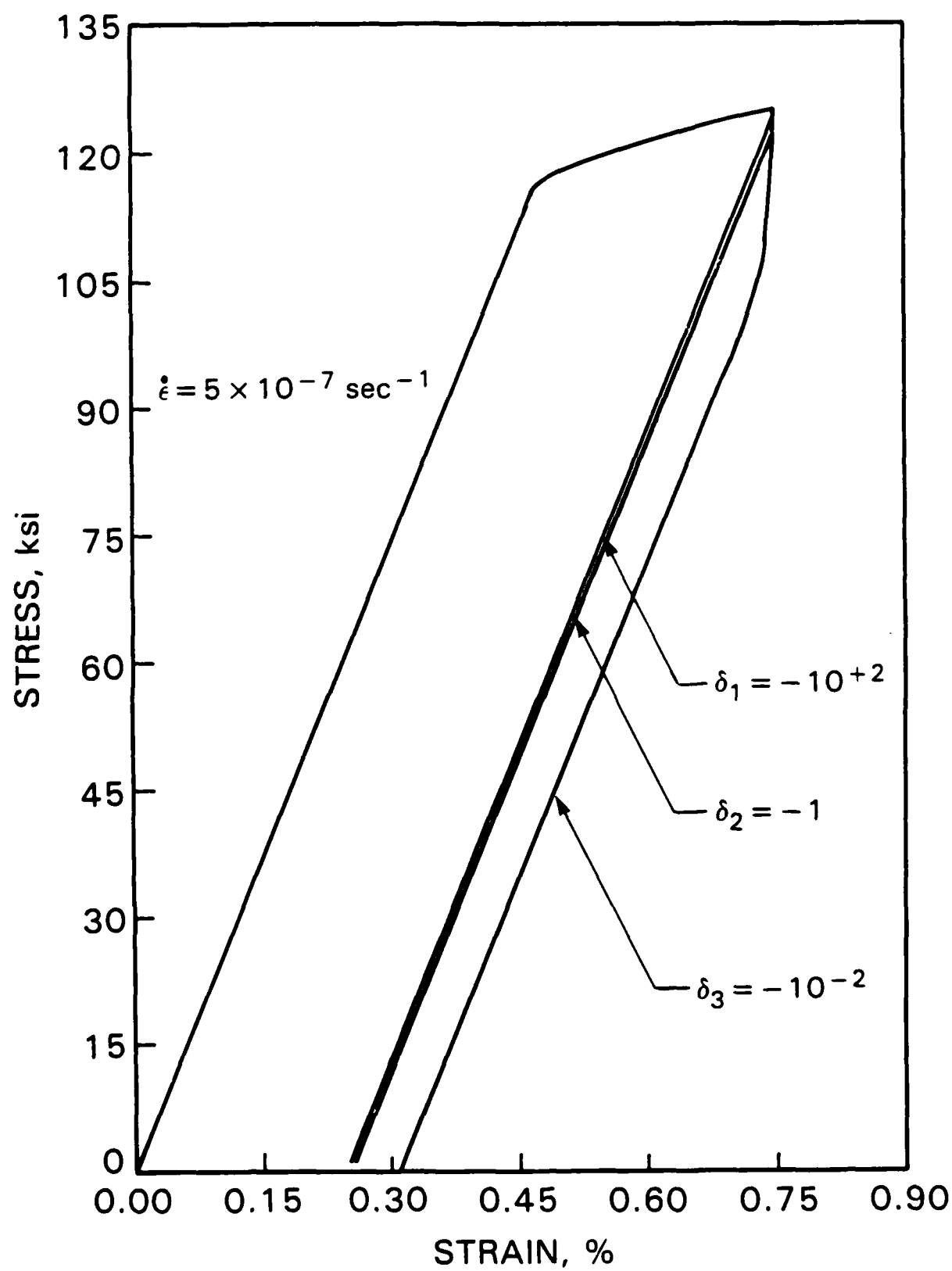
Fig. 13.13 — Predicted loading-unloading response at different unloading strain rates
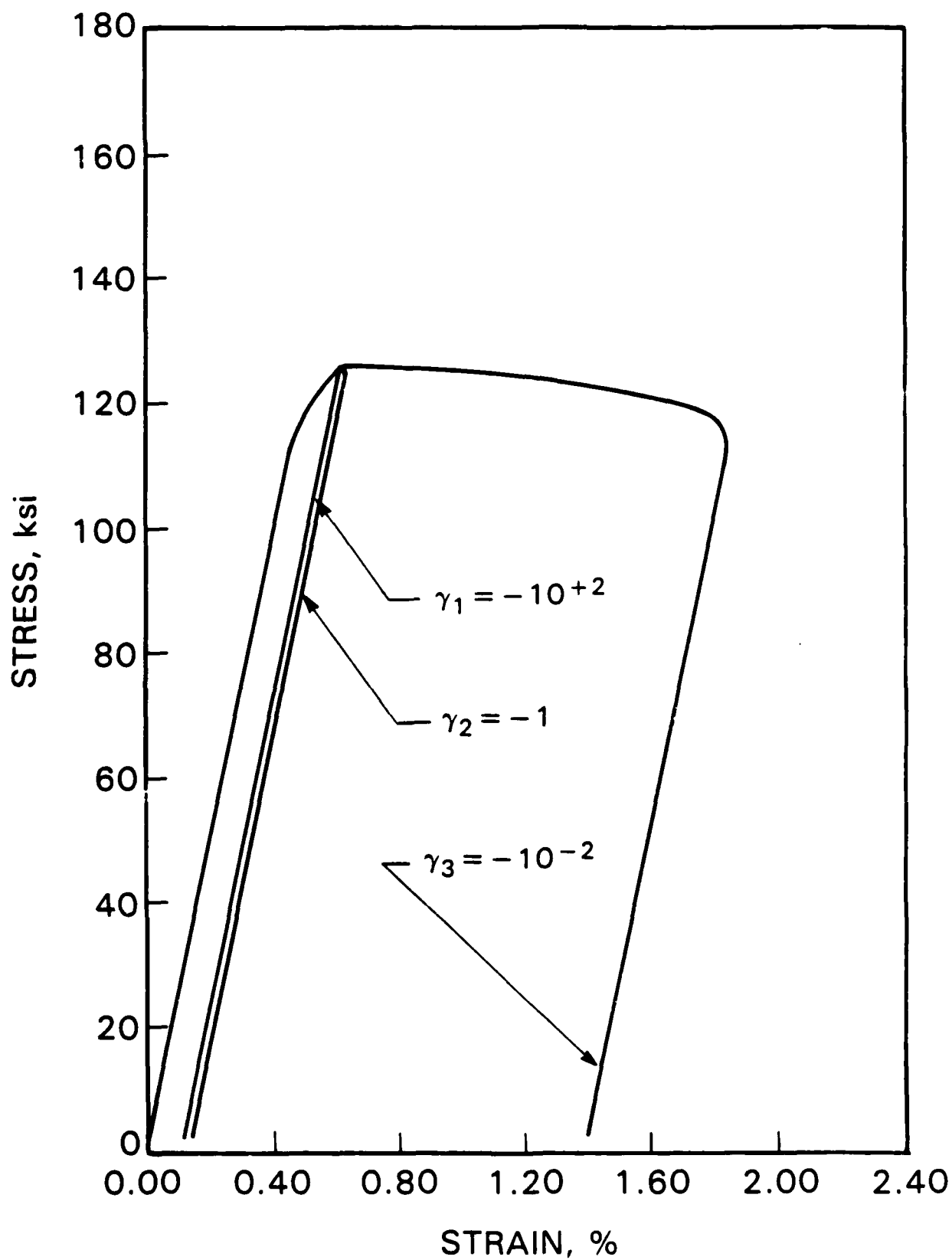by Bodner's theory.

Fig. 13.14 — Predicted loading-unloading response at different unloading stress rates by the modified Chaboche theory.
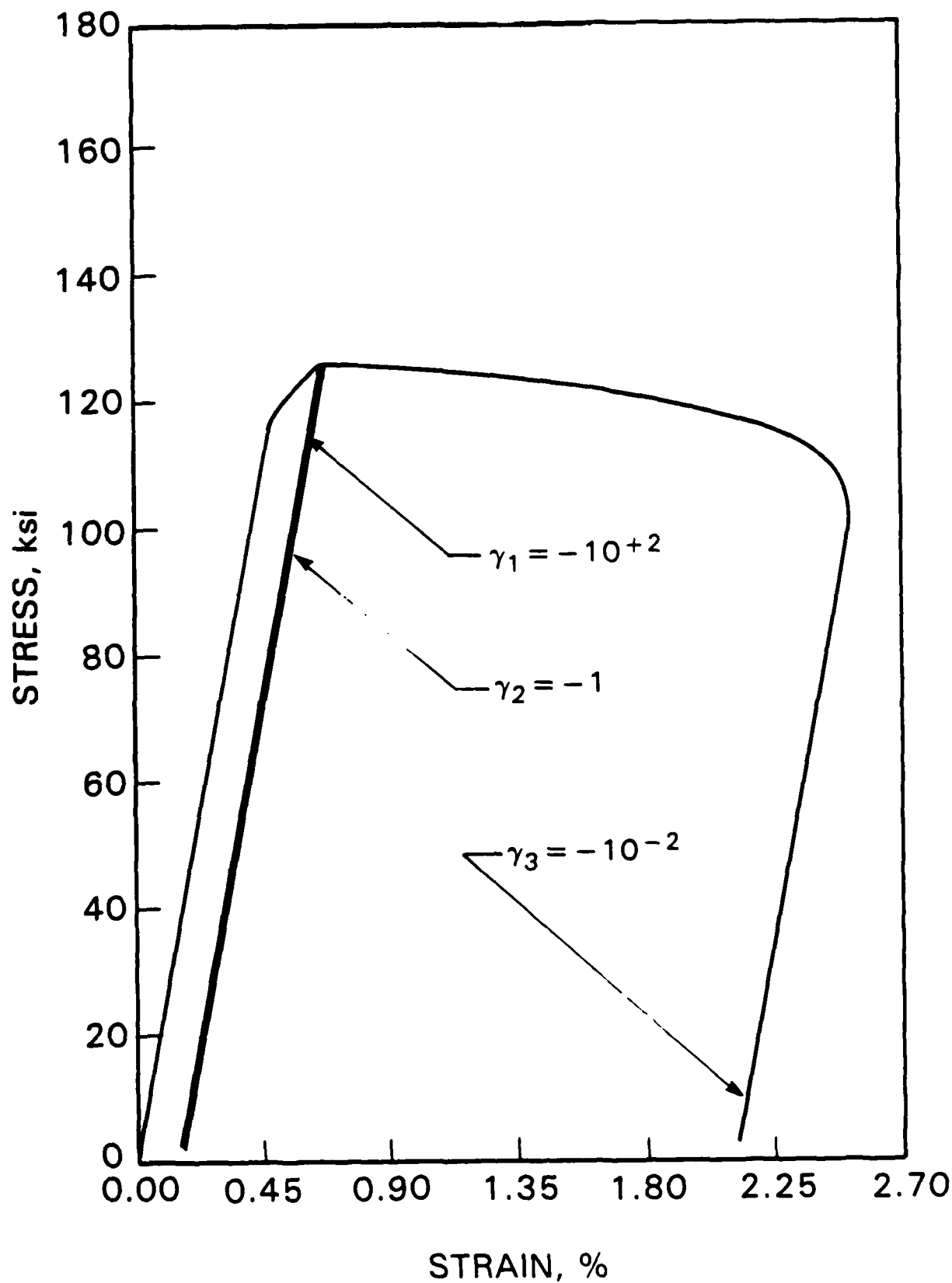
Fig. 13.15 — Predicted loading-unloading response at different unloading stress rates by Bodner's theory.

Figures 13.16 and 13.17 indicate qualitatively similar load-unload-reload viscoplastic behavior, and illustrate the effect of the unloading rate on the subsequent reloading. Shown are: (a) A load-unload-reload cycle at the same absolute strain rate $|\dot{\epsilon}| = 5 \times 10^{-2}$ sec^{-1}, and (b) loading and reloading at the same strain rate with unloading at a smaller strain rate. In both cases, the strain rate for loading was chosen at 5×10^{-5} sec^{-1} with an unloading rate of -5×10^{-7} sec^{-1} selected for the second case. In the first case, $\delta = -1$ at unloading, the initial unloading response is inelastic, approaching very rapidly, however, elastic unloading. The reloading response correspondingly does not exhibit any discernible difference when compared to the monotonic reloading at the same loading rate. In the second, $\delta = -10^{-2}$ at unloading, significant inelastic deformation occurs initially upon unloading over the vertical portion of the unload curve. Consequently, because of the accompanying strain hardening, the subsequent reloading is considerably different from the monotonic response at the original loading rate.

13g. Cyclic Load Behavior

At 1200 °F the cyclic behavior of Inconel shows that is softens for approximately 10% to 20% of the fatigue life prior to a period of cyclic stabilization. In the Chaboche theory cyclic softening and subsequent stabilization are modeled by means of the isotropic hardening variable R, with its saturation q below its initial yield value $R_0(\dot{\epsilon})$. For Bodner's theory the parameter q, which proportions the hardening into isotropic and directional parts must, because of the softening, be assigned a negative value. It was estimated in Ref [38] that for Inconel at 1200°F a value of q between -0.05 and -0.10 is appropriate.

The first few hystersis loops of a strain-controlled, fully reversed cyclic test at $|\dot{\epsilon}| = 4 \times 10^{-5}$ sec^{-1} strain rate and $\Delta\epsilon = 2.0\%$ strain range as predicted by the two theories are shown in Figs. 13.18 and 13.19. These curves show that the Chaboche theory simulates the actual softening quite realistically, as indicated by the continuous decrease in the stress range and a corresponding increase in the inelastic strain range. The predictions shown by the Bodner theory, on the other hand, are not typical of cyclic softening. The predicted hystersis loops have almost constant stress and strain amplitudes. The predicted curves of Fig. 13.19 are for a value of q = - 0.10. Other values, ranging form -0.05 to -0.20, were also tried with less favorable results.

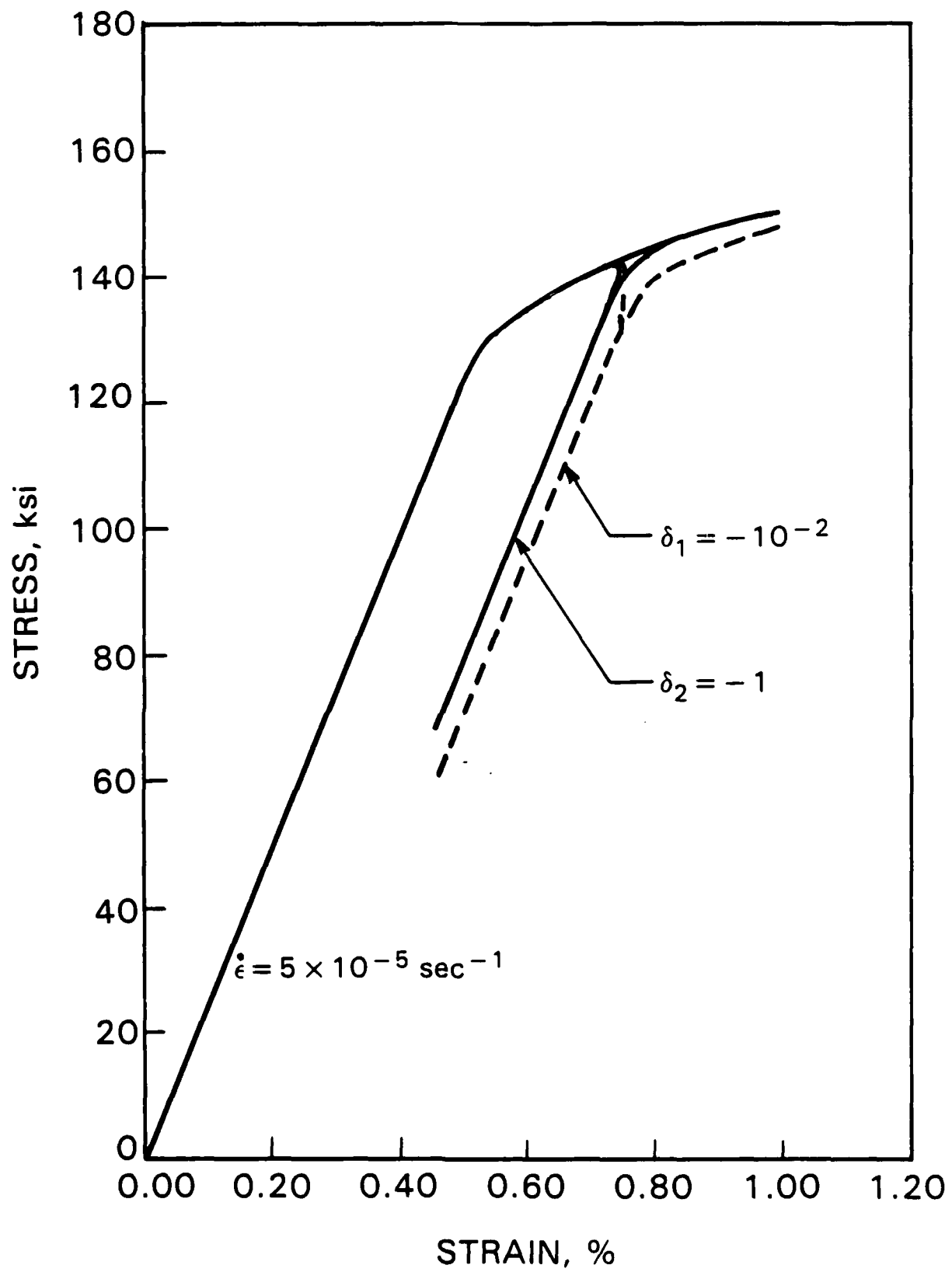| Parameter	Description	Value
E	Young's modulus	24.73×10^3 ksi
ν	Poisson's ratio	0.336
a	Saturation value of kinematic hardening variable	30.00 ksi
c	Kinematic hardening exponent	350.0
γ	Coefficient of hardening recovery	0.4×10^{-10}
m	Hardening recovery exponent	7.00
K	Overstress parameter	155.0 ksi sec$^{1/n}$
n	Strain rate sensitivity parameter	5.10
q	Saturation value of isotropic hardening variable	50.00 ksi
b	Isotropic hardening exponent	3.75

Fig. 13.16 — Predicted loading-unloading-reloading response at different unloading rates by the modified Chaboche theory.
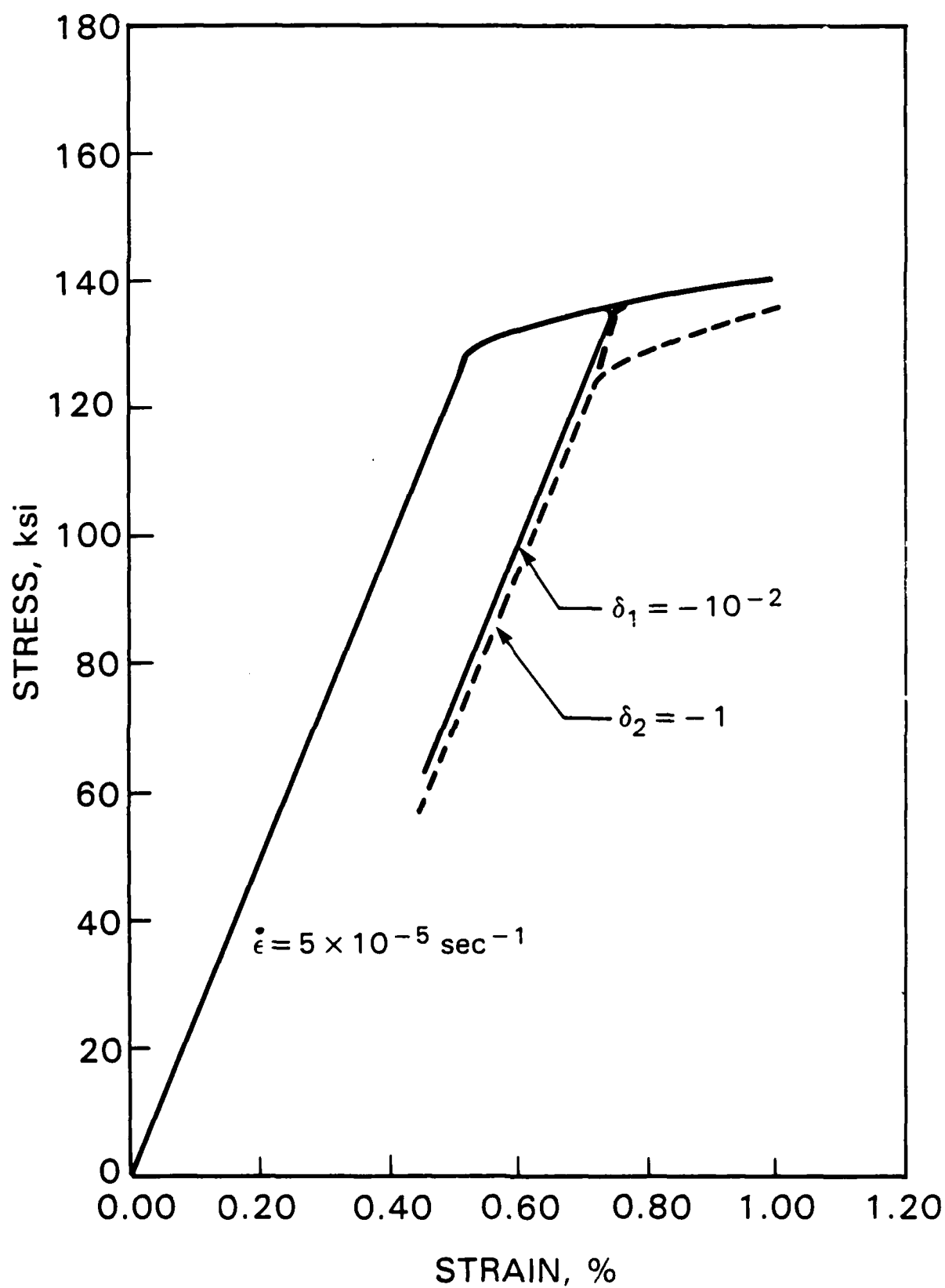
Fig. 13.17 — Predicted loading-unloading-reloading response at different unloading rates by Bodner's theory.
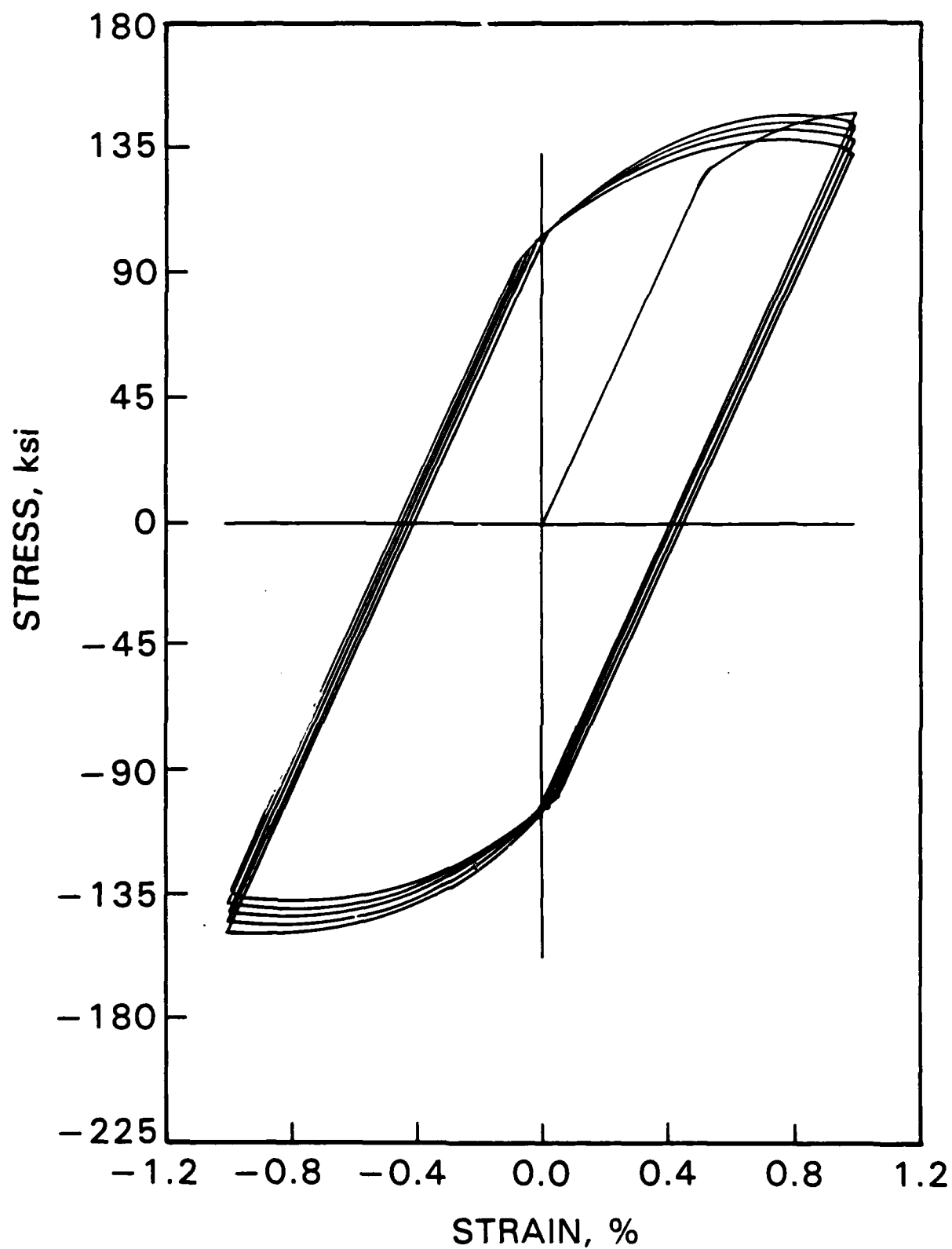
Fig. 13.18 — Predicted strain-controlled fully-reversed ($\dot{\epsilon} = 4 \times 10^{-3}$ sec^{-1}, $\Delta\epsilon = 2\%$) cyclic behavior by the modified Chaboche theory.
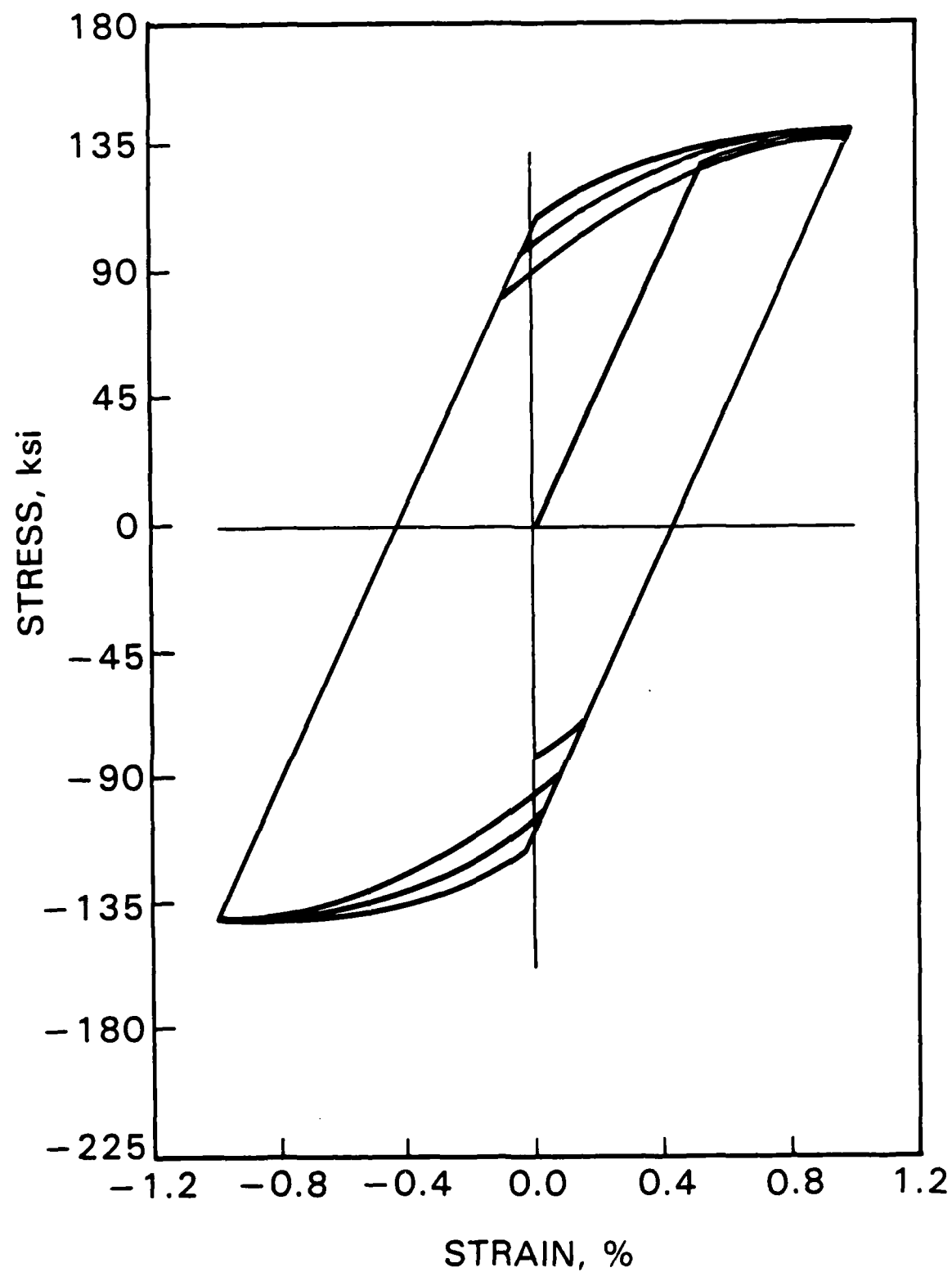
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Fig. 13.19 — Predicted strain-controlled fully-reversed ($\dot{\epsilon} = 4 \times 10^{-3}$ sec^{-1}, $\Delta\epsilon = 2\%$) cyclic behavior by Bodner's theory; q = -0.10.

CONCLUSION

There is no significant advantage to be gained by abandoning the yield condition-yield function-yield surface description of non-recoverable deformation, whether it be rate dependent or not. In a continuum mechanics description of inelastic behavior which views deformation on a gross phenomenological scale, the yield condition and the yield function are the means by which this manner of description (a) identifies that there has been a change in material response due to micro-structural changes, and (b) characterizes that change.

The seeming simplicity that was to result with not having to consider a yield condition and a yield function and its evolution with load history (hardening), actually turns out to provide no real simplification at all. On the contrary, in the absence of the simpler inherent theoretical structure that clearly links and runs through the continually generalized line of yield based viscoplastic theories of Part I, the proposed viscoplastic theories of Part II resort to a range of different, and at times spurious, arguments that serve as the foundations of each of the different theories. These include: incorporating microstructural models for polycrystalline inelastic deformation together with internal variables into the continuum theory, where the internal variables are loosely identified with the variables of the microstructural model; appeal to empirical data correlation formulae; modification of descriptive mechanical models for viscoelastic behavior; and use of undefined 'hidden' internal variables.

Several other viscoplastic constitutive theories have recently been proposed [67-71] that were not included in this study. The conclusion drawn above, however, would not have been altered had they been included.

A major difficulty confronting viscoplastic theories without the yield condition arises when attempting to model anisotropic hardening. As a material hardens, it is actually continuously changing its yield stress value as deformation proceeds. The yield surface offers a natural and simple geometrical representation of this phenomenon, which in turn provides the simplest means of analytically describing or modeling the hardening process. This is brought out by the tortuous and dubious attempt by Bodner and Stouffer to incorporate so-called directional (kinematic) hardening into the fabric of their constitutive theory. This is also shown explicitly by the difficulty that their theory encounters when attempting to model uniaxial load cycling. The difficulties experienced with uniaxial load cycling will probably become more pronounced for multiaxial cyclic behavior where cross-hardening effects enter, and for arbitrary cyclic loading.

The modified Chaboche viscoplastic theory appears to offer the most promise for being able to model reasonably well the entire range of inelastic material behavior for small deformation.

Extension of viscoplastic constitutive theory into the domain of large deformation is possible in principle. At the present time, however, a generalization of this kind that would include anisotropic hardening in explicit terms is not as yet clearly understood. Controversy (and thus confusion) exists over (a) how to properly describe kinematically large scale non-recoverable deformation, particularly when there is load cycling, and (b) the lack of frame indifference for the kinematic hardening description used for small deformation theory.

From the results of the comparative-qualitative study given in Part III, as well as from other studies, it appears as if it is now possible to reasonably model a wide range of inelastic time-dependent material behavior characteristics. Further development of viscoplastic constitutive theory has the potential for application to the problem of developing a general theory of fracture.

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Appendix

STRESS, DEVIATOR STRESS AND STRAIN MATRICES FOR UNIAXIAL LOADING

$$T_{11} = \sigma, \quad T_{22} = T_{33} = T_{12} = T_{13} = T_{23} = 0.$$

$$[T] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.1}$$

Correspondingly,

$$\overset{o}{T} = T - \left[\frac{1}{3} I_T\right] 1 - T_{jk} - \frac{1}{3}\left(T_{11} + T_{22} + T_{33}\right) \delta_{jk} \quad \text{from which}$$

$$\left[\overset{o}{T}\right] = \begin{bmatrix} \dfrac{2}{3}\sigma & 0 & 0 \\ 0 & -\dfrac{1}{3}\sigma & 0 \\ 0 & 0 & -\dfrac{1}{3}\sigma \end{bmatrix}. \tag{A.2}$$

$$\sqrt{II'_{\overset{o}{T}}} = [tr(\overset{o}{T} \cdot \overset{o}{T})]^{1/2} = (\overset{o}{T}_{jk} \overset{o}{T}_{kj})^{1/2} = \overset{o}{T}{}^2_{11} + \overset{o}{T}{}^2_{22} + \overset{o}{T}{}^2_{33} + 2\overset{o}{T}{}^2_{12} + 2\overset{o}{T}{}^2_{13} + 2\overset{o}{T}{}^2_{23} \tag{A.3}$$

$$= \sqrt{\frac{2}{3}}\sigma$$

For the elastic strain: $\Sigma_{11}' = \epsilon'$, $\Sigma_{22}' = \Sigma_{22}' = -\nu\Sigma_{11}' = -\nu\epsilon'$, $\Sigma_{12}' = \Sigma_{12}' = \Sigma_{23}' = 0$.

$$\left[\Sigma'\right] = \begin{bmatrix} \epsilon' & 0 & 0 \\ 0 & -\nu\epsilon' & 0 \\ 0 & 0 & -\nu\epsilon' \end{bmatrix}. \tag{A.4}$$

Assuming inelastic incompressibility, $\mathrm{T}_{\Sigma''} = \Sigma_{11}'' + \Sigma_{22}'' + \Sigma_{33}'' = 0$. Then as $\Sigma_{22}'' + \Sigma_{33}'' = -\Sigma_{11}''$ and $\Sigma_{22}'' = \Sigma_{33}''$ (symmetry of loading), $-\Sigma_{22}'' = \Sigma_{33}'' = -\frac{1}{2}\Sigma_{11}'' = -\frac{1}{2}\epsilon''$. Also, because of the lack of shear stress components $\Sigma_{12}'' = \Sigma_{13}'' = \Sigma_{23}'' = 0$.

$$\left[\Sigma''\right] = \begin{bmatrix} \epsilon'' & 0 & 0 \\ 0 & -\dfrac{1}{2}\epsilon'' & 0 \\ 0 & 0 & -\dfrac{1}{2}\epsilon'' \end{bmatrix} . \tag{A.5}$$

Because of the inelastic incompressibility

$$\overset{0}{\Sigma}'' = \Sigma'' - \left[\frac{1}{3}I_{\Sigma''}\right]\mathbf{1} = \Sigma'' , \tag{A.6}$$

and

$$\sqrt{II'_{\overset{0}{\Sigma}''}} = \sqrt{II'_{\Sigma''}} = \left[tr\left(\Sigma'' \cdot \Sigma''\right)\right]^{1/2} = \left(\Sigma''_{jk}\,\Sigma''_{kj}\right)^{1/2}$$
$$= \Sigma_{11}^{2}{}'' + \Sigma_{22}^{2}{}'' + \Sigma_{33}^{2}{}'' + 2\Sigma_{12}^{2}{}'' + 2\Sigma_{13}^{2}{}'' + 2\Sigma_{23}^{2}{}'' = \sqrt{\frac{3}{2}}\,\epsilon'' . \tag{A.7}$$

END

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