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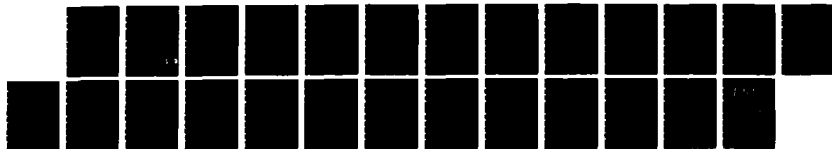
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HILL CENTER FOR STOCHASTIC PROC R L KARANDIKAR
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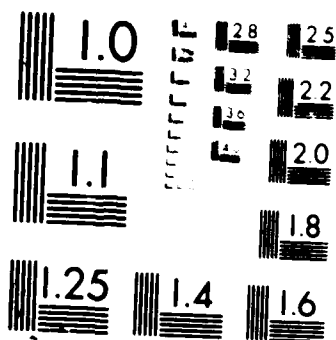
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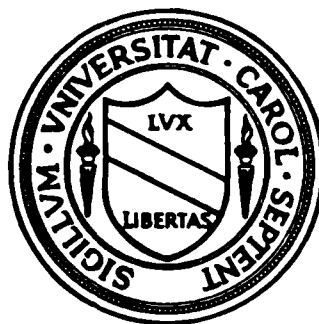
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ON THE FEYNMAN-KAC's FORMULA AND ITS APPLICATIONS TO FILTERING THEORY

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ON THE FEYNMAN-KAC's. FORMULA AND ITS APPLICATIONS TO FILTERING THEORY

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ON THE FEYNMAN-KAC'S. FORMULA AND ITS APPLICATIONS TO FILTERING THEORY

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1. Introduction : Let (X_t) be a Markov process, not assumed to be time homogeneous. It is well known that $\hat{X}_t = (t, X_t)$ is a time homogeneous Markov process. Let A be its generator. The Feynman-Kac's formula for X_t takes the following form if the equation

$$(1.1) \quad Av + cv = 0$$

admits a solution v , then v has the representation, for $s < t$

$$(1.2) \quad v(s, X_s) = E \left[v(t, X_t) \exp \left(\int_s^t c(u, X_u) du \right) \middle| \sigma(X_s) \right].$$

We prove this under general conditions on (X_t) .

Then we come to the question of existence of solution to (1.1). We show that under some regularity conditions on (X_t) , (1.1) has a solution for a rich class of boundary conditions. This implies that the 'dual' equation to (1.1) admits a unique solution. The 'dual' equation is an equation for measures on the state space of (X_t) and its unique solution is the distribution of X_t under an absolutely continuous change of the underlying probability measure by a multiplicative functional.

These results on the measure valued equations significantly extend results given in [3] on the conditional distributions for the nonlinear filtering problem (in the white noise approach).

2. Let $(S, \underline{S})$ be a measurable space. Let (X_t) be an $(S, \underline{S})$ valued Markov process on a probability space $(\Omega, \underline{A}, \pi)$ with transition probability function P , i.e.

$$\{\omega : X_t(\omega) \in B\} \in \underline{A}$$

and

$$(2.1) \quad E_{\pi} \left[1_B(X_t) | \underline{F}_s^X \right] = P(s, X_s, t-s, B) \quad \text{a.s. } \pi$$

for all $0 \leq s \leq t < \infty$, $B \in \underline{S}$. Here, the function $P(s, x, t, B)$ on $\{0 \leq s < \infty, t \geq 0, x \in S, B \in \underline{S}\}$ is assumed to satisfy the following conditions.

(2.2) For $s \geq 0, t \geq 0, x \in S$; $P(s, x, t, \cdot)$ is a countably additive probability measure on $(S, \underline{S})$.

(2.3) For $s \geq 0, x \in S, B \in \underline{S}$; $P(s, x, 0, B) = 1_B(x)$.

(2.4) For $t \geq 0, B \in \underline{S}$; $(s, x) \rightarrow P(s, x, t, B)$ is a $B([0, \infty)) \otimes \underline{S}$ measurable function ($\underline{B}(E)$ denotes the Borel σ -field of a topological space E and $\otimes$ denotes the product of σ -fields).

(2.5) For $s \geq 0, u \geq 0, t \geq 0, x \in S, B \in \underline{S}$; we have

$$\int_S P(s+t, z, u, B) P(s, x, t, dz) = P(s, x, t+u, B)$$

Throughout, $\underline{F}_t^X$ denotes the smallest σ -field with respect to which the family $\{X_u : 0 \leq u \leq t\}$ is measurable. We also assume that

(2.6) the process (X_t) is $\underline{F}_t^X$ -progressively measurable, i.e. for all $t_0 < \infty$, the mapping $(t, \omega) \rightarrow X_t(\omega)$ from $[0, t_0] \times \Omega \rightarrow S$ is $\underline{B}([0, t_0]) \otimes \underline{F}_{t_0}^X$ measurable.

: 3 :

Let $\hat{S} = [0, \infty) \times S$, $\hat{S} = \underline{B}([0, \infty)) \otimes \underline{S}$ and $\underline{J}$ be the class of bounded real valued $\hat{S}$ measurable functions on $\hat{S}$.

Definition : A sequence $\{f_k\} \subseteq \underline{J}$ is said to converge weakly to $f \in \underline{J}$, written as $w\text{-}\lim_{k \rightarrow \infty} f_k = f$, if $f_k(x)$ is uniformly bounded and for each $x \in S$, $f_k(x)$ converges to $f(x)$.

For $f \in \underline{J}$, $t \geq 0$, let $T_t f : S \rightarrow \mathbb{R}$ be defined by

$$(2.7) \quad (T_t f)(s, x) = \int f(s+t, z) P(s, x, t, dz), \quad (s, x) \in \hat{S}.$$

Using the properties of P , it can be checked that $T_t f \in \underline{J}$ and that for $u \geq 0$, $t \geq 0$,

$$(2.8) \quad T_u [T_t f] = T_{t+u} f, \quad f \in \underline{J}.$$

Thus $\{T_t : t \geq 0\}$ is a semigroup of operators (from $\underline{J}$ into itself).

Remark : It is well known and easy to check that $\hat{X}_t = (t, X_t)$ is a Markov process with stationary transition probability function $\hat{P}$ given by

$$\hat{P}(t, (s, x), B) = P(s, x, t, B^{(s+t)}) , \quad B \in \hat{\mathcal{S}}$$

where B^u denotes the u -section of $B \subseteq S$. The semigroup $\{T_t : t \geq 0\}$ defined above is the usual semigroup associated with the transition function P (as in [2], section 2.1).

We will now recall the definition and some properties of the weak generator A of $\{T_t : t \geq 0\}$. Let $\underline{J}_0$ be given by

$$\underline{J}_0 = \{f \in \underline{J} ; \lim_{t \downarrow 0} T_t f = f\}$$

Definition : Let $\underline{D}_A$ be the class of $f \in \underline{J}$ for which the

$$(2.9) \quad \lim_{t \downarrow 0} \frac{T_t f - f}{t} = g$$

exists and belongs to $\underline{J}_0$ and for $f \in \underline{D}_A$, define $Af = g$, where g is given by (2.9).

The following properties are easy to prove. We will only state them here. For a proof see chapter 1 in [2].

$$(2.10) \quad T_t(\underline{D}_A) \subseteq \underline{D}_A \quad \text{and for } f \in \underline{D}_A, \quad A(T_t f) = T_t Af$$

$$(2.11) \quad \text{For } f \in \underline{J}_0, \quad t \mapsto (T_t f)(s, x) \text{ is a right continuous function for all } (s, x) \in \hat{S}.$$

$$(2.12) \quad \text{For } f \in \underline{D}_A, \text{ we have, for all } (s, x) \in \hat{S}, \quad t \geq 0$$

$$(T_t f)(s, x) = f(s, x) + \int_0^t (T_u Af)(s, x) du$$

(2.13) Given $f \in \underline{J}_0$ there exists a sequence $\{f_k\} \subseteq \underline{D}_A$ such that

$$\lim_{k \rightarrow \infty} f_k = f.$$

In (2.13) above, f_k can be taken to be

$$f_k(s, x) = \int_0^\infty k e^{-kt} (T_t f)(s, x) dx.$$

The property (2.12) has the following important consequence.

Proposition 1 : For $f \in \underline{D}_A$, M_t given by

$$(2.14) \quad M_t(\omega) = f(t, X_t(\omega)) - \int_0^t (Af)(u, X_u(\omega)) du$$

is a martingale with respect to the σ -fields $\underline{F}_t^X$.

Proof : The progressive measurability of (X_t) implies the $\underline{F}_t^X$ measurability of M_t . Since $f, Af \in \underline{J}$, they are bounded and hence M_t is itself bounded for each t . Now (2.1) implies

$$(2.15) \quad \begin{aligned} E_\pi [f(t, X_t) | \underline{F}_s^X] &= \int f(t, z) P(s, X_s, t-s, dz) \\ &= (T_{t-s} f)(s, X_s) \end{aligned}$$

for $s \leq t$. Similarly for $s \leq u$, we have

$$(2.16) \quad E_\pi [(Af)(u, X_u) | \underline{F}_s^X] = (T_{u-s} Af)(s, X_s).$$

Using (2.11), (2.12), (2.15) and (2.16), it can be checked that

$$E_\pi [M_t - M_s | \underline{F}_s^X] = 0.$$

□

We now turn our attention to the Feynman-Kac's formula. Our next result is a step in this direction.

Let $g : [0, t_0] \times S \rightarrow \mathbb{R}$ be a $\mathbb{B}([0, t_0]) \times \mathcal{S}$ measurable function such that

$$(2.17) \quad E_{\pi} \left[\int_0^{t_0} |g(u, X_u)| du \right] < \infty$$

and for a positive integrable function $a : [0, t_0] \rightarrow \mathbb{R}$,

$$(2.18) \quad g(u, x) \leq a(u) \quad \text{for all } x \in S, u \in [0, t_0].$$

Fix $0 \leq s \leq t_0$ and let

$$(2.19) \quad B_t(\omega) = \exp\left(\int_s^t g(u, X_u(\omega)) du\right).$$

Then we have

Theorem 2 : Let $f \in \underline{D}_A$ and g satisfy (2.17), (2.18). Then

$$(2.20) \quad Z_t = f(t, X_t) \cdot B_t - \int_s^t [(Af)(u, X_u) + g(u, X_u)] \cdot B_u du$$

is an $\mathbb{F}_t^X$ martingale for $t \geq s$ (where B is given by (2.19)).

Proof : It is easy to see that Z_t is $\mathbb{F}_t^X$ measurable. The condition (2.18) implies that B_t is bounded. Since f, Af are also bounded the condition (2.17) gives the integrability of Z_t . To prove the martingale property, suffices to prove that for $s \leq r \leq t$, $C \in \mathbb{F}_r^X$,

$$(2.20) \quad E_{\pi} [(Z_t - Z_r) \cdot 1_C] = 0.$$

Let $f_1(t, \omega) = f(t_0, X_{t_0}(\omega)) - \int_t^{t_0} (Af)(u, X_u(\omega)) du$. Then by Proposition 1, it follows that for $0 \leq t \leq t_0$

: 7 :

$$(2.21) \quad E_{\pi} \left[f_1(t, \cdot) | \mathcal{F}_t^X \right] = f(t, X_t)$$

and hence

$$(2.22) \quad E_{\pi} \left[1_C \cdot (Z_t - Z_r) \right] = E_{\pi} 1_C \cdot \{ f_1(t, \cdot) B_t - f_1(r, \cdot) B_r - \int_r^t \{ (Af + gf)(u, X_u) du \} \}.$$

Now for each ω , $f_1(t, \omega)$, $B_t(\omega)$ are absolutely continuous functions and hence

$$\begin{aligned} f_1(t, \omega) B_t(\omega) - f_1(r, \omega) B_r(\omega) &= \int_r^t \frac{d}{du} [f_1(u, \omega) B_u(\omega)] du \\ &= \int_r^t \{ f_1(u, \omega) \cdot g(u, X_u(\omega)) B_u(\omega) \\ &\quad + (Af)(u, X_u(\omega)) \cdot B_u(\omega) \} du. \end{aligned}$$

Thus

$$\begin{aligned} (2.23) \quad E_{\pi} [f_1(t, \cdot) B_t - f_1(r, \cdot) B_r] &= E 1_C \int_r^t f_1(u, \cdot) g(u, X_u) B_u du \\ &\quad + E 1_C \int_r^t (Af)(u, \cdot) B_u du \\ &= E 1_C \int_r^t (Af + gf)(u, X_u) du \end{aligned}$$

using (2.21) once again. Now (2.22) and (2.23) give the required equality

$$E [1_C (Z_t - Z_r)] = 0.$$

Remark : It can be verified that

$$Z_t = M_t B_t - \int_s^t M_u dB_u$$

where M is given by (2.14). Hence if M were right continuous,

it would follow from the "integration by parts formula for martingale"

(See [5]) that $(Z_t, \mathbb{F}_t^X)$ is a martingale. However, in general M_t need not be right continuous and hence we have given a direct proof.

The following is the Feynman-Kac's formula for a time inhomogeneous Markov process.

Theorem 3 : Let $0 < t_0 < \infty$ be fixed. Let $c : [0, t_0] \times S \rightarrow \mathbb{R}$ and $g : S \rightarrow \mathbb{R}$ be bounded measurable functions. Suppose that $v \in \mathcal{D}_A$ is a solution to

$$(2.24) \quad [Av + cv](u, x) 1_{\{u < t_0\}} = 0$$

and

$$(2.25) \quad v(t_0, x) = g_0(x).$$

Then v admits a representation, for $s < t_0$

$$(2.26) \quad v(s, X_s) = E_{\pi} \left[g_0(X_{t_0}) \exp \left(\int_s^{t_0} c(u, X_u) du \right) \middle| \mathbb{F}_s^X \right] \quad \text{a.e. } \pi.$$

Proof : Fix $s < t_0$. Take $f = v$ and $g = c$ in Theorem 2 to obtain that $(Z_t, \mathbb{F}_t^X)_{s \leq t \leq t_0}$ is a martingale, where

$$(2.27) \quad Z_t = v(t, X_t) \exp \left(\int_s^t v(u, X_u) du \right).$$

Here we have used the fact that v satisfies (2.24) so that the second term appearing in the expression for Z_t is zero. Thus

$$E_{\pi} [Z_{t_0} | \mathbb{F}_s^X] = Z_s \quad \text{a.s. } \pi$$

This is same as (2.26) since $v(t_0, x) = g_0(x)$.

□

3. In this section we consider the question as to under what conditions in $c, g, (X_t)$ does the problem (2.24), (2.25) admit a solution. Of course, if the solution exists, it has to satisfy (2.26) and this gives a clue as to what conditions one should put on $c, g, (X_t)$.

Suppose that S is a topological space, $\mathcal{S}$ is its Borel σ field. Let $\mathcal{X}$ be the space of all right continuous mappings $\underline{X}$ from $[0, \infty)$ into S . We will denote by X_t the value of $\underline{X}$ at t . Let $\mathcal{F}_t^S = \sigma(\underline{X}_u : s \leq u \leq t)$. We assume that

$$(3.1) \quad \text{for all } \omega, X_s(\omega) \in \mathcal{X}$$

and that for all $(s, x) \in \hat{S}$, there exists a probability measure $P_{s, x}$ on $(\mathcal{X}, \mathcal{F}_\infty^S)$ such that for $0 \leq t_0 \leq t_1 \leq \dots \leq t_k, y_i \in S, A_1, A_2, \dots, A_k \in \mathcal{S}$; $k \geq 1$, we have

$$(3.2) \quad P_{t_0, y} (X_{t_i} \in A_i : 1 \leq i \leq k) = \int \dots \int \prod_{i=1}^k 1_{A_i}(y_i) P(t_{i-1}, y_{i-1}, t_i - t_{i-1}, dy_i).$$

Remark The main thrust of this assumption is that $P_{s, x}$ is realized on $\mathcal{X}$. The relation (2.1) and (3.2) imply that for $\{t_i\}, \{A_i\}$ as in (3.2), we have

$$(3.3) \quad E_\pi \left[\prod_{i=1}^k 1_{A_i}(X_{t_i}) \middle| \mathcal{F}_{t_0}^X \right] = P_{t_0, X_{t_0}} (X_{t_i} \in A_i : 1 \leq i \leq k) \text{ a.s. } \pi$$

and hence by standard arguments, we have for $B \in \mathcal{A}_{\infty}^{t_0}$,

$$(3.4) \quad \pi(X \in B | \mathcal{F}_{t_0}^X) = P_{t_0, X_{t_0}}(B) \quad \text{a.s. } \pi.$$

Similarly, it can be proved that for $s < t, B \in \mathcal{A}_{\infty}^t, x \in S$,

$$(3.5) \quad P_{s, x}(B | \mathcal{A}_t^S) = P_{t, X_t}(B) \quad \text{a.s. } P_{s, x}.$$

We are now in a position to prove a 'converse' to the Feynman-Kac's formula.

Theorem 4 : Let $0 < t_0 < \infty$ be fixed. Let $c : [0, t_0] \times S \rightarrow \mathbb{R}$ be a bounded continuous function. Let $f \in \underline{D}_A$. Let $v : \hat{S} \rightarrow \mathbb{R}$ be defined by

$$(3.6) \quad \begin{aligned} v(s, x) &= E_{P_{s, x}} \left[f(t_0, X_{t_0}) \exp \left(\int_s^{t_0} c(u, X_u) du \right) \right], \quad s < t_0 \\ &= f(s, x), \quad s \geq t_0. \end{aligned}$$

Then $v \in \underline{D}_A$ and $Av = f_1$ where

$$(3.7) \quad \begin{aligned} f_1(s, x) &= -c(s, x)v(s, x), \quad s < t_0 \\ &= (Af)(s, x), \quad s \geq t_0. \end{aligned}$$

Proof Since $v(s, x) = f(s, x)$ for $s \geq t_0$, we have

$$(T_t v)(s, x) = (T_t f)(s, x)$$

for $s \geq t_0$, $x \in S$, $t \geq 0$. Hence for $s \geq t_0$, $x \in S$.

$$(3.8) \quad \lim_{t \downarrow 0} \frac{(T_t v)(s, x) - v(s, x)}{t} = (Af)(s, x) = f_1(s, x).$$

For $x \in X$, $s \leq t_0$ let us define

$$(3.9) \quad C_s(x) = \exp \left(\int_s^{t_0} c(u, X_u) du \right).$$

Then for $s \leq t_0$, we have

$$v(s, x) = E_{P_{s, x}} \left[f(t_0, X_{t_0}) C_s(x) \right].$$

: 11 :

For $s < t_0$, $s + t < t_0$, we thus have

$$\begin{aligned}
 (3.10) \quad (T_t v)(s, x) &= \int v(s+t, z) P(s, x, t, dz) \\
 &= E_{P_{s, x}} \left[v(s+t, \underline{X}_{s+t}) \right] \\
 &= E_{P_{s, x}} \left[E_{P_{s+t, \underline{X}_{s+t}}} (f(t_0, \underline{X}_{t_0}) C_{s+t}(\underline{X})) \right] \\
 &= E_{P_{s, x}} \left[f(t_0, \underline{X}_{t_0}) C_{s+t}(\underline{X}) \right].
 \end{aligned}$$

by (3.5). Hence, for $s < t_0$, $x \in S$, $s + t < t_0$ we have

$$(3.11) \quad \frac{(T_t v)(s, x) - v(s, x)}{t} = E_{P_{s, x}} \left[f(t_0, \underline{X}_{t_0}) \cdot \frac{C_{s+t}(\underline{X}) - C_s(\underline{X})}{t} \right].$$

For all $\underline{X} \in \mathcal{X}$, we have from (3.9)

$$(3.12) \quad \lim_{t \rightarrow 0} \frac{C_{s+t}(\underline{X}) - C_s(\underline{X})}{t} = -c(s, \underline{X}_s) \cdot C_s(\underline{X}).$$

Further

$$(3.13) \quad \left| \frac{C_{s+t}(\underline{X}) - C_s(\underline{X})}{t} \right| = \left| -c(t, \underline{X}_t) \cdot C_t(\underline{X}) \right|$$

$$\leq K$$

where K depends only on t_0 and the upper bound of $|c|$. The dominated convergence theorem gives that for $s < t_0$

$$\begin{aligned}
 (3.14) \quad \lim_{t \rightarrow 0} \frac{(T_t v)(s, x) - v(s, x)}{t} &= E_{P_{s, x}} \left[f(t_0, \underline{X}_{t_0}) \{-c(s, \underline{X}_s) C_s(\underline{X})\} \right] \\
 &= -c(s, x) v(s, x) \\
 &= f_1(s, x)
 \end{aligned}$$

as $P_{s,x}(X_s = x) = 1$. Also, (3.13) implies that the left hand expression in (3.11) is uniformly bounded (in s, x, t). Thus we have

$$(3.15) \quad \lim_{t \downarrow 0} \frac{(T_t v)(s, x) - v(s, x)}{t} = f_1(s, x).$$

Remains to prove that $f_1 \in J_0$. This will prove that $v \in D_A$ and that $Av = f_1$. If $s \geq t_0$, $f_1(s, x) = (Af)(s, x)$ and hence for $s \geq t_0$, $t \geq 0$, $(T_t f_1)(s, x) = (T_t Af)(s, x)$. Since $Af \in J_0$, this gives

$$(3.16) \quad \lim_{t \downarrow 0} (T_t f_1)(s, x) = f_1(s, x) \quad \text{for } s \geq t_0, x \in S.$$

For $s < t_0$, we have

$$\begin{aligned} T_t f_1(s, x) - f_1(s, x) &= -E_{P_{s,x}} \left[c(s+t, X_{s+t}) v(s+t, X_{s+t}) - c(s, x) v(s, x) \right] \\ &= -E_{P_{s,x}} \left[v(s+t, X_{s+t}) \{c(s+t, X_{s+t}) - c(s, x)\} \right] \\ &\quad - c(s, x) E_{P_{s,x}} [v(s+t, X_{s+t}) - v(s, x)]. \end{aligned}$$

Now as $t \downarrow 0$, $c(s+t, X_{s+t}) \rightarrow c(s, X_s) = c(s, x)$ a.e. $P_{s,x}$, as c is continuous and X_s is right continuous. Hence by the dominated convergence theorem,

$$\lim_{t \downarrow 0} E_{P_{s,x}} [v(s+t, X_{s+t}) \{c(s+t, X_{s+t}) - c(s, x)\}] = 0.$$

The relation (3.14) implies that

$$-c(s, x) E_{P_{s,x}} [v(s+t, X_{s+t}) - v(s, x)] = -c(s, x) [(T_t v)(s, x) - v(s, x)]$$

+ 0

as $t \downarrow 0$. These observations give

: 13 :

$$(3.17) \quad \lim_{t \rightarrow 0} (T_t f_1)(s, x) = f_1(s, x) \quad \text{for } s < t_0, x \in S.$$

Now (3.16), (3.17) and the fact that $T_t f_1$ is uniformly bounded yield

$$w\text{-}\lim_{t \rightarrow 0} (T_t f_1) = f_1.$$

□

Remark : Under the conditions assumed in this section and Theorem 4, the equations (2.24), (2.25) for $g_0(x) = f(t_0, x)$ have a unique solution v on $[0, t_0] \times S$ which is given by (3.6). To see this, let v' be any solution. Apply Theorem 3 to the process $\{X_t : t \geq s\}$ on the probability space $(X, \underline{A}_\infty^s, P_{s,x})$ to obtain, for $s < t_0$,

$$v'(s, X_s) = E_{P_{s,x}} \left[f(t_0, X_{t_0}) \mid \underline{A}_s^s \right] \quad \text{a.s. } P_{s,x}.$$

Since under $P_{s,x}$ any set in $\underline{A}_s^s$ has measure zero or one, the conditional expectation appearing above is the unconditional expectation and thus equals $v(s, x)$. Also $X_s = x$ a.s. $P_{s,x}$. Hence we have

$$v'(s, X_s) = v(s, x) \quad \text{a.s. } P_{s,x}.$$

These observations imply

$$v'(s, x) = v(s, x).$$

4. We now consider an equation dual to (2.24), namely

$$\frac{d}{dt} K_t = A^* K_t + g(t, \cdot) K_t$$

where $\{K_t\} \subseteq \underline{M}(S)$ - the class of finite signed measures on $(S, \underline{S})$. The equation (4.1) is purely formal and is to be interpreted as

$$(4.2) \quad \langle f(t, \cdot), K_t \rangle = \langle f(0, \cdot), K_0 \rangle + \int_0^t \langle Af(u, \cdot), K_u \rangle du + \int_0^t \langle g(u, \cdot) f(u, \cdot), K_u \rangle du$$

for $f \in \underline{D}_A$. Here, $\langle \theta, \mu \rangle$ denotes $\int \theta d\mu$ for $\mu \in \underline{M}(S)$ and a function $\theta: S \rightarrow \mathbb{R}$. Thus $\langle f(t, \cdot), \mu \rangle = \int f(t, x) d\mu(x)$ for $f \in \underline{J}$. We will show that this equation with boundary condition

$$(4.3) \quad K_0 = \Pi \circ X_0^{-1}$$

admits a unique solution which is given by

$$(4.4) \quad K_t(B) = E_\pi \left[1_B(X_t) \exp \left(\int_0^t g(u, X_u) du \right) \right], \quad B \in \underline{S}.$$

The uniqueness will be proved in the class of $\{K_t\}$ satisfying

$$(4.5) \quad \{K_t\} \subseteq \underline{M}(S), \quad t \mapsto K_t(B) \text{ is a Borel measurable function}$$

for all $B \in \underline{S}$ and $K_t \ll \Pi \circ X_t^{-1}$ with

$$\left| \frac{dK_t}{d\Pi \circ X_t^{-1}} \right| \leq M$$

for all t , for a fixed constant M .

We continue to assume that the conditions imposed on (X_t) in Section 3, are valid. We further assume that S is a complete separable metric space. We begin with a Lemma.

Lemma 5 : Let $0 < t < \infty$ be fixed. Let $\mu \in \underline{M}(S)$ be such that

$$(4.6) \quad \langle f(t, \cdot), \mu \rangle = 0 \quad \forall f \in \underline{D}_A.$$

Then $\mu \equiv 0$.

Proof : Let $\underline{E}$ be the class of $f \in \underline{J}$ for which (4.6) holds. Easy to see that if $f_k \in \underline{E}$, $w\text{-}\lim_{k \rightarrow \infty} f_k = f$, then $f \in \underline{E}$. Hence by (2.13), $\underline{J}_0 \subseteq \underline{E}$.

For $f \in C_b(\hat{S})$, (i.e. $f : \hat{S} \rightarrow \mathbb{R}$ is bounded continuous), we have

$$(T_t f)(s, x) = E_{P_{s, x}} f(s+t, X_{s+t}) \rightarrow f(s, x) \text{ as } t \downarrow 0,$$

since X_u is right continuous. Thus $C_b(\hat{S}) \subseteq \underline{J}_0 \subseteq \underline{E}$.

Given $f_0 \in C_b(S)$, taking $f(s, x) = f_0(x)$, we have $f \in C_b(\hat{S}) \subseteq \underline{E}$ and hence

$$(4.7) \quad \langle f_0, \mu \rangle = 0.$$

The validity of (4.7) for all $f_0 \in C_b(S)$ implies $\mu = 0$ because $\underline{S}$ - the Borel σ field - is also the smallest σ field with respect to which $C_b(S)$ is measurable.

□

We are now in a position to prove the assertions made at the beginning of this section. This result may be considered as a dual Feynman-Kac's formula.

Theorem 6 : Suppose that g satisfies (2.17) and (2.18). Then the equation (4.2) with boundary condition (4.3) admits a unique solution in the class of $\{K_t\}$ satisfying (4.5). The unique solution is given by (4.4).

Proof : First we will prove that $\{K_t\}$ defined by (4.4) satisfies (4.2). Let $\{K_t\}$ be defined by (4.4). Easy to see that (4.3) and (4.5) are satisfied.

Taking $s = 0$ in Theorem 2, it follows from the martingale property of Z_t that $E_\pi Z_t = E_\pi Z_0$. Here, Z_t is given by (2.20) where in turn B_t is given by (2.19), with $s = 0$. Noting that with these notations,

$$\langle \theta, K_t \rangle = E_\pi \theta(X_t) B_t$$

we conclude from the relation $E_\pi Z_t = E_\pi Z_0$ that

$$\langle f(t, \cdot), K_t \rangle - \int_0^t \langle (Af + \sigma f)(u, \cdot), K_u \rangle du = \langle f(0, \cdot), K_0 \rangle.$$

Hence $\{K_t\}$ satisfies (4.2).

To prove the uniqueness part, we will prove the following. Suppose $\{K_t\}$ satisfies (4.2), (4.5) and $K_0 \equiv 0$. Then $K_t \equiv 0$, $t \geq 0$.

For this fix $t_0 < \infty$ and $f \in \underline{D}_A$. Let ν be the measure defined on $S' = [0, t_0] \times S$ by

$$(4.8) \quad \nu(B) = E_\pi \int_0^{t_0} 1_B(u, X_u) du, \quad B \in \underline{S}' = \underline{S}(S').$$

Then note that (2.17) implies $\int_{S'} |g| d\nu < \infty$. Hence if $g_k : S' \rightarrow \mathbb{R}$ is defined by

$$(4.9) \quad g_k(s, x) = g(s, x) 1_{\{|g(s, x)| \leq k\}}$$

then we have

$$(4.10) \quad \int_{S'} |g_k - g| d\nu \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

For each k , g_k is bounded by k . By Lusin's theorem (see [1], p. 187) we can get $c_{k,i} \in C_b(S')$, bounded by k , such that

$$(4.11) \quad c_{k,i} \rightarrow g_k \quad \text{a.e. } \nu \quad \text{as } i \rightarrow \infty.$$

Hence

$$(4.12) \quad \lim_{i \rightarrow \infty} \int |c_{k,i} - g_k| dv = 0.$$

Let $v_{k,i}$ be given by (3.6) for $c = c_{k,i}$. Then $A v_{k,i} = -c_{k,i} v_{k,i}$ on $[0, t_0) \times S$ by Theorem 4. Using (4.2) for $v_{k,i}$ and recalling that $K_0 = 0$, we have

$$\begin{aligned} (4.13) \quad \langle f(t_0, \cdot), K_{t_0} \rangle &= \langle v_{k,i}(t_0, \cdot), K_{t_0} \rangle \\ &= \int_0^{t_0} \langle (A v_{k,i} + g v_{k,i})(u, \cdot), K_u \rangle du \\ &= \int_0^{t_0} \langle (g - c_{k,i}) v_{k,i}(u, \cdot), K_u \rangle du. \end{aligned}$$

Thus

$$\begin{aligned} (4.14) \quad |\langle f(t_0, \cdot), K_{t_0} \rangle| &\leq M \int_0^{t_0} \langle |(g - c_{k,i}) v_{k,i}|(u, \cdot), \pi \circ X_u^{-1} \rangle du \\ &= M E_\pi \int_0^{t_0} |(g - c_{k,i}) v_{k,i}|(u, X_u) du. \\ &= M \int |g - c_{k,i}| \cdot |v_{k,i}| dv. \end{aligned}$$

As $i \rightarrow \infty$, $v_{k,i}$ converges pointwise to v_k and is bounded by k , where v_k is given by (3.6) for $c = g_k$. This and (4.11), (4.12), (4.14) imply

$$|\langle f(t_0, \cdot), K_{t_0} \rangle| \leq M \int |g - g_k| \cdot |v_k| dv.$$

Since (4.9) implies $c_k(u, x) \leq a(u)$, it follows that

$$|v_k| \leq M_1 \cdot \exp\left(\int_0^{t_0} a(u) du\right) = M_2$$

where $|f| \leq M_1$. Hence

$$|\langle f(t_0, \cdot), K_{t_0} \rangle| \leq M \cdot M_2 \int |g - g_k| dv.$$

This and (4.10) imply $\langle f(t_0, \cdot), K_{t_0} \rangle = 0$. Since $f \in \mathcal{D}_A$ is arbitrary, Lemma 5 gives $K_{t_0} = 0$. This completes the proof. $\square$

We will briefly consider the equation for normalized measures

$$(4.15) \quad N_t(B) = \frac{K_t(B)}{K_t(S)}, \quad B \in \mathcal{S}$$

where K_t is given by (4.4). It is easy to see, using (4.2) that $\{N_t\}$ satisfies.

$$(4.16) \quad \langle f(t, \cdot), N_t \rangle = \langle f(0, \cdot), N_0 \rangle + \int_0^t \langle (Af + gf)(u, \cdot), N_u \rangle du - \int_0^t \langle f(u, \cdot), N_u \rangle \langle g(u, \cdot), N_u \rangle du$$

We will now prove that $\{N_t\}$ is the unique solution to this equation.

Theorem 7 : The equation (4.16) with boundary condition $N_0 = \mathbb{W} \circ X_0^{-1}$ admits a unique solution in the class of $\{N_t\}$ satisfying (4.5). The solution is given by (4.15).

Proof : We need to prove uniqueness of the solution. Let N_t^i be any other solution, i.e. satisfying (4.5), (4.16) and $N_0^i = \mathbb{W} \circ X_0^{-1}$. Then it can be checked that $N_t^i(S) = 1$ for all $t \geq 0$. Further, if K_t^i is defined by

$$(4.17) \quad K_t^i(B) = N_t^i(B) \cdot \exp\left(\int_0^t \langle g(u, \cdot), N_u^i \rangle du\right),$$

then K_t^i is a solution to (4.2) and that it satisfies (4.3), (4.5).

Hence by Theorem 6, $K_t^i = K_t$. This and the observation that

$$N_t^i(B) = \frac{K_t^i(B)}{K_t^i(S)}$$

give us the required equality, namely $N_t^i = N_t$. $\square$

5. We will now give applications of the results in the previous sections to filtering theory.

We refer the reader to [4] for a detailed discussion and background on the white noise approach to filtering theory.

We assume that the signal process (X_t) is a Markov process satisfying the conditions imposed in the previous sections.

Let $\underline{K}$ be a separable Hilbert space. Let $h : [0, T] \times S \rightarrow \underline{K}$ be a measurable function such that

$$(5.1) \quad E_{\pi} \left[\int_0^T \|h_u(X_u)\|_{\underline{K}}^2 du \right] < \infty$$

Let $H = L^2([0, T], \underline{K})$ and let $\xi: \Omega \rightarrow H$ be defined by

$$(\xi(\omega))_u = h_u(X_u(\omega)), \quad 0 \leq u \leq T.$$

Consider the model

$$y = \xi + e$$

where $e = (e_t)$ is $\underline{K}$ -valued white noise independent of (X_t) . Here y is the observation process and y, ξ, e are realised on a Quasi cylinder probability space $(E, \underline{E}, \alpha)$ (See [4] section 6). We now state the Bayes formula. For the relevant definitions and proof, see [4].

Theorem 8 : For $g : S \rightarrow \mathbb{R}$ bounded, measurable,

$$(5.2) \quad E_{\alpha}(g(X_t) | y_u : u \leq t) = \int_S g(x) dF_t(y)(x)$$

where

$$(5.3) \quad F_t(y)(B) = E_{\pi} \left[1_B(X_t) \exp \left(\int_0^t (h_u(X_u), y_u)_{\underline{K}} du - \frac{1}{2} \int_0^t \|h_u(X_u)\|_{\underline{K}}^2 du \right) \right]$$

and

$$(5.4) \quad F_t(y)(B) = \frac{\Gamma_t(y)(B)}{\Gamma_t(y)(S)}$$

for $0 \leq t \leq T$, $y \in H$, $B \in \underline{S}$.

$\Gamma_t(y)$, $F_t(y)$ are known as unnormalized and normalized conditional distribution of X_t given $\{y_u : u \leq t\}$, respectively.

The following is an immediate consequence of Theorems 6,7. Let $\varepsilon_y(t,x) = (h_t(x), y_t)_{\underline{K}} - \frac{1}{2} \|h_t(x)\|_{\underline{K}}^2$, $(t,x) \in [0,T] \times S$, $y \in H$.

Theorem 9 : Let h satisfy (5.1). (i) For all $y \in H$, $\Gamma_t(y)$ is the unique solution to the equation

$$(5.5) \quad \langle f(t, \cdot), \Gamma_t(y) \rangle = \langle f(0, \cdot), \Gamma_0(y) \rangle + \int_0^t \langle (Af + \varepsilon_y f)(u, \cdot), \Gamma_u(y) \rangle du,$$

$f \in \underline{D}_A$

with the condition $\Gamma_0(y) = \pi \circ X_0^{-1}$ in the class of $\{K_t\}$ satisfying (4.5).

(ii) For all $y \in H$, $F_t(y)$ is the unique solution to the equation

$$(5.6) \quad \langle f(t, \cdot), F_t(y) \rangle = \langle f(0, \cdot), F_0(y) \rangle + \int_0^t \langle (Af + \varepsilon_y f)(u, \cdot), F_u(y) \rangle du$$

$$- \int_0^t \langle f(u, \cdot), F_u(y) \rangle \langle \varepsilon_y(u, \cdot), F_u(y) \rangle du, \quad f \in \underline{D}_A$$

with the initial condition $F_0(y) = \pi \circ X_0^{-1}$ in the class of $\{K_t\}$ satisfying (4.5).

Proof : Since

$$|\varepsilon_y(t,x)| \leq \|h_t(x)\|_{\underline{K}}^2 + \frac{1}{2} \|y_t\|_{\underline{K}}^2$$

and

$$\varepsilon_y(t,x) \leq \frac{1}{2} \|y_t\|_{\underline{K}}^2,$$

it follows that for all $y \in H$, ε_y satisfies (2.17) and (2.18). Thus (i) follows from Theorem 6 and (ii) from Theorem 7.

Remark : Theorem 9 was proved in [3] under the much stronger condition

$$(5.7) \quad ||h_t(x)|| \leq a_t \quad \text{with} \quad \int_0^T a_t^2 dt < \infty .$$

The equations (5.5) and (5.6) are analogues of the Zakai and Fujisaki-Kallianpur-Kunita equations. In [3], $r_t(y)$ and $F_t(y)$ were also characterized as unique solutions to another type of equations (equations (3.4) and (3.11) in [3]) under the condition (5.7). With a little bit of work, it can be shown that (5.7) can be replaced by (5.1) in these results as well.

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