UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY 2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR 7785	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Underwater Systems Center 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) New London Laboratory New London, CT 06320	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING / SPONSORING ORGANIZATION 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
WEIGHTED LEAST SQUARES FIT OF A REAL TONE TO DISCRETE DATA, BY MEANS OF AN EFFICIENT FAST FOURIER TRANSFORM SEARCH

12. PERSONAL AUTHOR(S)
Albert H. Nuttall

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
	FROM ____ TO ____	27 August 1986	42

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Weighted Least Squares Fit; Real Tone; FFT Procedure; Fine-Grain Search; Maximum Likelihood; Discrete-Time Data

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The weighted least squares fit of a real tone with arbitrary amplitude, frequency, and phase, to a given set of real discrete data, is reduced to a one-dimensional maximization of a function of frequency only. This function is manipulated into a form that can be efficiently calculated by one FFT of a complex sequence that is related to the available real data and the arbitrary real weight sequence utilized. The decoupling of the complex FFT outputs, to yield the two functions that are necessary to conduct the coarse search in frequency, is accomplished in an extremely simple fashion. A refined interpolation procedure then fits a parabola in the region near the maximum and gives a fine-grained estimate of frequency.

An explanation of the apparently anomalous behavior near zero and Nyquist

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL Albert H. Nuttall	22b. TELEPHONE (Include Area Code) (203) 440-4618	22c. OFFICE SYMBOL Code 3314

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

18. (Cont'd.)

Windows
Spillover

19. (Cont'd.)

frequencies is given, which shows that in the limit, a constant plus linear trend is being fitted to the discrete data. A program is presented for the complete procedure, including evaluation of the best frequency, amplitude, and phase of the fitted tone. The technique is applicable to short data records, without any approximations, and for arbitrary weight sequences.

NUSC Technical Report 7785
27 August 1986

Weighted Least Squares Fit of a Real Tone to Discrete Data, by Means of an Efficient Fast Fourier Transform Search

Albert H. Nuttall
Surface Ship Sonar Department

Naval Underwater Systems Center
Newport, Rhode Island / New London, Connecticut

Preface

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. Albert H. Nuttall (Code 3314), sponsored by the NUSC In-House Independent Research Program, Program Manager Mr. W. R. Hunt, Director of Navy Laboratories (SPAWAR 05).

The technical reviewer for this report was Dr. William I. Roderick (Code 32).

Reviewed and Approved: 27 August 1986

WA Von Winkle

W. A. Von Winkle
Associate Technical Director
for Technology

TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	iii
LIST OF SYMBOLS	iv
INTRODUCTION	1
ERROR MINIMIZATION	2
Complex Tone	2
Real Tone	3
MANIPULATIONS OF B(a)	6
Alternative Form for B(a)	6
Properties of B(a)	7
Value of B(a) as $a \to 0$	8
Value of B(a) as $a \to \pi$	9
Example of Equal Weights	10
IN-PHASE AND QUADRATURE COEFFICIENTS	12
Normalization of Weights	13
Interpretation of (43)	14
Value of Coefficients as $a \to 0$	15
FFT REALIZATION	16
Manipulation into FFT Forms	16
Two Real FFTs via One Complex FFT	17
Selection of FFT Size M	18
Interpolation Procedure	18
RESULTS	21
SUMMARY	29

TABLE OF CONTENTS (Cont'd)

Page

APPENDIX. PROGRAM FOR ESTIMATION OF TONE PARAMETERS 31

REFERENCES 37

LIST OF ILLUSTRATIONS

Figure		Page
1.	Flat Weighting, No Noise	23
2.	Hanning Weighting, No Noise	24
3.	Flat Weighting, Additive Noise	25
4.	Hanning Weighting, Additive Noise	26
5.	Flat Weighting, Two Tones	27
6.	Hanning Weighting, Two Tones	28

LIST OF SYMBOLS

N	Number of data points
x_k	Data value at sample number k
Δ	Sample increment
E	Weighted squared error, (7)
w_k	Weight at sample number k
ω	Radian frequency of tone
α,β	Coefficients of in-phase and quadrature components, (5)
FFT	Fast Fourier Transform
a	Normalized frequency $\omega\Delta$, (6),(7)
W	Window for weights $\{w_k\}$, (8)
L	Window for weighted data, (8)
sub r,i	Real and imaginary parts, respectively
A_{mn}	Auxiliary quantities, (9),(10)
$B(a)$	Maximizing function, (12),(13)
Re	Real part
W_n	n-th moment of weights, (24)
L_n, $\tilde{L}_n$	n-th moment of weighted data, (24),(31)
μ,ν	Constant and linear trend, (26)
$\tilde{a}$	Perturbation of a about π, (29)
M	Size of FFT, (53),(54)
m	Frequency index, (53)
$\mathcal{L}(m)$	FFT of weighted data, (54)
d_j	Auxiliary sequence, (56)
$\mathcal{D}(m)$	FFT of $\{d_j\}$, (57)
z_k	Complex FFT input, (59)
$Z(m)$	Complex FFT output, (60)
S,D	Sum and difference variables, (63)
X,Y	Real and imaginary parts of Z, (64)

WEIGHTED LEAST SQUARES FIT OF A REAL TONE TO DISCRETE DATA, BY MEANS OF AN EFFICIENT FAST FOURIER TRANSFORM SEARCH

INTRODUCTION

Estimation of the parameters of a tone with unknown amplitude, frequency, and/or phase has attracted considerable attention; see, for example, [1-9]. However, fitting data with a single pure complex tone leads to a simpler search problem than fitting with a real tone (as will be demonstrated in the next section). In particular, fitting with a complex tone was considered in [1-5, 7], while fitting with real tones has been the subject of [6, 8, 9]. However, the frequency of the tone was assumed known in [6, 8], whereas it had to be estimated in [9].

Here we will extend the results in several directions for the case of fitting real data with a real tone. First, arbitrary real weighting of the errors at each discrete instant are incorporated. Second, the function that must be searched for a maximum is manipulated into a form which requires that only two FFTs of two real sequences be conducted. Third, these two operations are combined into one FFT of a complex sequence, the outputs of which are decoupled in a very efficient manner, in order to yield the desired search function. Fourth, parabolic interpolation of the three outputs in the neighborhood of the search maximum is employed in order to give a refined estimate of the tone frequency. Finally, a minute search for the best tone frequency is conducted, the extent of which is left up to the user. The end result of this investigation is a program for conducting an efficient and fast fine-grained search for the determination of the unknown amplitude, frequency, and phase of the best-fitting real tone to a given set of discrete real data and subject to any error weighting of interest.

This procedure is applicable to arbitrary data record lengths. Also, no assumptions about the statistics of any additive noise, that may be present in the data record, are made. However, when the available data record is the result of a pure tone and additive zero-mean Gaussian noise, the procedure can be interpreted as maximum likelihood estimation [9].

ERROR MINIMIZATION

Before we begin the detailed investigation of fitting a real tone to real data, we first consider the simpler problem of fitting a pure complex tone. This will serve as a comparison procedure and will back up the statement made in the Introduction.

COMPLEX TONE

The discrete data available consist of N values $\{x_k\}$, taken at increment Δ. If the data are complex and we fit the data with a pure complex tone, we must address the problem of minimizing the weighted squared error

$$E = \sum_k w_k \left| x_k - \alpha \exp(i\omega k\Delta)\right|^2 , \tag{1}$$

where the summation on k is taken over all nonzero summands. Normally, the data $\{x_k\}$ and real weights $\{w_k\}$ will be taken to be nonzero over the range $1 \leq k \leq N$; however, the presentation allows for any range of the variable k. The parameter α in (1) is the complex amplitude, and ω is the pure tone (radian) frequency, which is presumed real.

If we consider ω given for the moment in (1), the best choice of α to minimize error E is given by

$$\alpha_o = \sum_k w_k \, x_k \exp(-i\omega k\Delta) \Big/ \sum_k w_k . \tag{2}$$

Substitution of this result for α in (1) results in error

$$E(\omega) = \sum_k w_k |x_k|^2 - \left|\sum_k w_k \, x_k \exp(-i\omega k\Delta)\right|^2 \Big/ \sum_k w_k . \tag{3}$$

This error is minimized by choosing frequency ω to maximize the quantity

$$\left| \sum_k w_k \; x_k \; \exp(-i\omega k\Delta) \right|^2 , \tag{4}$$

which is the standard magnitude-squared Fourier transform of the weighted data. Thus, direct application of an FFT is a good procedure to apply to this problem and has been so employed in the past [5]. Since (4) has period $2\pi/\Delta$ in ω, there is no need to compute (4) except for the range $-\pi \leq \omega\Delta \leq \pi$.

REAL TONE

We now restrict consideration to the case of major interest here, namely, real data $\{x_k\}$, and attempt to fit it with samples of a pure real tone, that is,

$$\alpha \cos(\omega k\Delta) + \beta \sin(\omega k\Delta) . \tag{5}$$

Here, α and β are the real coefficients of the in-phase and quadrature components of the tone. If we let "normalized frequency"

$$a = \omega\Delta , \tag{6}$$

the weighted squared error to be minimized is

$$E = \sum_k w_k \; [x_k - \alpha \cos(ak) - \beta \sin(ak)]^2 . \tag{7}$$

For later use, we define the two Fourier series:

$$W(u) = \sum_k w_k \exp(-iuk) ,$$

$$L(u) = \sum_k w_k \; x_k \; \exp(-iuk) . \tag{8}$$

The first is the window associated with weights $\{w_k\}$, while the latter is the Fourier transform of the weighted data.

The variable a appearing in (7) will be called the "frequency" of the tone. If we consider frequency a given for the moment, setting the partial derivatives of error E with respect to α and β, both equal to zero, results in the pair of simultaneous linear equations for their optimum values:

$$A_{11}\,\alpha_o + A_{12}\,\beta_o = L_r(a)\ ,$$

$$A_{12}\,\alpha_o + A_{22}\,\beta_o = -L_i(a)\ . \tag{9}$$

Here sub r and i denote real and imaginary parts, respectively. We also have the scale factors expressible in the forms

$$A_{11} = \sum_k w_k \cos^2(ak) = \frac{1}{2}\left[W(0) + W_r(2a)\right]\ ,$$

$$A_{22} = \sum_k w_k \sin^2(ak) = \frac{1}{2}\left[W(0) - W_r(2a)\right]\ ,$$

$$A_{12} = \sum_k w_k \cos(ak)\sin(ak) = -\frac{1}{2}\,W_i(2a)\ , \tag{10}$$

where we have made extensive use of (8). Solution of (9) yields for the tone coefficients,

$$\alpha_o = \frac{A_{22}L_r(a) + A_{12}L_i(a)}{A_{11}A_{22} - A_{12}^2}\ ,$$

$$\beta_o = \frac{-A_{11}L_i(a) - A_{12}L_r(a)}{A_{11}A_{22} - A_{12}^2}\ . \tag{11}$$

The use of (9)-(11) in error (7) now results in modified error

$$E(a) \equiv \sum_k w_k \left[x_k - \alpha_o \cos(ak) - \beta_o \sin(ak)\right]^2 =$$

$$= \sum_k w_k \left[x_k - \alpha_o \cos(ak) - \beta_o \sin(ak)\right] x_k =$$

$$= \sum_k w_k x_k^2 - \alpha_o L_r(a) + \beta_o L_i(a) =$$

$$= \sum_k w_k x_k^2 - B(a) , \qquad (12)$$

where we define real function

$$B(a) = \alpha_o L_r(a) - \beta_o L_i(a) =$$

$$= \frac{A_{22}L_r^2(a) + 2A_{12}L_r(a)L_i(a) + A_{11}L_i^2(a)}{A_{11}A_{22} - A_{12}^2} . \qquad (13)$$

This quantity, which must now be <u>maximized</u> by choice of a, was previously encountered in [9; (10)], but limited there to the case of equal weights $\{w_k\}$. We concentrate henceforth on function B(a), aware that we can always return to error E(a) by means of (12).

MANIPULATIONS OF B(a)

In this section, we derive alternative forms, properties, and interpretations of the function B(a). The weighted squared error is directly related to B(a) by means of (12).

ALTERNATIVE FORM FOR B(a)

A more useful and compact form for B(a) in (13) is possible. Reference to (10) reveals that the denominator of (13) is simply

$$\frac{1}{4}\left[W^2(0) - \left|W^2(2a)\right|\right] . \qquad (14)$$

Similarly, use of (10) allows development of the numerator of (13) according to

$$\frac{1}{2}\left[W(0) - W_r(2a)\right] L_r^2(a) - W_i(2a)L_r(a)L_i(a) + \frac{1}{2}\left[W(0) + W_r(2a)\right] L_i^2(a) =$$

$$= \frac{1}{2} W(0)\left|L^2(a)\right| - \frac{1}{2}\left[W_r(2a)L_r^2(a) + 2W_i(2a)L_r(a)L_i(a) - W_r(2a)L_i^2(a)\right] =$$

$$= \frac{1}{2} W(0)\left|L^2(a)\right| - \frac{1}{2} \mathrm{Re}\left\{W^*(2a)L^2(a)\right\} . \qquad (15)$$

Coupling (14) and (15) together, the expression in (13) becomes

$$B(a) = 2\,\frac{W(0)\left|L^2(a)\right| - \mathrm{Re}\left\{W^*(2a)L^2(a)\right\}}{W^2(0) - \left|W^2(2a)\right|} . \qquad (16)$$

The required quantities here are available from (8) as

$$W(2a) = \sum_k w_k \exp(-i2ak),$$

$$L(a) = \sum_k w_k\, x_k \exp(-iak) . \qquad (17)$$

It is immediately obvious from (16) that $B(a)$ can never be negative (presuming that the weights are nonnegative).

The general result for $B(a)$ in (16) is the quantity that must be maximized by choice of frequency a. However, it is interesting to observe that for frequencies where the window is small, that is,

$$|W(2a)| \ll W(0) \ , \tag{18}$$

then (16) simplifies to

$$B(a) \approx \frac{2}{W(0)}\left|L^2(a)\right| = \frac{2}{W(0)}\left|\sum_k w_k\ x_k\ \exp(-iak)\right|^2 , \tag{19}$$

which is identical to (4). Thus, for those frequencies where (18) is true, the function $B(a)$ is approximately the magnitude-squared Fourier transform of the weighted data; this corresponds to values of a not near multiples of π.

PROPERTIES OF B(a)

Since $W(2a)$ has period π in a, while $L(a)$ has period 2π in a, the function $B(a)$ in (16) must have period 2π in a; that is,

$$B(a + 2\pi) = B(a) \ . \tag{20}$$

But at the same time, we have even property

$$B(-a) = B(a) \ , \tag{21}$$

because $L(-a) = L^*(a)$, $W(-2a) = W^*(2a)$, using the realness of sequences $\{w_k\}$ and $\{x_k\}$. What this means is that we only need to

$$\text{compute } B(a) \text{ for } 0 \le a \le \pi \ , \tag{22}$$

since all other values can be obtained therefrom. Reference to (6) reveals that ω is being varied over the range $(0, \pi/\Delta)$, or that cyclic frequency $f = \omega/(2\pi)$ is varying over $(0, .5/\Delta)$. This latter range extends up to the Nyquist frequency, as expected.

VALUE OF B(a) AS $a \to 0$

If we substitute $a = 0$ in (16), we get $B(0) = 0/0$, which is indeterminate. Hence, for small a, we expand (17) according to

$$W(2a) \sim W_0 - i2aW_1 - 2a^2W_2 ,$$

$$L(a) \sim L_0 - iaL_1 - \frac{1}{2}a^2L_2 , \tag{23}$$

where n-th order "moments"

$$W_n = \sum_k w_k k^n ,$$

$$L_n = \sum_k w_k x_k k^n . \tag{24}$$

Substitution of (23) in (16) and simplification yields

$$\lim_{a\to 0} B(a) = \frac{W_0L_1^2 - 2W_1L_1L_0 + W_2L_0^2}{W_0W_2 - W_1^2} . \tag{25}$$

This limiting result is the same value that is attained as if we minimized weighted error

$$E = \sum_k w_k [x_k - \mu - \nu k]^2 , \tag{26}$$

by choice of constant value μ and linear trend νk. In fact, direct minimization of (26) yields optimum coefficients

$$\mu_0 = \frac{W_2L_0 - W_1L_1}{W_0W_2 - W_1^2} , \quad \nu_0 = \frac{W_0L_1 - W_1L_0}{W_0W_2 - W_1^2} , \tag{27}$$

and associated minimum error

$$E_0 = \sum_k w_k\ x_k^2 - \frac{W_0 L_1^2 - 2W_1 L_1 L_0 + W_2 L_0^2}{W_0 W_2 - W_1^2} . \tag{28}$$

As claimed above, the last term in (28) is precisely the result given by (25); see (12) also. Thus, the limit, as $a \to 0$, of model fit (7) is the best-fitting constant plus linear trend to the given data. This can be obtained from (7) only if quadrature coefficient β behaves as $1/a$ as $a \to 0$. Indeed, in a later section, we will show that this is precisely the behavior of β in this limit. Thus, setting $a = 0$ in (7) and keeping β finite does not lead to the result in (25) and (28), but instead gives only the best fitting constant. We will allow the more general fit afforded by (26) here, and will utilize the value achieved by (25) in the limit, as $a \to 0$.

VALUE OF B(a) AS $a \to \pi$

If we substitute $a = \pi$ in (16), there follows $B(\pi) = 0/0$, which is indeterminate. However, if we let $a = \pi + \tilde{a}$, we see from (17) that

$$W(2a) = W(2\pi + 2\tilde{a}) = W(2\tilde{a}) ,$$

$$L(a) = L(\pi + \tilde{a}) = \sum_k w_k\ (-1)^k\ x_k\ \exp(-i\tilde{a}k) . \tag{29}$$

Thus, $W(2\tilde{a})$ behaves the same about $\tilde{a} = 0$ as $W(2a)$ does about $a = 0$. Also, the last term in (29) behaves the same about $\tilde{a} = 0$ as $L(a)$ does about $a = 0$, provided that each data element x_k is replaced by $(-1)^k\ x_k$. Thus, (25) can be immediately utilized to yield the result

$$\lim_{a \to \pi} B(a) = \frac{W_0 \tilde{L}_1^2 - 2W_1 \tilde{L}_1 \tilde{L}_0 + W_2 \tilde{L}_0^2}{W_0 W_2 - W_1^2} , \tag{30}$$

where moments (24) have been replaced by

$$\tilde{L}_n = \sum_k (-1)^k\ w_k\ x_k\ k^n . \tag{31}$$

Physical interpretation of result (30) is similar to that given earlier for $a \to 0$ in (25)-(28). Namely, in the limit as $a \to \pi$, the best constant plus linear trend is fitted to alternating data $\{(-1)^k x_k\}$. Again, this requires quadrature coefficient β in fit (7) to behave like $1/(a - \pi)$ as $a \to \pi$.

EXAMPLE OF EQUAL WEIGHTS

Let weights

$$w_k = \frac{1}{N} \quad \text{for} \quad 1 \le k \le N \ . \tag{32}$$

Then window (17) becomes

$$W(2a) = \frac{\sin(Na)}{N \sin(a)} \exp(-i(N + 1)a) \ . \tag{33}$$

This is the example considered in [9].

The moments (24) for this case are given by

$$W_0 = 1, \ W_1 = \frac{1}{2}(N + 1), \ W_2 = \frac{1}{6}(N + 1)(2N + 1) \ ,$$

$$W_0 W_2 - W_1^2 = \frac{1}{12}(N^2 - 1) \ . \tag{34}$$

The numerator of (25) is then

$$L_1^2 - (N + 1)L_1 L_0 + \frac{1}{6}(N + 1)(2N + 1)L_0^2 =$$

$$= \left(L_1 - \frac{N + 1}{2} L_0\right)^2 + \frac{1}{12}(N^2 - 1)L_0^2 =$$

$$= \frac{1}{N^2} \left[\sum_k x_k \left(k - \frac{N + 1}{2}\right)\right]^2 + \frac{N^2 - 1}{12N^2} \left[\sum_k x_k\right]^2 \ . \tag{35}$$

Then (12), (25), and (34) yield

$$\lim_{a \to 0} E(a) = \frac{1}{N} \sum_k x_k^2 - \frac{1}{N^2} \left[\sum_k x_k \right]^2 - \frac{12}{N^2(N^2 - 1)} \left[\sum_k x_k \left(k - \frac{N + 1}{2} \right) \right]^2 , \tag{36}$$

which can be recognized as the minimum error for the best-fitting constant plus linear trend to data $\{x_k\}$.

IN-PHASE AND QUADRATURE COEFFICIENTS

Since the modeling waveform in (7) is

$$\alpha \cos(ak) + \beta \sin(ak) = \mathrm{Re}\{(\alpha - i\beta) \exp(iak)\} , \qquad (37)$$

the complex coefficient or strength of pure complex tone exp(iak) is $\alpha - i\beta$. From (11) and (10), the numerator of $\alpha_0 - i\beta_0$ is expressible as

$$A_{22}L_r(a) + A_{12}L_i(a) + iA_{11}L_i(a) + iA_{12}L_r(a) =$$

$$= \frac{1}{2}\left[W(0) - W_r(2a)\right]L_r(a) + i\frac{1}{2}\left[W(0) + W_r(2a)\right]L_i(a) +$$

$$+ i\left(-\frac{1}{2}\right)W_i(2a)\left[L_r(a) - iL_i(a)\right] =$$

$$= \frac{1}{2} W(0)L(a) - \frac{1}{2} W(2a)L^*(a) . \qquad (38)$$

Combining this with the denominator previously computed in (14), we have for the optimum complex coefficient,

$$\alpha_0 - i\beta_0 = 2 \frac{W(0)L(a) - W(2a)L^*(a)}{W^2(0) - \left|W^2(2a)\right|} . \qquad (39)$$

For frequencies a such that the window is small relative to the origin value (see (18)), (39) simplifies to the approximate result

$$\alpha_0 - i\beta_0 \approx 2L(a) = 2 \sum_k w_k \, x_k \exp(-iak) , \qquad (40)$$

which is just the Fourier transform of the weighted data.

If only the phase of the real tone is of interest, (39) indicates that

$$\arg(\alpha_0 - i\beta_0) = \arg\Big(W(0)L(a) - W(2a)L^*(a)\Big) \; . \tag{41}$$

If frequency a is known, this result is directly applicable; but if a is unknown, the value a that maximizes (16) must be used.

NORMALIZATION OF WEIGHTS

Without loss of generality, the sum of the weights $\{w_k\}$ can be set equal to unity; that is, set

$$W(0) = \sum_k w_k = 1 \; . \tag{42}$$

Then the complex coefficient in (39) reduces to

$$\alpha_0 - i\beta_0 = 2\,\frac{L(a) - W(2a)L^*(a)}{1 - \left|W^2(2a)\right|} \; , \tag{43}$$

while the maximizing function B(a) in (16) becomes

$$B(a) = 2\,\frac{\left|L^2(a)\right| - \mathrm{Re}\left\{W^*(2a)L^2(a)\right\}}{1 - \left|W^2(2a)\right|} \; . \tag{44}$$

This slightly reduces the number of computations that have to be conducted and has been adopted in the program written here. This scaling is also retained in the following subsection.

INTERPRETATION OF (43)

An alternative form for coefficient (43) is

$$\alpha_0 - i\beta_0 = 2\,\frac{L(a) - W(2a)L(-a)}{1 - \left|W^2(2a)\right|} \; , \tag{45}$$

where we utilized the realness of data $\{x_k\}$ and weights $\{w_k\}$. This result can be interpreted as follows: the term

$$2L(a) = 2 \sum_k w_k \; x_k \; \exp(-iak) \tag{46}$$

is an estimate of the complex strength of the positive-frequency complex exponential exp(iak) in the real data $\{x_k\}$, as modified by the weights. Similarly, 2L(-a) estimates the strength of the term exp(-iak) in the real data. The window W(2a) measures the amount of spillover from frequency -a to frequency a, that is, at separation 2a, due to the weights $\{w_k\}$. This fraction (including phase information) of the spillover from negative frequencies to positive frequencies is subtracted from strength 2L(a). Finally, the denominator factor $1 - |W^2(2a)|$ renormalizes the remainder according to the fractional spillover.

To justify this last scale factor, suppose that the data $\{x_k\}$ contain a pure real tone at precisely the frequency a; that is, let

$$x_k = \alpha_0 \cos(ak) + \beta_0 \sin(ak) =$$

$$= \mathrm{Re}\left\{(\alpha_0 - i\beta_0)\exp(iak)\right\} . \tag{47}$$

Then (46) yields

$$2L(a) = \sum_k w_k[\alpha_0(\exp(iak) + \exp(-iak)) - i\beta_0(\exp(iak) - \exp(-iak))]\exp(-iak) =$$

$$= \alpha_0[1 + W(2a)] - i\beta_0[1 - W(2a)] . \tag{48}$$

Therefore,

$$2L(-a) = 2L^*(a) = \alpha_0[1 + W^*(2a)] + i\beta_0[1 - W^*(2a)] . \tag{49}$$

Therefore, the numerator of $\alpha_0 - i\beta_0$ in (45) is

$$2L(a) - W(2a)2L(-a) =$$

$$= \alpha_0(1 + W) - i\beta_0(1 - W) - W[\alpha_0(1 + W^*) + i\beta_0(1 - W^*)] =$$

$$= (\alpha_0 - i\beta_0)(1 - |W^2|) = (\alpha_0 - i\beta_0)[1 - |W^2(2a)|] , \qquad (50)$$

where we adopted the notational simplification $W = W(2a)$ during the manipulations. Thus, the denominator factor $1 - |W^2(2a)|$ in (45) is necessary to scale the amplitude back up to its correct value of $\alpha_0 - i\beta_0$.

VALUE OF COEFFICIENT AS $a \to 0$

We want to investigate the behavior of coefficient $\alpha_0 - i\beta_0$ in (39) as $a \to 0$. (If we try to set $a = 0$, we get $\alpha_0 - i\beta_0 = 0/0$, which is indeterminate.) Accordingly, substitute expansions (23)-(24) into (39) and simplify to obtain the expression

$$\alpha_0 - i\beta_0 \sim \frac{W_2L_0 - W_1L_1}{W_0W_2 - W_1^2} - \frac{i}{a}\,\frac{W_0L_1 - W_1L_0}{W_0W_2 - W_1^2} \quad \text{as } a \to 0 . \qquad (51)$$

This result corroborates the claim made under (28) that β_0 behaves as $1/a$ as $a \to 0$. That is, the optimum quadrature coefficient of the pure real tone gets arbitrarily large as frequency a tends to zero.

If we combine (51) with the modelling function in (7), we have

$$\alpha_0 \cos(ak) + \beta_0 \sin(ak) \sim \alpha_0 + \beta_0 ak \sim$$

$$\sim \frac{W_2L_0 - W_1L_1}{W_0W_2 - W_1^2} + \frac{W_0L_1 - W_1L_0}{W_0W_2 - W_1^2}\,k \qquad \text{as} \quad a \to 0 , \qquad (52)$$

which is precisely (26) and (27). Thus, the limit, as $a \to 0$, of modeling (7) is to fit the best constant plus linear trend to the data.

FFT REALIZATION

For purposes of minimizing computations, we henceforth assume that the weights have been normalized according to (42); that is, their sum equals unity. This feature is incorporated in the following equations and the resultant program.

MANIPULATION INTO FFT FORMS

According to (22), we are interested in evaluating B(a) in (44) over the range $0 \le a \le \pi$, where functions W and L are given by (17). Suppose then that we focus attention on values of frequency a given by

$$a = m\,\frac{2\pi}{M} \quad \text{for} \quad 0 \le m \le \frac{M}{2}\,. \tag{53}$$

Integer M will be chosen to be a power of 2, and is unrelated to N, the number of data points. Then (17) yields

$$L\left(m\,\frac{2\pi}{M}\right) = \sum_k w_k\, x_k\, \exp(-i2\pi mk/M) \equiv \mathcal{L}(m)\ , \tag{54}$$

which is recognized as an M-size FFT of N nonzero real weighted data values $\{w_k\, x_k\}$.

At the same time, (17) also gives for the window

$$W\left(2m\,\frac{2\pi}{M}\right) = \sum_k w_k\, \exp(-i2\pi 2mk/M) =$$

$$= \sum_{j\ \text{even}} w_{j/2}\, \exp(-i2\pi mj/M)\ , \tag{55}$$

where we let $j = 2k$. Now if we define sequence

$$d_j = \begin{Bmatrix} w_{j/2} & \text{for} & j & \text{even} \\ 0 & \text{for} & j & \text{odd} \end{Bmatrix}, \tag{56}$$

then (55) becomes

$$W\left(2m \frac{2\pi}{M}\right) = \sum_j d_j \exp(-i2\pi mj/M) \equiv \mathcal{D}(m) , \tag{57}$$

which is an M-size FFT of sequence $\{d_j\}$.

Direct employment of (54) and (57) in (44) yields

$$B\left(m \frac{2\pi}{M}\right) = 2 \frac{\left|\mathcal{L}^2(m)\right| - \mathrm{Re}\left\{\mathcal{D}^*(m)\,\mathcal{L}^2(m)\right\}}{1 - \left|\mathcal{D}^2(m)\right|} . \tag{58}$$

Thus, if we evaluate the two FFTs for $\{\mathcal{L}(m)\}$ and $\{\mathcal{D}(m)\}$ in (54) and (57), respectively, we have all the quantities necessary to determine $B(m2\pi/M)$ for $0 \leq m \leq M/2$.

TWO REAL FFT's VIA ONE COMPLEX FFT

Since (54) and (57) constitute FFTs of real sequences, they are not making full use of the capabilities of an FFT. To exploit the inherently complex nature of this tool, let

$$z_k = w_k x_k + i d_k \quad \text{for} \quad 1 \leq k \leq 2N , \tag{59}$$

where sequence $\{d_k\}$ was defined in (56). (Half of the real terms and half of the imaginary terms are zero in (59).) Then the FFT of size M of sequence (59) is

$$Z(m) = \sum_k z_k \exp(-i2\pi mk/M) = \mathcal{L}(m) + i\,\mathcal{D}(m) , \tag{60}$$

where we presume that $M > 2N$. (Methods of circumventing this limitation are given in [10].)

Using the realness of sequences $\{x_k\}$, $\{w_k\}$, $\{d_k\}$, it follows from (54) and (57) that

$$Z^*(M - m) = \mathcal{L}^*(M - m) - i\mathcal{D}^*(M - m) = \mathcal{L}(m) - i\mathcal{D}(m) \ . \qquad (61)$$

Now combining (60) and (61), we have

$$2\mathcal{L}(m) = Z(m) + Z^*(M - m) = S_x(m) + i\ D_y(m) \ ,$$

$$2\mathcal{D}(m) = -i[Z(m) - Z^*(M - m)] = S_y(m) - i\ D_x(m) \ , \qquad (62)$$

where the real sum and difference functions are defined as

$$\begin{aligned} S_x(m) &= X(m) + X(M - m), \\ S_y(m) &= Y(m) + Y(M - m), \\ D_x(m) &= X(m) - X(M - m), \\ D_y(m) &= Y(m) - Y(M - m), \end{aligned} \qquad (63)$$

in terms of the real and imaginary parts of FFT output $Z(m)$ in (60), namely,

$$Z(m) = X(m) + i\ Y(m) \ . \qquad (64)$$

Equation (62) accomplishes the decoupling of the FFT output $Z(m)$ so as to yield the two desired FFTs $\mathcal{L}(m)$ and $\mathcal{D}(m)$ indicated in (54) and (57). However, it is advantageous to continue with the breakdown of these two complex sequences $\mathcal{L}(m)$ and $\mathcal{D}(m)$, as done in (62), in terms of all the purely real quantities given in (63). For upon substitution of (62) in desired quantity $B(m2\pi/M)$ in (58), we obtain the simplified form

$$B\left(m\,\frac{2\pi}{M}\right) = \frac{S_x^2(m)[2 - S_y(m)] + D_y^2(m)[2 + S_y(m)] + 2S_x(m)D_x(m)D_y(m)}{4 - \left(S_y^2(m) + D_x^2(m)\right)} \ . \qquad (65)$$

This latter form, which utilizes only real arithmetic, can be used only for $0 < m < M/2$. The values for $B(0+)$ and $B(\pi-)$ must come from (25) and (30), respectively, with $W_0 = 1$. A program for calculation of $\{B(m2\pi/M)\}$ by means of (56), (59), (60), (63), and (65) is furnished in the appendix.

SELECTION OF FFT SIZE M

It was presumed in (59) and (60) that FFT size $M > 2N$, where N is the number of data points, in order that there be an array element in location $M-1$ available to receive data element id_{2N}. However, there is an additional reason for choosing M this large, having to do with the rate at which $B(a)$ varies. The function $B(a)$ in (44) depends critically on window function $W(2a)$. For equal weights, the results in (32) and (33) indicate that $W(2a)$ changes significantly in an interval of length π/N; in fact, this is the separation between zero crossings. If order to track this rapid variation in W, the increment $2\pi/M$ in frequency a in (53) and (58) must be smaller than π/N. Thus, requirement $M > 2N$ is a minimal requirement; in fact, it may be advantageous to consider M several times larger than N, if storage and FFT execution time are not excessive. Of course, the larger M is taken, the less fine-grain interpolation will be required later.

For other weightings than flat, such as Hanning, where the effective length of the weighting is foreshortened due to taper at the edges, the window function W is broader, and the condition on M is alleviated somewhat. However, $M > 2N$ is a good rule of thumb to use in most cases.

INTERPOLATION PROCEDURE

When the complete set of values of $B(m2\pi/M)$ for $0 \leq m \leq M/2$ are available, they are searched to find the maximum value. This maximum value and the two neighboring bin outputs (m values) are then used in a parabolic interpolation procedure to refine the estimate of the location of the best value of frequency a and the corresponding maximum value of $B(a)$.

Finally, this latter value of a can be used as a starting value for a fine-grained search, again by means of parabolic interpolation, in the neighborhood of this peak. These features are all incorporated in the accompanying program for this search procedure, where direct use of (44) is made; the previous FFT results are of no use in this final vernier estimation. Along with each estimated frequency a, the corresponding coefficient $\alpha_0 - i\beta_0$ is also estimated and printed out. A few stages of the vernier analysis suffice to give stable frequency estimates within the accuracy of the computer used here.

RESULTS

An example of $N = 25$ data points with FFT size $M = 1024$ is displayed in figure 1, for the data sequence

$$x_k = \cos(k) + \frac{1}{2}\sin(k) \quad \text{for} \quad 1 \leq k \leq N \tag{66}$$

and for the equal (or flat) weighting case of (32). The abscissa is normalized frequency $a = \omega\Delta$, and the ordinate is $B(a)$ normalized relative to its peak value. The low-level sidelobes in figure 1 are due to the nature of the window $W(2a)$, given by (33) for this case.

The line labeled INITIAL gives the bin number Js in which the peak is located. This bin and the two adjacent ones are then interpolated by means of a parabola to yield the initial value for $B(a)$ labeled Big and the abscissa estimate $a = 1.0000445$. This value of a is then employed in subroutine SUB B to give the corresponding value $B(a)$.

In the next two lines of the print out, the above value of a is perturbed by ± Delta, the function $B(a)$ is computed, and parabolic interpolation is again used on these three points to give the estimates labeled as REFINED values. Then this refined a value is used to recompute $B(a)$ and indicated as the MAXIMUM value in the print out. Finally, the coefficient estimates α and β, along with the minimum error, E_{min}, are printed out.

The whole cycle of perturbation and parabolic interpolation is repeated in the next separated four lines of print out, but this time with Delta decreased by a factor of 10. This cycle is repeated one final time in each of the figures presented. Prolonged repetition would result in excessive round-off error, due to the differencing of similar function values.

If the weighting is changed to Hanning,

$$w_k = 1 - \cos\left(\frac{2\pi k}{N + 1}\right) \quad \text{for} \quad 1 \leq k \leq N , \tag{67}$$

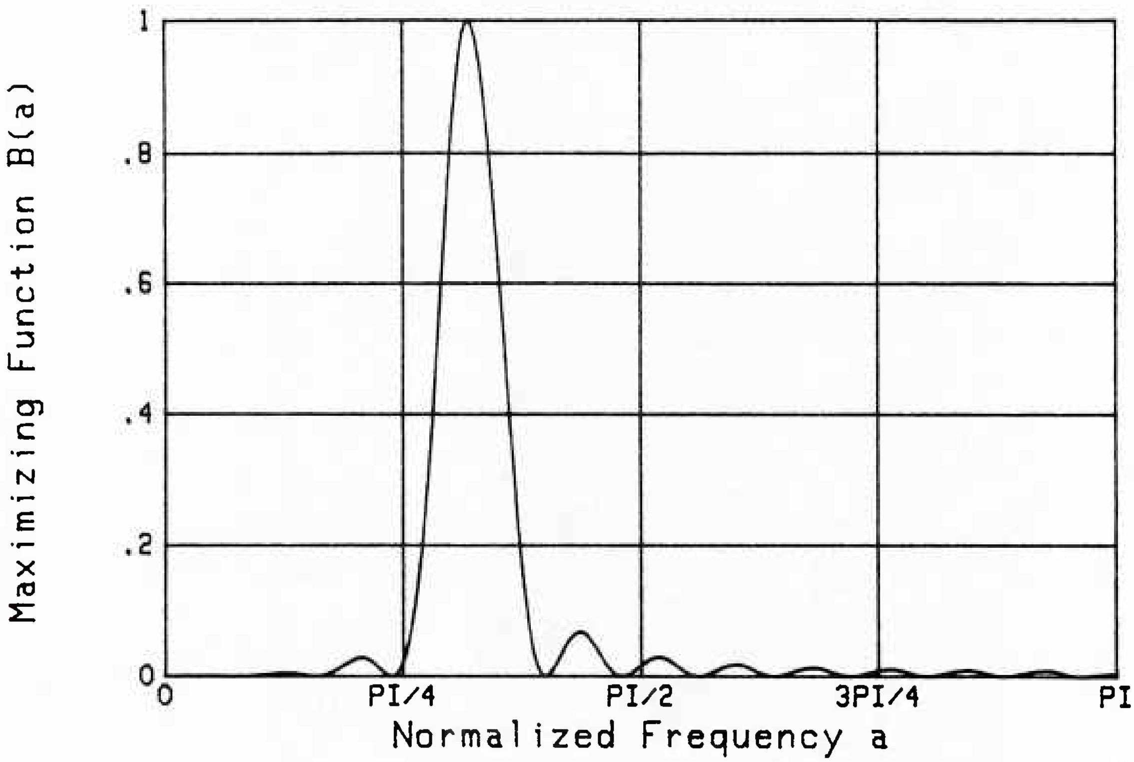
the corresponding results are displayed in figure 2. The initial estimate is $a = .99999970$, which is then refined to $a = 1$. The coefficients converge rapidly to the correct values, and the figure displays no visible sidelobes for this case of Hanning weighting. However, the window is broader.

The results of figures 3 and 4 correspond to figures 1 and 2, respectively, except that white noise of power 1/12 has been added to the waveform of (66). Now the Hanning weighting result in figure 4 also displays sidelobes, due to random fitting of the particular noise samples utilized. The refined values of a converge to $a = .99318$ and $a = 1.00208$, respectively, which are not exactly correct, due to the additive noise. Also, the coefficient α and β are considerably off their correct values, although the Hanning results in figure 4 are better than for the flat weighting used in figure 3.

Figures 5 and 6 are conducted for the two-tone data sequence

$$x_k = \cos(k) + \frac{1}{2}\sin(k) + \cos(2k) \ , \tag{68}$$

with no additive noise. The second peak near $a = 2$ in these figures is due to the attempted match of model (7) to the data, when a is near 2. The program locks onto the stronger tone and indicates its frequency as $a = .99447$ and $a = 1.00128$, respectively. Again, the estimates of frequency a and coefficients α and β are better for the Hanning weighting in figure 6 than for the flat weighting in figure 5. This is due to the lower sidelobes of window $W(2a)$ in the Hanning case.

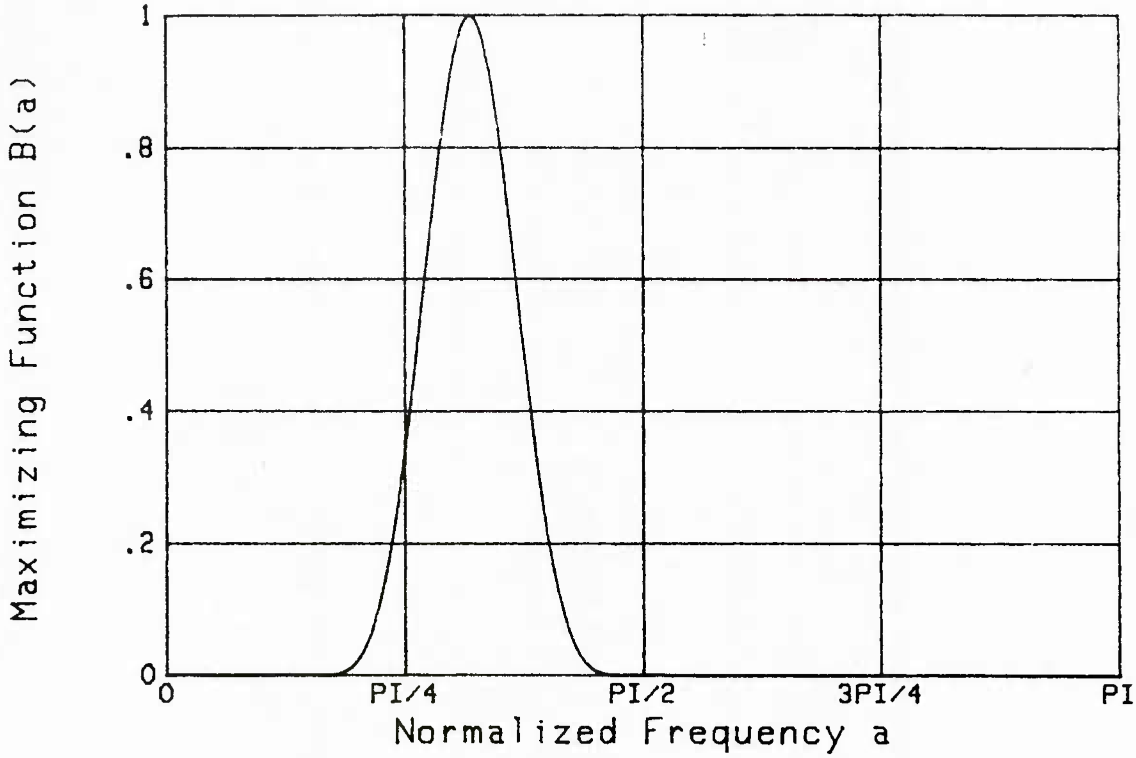



```
NUMBER OF DATA POINTS N = 25
SIZE OF FFT M = 1024
INITIAL: Js = 163   Big = .621074542628   a = 1.00004447034   B(a) = .62107486
REFINED a = 1.00000000426
REFINED B(a) = .621074930024
MAXIMUM B(a) = .621074930049
Alpha =  .999999969685    Beta =  .500000054303    Emin = 5.55111512313E-16

REFINED a = 1.00000000011
REFINED B(a) = .621074930049
MAXIMUM B(a) = .621074930049
Alpha =  .999999999199    Beta =  .500000001436    Emin = 1.11022302463E-16

REFINED a = 1
REFINED B(a) = .621074930049
MAXIMUM B(a) = .621074930049
Alpha =  1    Beta =  .499999999994    Emin = 1.11022302463E-16
```

Figure 1. Flat Weighting, No Noise

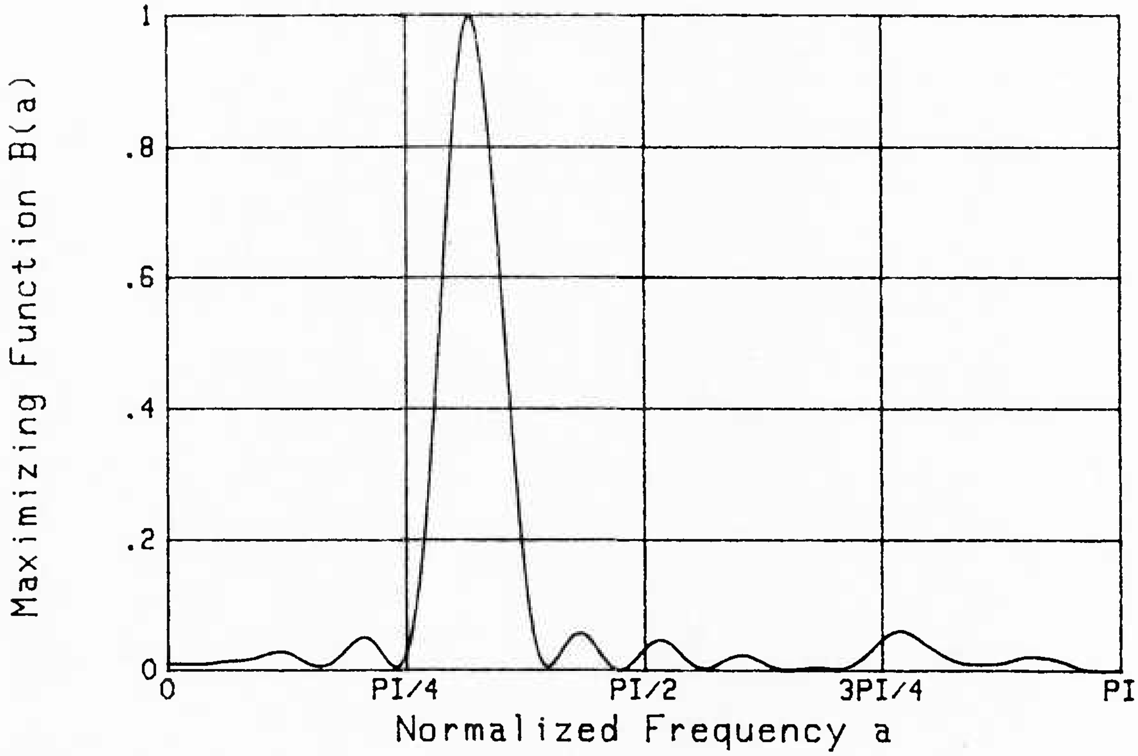



```
NUMBER OF DATA POINTS N = 25
SIZE OF FFT M = 1024
INITIAL: Js = 163   Big = .624753874556   a = .999999700554   B(a) = .62475387
REFINED a = .999999999908
REFINED B(a) = .624753873395
MAXIMUM B(a) = .624753873395
Alpha =  1.00000000059    Beta =   .49999999881    Emin = 0

REFINED a = .999999999999
REFINED B(a) = .624753873395
MAXIMUM B(a) = .624753873395
Alpha =  1.00000000001    Beta =   .499999999986    Emin =-1.11022302463E-16

REFINED a = .999999999997
REFINED B(a) = .624753873395
MAXIMUM B(a) = .624753873395
Alpha =  1.00000000002    Beta =   .49999999996    Emin = 1.11022302463E-16
```

Figure 2. Hanning Weighting, No Noise

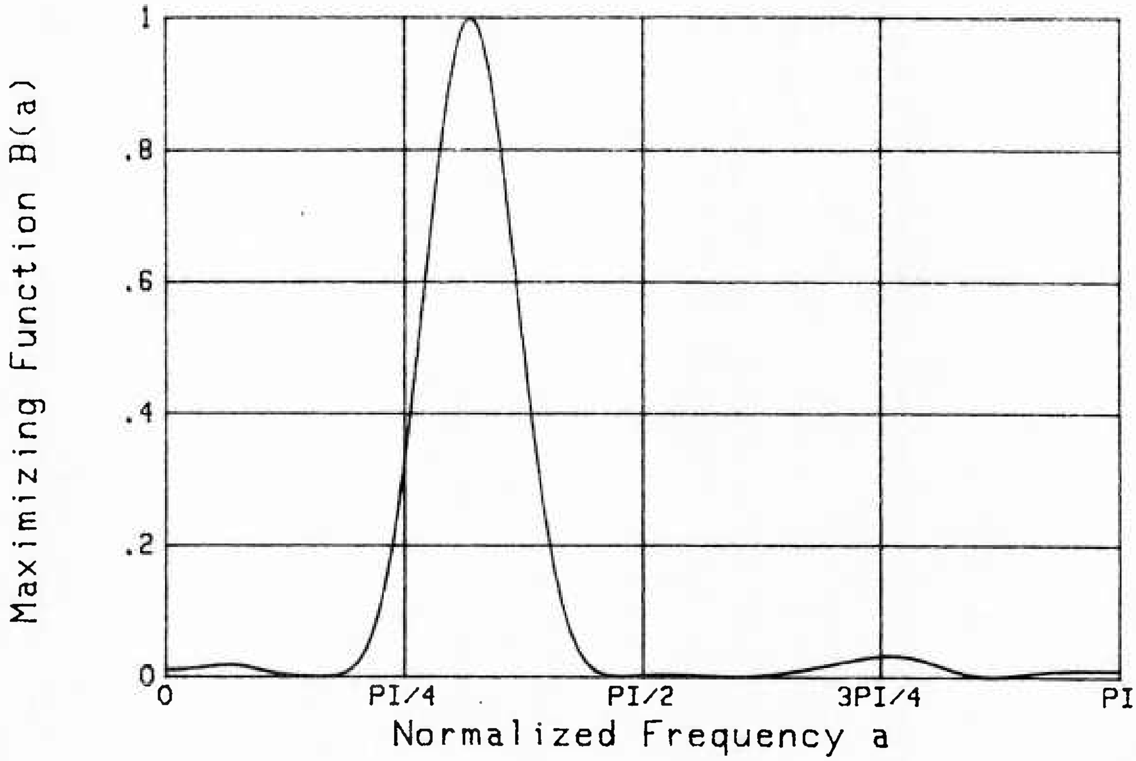


NUMBER OF DATA POINTS N = 25
SIZE OF FFT M = 1024
INITIAL: Js = 162 Big = .657123186854 a = .993221368141 B(a) = .65712576
REFINED a = .993178780656
REFINED B(a) = .657125832748
MAXIMUM B(a) = .657125832776
Alpha = 1.1193876303 Beta = .282608428946 Emin = 6.39949448814E-02

REFINED a = .99317877595
REFINED B(a) = .657125832776
MAXIMUM B(a) = .657125832776
Alpha = 1.11938765005 Beta = .282608363445 Emin = 6.39949448814E-02

REFINED a = .99317877584
REFINED B(a) = .657125832776
MAXIMUM B(a) = .657125832776
Alpha = 1.11938765051 Beta = .282608361911 Emin = 6.39949448814E-02

Figure 3. Flat Weighting, Additive Noise

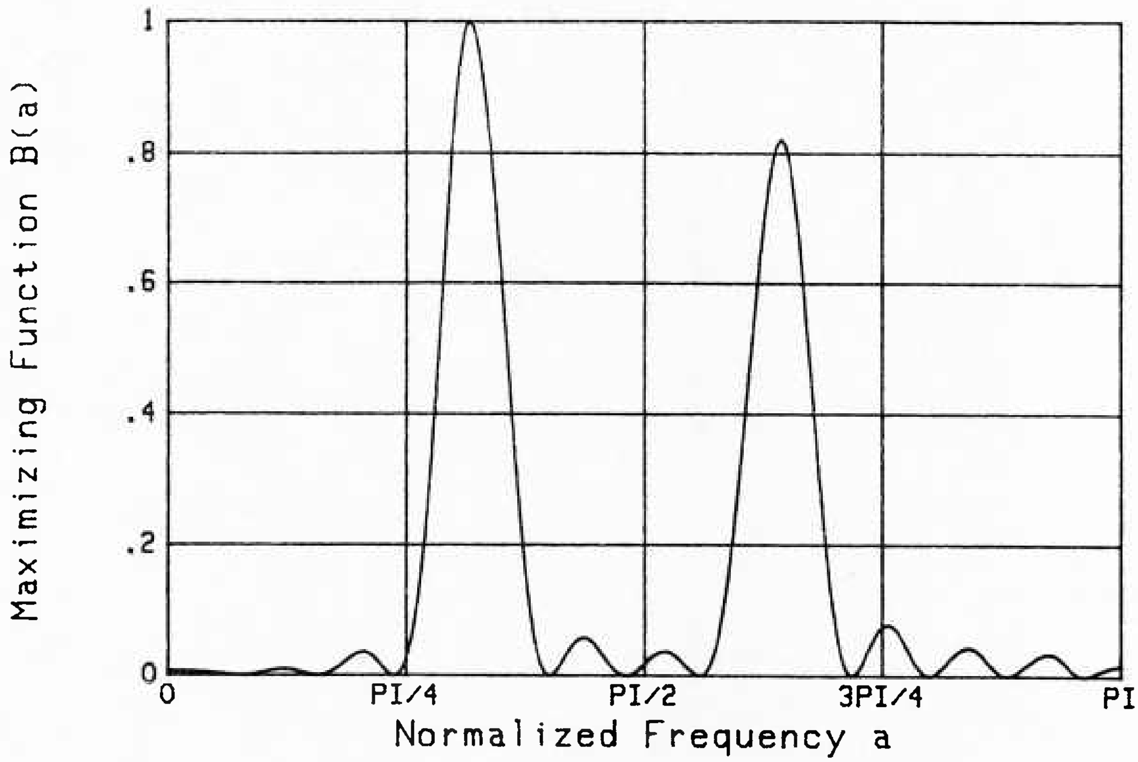



```
NUMBER OF DATA POINTS N = 25
SIZE OF FFT M = 1024
INITIAL: Js = 163  Big = .643943182613  a = 1.00208436495  B(a) = .64394317
REFINED a = 1.00208395996
REFINED B(a) = .643943178345
MAXIMUM B(a) = .643943178345
Alpha =  1.0636516553    Beta =  .396291769342    Emin = 5.15554013299E-02

REFINED a = 1.00208395965
REFINED B(a) = .643943178345
MAXIMUM B(a) = .643943178345
Alpha =  1.06365165685    Beta =  .396291765184    Emin = 5.15554013299E-02

REFINED a = 1.00208395966
REFINED B(a) = .643943178345
MAXIMUM B(a) = .643943178345
Alpha =  1.06365165684    Beta =  .396291765211    Emin = 5.15554013299E-02
```

Figure 4. Hanning Weighting, Additive Noise

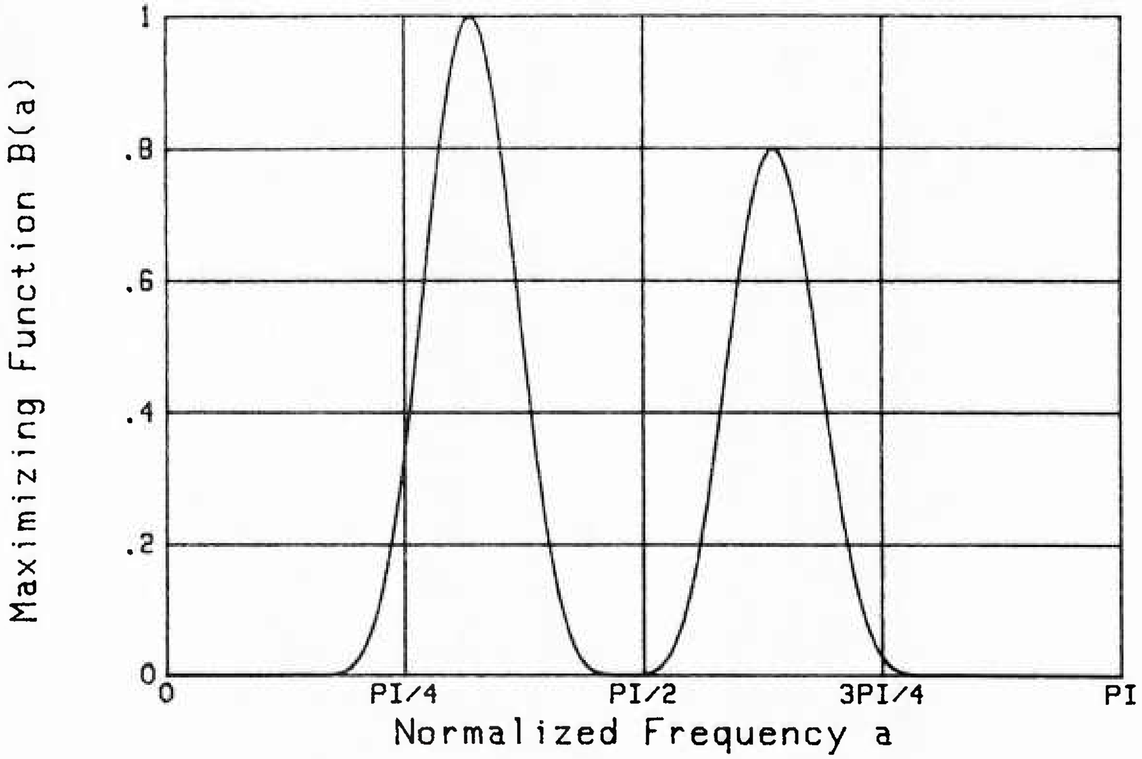



```
NUMBER OF DATA POINTS N = 25
SIZE OF FFT M = 1024
INITIAL: Js = 162  Big = .612307134756  a = .994519907011  B(a) = .6123056
REFINED a = .994474013322
REFINED B(a) = .612305684026
MAXIMUM B(a) = .612305684054
Alpha =  1.03244095052    Beta =  .417726698818    Emin = .499913742067

REFINED a = .994474009397
REFINED B(a) = .612305684054
MAXIMUM B(a) = .612305684054
Alpha =  1.03244097595    Beta =  .417726642805    Emin = .499913742067

REFINED a = .994474009277
REFINED B(a) = .612305684054
MAXIMUM B(a) = .612305684054
Alpha =  1.03244097673    Beta =  .417726641086    Emin = .499913742067
```

Figure 5. Flat Weighting, Two Tones




```
NUMBER OF DATA POINTS N = 25
SIZE OF FFT M = 1024
INITIAL: Js = 163  Big = .623356825941  a = 1.00127460591  B(a) = .623356
REFINED a = 1.00127551942
REFINED B(a) = .623356862582
MAXIMUM B(a) = .623356862582
Alpha =  .989782416069    Beta =  .51724932123    Emin = .499950569458

REFINED a = 1.00127551958
REFINED B(a) = .623356862582
MAXIMUM B(a) = .623356862582
Alpha =  .989782415052    Beta =  .517249323177    Emin = .499950569458

REFINED a = 1.00127551958
REFINED B(a) = .623356862582
MAXIMUM B(a) = .623356862582
Alpha =  .989782415038    Beta =  .517249323204    Emin = .499950569458
```

Figure 6. Hanning Weighting, Two Tones

SUMMARY

An automatic procedure for determining the best frequency, amplitude, and phase of a real tone fitted to discrete real data has been devised and programmed. It employs a single complex FFT for the initial search and then refines the estimates by simple parabolic interpolation procedures. The size M of the FFT is unrelated to the number N of data points, but should generally be taken at least equal to 2N in order to guarantee adequate sampling in the frequency search. The user can input any real weights $\{w_k\}$ of his choosing into the program; these are then automatically normalized to make their sum equal to unity.

The procedure is applicable to data records of any length N, without any approximations. However, if there is considerable noise in the data, then large N will be required in order to attain accurate estimates of the tone frequency, amplitude, and phase. This is not a drawback of the least squares procedure or program, but is a fundamental limitation of estimation capability in the presence of noise.

No derivatives of any of the error functions to be extremized are required in this approach. Instead, direct parabolic interpolation of the appropriate sampled functions is employed and can be carried through several stages to the desired degree of accuracy or until round-off error dominates. For a very near 0 or π, the approximation of B(a) by a parabola may not be adequate; special techniques may be required at these limits.

APPENDIX

PROGRAM FOR ESTIMATION OF TONE PARAMETERS

Inputs required of the user are in

line 10: N, number of data points,
line 20: M, size of FFT.

The program is configured to accept up to N = 8000 data points and an FFT size up to M = 16384. The user can also change the weighting from flat (in line 200) to whatever weighting is of interest. The appropriate window FFT is undertaken automatically, by means of lines 290 and 560. The initial estimate of a and the plot of B(a) are completed by line 1010. If refined estimates of a are desired, CONT EXECUTE must be performed, and can be repeated for additional refinement.

The terminology DOUBLE denotes INTEGER variables in BASIC on the HP 9000 computer. Subroutine SUB B computes B(a) and coefficients α and β at any frequency a of interest. Generation of data for the examples here is accomplished in SUB Data, which must be replaced by the user to bring in his own data.

```
 10    N=25                                   !  NUMBER OF DATA POINTS
 20    M=1024                                 !  SIZE OF FFT; M > 2N REQUIRED
 30    PRINT "NUMBER OF DATA POINTS N =";N
 40    PRINT "SIZE OF FFT M =";M
 50    DIM W(1:8000),Xd(1:8000),X(0:16383),Y(0:16383),Cos(0:4096)
 60    REDIM W(1:N),Xd(1:N),X(0:M-1),Y(0:M-1),Cos(0:M/4)
 70    DOUBLE N,M,Ms,Ks,Js,M2                 !  INTEGERS
 80    IF M>2*N THEN 120
 90    BEEP
100    PRINT "M <= 2N; INCREASE M OR DECREASE N."
110    PAUSE
120    T=2.*PI/M
130    FOR Ms=0 TO M/4
140    Cos(Ms)=COS(T*Ms)                      !  QUARTER-COSINE TABLE
150    NEXT Ms
160    MAT X=(0.)
170    MAT Y=(0.)
180    S=0.
190    FOR Ks=1 TO N
200    Wk=1.                                  !  SPECIFY WEIGHTS, k=1:N
210    W(Ks)=Wk
220    S=S+Wk
230    NEXT Ks
240    S=1./S
250    W1=W2=0.
260    FOR Ks=1 TO N
270    T=W(Ks)*S                              !  SCALE WEIGHTS
280    W(Ks)=T                                !  SO THAT SUM = 1
290    Y(Ks+Ks)=T
300    T=T*Ks
310    W1=W1+T                                !  MOMENTS OF
320    W2=W2+T*Ks                             !  WEIGHTS
330    NEXT Ks
340    CALL Data(N,Xd(*))                     !  FILL UP DATA ARRAY Xd(1:N)
350    So=To=Se=Te=En=0.
360    FOR Ks=1 TO N STEP 2
370    T1=Xd(Ks)
380    T2=W(Ks)*T1
390    X(Ks)=T2
400    So=So+T2
410    To=To+T2*Ks
420    En=En+T1*T2
430    NEXT Ks
440    FOR Ks=2 TO N STEP 2
450    T1=Xd(Ks)
460    T2=W(Ks)*T1
470    X(Ks)=T2
480    Se=Se+T2
490    Te=Te+T2*Ks
500    En=En+T1*T2                            !  TOTAL WEIGHTED ENERGY
510    NEXT Ks
```

```
 520    L0=Se+So                        !  MOMENTS
 530    L1=Te+To                        !     OF
 540    L0t=Se-So                       !  WEIGHTED
 550    L1t=Te-To                       !    DATA
 560    CALL Fft14(M,Cos(*),X(*),Y(*))
 570    M2=M/2
 580    T=W2-W1*W1                      !  EVALUATE B(a) IN X(0:M/2)
 590    X(0)=(L1*L1-2.*W1*L1*L0+W2*L0*L0)/T
 600    X(M2)=(L1t*L1t-2.*W1*L1t*L0t+W2*L0t*L0t)/T
 610    FOR Ms=1 TO M2-1
 620    T1=X(Ms)
 630    T2=X(M-Ms)
 640    Sx=T1+T2
 650    Dx=T1-T2
 660    T1=Y(Ms)
 670    T2=Y(M-Ms)
 680    Sy=T1+T2
 690    Dy=T1-T2
 700    T1=Dx*Sx*Dy
 710    T2=4.-(Sy*Sy+Dx*Dx)
 720    X(Ms)=(Sx*Sx*(2.-Sy)+Dy*Dy*(2.+Sy)+T1+T1)/T2
 730    NEXT Ms
 740    Big=X(0)                        !  SEARCH FOR MAXIMUM
 750    Js=0
 760    FOR Ms=1 TO M2
 770    T=X(Ms)
 780    IF T<=Big THEN 810
 790    Big=T                           !  MAXIMUM VALUE AND
 800    Js=Ms                           !  LOCATION IN ARRAY
 810    NEXT Ms
 820    IF Js>0 AND Js<M2 THEN 850
 830    T=0.
 840    GOTO 890
 850    T1=X(Js+1)
 860    T2=X(Js-1)
 870    T=.5*(T1-T2)/(Big+Big-T1-T2) !  PARABOLIC INTERPOLATION
 880    Big=Big+.25*(T1-T2)*T           !  FOR MAXIMUM VALUE
 890    As=(Js+T)*2.*PI/M               !  AND LOCATION OF MAXIMUM
 900    CALL B(N,As,W(*),Xd(*),Alpha,Beta,Ba)
 910    PRINT "INITIAL: ";"Js =";Js;" Big =";Big;" a =";As;" B(a) =";Ba
 920    GINIT
 930    PLOTTER IS "GRAPHICS"
 940    GRAPHICS ON
 950    WINDOW 0.,M2,0.,Ba
 960    GRID M2/8.,Ba/10.
 970    FOR Ms=0 TO M2
 980    PLOT Ms,X(Ms)                   !  PLOT B(a) OVER [0,PI]
 990    NEXT Ms
1000    PENUP
1010    PAUSE
1020    GRAPHICS OFF
```

```
1030    Delta=1./M                        !  INITIAL SEARCH INCREMENT
1040    Delta=Delta*.1                    !  FINE-GRAIN SEARCH
1050    CALL B(N,As-Delta,W(*),Xd(*),Alpha,Beta,Bam)
1060    CALL B(N,As+Delta,W(*),Xd(*),Alpha,Beta,Bap)
1070    T=.5*(Bap-Bam)/(Ba+Ba-Bam-Bap)
1080    IF ABS(T)<1. THEN 1100
1090    PRINT "REFINED INTERPOLATION IS BEYOND EDGES OF SEARCH INCREMENT: ";T
1100    As=As+T*Delta
1110    Ba=Ba+.25*(Bap-Bam)*T
1120    PRINT "REFINED a =";As
1130    PRINT "REFINED B(a) =";Ba
1140    CALL B(N,As,W(*),Xd(*),Alpha,Beta,Ba)
1150    PRINT "MAXIMUM B(a) =";Ba
1160    Emin=En-Ba                        !  MINIMUM ENERGY
1170    PRINT "Alpha = ";Alpha;"    Beta = ";Beta;"    Emin =";Emin
1180    PRINT
1190    PAUSE
1200    GOTO 1040
1210    END
1220    !
1230    SUB B(DOUBLE N,REAL As,W(*),Xd(*),Alpha,Beta,Ba)
1240    DOUBLE Ks
1250    A2=As+As
1260    Wr=Wi=Lr=Li=0.
1270    FOR Ks=1 TO N
1280    Tw=W(Ks)
1290    Tx=Tw*Xd(Ks)
1300    T1=As*Ks
1310    T2=A2*Ks
1320    Wr=Wr+Tw*COS(T2)
1330    Wi=Wi-Tw*SIN(T2)
1340    Lr=Lr+Tx*COS(T1)
1350    Li=Li-Tx*SIN(T1)
1360    NEXT Ks
1370    T1=(1.-Wr)*Lr
1380    T2=(1.+Wr)*Li
1390    T=2./(1.-(Wr*Wr+Wi*Wi))
1400    Alpha=(T1-Wi*Li)*T
1410    Beta=(Wi*Lr-T2)*T
1420    Ba=Wi*Lr*Li
1430    Ba=(T1*Lr+T2*Li-Ba-Ba)*T
1440    SUBEND
1450    !
```

```
1460    SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
1470    DOUBLE N1,N2,N3,N4,Log2n,J,K    !  INTEGERS < 2^31 = 2,147,483,648
1480    DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1490    IF N=1 THEN SUBEXIT
1500    IF N>2 THEN 1580
1510    A=X(0)+X(1)
1520    X(1)=X(0)-X(1)
1530    X(0)=A
1540    A=Y(0)+Y(1)
1550    Y(1)=Y(0)-Y(1)
1560    Y(0)=A
1570    SUBEXIT
1580    N1=N/4
1590    N2=N1+1
1600    N3=N2+1
1610    N4=N3+N1
1620    Log2n=1.4427*LOG(N)
1630    FOR I1=1 TO Log2n
1640    I2=2^(Log2n-I1)
1650    I3=I2+I2
1660    I4=N/I3
1670    FOR I5=1 TO I2
1680    I6=(I5-1)*I4+1
1690    IF I6<=N2 THEN 1730
1700    A1=-Cos(N4-I6-1)
1710    A2=-Cos(I6-N1-1)
1720    GOTO 1750
1730    A1=Cos(I6-1)
1740    A2=-Cos(N3-I6-1)
1750    FOR I7=0 TO N-I3 STEP I3
1760    I8=I7+I5-1
1770    I9=I8+I2
1780    T1=X(I8)
1790    T2=X(I9)
1800    T3=Y(I8)
1810    T4=Y(I9)
1820    A3=T1-T2
1830    A4=T3-T4
1840    X(I8)=T1+T2
1850    Y(I8)=T3+T4
1860    X(I9)=A1*A3-A2*A4
1870    Y(I9)=A1*A4+A2*A3
1880    NEXT I7
1890    NEXT I5
1900    NEXT I1
```

```
1910    I1=Log2n+1
1920    FOR I2=1 TO 14
1930    L(I2-1)=1
1940    IF I2>Log2n THEN 1960
1950    L(I2-1)=2^(I1-I2)
1960    NEXT I2
1970    K=0
1980    FOR I1=1 TO L(13)
1990    FOR I2=I1 TO L(12) STEP L(13)
2000    FOR I3=I2 TO L(11) STEP L(12)
2010    FOR I4=I3 TO L(10) STEP L(11)
2020    FOR I5=I4 TO L(9) STEP L(10)
2030    FOR I6=I5 TO L(8) STEP L(9)
2040    FOR I7=I6 TO L(7) STEP L(8)
2050    FOR I8=I7 TO L(6) STEP L(7)
2060    FOR I9=I8 TO L(5) STEP L(6)
2070    FOR I10=I9 TO L(4) STEP L(5)
2080    FOR I11=I10 TO L(3) STEP L(4)
2090    FOR I12=I11 TO L(2) STEP L(3)
2100    FOR I13=I12 TO L(1) STEP L(2)
2110    FOR I14=I13 TO L(0) STEP L(1)
2120    J=I14-1
2130    IF K>J THEN 2200
2140    A=X(K)
2150    X(K)=X(J)
2160    X(J)=A
2170    A=Y(K)
2180    Y(K)=Y(J)
2190    Y(J)=A
2200    K=K+1
2210    NEXT I14
2220    NEXT I13
2230    NEXT I12
2240    NEXT I11
2250    NEXT I10
2260    NEXT I9
2270    NEXT I8
2280    NEXT I7
2290    NEXT I6
2300    NEXT I5
2310    NEXT I4
2320    NEXT I3
2330    NEXT I2
2340    NEXT I1
2350    SUBEND
2360    !
2370    SUB Data(DOUBLE N,REAL Xd(*))
2380    DOUBLE Ks
2390    FOR Ks=1 TO N
2400    Xd(Ks)=COS(Ks)+.5*SIN(Ks)
2410    NEXT Ks
2420    SUBEND
```

REFERENCES

1. L. C. Palmer, "Coarse Frequency Estimation Using the Discrete Fourier Transform," IEEE Trans. on Information Theory, vol. IT-20, Jan. 1974, pp. 104-109.

2. D. C. Rife, "Digital Tone Parameter Estimation in the Presence of Gaussian Noise," Ph.D. dissertation, Polytechnic Institute of Brooklyn, June 1973.

3. D. C. Rife and R. R. Boorstyn, "Single Tone Parameter Estimation From Discrete-Time Observations," IEEE Trans. on Information Theory, vol. IT-20, Sept. 1974, pp. 591-598.

4. D. C. Rife and R. R. Boorstyn, "Multiple Tone Parameter Estimation From Discrete Time Observations," BSTJ, vol. 55, no. 9, Nov. 1976, pp. 1389-1410.

5. T. Abatzoglou, "A Fast Maximum Likelihood Algorithm for Frequency Estimation of a Sinusoid Based on Newton's Method," IEEE Trans. on Acoustics, Speech and Signal Processing, vol. ASSP-33, Feb. 1985, pp. 77-89.

6. E. Plotkin et al., "Estimation of an Initial Phase Over Extremely Short Record-Length of a Sinewave Signal," IEEE Trans. on Instrumentation and Measurement, vol. IM-34, Dec. 1985, pp. 624-629.

7. R. Barrett and D. McMahon, "Comparison of Frequency Estimators for Underwater Acoustic Data," Journal of the Acoustical Society of America, vol. 79, no. 5, May 1986, pp. 1461-1471.

8. R. Kenefic, "On the Estimation of an Initial Phase Over Extremely Short Record-Length of a Sinewave Signal," submitted to the IEEE Trans. on Instrumentation and Measurement, August 1986.

REFERENCES (Cont'd)

9. R. Kenefic and A. H. Nuttall, "Maximum Likelihood Estimation of the Parameters of a Tone Using Real Discrete Data," Journal of Oceanic Engineering, vol. OE-12, no. 1, Special Issue on Underwater Acoustic Signal Processing, January 1987.

10. A. H. Nuttall, Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly from Characteristic Functions, NUSC Technical Report 7023, 1 October 1983.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
ASN (RE&S)	1
OUSDR&E (Research and Advanced Technology)	2
DEPUTY USDR&E (Res & Adv Tech)	1
DEPUTY USDR&E (Dir Elect & Phys Sc)	1
ONR, ONR-100, -102, -200, -400, -410, 411 (N. Gerr), -422, -425AC, -430	9
COMSPAWARSYSCOM, SPAWAR 05 (W. R. Hunt)	2
CNO, OP-098, OP-941, OP-951	3
DIA (DT-2C)	10
NRL, Code 5132 (Dr. P. B. Abraham), Code 5370 (W. Gabriel), Code 5135 (N. Yen), (A. A. Gerlach)	5
USRD	1
NORDA, Code 345 (R. Wagstaff)	2
USOC, Code 240, Code 241	2
NAVSUBSUPACNLON	1
NAVOCEANO, Code 02	2
NAVELECSYSCOM, ELEX 03, 310	2
NAVSEASYSCOM, SEA-00, -05R, -06F, 63R, -92R	5
NAVAIRDEVCEN, Warminster	1
NAVAIRDEVCEN, Key West	1
NOSC, Code 8302, Code 6565 (Library), Code 713 (F. Harris)	3
NAVWPNSCEN	1
NCSC, Code 724	1
NAVCIVENGRLAB	1
NAVSWC	1
NAVSURFWPNCEN, Code U31	1
NISC,	1
CNET, Code 017	1
CNTT	1
NAVSUBSCOL	1
NAVTRAEQUIPCENT, Technical Library	1
NAVPGSCOL	1
NAVWARCOL	1
NETC	1
APL/UW, SEATTLE	1
ARL/PENN STATE, STATE COLLEGE	1
CENTER FOR NAVAL ANALYSES (ACQUISITION UNIT)	1
DTIC	12
DARPA, A. Ellinthorpe	2
NOAA/ERL	1
NATIONAL RESEARCH COUNCIL	1
WOODS HOLE OCEANOGRAPHIC INSTITUTION (Dr. R. C. Spindel)	2
ENGINEERING SOCIETIES LIB, UNITED ENGRG CTR	1
NATIONAL INSTITUTE OF HEALTH	1
ARL, UNIV OF TEXAS	1

INITIAL DISTRIBUTION LIST (Cont'd)

Addressee	No. of Copies
MARINE PHYSICAL LAB, SCRIPPS	1
UNIVERSITY OF CALIFORNIA, SAN DIEGO	1
NAVSURWEACTR	1
DELSI	1
DIRECTOR SACLANT ASW RES CEN	1
COM SPACE & NAV WAR SYS COM	1
COM NAVAL PERSONNEL R&D CENTER	1
COM NAV SUB COLLEGE	1
B-K DYN INC	1
BBN, Arlington, VA (Dr. H. Cox)	1
BBN, Cambridge, MA (H. Gish)	1
BBN, New London, CT (Dr. P. Cable)	1
EWASCTRI	1
MAR, INC, East Lyme, CT	1
HYDROINC (D. Clark)	1
SUMRESCR (M. Henry)	1
ANALTECHNS, N. Stonington, CT (H. Jarvis)	1
ANALTECHNS, New London, CT	1
EDOCORP (J. Vincenzo)	1
TRA CORP., Austin, TX (Dr. T. Leih, J. Wilkinson)	2
TRA CORP., Groton, CT	1
NETS (R. Medeiros)	1
GESY, D. Bates	1
SONALYSTS, Waterford, CT (J. Morris)	1
ORI CO, INC. (G. Assard)	1
HUGHES AIRCRAFT CO. (S. Autrey)	1
MIT (Prof. A. Baggaroer)	1
RAYTHEON CO. (J. Bartram)	1
Dr. Julius Bendat, 833 Moraga Dr, Los Angeles, CA	1
COOLEY LABORATORY (Prof. T. Birdsall)	1
PROMETHEUS, INC. (Dr. J. S. Byrnes)	1
ROYAL MILITARY COLLEGE OF CANADA (Prof. Y. T. Chan)	1
UNIV. OF FLORIDA (D. C. Childers)	1
SANDIA NATIONAL LABORATORY (J. Claasen)	1
COGENT SYSTEMS, INC. (J. P. Costas)	1
IBM CORP. (G. Demuth)	1
UNIV. OF STRATHCLYDE, CLASGOW, SCOTLAND (Prof. T. Durrani)	1
ROCKWELL INTERNATIONAL CORP. (L. T. Einstein and Dr. D. F. Elliott)	2
GENERAL ELECTRIC CO. (Dr. M. Fitelson)	1
HONEYWELL, INC. (D. M. Goodfellow)	1
UNIV. OF TECHNOLOGY, LOUGHBOROUGH, LEICESTERSHIRE, ENGLAND (Prof. J. W. R. Griffiths)	1
HARRIS SCIENTIFIC SERVICES (B. Harris)	1
UNIV OF CALIFORNIA, SAN DIEGO (Prof. C. W. Helstrom)	1
EG&G (Dr. J. Hughen)	1

INITIAL DISTRIBUTION LIST (Cont'd)

Addressee	No. of Copies
BELL COMMUNICATIONS RESEARCH (J. F. Kaiser)	1
UNIV. OF RI (Prof. S. Kay, Prof. L. Scharf, Prof. D. Tufts)	3
DREXEL UNIV. (Prof. S. Kesler)	1
UNIV. OF CT (Prof. C. H. Knapp)	1
APPLIED SEISMIC GROUP (R. Lacoss)	1
ADMIRALTY RESEARCH ESTABLISHMENT, ENGLAND (Dr. L. J. Lloyd)	1
NAVAL SYSTEMS DIV., SIMRAD SUBSEA A/S, NORWAY (E. B. Lunde)	1
DEFENCE RESEARCH ESTAB. ATLANTIC, DARTMOUTH, NOVA SCOTIA CANADA (B. E. Mackey, Library)	1
MARTIN MARIETTA AEROSPACE (S. L. Marple)	1
PSI MARINE SCIENCES (Dr. R. Mellen)	1
Dr. D. Middleton, 127 E. 91st St. NY, NY 10128	1
Dr. P. Mikhalevsky, 803 W. Broad St. Falls Church, VA 22046	1
CANBERRA COLLEGE OF ADV. EDUC., AUSTRALIA 2616 (P. Morgan)	1
NORTHEASTERN UNIV., (Prof. C. L. Nikias)	1
ASTRON RESEARCH & ENGR, (Dr. A. G. Piersol)	1
WESTINGHOUSE ELEC. CORP. (Dr. H. L. Price)	1
M/A-COM GOVT SYSTEMS, (Dr. R. Price)	1
DALHOUSIE UNIV. HALIFAX, NOVA SCOTIA (Dr. B. Ruddick)	1
NATO SACLANT ASW RESEARCH CENTER, LIBRARY, APO NY, NY 09019	1
YALE UNIV. (Prof. P. M. Schultheiss)	1
NATIONAL RADIO ASTRONOMY OBSERVATORY (F. Schwab)	1
DEFENSE SYSTEMS, INC (Dr. G. S. Sebestyen)	1
NATO SACLANT ASW RESEARCH CENTRE (Dr. E. J. Sullivan)	1
PENN STATE UNIV, APPLIED RESEARCH LAB. (F. W. Symons)	1
NAVAL PG SCHOOL, (Prof. C. W. Therrien)	1
DEFENCE RESEARCH ESTABLISHMENT PACIFIC, VICTORIA, B.C. CANADA VOS 1BO (Dr. D. J. Thomson)	1
Robert J. Urick, 11701 Berwick Rd, Silver Spring, MD 20904	1
RCA CORP (H. Urkowitz)	1
USEA S.P.A. LA SPEZIA, ITALY (H. Van Asselt)	1
TEL-AVIV UNIV. ISRAEL (Prof. E. Weinstein)	1
COAST GUARD ACADEMY (Prof. J. J. Wolcin)	1
SPACE PHYSICS LAB, UNIV OF ALBERTA, EDMONTON, CANADA (K. L. Yeung)	1
COLUMBIA RESEARCH CORP. (W. Hahn)	1
SAIC, NLON (Dr. F. DiNapoli)	1