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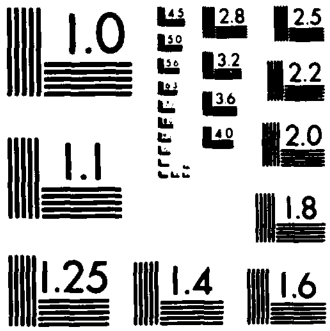

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Optimal Repeated Measurements Designs for Comparing Test Treatments with a Control

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ABSTRACT

A-optimal and MV-optimal repeated measurements designs are given both for direct and residual treatment effects, for comparing several test treatments with a control. The models considered are basically of two types: without preperiods and the circular model. It is shown that some known balanced and strongly balanced uniform repeated measurements designs can be modified to obtain optimal designs for this problem. Some other methods of finding optimal designs are also given.

1. INTRODUCTION

We consider the problem of finding optimal experimental designs for comparing t test treatments with a control, using n experimental units which are observed in each of p periods; the model is the homoscedastic, additive, repeated measurements linear model. Recently, this problem has been considered by Pigeon (1984) and Pigeon and Raghavarao (1987). They have defined classes of 'good' designs and investigated their efficiencies. Though they did not consider the problem of finding optimal designs, we shall show that some of their designs are indeed optimal.

The problem of determining optimal designs in repeated measurements models where all the elementary treatment contrasts are equally important, i.e., there is no special treatment like a control, has been investigated by several authors. Recent research includes

Hedayat and Afsarinejad (1975, 1978), Cheng and Wu (1980), Magda (1980), Kunert (1983, 1984a, 1984b, 1985), Mukhopadhyay and Saha (1983), Sen and Mukherjee (1987), Afsarinejad (1983, 1985) and Constantine and Hedayat (1982). Hedayat and Afsarinejad (1975) has an excellent bibliography of eariler literature.

The existing literature on optimal designs for comparing treatments with a control deal only with additive, homoscedastic models eliminating 0, 1, or 2 ways of heterogeneity, with no residual effects. Even though this literature does not have any optimal design for comparing treatments with a control in repeated measurements models, some methods which have been used there will prove to be valuable in the present context. One of the relevant methods is due to Notz (1985). He showed that if one starts with a latin square of order $t^2 + t$, using the symbols $1, 2, \ldots, t^2 + t$, and redefines each of the symbols $t^2 + 1, \ldots, t^2 + t$ as 0, which denotes the control, and calls the symbols $1, \ldots t^2$ the test treatments, then the resulting design is optimal for comparing t^2 test treatments with a control in additive models eliminating heterogeneity along $t^2 + t$ rows and $t^2 + t$ columns. It is interesting to note that the initial latin square was itself universally optimal (Kiefer, 1975) for comparing $t^2 + t$ treatments among themselves; all treatments being equally important in the criterion.

In section 3 of this paper we successfully use this type of a technique in the more complicated setup of repeated measurements models. More precisely, we show that, in these models, an optimal design for comparing several treatments with a control can sometimes be obtained by starting with a suitable optimal design for the traditional setup of comparing all treatments among themselves (with no special treatment like a control) and then redefining some treatment symbols to be the control. Many optimal designs to be given in section 3 will be of this type. It is interesting to note that Pigeon (1984) had suggested this as one of the methods of getting 'efficient' designs.

Our method of proof is different from that of Notz (1985). We shall essentially couple some orthogonality type conditions with optimality in simpler models to prove our results. This technique had been used by Kiefer (1975) for proving the optimality of regular generalized Youden designs, by Magda (1980), Kunert (1983, 1984a, 1984b) and Sen and Mukherjee (1986) for repeated measurements models and by Jacroux (1987b) and

Majumdar (1986) for comparing treatments with controls in additive classification models eliminating heterogeneity in two directions. In the repeated measurements setup, the most extensive use of this technique has been made by Kunert in the papers mentioned above.

We devote section 2 to preliminary notations, definitions and basic lemmas. The optimal designs for repeated measurements models are given in sections 3 and 4. Section 3 contains optimal designs which can be obtained by the technqiue due to Notz, while section 4 briefly outlines some other techniques. Our results are illustrated by examples.

2. PRELIMINARIES

We are interested in an experiment in which n individuals or units are to be observed in each of p periods upon application of one of t treatments in each period. An allocation of the treatments is called a repeated measurements design, to be denoted by RMD(t,n,p). The set of all RMD(t,n,p) will be denoted by $\Omega(t,n,p)$. If d is an RMD(t,n,p), then $d(i,j)$ will denote the treatment assigned to period i of unit j. The corresponding observation will be denoted by Y_{dij}.

Suppose t test treatments are to be compared with a control, which is a standard treatment, and a design d is selected from $\Omega(t+1,n,p)$. Then the model is

$$Y_{dij} = \mu + \alpha_i + \beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)} + \epsilon_{ij}. \tag{2.1}$$

Here $i = 1,\ldots,p, j = 1,\ldots,n$; $d(i,j) = 0,1,\ldots,t$ with 0 denoting the control and $1,\ldots,t$ denoting the test treatments. The usual assumptions regarding ϵ holds in this model, i.e., $E(\epsilon) = 0$, $V(\epsilon) = \sigma^2$, ϵ's are uncorrelated. The symbols μ stands for a general effect, α for the period effect, β for the unit effect, τ for the direct effect of the treatment and ρ for the residual effect of the treatment applied in the previous period. When $\rho_{d(o,j)} = 0$ for all j, then this the is *model without preperiods* (see Hedayat and Afsarinejad (1975, 1978), Cheng and Wu (1980)). On the other hand it is sometimes convenient to have a period preceeding period 1. The observations in this period are not used in the analysis; the treatments are applied principally to generate residual effects in period 1. Among models with preperiods, we shall consider only the *circular model* proposed by Magda (1980). Here $\rho_{d(o,j)} = \rho_{d(p,j)}$ for all j.

If Y_d denotes the vector of all np observations, then as in Kunert (1983), we may write

$$Y_d = 1_{np}\mu + P\alpha + U\beta + T_d\tau + F_d\rho + \epsilon \tag{2.2}$$

where 1_a denotes an $a \times 1$ vector of unities, $\alpha = (\alpha_1, \ldots, \alpha_p)'$, $\beta = (\beta_1, \ldots, \beta_n)'$, $\tau = (\tau_0, \ldots, \tau_t)'$, $\rho = (\rho_0, \ldots, \rho_t)'$, $\epsilon = (\epsilon_1, \ldots, \epsilon_{np})'$; P, U, T_d and F_d having obvious interpretations. Let us define,

n_{diu} = number of appearances of treatments i on unit u

$\tilde{n}_{diu}$ = number of appearances of treatment i on unit u in the first $p-1$ periods

ℓ_{dik} = number of appearances of treatment i in period k

m_{dij} = number of appearances of treatment i preceeded by treatment j in the same unit

r_{di} = number of appearances of treatment i

$\tilde{r}_{di}$ = number of appearances of treatment i in the first $p-1$ periods.

Before proceeding to find optimal designs for comparing treatments with a control, we have to define some designs which are optimal in the setup where all treatments are equally important, without any special treatment like a control. To do this, we briefly (for Definitions 2.1, 2.2 and 2.3 only) depart from our established notation and number the treatments in $\Omega(t,n,p)$ by $1, \ldots, t$; 0 is absent here since there is no control.

DEFINITION 2.1. *(Hedayat and Afsarinejad (1975)). An RMD(t,n,p) is a balanced uniform repeated measurements design, denoted by BURMD(t,n,p), if*

$$r_{d1} = \ldots = r_{dt} \tag{2.3}$$

$$n_{di1} = \ldots = n_{din} \text{ for each } i, \tag{2.4}$$

$$\ell_{di1} = \ldots = \ell_{dip} \text{ for each } i, \tag{2.5}$$

and

$$m_{dij} = m_{d12} \text{ for all } i, j, i \neq j.$$

DEFINITION 2.2. *(Cheng and Wu (1980)). An RMD(t,n,p) is a strongly balanced uniform repeated measurements design, denoted by SBURMD(t,n,p), if (2.3), (2.4), (2.5) hold and*

$$m_{dij} = m_{d11} \text{ for all } i, j.$$

DEFINITION 2.3. *(Magda (1980)). An RMD(t,n,p) is a circular strongly balanced uniform repeated mesurements design, denoted by CSBURMD(t,n,p), if (2.3), (2.4), (2.5) hold and*

$$e_{dij} = e_{d11}, \text{ for all } i, j,$$

where

$$e_{dij} = m_{dij} + f_{dij}$$

with

$$f_{dij} = \text{ number of units } u \text{ with } d(1,u) = i \text{ and } d(p,u) = j.$$

There are several results establishing the universal optimality for direct as well as residual treatments effects of these RMD's. Hedayat and Afsarinejad (1978) proved that a BURMD is universally optimal in the class of all designs with the qualification that no treatment appears in consecutive periods. Cheng and Wu (1980) proved that a design obtained by repeating the observations in the last period of a certain BURMD is universally optimal. Cheng and Wu (1980) also proved the universal optimality of an SBURMD. All these results are for a model without preperiods. On the other hand, Magda (1980) proved the universal optimality of a CSBURMD in a circular model. There are many more results on optimality of RMD's in these and the other papers mentioned in the introduction.

In case all the treatments are equally important, the parametric functions of interest are usually an orthonormal basis of the treatment contrasts. On the other hand, in the case where several test treatments $(1, \ldots, t)$ are being compared with a control (o), the parametric functions of direct treatment effects which are of interest are

$(\tau_0 - \tau_1), \ldots, (\tau_0 - \tau_t)$. These are estimated by their BLUE's $(\hat{\tau}_{d0} - \hat{\tau}_{d1}), \ldots, (\hat{\tau}_{d0} - \hat{\tau}_{dt})$, under the design d. The problem here is to choose a design for which these estimates are "most accurate". Sometimes the object of an experiment may be to estimate the corresponding residual treatment contrasts, $(\rho_0 - \rho_1), \ldots, (\rho_0 - \rho_t)$, which are estimated by $(\hat{\rho}_{d0} - \hat{\rho}_{d1}), \ldots, (\hat{\rho}_{d0} - \hat{\rho}_{dt})$. We suspect that in most experiments, these residual treatment contrasts have less importance than the direct treatment contrasts.

Not all individual criteria belonging to the class which forms universal optimality may be statistically meaningful in the context of comparing test treatments with a control. There are, however, two criteria, A-optimality and MV-optimality, which do possess natural statistical interpretations, and have received considerable attention in the case of optimal block designs and optimal row-column designs. These are given in Definitions 2.4 and 2.5.

DEFINITION 2.4. *(i) $d_0 \in \Omega(t+1, n, p)$ is A-optimal for the direct treatment effects if*

$$\sum_{i=1}^{t} \mathrm{Var}(\hat{\tau}_{d_0 0} - \hat{\tau}_{d_0 i}) \leq \sum_{i=1}^{t} \mathrm{Var}(\hat{\tau}_{d0} - \hat{\tau}_{di}) \tag{2.6}$$

for all $d \in \Omega(t+1, n, p)$.

(ii) $d_0 \in \Omega(t+1, n, p)$ is A-optimal for the residual effects of the treatments if (2.6) hold for all $d \in \Omega(t+1, n, p)$ with $\hat{\tau}$'s replaced by $\hat{\rho}$'s.

DEFINITION 2.5. *(i) $d_0 \in \Omega(t+1, n, p)$ is MV-optimal for the direct treatment effects if*

$$\mathrm{Max}\{\mathrm{Var}(\hat{\tau}_{d_0 0} - \hat{\tau}_{d_0 i}) : i = 1, \ldots, t\} \leq \mathrm{Max}\{\mathrm{Var}(\hat{\tau}_{d0} - \hat{\tau}_{di} : i = 1, \ldots, t\} \tag{2.7}$$

for all $d \in \Omega(t+1, n, p)$.

(ii) $d_0 \in \Omega(t+1, n, p)$ is MV-optimal for the residual effects of the treatments if (2.7) holds for all $d \in \Omega(t+1, n, p)$ with $\hat{\tau}$'s replaced by $\hat{\rho}$'s.

The purpose of this paper is to find A- and MV-optimal designs. We shall start with two basic lemmas. Consider a class of designs $\mathcal{D}$ and two Gauss-Markov models for a $d \in \mathcal{D}$;

$$E(Y_d) = X_{1d}\theta_1 + X_{2d}\theta_2 + X_{3d}\theta_3 \quad V(Y_d) = \sigma^2 I \tag{2.8}$$

$$E(Y_d) = X_{1d}\theta_1 + X_{2d}\theta_2 \quad V(Y_d) = \sigma^2 I \tag{2.9}$$

The model (2.8) has been called 'finer' than (2.9) by Kunert (1983).

Our parameter vector of interest is θ_1; more precisely, we would like to estimate the vector $Q\theta_1$ where Q is a matrix. The vectors θ_2, θ_3 and the scalar σ^2 are nuisance parameters. Let

$$A_d = X'_{1d}(I - X_{2d}(X'_{2d}X_{2d})^{-}X'_{2d})X_{1d}$$

and

$$C_d = X'_{1d}(I - Z_d(Z'_dZ_d)^{-}Z'_d)X_{1d}$$

where

$$Z_d = (X_{2d} \ : \ X_{3d}).$$

Suppose ψ is a real valued function defined on nonnegative definite matrices. Then $d_0 \in \mathcal{D}$ is ψ-optimal for estimating $Q\theta_1$ under (2.8) if

$$\psi(QC_{d_0}^{-}Q') \leq \psi(QC_d^{-}Q') \text{ for all } d \in \mathcal{D},$$

while it is ψ-optimal for estimating $Q\theta_1$ under (2.9) if

$$\psi(QA_{d_0}^{-}Q') \leq \psi(QA_d^{-}Q') \text{ for all } d \in \mathcal{D}.$$

To avoid trivialities, we shall assume that $Q\theta_1$ is estimable for each $d \in \mathcal{D}$ under (2.8). Hence $Q\theta_1$ is estimable under (2.9) also. We also assume that

$$C_d 1 = A_d 1 = 0 \text{ for all } d \in \mathcal{D},$$

where 1 is a vector of unities. Consequently $Q1 = 0$.

The question is: when is a design which is ψ-optimal under (2.9), ψ-optimal under (2.8) also? An answer is given in Lemma 2.1.

LEMMA 2.1. *Let ψ have the property that $\psi(A - B) \leq \psi(A)$ whenever A, B and $A - B$ are nonnegative definite. Suppose $d_0 \in \mathcal{D}$ satisfies*

$$X'_{1d_0} X_{3d_0} = X'_{1d_0} X_{2d_0} (X'_{2d_0} X_{2d_0})^{-} X'_{2d_0} X_{3d_0} \tag{2.10}$$

and

$$\psi(Q A^{-}_{d_0} Q') \leq \psi(Q A^{-}_{d} Q') \text{ for all } d \in \mathcal{D} \tag{2.11}$$

then d_0 is ψ-optimal for estimating $Q\theta_1$ under (2.8).

PROOF: By Lemma 3.1 of Magda (1980) (also reproduced in Proposition 2.3 of Kunert (1983)), (2.10) implies

$$C_{d_0} = A_{d_0}.$$

Hence

$$\begin{aligned} \psi(Q C^{-}_{d_0} Q') &= \psi(Q A^{-}_{d_0} Q') \\ &\leq \psi(Q A^{-}_{d} Q') \text{ by (2.11)} \\ &\leq \psi(Q C^{-}_{d} Q') \text{ for all } d \in \mathcal{D} \end{aligned}$$

since there exists g-inverses such that $C^{-}_{d} - A^{-}_{d}$ is nonnegative definite, because $A_d - C_d$ is nonnegative definite (cf. Wu (1980)). Hence the lemma.

For the special case when $X_{2d} = 0$ for all $d \in \mathcal{D}$ (that is, θ_2 is absent in (2.8) and (2.9)), and $X'_{1d} X_{1d}$ is nonsingular for all $d \in \mathcal{D}$, a somewhat stronger result is available in Theorem 3.1 of Majumdar (1986). We reproduce this as Lemma 2.2, for the sake of completeness.

LEMMA 2.2. *Let ψ have the property that $\psi(A-B) \leq \psi(A)$ whenever A, B and $A-B$ are nonnegative definite matrices. Suppose $d_0 \in \mathcal{D}$ satisfies*

$$X'_{3d_0} X_{1d_0} (X'_{1d_0} X_{1d_0})^{-1} Q' = 0 \tag{2.12}$$

and

$$\psi(Q(X'_{1d_0} X_{1d_0})^{-1} Q') \leq \psi(Q(X'_{1d} X_{1d})^{-1} Q'), \text{ for all } d \in \mathcal{D}. \tag{2.13}$$

Then d_0 is ψ-optimal for estimating $Q\theta_1$ under (2.8).

Examples of functions ψ which satisfy the conditions of Lemmas 2.1 and 2.2 are: ψ = trace, giving the A-optimality criterion and ψ = maximum diagonal element, giving the MV-optimality criterion.

3. OPTIMAL DESIGNS

In this section we first apply Lemma 2.2 to the repeated measurements model (2.1) without preperiods. Here

$$\mathcal{D} = \Omega(t+1, n, p)$$

$$Q = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & -1 \end{pmatrix}$$

a $t \times t+1$ matrix; and equating models (2.2) and (2.8) we substitute,

$$X_{1d} = T_d, \ X_{3d} = [1_{np} \ : \ P \ : \ U \ : \ F_d].$$

Recall that in our notation $X_{2d} = 0$ for all $d \in \mathcal{D}$. To find A-optimal designs, we choose ψ = trace. Hence, condition (2.13) is satisfied if

$$r_{d_0 1} = \dots = r_{d_0 t}, \ r_{d_0 0} = r_{d_0 1} \sqrt{t}. \tag{3.1}$$

Condition (2.14) tells us that if we denote

$$H = (\mathrm{Diag}(r_{d_0 0}^{-1}), r_{d_0 1}^{-1}, \ldots, r_{d_0 t}^{-1})Q',$$

then

$$1'_{np}T_{d_0}H = 0,\ P'T_{d_0}H = 0,\ U'T_{d_0}H = 0,\ F'_{d_0}T_{d_0}H = 0.$$

These conditions lead us to the following relations:

$$\ell_{d_0 1k} = \ldots = \ell_{d_0 tk}, \ell_{d_0 0k} = \ell_{d_0 1k}\sqrt{t} \text{ for } k = 1, \ldots, p \tag{3.2}$$

$$n_{d_0 1u} = \ldots = n_{d_0 tu}, n_{d_0 0u} = n_{d_0 1u}\sqrt{t} \text{ for } u = 1, \ldots, n \tag{3.3}$$

$$m_{d_0 10} = \ldots = m_{d_0 t0}, m_{d_0 00} = m_{d_0 10}\sqrt{t} \tag{3.4}$$

$$m_{d_0 1i} = \ldots = m_{d_0 ti}, m_{d_0 0i} = m_{d_0 1i}\sqrt{t} \text{ for } i = 1, \ldots, t. \tag{3.5}$$

This shows that any RMD(t+1,n,p) satisfying relations (3.1) - (3.5) is A-optimal for direct treatment effects in $\Omega(t+1, n, p)$ for a model without preperiods. These designs are MV-optimal also, since (3.1) satisfies (2.13) with ψ = maximum diagonal element. One way to obtain these optimal designs is given in Theorem 3.1 which can be established by verifying conditions (3.1) - (3.5).

THEOREM 3.1. *Let d^* be a strongly balanced uniform repeated measurements design SBURMD $(w^2 + w, n, p)$, where w is a positive integer. Let d_0 be obtained from d^* by changing each of the treatment symbols $w^2 + 1, \ldots, w^2 + w$ to the symbol 0 which denotes the control, and keeping everything else unchanged. Then d_0 is A- and MV-optimal for direct treatment effects for comparing w^2 test treatments with a control in $\Omega(w^2 + 1, n, p)$ for a model without preperiods.*

Example 3.1. Methods for constructing SBURMD's have been given in Cheng and Wu (1980). Using these, we give an A- and MV-optimal design for direct treatment effects in $\Omega(5, 36, 12)$ in Table 1. Here $w = 2, n = 36, p = 12$; 0 denotes the control and $1, 2, 3, 4$ the test treatments.

Table 1: Showing the optimal design of Example 3.1

Periods

	1	1	2	2	3	3	4	4	0	0	0	0
	1	2	2	3	3	4	4	0	0	0	0	1
	1	3	2	4	3	0	4	0	0	1	0	2
	1	4	2	0	3	0	4	1	0	2	0	3
	1	0	2	0	3	1	4	2	0	3	0	4
	1	0	2	1	3	2	4	3	0	4	0	0
	2	1	3	2	4	3	0	4	0	0	1	0
	2	2	3	3	4	4	0	0	0	0	1	1
	2	3	3	4	4	0	0	0	0	1	1	2
	2	4	3	0	4	0	0	1	0	2	1	3
	2	0	3	0	4	1	0	2	0	3	1	4
	2	0	3	1	4	2	0	3	0	4	1	0
	3	1	4	2	0	3	0	4	1	0	2	0
	3	2	4	3	0	4	0	0	1	0	2	1
Units	3	3	4	4	0	0	0	0	1	1	2	2
	3	4	4	0	0	0	0	1	1	2	2	3
	3	0	4	0	0	1	0	2	1	3	2	4
	3	0	4	1	0	2	0	3	1	4	2	0
	4	1	0	2	0	3	1	4	2	0	3	0
	4	2	0	3	0	4	1	0	2	0	3	1
	4	3	0	4	0	0	1	0	2	1	3	2
	4	4	0	0	0	0	1	1	2	2	3	3
	4	0	0	0	0	1	1	2	2	3	3	4
	4	0	0	1	0	2	1	3	2	4	3	0
	0	1	0	2	1	3	2	4	3	0	4	0
	0	2	0	3	1	4	2	0	3	0	4	1
	0	3	0	4	1	0	2	0	3	1	4	2
	0	4	0	0	1	0	2	1	3	2	4	3
	0	0	0	0	1	1	2	2	3	3	4	4
	0	0	0	1	1	2	2	3	3	4	4	0
	0	1	1	2	2	3	3	4	4	0	0	0
	0	2	1	3	2	4	3	0	4	0	0	1
	0	3	1	4	2	0	3	0	4	1	0	2
	0	4	1	0	2	0	3	1	4	2	0	3
	0	0	1	0	2	1	3	2	4	3	0	4
	0	0	1	1	2	2	3	3	4	4	0	0

Let us now turn our attention to the estimation of residual treatment effects in models without preperiods. Let us substitute in Lemma 2.2,

$$X_{1d} = F_d,\ X_{3d} = [1_{np}\ :\ P\ :\ U\ :\ T_d].$$

Condition (2.13) is satisfied for both A- and MV-optimality if

$$\tilde{r}_{d_0 1} = \ldots = \tilde{r}_{d_0 t}, \quad \tilde{r}_{d_0 0} = \tilde{r}_{d_0 1}\sqrt{t} \tag{3.6}$$

Then condition (2.12) gives us the following relations

$$\ell_{d_0 1k} = \ldots = \ell_{d_0 tk}, \ell_{d_0 0k} = \ell_{d_0 1k}\sqrt{t} \text{ for } k = 1, \ldots, p-1 \tag{3.7}$$

$$\tilde{n}_{d_0 iu} = \ldots = \tilde{n}_{d_0 tu}, \tilde{n}_{d_0 0u} = \tilde{n}_{d_0 1u}\sqrt{t} \text{ for } u = 1, \ldots, n \tag{3.8}$$

$$m_{d_0 01} = \ldots = m_{d_0 0t}, m_{d_0 00} = m_{d_0 01}\sqrt{t} \tag{3.9}$$

$$m_{d_0 i1} = \ldots = m_{d_0 it}, m_{d_0 i0} = m_{d_0 i1}\sqrt{t},\ i = 1, \ldots, t \tag{3.10}$$

Hence the conditions (3.6) - (3.10) are sufficient for a design d_0 to be A- and MV-optimal for residual effects in $\Omega(t+1, n, p)$. The design d_0 of Theorem 3.1 does not satisfy (3.8). A family of optimal designs which do satisfy these conditions are given in Theorem 3.2.

THEOREM 3.2. *Let d^* be obtained by repeating the observations in the last period of a balanced uniform repeated measurements design BURMD($w^2 + w, \lambda(w^2 + w), w^2 + w$), where λ and w are two arbitrary positive integers. Let d_0 be obtained from d^* by changing each of the treatment symbols $w^2 + 1, \ldots, w^2 + w$ to the symbol o which denotes the control and keeping everything else unchanged. Then d_0 is A- and MV-optimal for residual treatment effects for comparing w^2 test treatments with a control in $\Omega(w^2 + 1, \lambda(w^2 + w), w^2 + w + 1)$ for a model with preperiods.*

To verify conditions (3.6) - (3.10) it helps to realize that in the original BURMD($w^2 + w, \lambda(w^2 + w), w^2 + w$) each treatment occurs λ times in each period and each treatment is preceeded by every other treatment λ times (see Hedayat and Afsarinejad (1975, p. 231). Cheng and Wu (1980,Corollary 3.3.1) established optimality properties of a BURMD augmented by it last period, for comparing all treatments among themselves, no treatment being a control.

Example 2.3. Starting with the BURMD(6,6,6) from Hedayat and Afsarinejad (1975, p. 232) we give an A- and MV-optimal design for residual treatment effects in $\Omega(5, 6, 7)$. Here the control is denoted by 0 and the 4 test treatments by 1, 2, 3 and 4; $w = 2$ and $\lambda = 1$.

	Units					
	1	2	3	4	0	0
	0	1	2	3	4	0
	2	3	4	0	0	1
Periods	0	0	1	2	3	4
	3	4	0	0	1	2
	4	0	0	1	2	3
	4	0	0	1	2	3

Finally, let us consider a repeated measurements model which is circular. For direct treatment effects we again start with the substitution $X_{1d} = T_d$ and $X_{3d} = [1_{np} : P : U : F_d]$. It is clear that a sufficient condition for d_0 to be A- and MV-optimal is that the conditions (3.1) - (3.5) are satisfied, with the exception that m_{dij} is replaced by e_{dij}. The symbol e_{dij} has been defined in Definition 2.3 - it is the number of times treatment i follows treatment j, counting in a circular fashion. On the other hand, we substitute $X_{1d} = F_d$ and $X_{3d} = [1_{np} : P : U : T_d]$ to get optimal designs for residual treatment effects. The sufficient conditions for the optimality of d_0 are (3.1), (3.2), (3.3) and (3.9), (3.10) with m_{dij}'s replaced by e_{dij}'s. Families of optimal designs are given in the following theorem, which is easily established by verifying the sufficient conditions mentioned above.

THEOREM 3.3. *Let d^* be a circular strongly balanced uniform repeated measurements design CSBURMD($w^2 + w, n, p$), where w is a positive integer. Let d_0 be obtained from*

d^* by changing each of the treatment symbols $w^2+1,\ldots,w^2+w$ to the symbol o which denotes the control, and keeping everything else unchanged. Then d_0 is A- and MV-optimal for direct as well as residual treatment effects for comparing w^2 test treatments with a control in $\Omega(w^2+1,n,p)$ for a circular model.

Example 3.3. Using the methods for constructing CSBURMD's given by Sen and Mukherjee (1985), we give an A- and MV-optimal design in $\Omega(5,6,12)$, under the circular model for direct as well as residual treatment effects. Here $w=2$; 0 denotes the control, while 1,2,3,4 denote the 4 test treatments

	Periods											
	1	0	2	0	3	4	4	3	0	2	0	1
	2	1	3	0	4	0	0	4	0	3	1	2
Units	3	2	4	1	0	0	0	0	1	4	2	3
	4	3	0	2	0	1	1	0	2	0	3	4
	0	4	0	3	1	2	2	1	3	0	4	0
	0	0	1	4	2	3	3	2	4	1	0	0

4. SOME MORE OPTIMAL DESIGNS

In section 3 we applied Lemma 2.2 to obtain optimal designs. In this section we shall apply Lemma 2.1. The optimal designs in this section cannot be obtained by Notz-type methods, in general.

The first step consists in equating models (2.2) and (2.8). This means that we have to partition the set of matrices

$$\{1_{np}, P, U, T_d, F_d\}$$

as

$$\{X_{1d}, X_{2d}, X_{3d}\}.$$

There are many possible choices, depending on how we partition. We shall consider only one partition. This application will demonstrate the use of Lemma 2.1.

Let,

$$X_{1d} = T_d, X_{2d} = P \text{ and } X_{3d} = [1_{np} \ : U \ : \ F_d].$$

Thus we are looking for A- and MV-optimal designs for direct treatment effects. We shall consider a circular model for the repeated measurements.

For a $d_0 \in \Omega(t+1, n, p)$, condition (2.10) gives us

$$n_{d_0 i1} = \ldots = n_{d_0 in} \text{ for each } i = 0, 1, \ldots, t \tag{4.1}$$

and

$$e_{d_0 ij} = n^{-1}[\ell_{d_0 i1}\ell_{d_0 jp} + \ell_{d_0 i2}\ell_{d_0 j1} + \ldots + \ell_{d_0 ip}\ell_{d_0 jp-1}]$$
$$\text{for each } i = 0, 1, \ldots, t \text{ and } j = 0, 1, \ldots, t, \tag{4.2}$$

where the notation e was defined in Definition 2.3.

For ψ = trace or ψ = maximum diagonal element, condition (2.11) says that d_0 is A- or MV-optimal for comparing test treament with a control in the model

$$E(y_{ik}) = \tau_i + \rho_k, \quad V(y_{ik}) = \sigma^2, \ y_{ik}\text{'s uncorrelated.}$$

This is an additive, homoscedastic model with treatment and block effects, the blocks being the periods. Many results giving optimal block designs for comparing test treatments with a control are available in the literature. To get started in this area the reader may look at Bechhofer and Tamhane (1981), Majumdar and Notz (1983), Giovagnoli and Wynn (1985), Constantine(1983), Hedayat and Majumdar (1985), and Jacroux (1987a). This is a list of only a few, not all, important papers. It is interesting to note that conditions (2.11) and (4.1) show that d_0 is an optimal design for comparing test treatments with a control in an additive model eliminating 2 ways of heterogeneity along periods and units (see Jacroux (1987b, Theorem 3.4)).

Example 4.1. Let us give an example to illustrate this method. We consider the class $\Omega(4, 2, 432)$. There are 3 test treatments (labelled 1, 2 and 3) and a control (labelled o). To display the design as an array let us define a 2×24 matrix using the symbols a, b, c, d:

$$B(a,b,c,d) = \begin{pmatrix} a & b & b & c & d & d & b & a & a & c & d & d & a & b & b & d & c & c & b & a & a & d & c & c \\ b & a & a & d & c & c & a & b & b & d & c & c & b & a & a & c & d & d & a & b & b & c & d & d \end{pmatrix}$$

Now consider the 2×432 array given as

$$A = (D,\ E,\ F)$$

where

$$D = (B(1,0,2,0), B(1,0,2,0), B(1,0,2,0), B(1,0,3,0), B(1,0,3,0), B(1,0,3,0))$$
$$E = (B(2,0,3,0), B(2,0,3,0), B(2,0,3,0), B(1,0,1,2), B(1,0,1,3), B(1,0,2,3))$$
$$F = (B(2,0,1,2), B(2,0,1,3), B(2,0,2,3), B(3,0,1,2), B(3,0,1,3), B(3,0,2,3)).$$

The array A, with rows denoting units and columns denoting periods gives an A- and MV-optimal design for direct treatment effects for comparing 3 test treatments with a control in $\Omega(4,2,432)$ for a circular model. To prove the optimality of A, first observe that condition (2.11) for this setup follows from Theorem 3.1 of Hedayat and Majumdar (1984). Finally, the conditions (4.1) and (4.2) can be established directly. It can be seen that for this design d_0, the matrix

$$((e_{d_0 ij})) = \begin{pmatrix} 144 & 60 & 60 & 60 \\ 60 & 68 & 26 & 26 \\ 60 & 26 & 68 & 26 \\ 60 & 26 & 26 & 68 \end{pmatrix}$$

where the rows and columns are ordered 0,1,2,3.

An optimal design given by conditions (2.11), (4.1) and (4.2) enjoy the model robustness property that they remain optimal even when the unit effects and/or the residual effects are zero. The designs in section 3 also have similar properties. For example, the optimal design d_0 of Theorem 3.1 remains optimal when some or all of the following hold:

$$\alpha_1 = \ldots = \alpha_p,\ \beta_1 = \ldots = \beta_n \text{ and } \rho_0 = \ldots = \rho_t.$$

In conclusion we observe that this is the first paper which explicitly gives optimal designs for comparing test treatments with a control. More research needs to be done to investigate classes $\Omega(t,n,p)$ not covered by this paper. In particular, attention needs to be directed at classes where the number of periods is small, since this is very useful in practice. Not many results are known for this situation, even in the case where there are no special treatments like the control, and all treatments are equally important.

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