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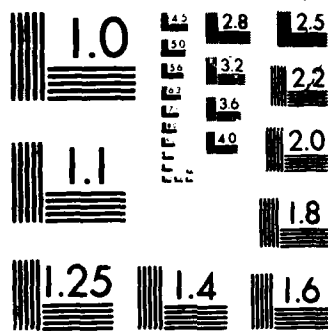
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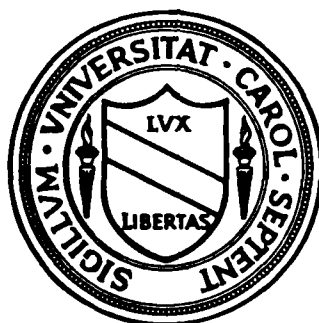
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Abstract

The moment problems minimize (maximize) $\{E\|X_1 - X_2\|^p; E\|X_i\|^{q_j} = a_{ij}, i = 1, 2, j = 1, \dots, n\}$, where X_i takes values in a separable norm space are investigated. Explicit solutions when $n = 1$ are given for all $p \geq 0$ and $q_j \geq 0$ ($0^0 := 0$). If $n = 2$, the minimization problem is solved for $1 \leq q_1 \leq p < q_2$ and the maximization problem for $0 < p \leq q_1, 1 \leq q_1 < q_2$ or $0 < q_1 < q_2 < p$. Possible generalizations and open problems are presented.

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1. Introduction

In probability theory the following two measure theoretic problems are well known, (see e.g. Levin and Miljutin (1978), Kellerer (1984), Kemperman (1983), Haneveld (1985), Rüschemdorf (1985), Rachev (1984) and references there):

A. Marginal problem: For fixed probability measures (laws) P_1 and P_2 on a measurable space U and a measurable function c on the product space $U^2 = U \times U$:

$$(1.1) \quad \text{minimize (maximize)} \int_{U^2} c(x,y) P(dx,dy),$$

where the laws P on U^2 have marginals P_1 and P_2 , i.e.

$$(1.2) \quad \pi_i P = P_i, \quad i = 1, 2, \dots$$

B. Moment problem: For fixed real numbers a_{ij} and real-valued continuous functions f_{ij} ($i = 1, 2, j = 1, \dots, n$).

$$(1.3) \quad \text{minimize (maximize)} \int_{U^2} c(x,y) P(dx,dy),$$

where the law P on U^2 satisfies the marginal moment conditions.

$$(1.4) \quad \int_U f_{ij} dP_i = a_{ij}, \quad i = 1, 2, j = 1, \dots, n.$$

Dual relationships and explicit solutions of the marginal problem for different spaces U and criterion functions c are given by Levin and Miljutin (1978), Kellerer (1984), Haneveld (1985), Cambanis, Simons and Stout (1976), Cambanis and Simons (1982), Rüschemdorf (1985) and Rachev (1984, 1985). In citing some results concerning duality and explicit solutions of A, we shall use the following notations:

(1.5) (U, d) is a separable metric space with metric d ;

(1.6) $P(U^k)$ is the space of all Borel probability measures on the Cartesian product U^k ;

(1.7) H is the class of all convex functions

$$H : [0, \infty) \rightarrow [0, \infty), H(0) = 0, K_H := \sup_{t \geq 0} H(2t)/H(t) < \infty;$$

(1.8) $P_H := \{P \in P(U) : \int_U H(d(x, a)) P(dx) < \infty\}, H \in H$;

(Note that P_H does not depend on $a \in U$ for any $H \in H$);

(1.9) $D(x, y) := H(d(x, y))$;

(1.10) $Lip(U) := \{f: U \rightarrow \mathbb{R}, \sup_{x \in U} |f(x)| < \infty, \exists \alpha(f) > 0: |f(x) - f(y)| \leq \alpha(f) d(x, y)$

for all $x, y \in U\}$;

(1.11) $\underline{G}_H := \{(f, g): f, g \in Lip(U), f(x) + g(y) \leq D(x, y), x, y \in U\}$;

(1.12) $\overline{G}_H := \{(f, g): f, g \in Lip(U), f(x) \geq 0, g(y) \geq 0, f(x) + g(y) \geq D(x, y), x, y \in U\}$.

Theorem A. i) (Duality solutions of A). Let (U, d) be a separable metric space $H \in H, P_1, P_2 \in P_H$. Then

$$(1.13) \hat{L}_H(P_1, P_2) := \inf_{U^2} \left\{ \int D(x, y) P(dx, dy) : P \in P(U^2), \pi_i P = P_i, i=1,2 \right\}$$

$$= \sup \left\{ \int_U f dP_1 + \int_U g dP_2 : (f, g) \in \underline{G}_H \right\}$$

and

$$(1.14) \check{L}_H(P_1, P_2) := \sup_{U^2} \left\{ \int D(x, y) P(dx, dy) : P \in P(U^2), \pi_i P = P_i, i=1,2 \right\}$$

$$= \inf \left\{ \int_U f dP_1 + \int_U g dP_2 : (f, g) \in \overline{G}_H \right\}.$$

(ii) (Explicit solutions of A). If $(U, d) = \mathbb{R}, |\cdot|$ and H is a convex function then

$$(1.15) \quad \hat{L}_H(P_1, P_2) = \int_0^1 D(F_1^{-1}(t), F_2^{-1}(t)) dt,$$

$$(1.16) \quad \check{L}_H(P_1, P_2) = \int_0^1 D(F_1^{-1}(t), F_2^{-1}(1-t)) dt,$$

where F_j is the distribution function corresponding to P_j and F_j^{-1} is its inverse, $j = 1, 2$.

Indication: i) see Rachev (1985). ii) see Cambanis, Simons and Stout (1976).

Kellerer (1984) provides duality solutions of A for general c ; however, in case $c = D$, his dual solutions are not as sharp as these in i).

The possible solutions of the marginal problems are related to the dual and explicit expressions for the so-called *minimal metrics* (see Zolotarev (1976, 1983)) and *maximal distances* (see Rachev (1985)) that are fruitful in the development of a considerable range of stability problems for stochastic models (see Zolotarev (1983), Rachev (1984)).

Owing to the number of its important applications (see Shohat and Tamarkin (1943), Ahiezer and Krein (1962), Karlin and Studden (1966), Hoeffding (1955), Hoeffding and Shrikhande (1956), Basu and Simons (1983), Kemperman (1972, 1983), Haneveld (1985)) the moment problem can also be treated as an approximation of the marginal problem. Indeed, if the laws P_1 and P_2 (see (1.2)) are not determined completely and if only some functionals of P_1 and P_2 are given (see (1.4)) then one has to solve the problem B instead of A. The significance of the moment problem for the theory of probability metrics was also stressed by Sholpo (1983) and Rachev (1985).

General dual representations of moment problems on a compact space U are given in Kemperman (1972,1983), Haneveld (1985). In a more general case of a completely regular topological space U , dual expressions are given in Kemperman (1983) under a "tightness" condition on the pairs (f_{ij}, a_{ij}) ($i=1,2, j=1,\dots,n$).

The present paper is devoted to the explicit solutions of some moment problems on separable metric space U with metric d . In this case, considering a "rich enough" probability space $(\Omega, \mathcal{A}, \Pr)$ without atoms and the space $X = X(U)$ of all U -valued random variables (rv's) X on $(\Omega, \mathcal{A}, \Pr)$ one can rewrite the moment problem B as follows:

$$(1.17) \quad \text{minimize (maximize)} \{ \mathbb{E} c(X_1, X_2) : \mathbb{E} f_{ij}(X_i) = a_{ij}, \quad i=1,2, j=1,2,\dots,n \}.$$

In fact, the above assumptions guarantee that the set of all Borel probability measures on U^2 coincides with the set of all joint distributions $\Pr_{X,Y}$ of pairs of rv's X, Y (see Rachev and Shortt (1987)).

The main reason for considering not arbitrary but separable space (U, d) is that we need the measurability of d . For example, if $c(x, y) = d(x, y)$ in (1.5) and (U, d) is a metric space of cardinality $> c$, then the metric $d: U \times U \rightarrow \mathbb{R}$ is not measurable with respect to the product σ -algebra $\mathcal{B}(U) \times \mathcal{B}(U)$ ($\mathcal{B}(U)$ is the Borel σ -algebra on (U, d)), see Billingsley (1968).

In Section 2 explicit solutions are given for the problem (1.17) in the case where $n=1$, U is a separable norm space with norm $\|\cdot\|$ and

$$(1.18) \quad c(x, y) = h(\|x - y\|), \quad f_{i1}(x) := g(\|x\|), \quad i=1,2.$$

In particular, for any $p \geq 0$ and $q \geq 0$ we solve (1.17) with

$$(1.19) \quad c(x,y) = \|x-y\|^p, \quad f_{i1}(x) = \|x\|^q.$$

(Here and in the sequel 0^0 means 0).

In Section 3 we assume that U is again separable norm space but $n \geq 2$. In this case, among other results, explicit solutions are given for the moment problem (1.17) with

$$(1.20) \quad c(x,y) = \|x-y\|^p, \quad f_{ij}(x) = \|x\|^{q_j}, \quad i=1,2, \quad j=1,2,$$

for $1 \leq q_1 \leq p < q_2$ when minimizing in (1.17) and

for $0 < p \leq q_1$, $1 \leq q_1 < q_2$ or $0 < q_1 < q_2 < p$ when maximizing in (1.17).

Here, we also give the explicit solution of the well known moment problem.

$$(1.21) \quad \text{minimize (maximize)} \{ \mathbb{E} \|X\| : \mathbb{E} \|X\|^{q_i} = a_i, \quad i=1,2, \}$$

for all nonnegative p, q_1 and q_2 .

In Section 4 we apply the results of Section 3 to obtain precise bounds for $\hat{L}_H(p_1, p_2)$ and $\check{L}_H(p_1, p_2)$ when the moments $\int \|x\|^{q_{p_i}}(dx)$ ($i=1,2$) are fixed. Some open problems are offered.

2. Moment problems with one fixed pair of marginal moments

Let U be a separable norm space with norm $\|\cdot\|$ and M be the class of all strictly increasing continuous functions $f: [0, \infty] \rightarrow [0, \infty]$, $f(0) = 0$, $f(\infty) = \infty$. In the present section we treat the explicit representations of the following extremal functionals:

$$(2.1) \quad I(h, g; a, b) := \inf\{\mathbb{E}h(\|X - Y\|) : X, Y \in X(U), \mathbb{E}g(\|X\|) = a, \mathbb{E}g(\|Y\|) = b\}$$

and

$$(2.2) \quad S(h, g; a, b) := \sup\{\mathbb{E}h(\|X - Y\|) : X, Y \in X(U), \mathbb{E}g(\|X\|) = a, \mathbb{E}g(\|Y\|) = b\},$$

where $a > 0$, $b > 0$, $h \in M$, $g \in M$. In particular, for all $p \geq 0$, $q \geq 0$ the values

$$(2.23) \quad I(p, q; a, b) := I(h, g; a, b) (h(t) = t^p, g(t) = t^q),$$

$$(2.4) \quad S(p, q; a, b) := S(h, g; a, b) (h(t) = t^p, g(t) = t^q)$$

are calculated. Note that here and in the sequel $\mathbb{E}\|X - Y\|^0$ means $\Pr(X \neq Y)$.

The scheme of the proofs of all statements here is as follows: first we prove the necessary inequalities that give us the required bounds and then we construct pairs of random variables which achieve the bounds or approximate them with arbitrary precision.

Let $f, f_1, f_2 \in M$ and consider the following conditions (here and in the sequel f^{-1} is the inverse function of $f \in M$):

$$A(f_1, f_2) : f_1 \circ f_2^{-1}(t) \quad (t \geq 0) \text{ is convex};$$

$$B(f) : f^{-1}(\mathbb{E}f(\|X + Y\|)) \leq f^{-1}(\mathbb{E}f(\|X\|)) + f^{-1}(\mathbb{E}f(\|Y\|)) \text{ for any } X, Y \in X;$$

$$C(f) : \mathbb{E}f(\|X + Y\|) \leq \mathbb{E}f(\|X\|) + \mathbb{E}f(\|Y\|) \text{ for any } X, Y \in X;$$

$$D(f_1, f_2) : \lim_{t \rightarrow \infty} f_1(t)/f_2(t) = 0;$$

$E(f_1, f_2): f_1 \circ f_2(t) \ (t \geq 0)$ is concave;

$F(f_1, f_2): f_1$ is concave and f_2 is convex;

$G(f_1, f_2): \lim_{t \rightarrow \infty} f_1(t)/f_2(t) = \infty$.

Obviously, if $h(t) = t^p$, $g(t) = t^q$ ($p > 0, q > 0$), then $A(h, g) \Leftrightarrow p \geq q$,
 $B(g) \Leftrightarrow q \geq 1$, $C(g) \Leftrightarrow q \leq 1$, $D(h, g) \Leftrightarrow q > p$, $E(h, g) \Leftrightarrow q \geq p$,
 $F(h, g) \Leftrightarrow p \leq 1 \leq q$, $G(h, g) \Leftrightarrow p > q$ and hence, the conditions A-G
cover all possible values of the pairs (p, q) .

In the next theorem we establish explicit solutions of the moment
problem (1.17) under the conditions (1.18) and combinations of require-
ments A-G on h and g .

Theorem 1. For any $a \geq 0$ and $b \geq 0$, $a + b > 0$ the following equalities
hold:

i)

$$(2.5) \quad I(h, g; a, b) = \begin{cases} h(|g^{-1}(a) - g^{-1}(b)|) & \text{if } A(h, g) \text{ and } B(g) \text{ hold,} \\ h \circ g^{-1}(|\alpha - \beta|) & \text{if } A(h, g) \text{ and } C(g) \text{ hold,} \\ 0 & \text{if } D(h, g) \text{ holds.} \end{cases}$$

ii) For any $u \in U$, $h \in M$ and $g \in M$

$$(2.6) \quad \inf\{\Pr\{X \neq Y\} : \mathbb{E}g(\|X\|) = a, \mathbb{E}g(\|Y\|) = b\} = 0,$$

$$(2.7) \quad \inf\{\mathbb{E}h(\|X - Y\|) : \Pr\{X \neq u\} = a, \Pr\{Y \neq u\} = b\} = 0 \quad (a, b \in [0, 1]).$$

iii)

$$(2.8) \quad S(h, g; a, b) = \begin{cases} h(g^{-1}(a) + g^{-1}(b)) & \text{if } F(h, g) \text{ holds or if } B(g) \text{ and } E(h, g) \text{ hold,} \\ h \circ g^{-1}(\alpha + \beta) & \text{if } C(g) \text{ and } E(h, g) \text{ hold,} \\ \infty & \text{if } G(h, g) \text{ holds.} \end{cases}$$

iv) For any $u \in U$, $h \in M$, $g \in M$,

$$(2.9) \quad \sup\{\Pr\{X \neq Y\} : \mathbb{E}g(\|X\|) = a, \mathbb{E}g(\|Y\|) = b\} = 1,$$

$$(2.10) \quad \sup\{\Pr\{X \neq Y\} : \Pr\{X \neq u\} = a, \Pr\{Y \neq u\} = b\} = \min(a+b, 1) \quad (a, b \in [0, 1]),$$

$$(2.11) \quad \sup\{\mathbb{E}h(\|X - Y\|) : \Pr\{X \neq u\} = a, \Pr\{Y \neq u\} = b\} = \infty.$$

Proof: i) *Case 1.* Let $A(h, g)$ and $B(g)$ be fulfilled. Denote

$$\phi(a, b) := h(|g^{-1}(a) - g^{-1}(b)|), \quad a \geq 0, b \geq 0.$$

Claim 1. $I(h, g, a, b) \geq \phi(a, b)$. By the Jensen's inequality and $A(h, g)$

$$(2.12) \quad h \circ g^{-1}(\mathbb{E}Z) \leq \mathbb{E}h \circ g^{-1}(Z).$$

Taking $Z = g(\|X - Y\|)$ and using $B(g)$ we obtain

$$h^{-1}(\mathbb{E}h(\|X - Y\|)) \geq g^{-1}(\mathbb{E}g(\|X - Y\|)) \geq |g^{-1}(\mathbb{E}g(\|X\|)) - g^{-1}(\mathbb{E}g(\|Y\|))|$$

for any $X, Y \in X$ which proves the claim.

Claim 2. There exists an "optimal" pair (X^*, Y^*) of rv's such that $\mathbb{E}g(\|X^*\|) = a$, $\mathbb{E}g(\|Y^*\|) = b$, $\mathbb{E}(\|X^* - Y^*\|) = \phi(a, b)$. Let \bar{e} here and in the sequel be a fixed point of U with $\|\bar{e}\| = 1$. Then the required pair (X^*, Y^*) is given by

$$(2.13) \quad X^* = g^{-1}(a)\bar{e}, \quad Y^* = g^{-1}(b)\bar{e}$$

which proves the claim.

Case 2. Let $A(h, g)$ and $C(g)$ be fulfilled. Denote $\phi_1(t) := h \circ g^{-1}(t)$, $t \geq 0$. As in Claim 1 we get $I(h, g; a, b) \geq \phi_1(|a - b|)$. Suppose that $a > b$ and for each $\varepsilon > 0$ define a pair $(X_\varepsilon, Y_\varepsilon)$ of rv's as follows:

$$\Pr\{X_\varepsilon = c_\varepsilon \bar{e}, Y_\varepsilon = \bar{0}\} = p_\varepsilon, \quad \Pr\{X_\varepsilon = d_\varepsilon \bar{e}, Y_\varepsilon = d_\varepsilon \bar{e}\} = 1 - p_\varepsilon,$$

where

$$(2.14) \quad \bar{0} := 0\bar{e}, p_\varepsilon := \frac{a-b}{a-b+\varepsilon}, c_\varepsilon := g^{-1}(a-b+\varepsilon), d_\varepsilon := g^{-1}\left(\frac{b}{1-p_\varepsilon}\right).$$

Then $(X_\varepsilon, Y_\varepsilon)$ enjoys the side conditions in (2.1) and

$$\mathbb{E}h(\|X_\varepsilon - Y_\varepsilon\|) = \phi_1(a-b+\varepsilon) \frac{a-b}{a-b+\varepsilon}.$$

Letting $\varepsilon \rightarrow 0$ we claim (2.5).

Case 3. Let $D(h, g)$ be fulfilled. In order to obtain (2.5) it is sufficient to define a sequence (X_n, Y_n) ($n \geq N$) such that $\lim_{n \rightarrow \infty} \mathbb{E}h(\|X_n - Y_n\|) = 0$, $\mathbb{E}g(\|X_n\|) = a$, $\mathbb{E}g(\|Y_n\|) = b$. An example of such sequence is the following one:

$$\Pr\{X_n = \bar{0}, Y_n = \bar{0}\} = 1 - c_n - d_n,$$

$$\Pr\{X_n = na\bar{e}, Y_n = \bar{0}\} = c_n,$$

$$\Pr\{X_n = \bar{0}, Y_n = nb\bar{e}\} = d_n,$$

where $c_n = a/g(na)$, $d_n = b/g(nb)$ and N satisfies $c_N + d_N < 1$.

ii) Define the sequence (X_n, Y_n) ($n = 2, 3, \dots$) such that

$$\Pr\{X_n = g^{-1}(na)\bar{e}, Y_n = g^{-1}(nb)\bar{e}\} = \frac{1}{n},$$

$$\Pr\{X_n = \bar{0}, Y_n = \bar{0}\} = \frac{n-1}{n}.$$

Hence, $\mathbb{E}g(\|X_n\|) = a$, $\mathbb{E}g(\|Y_n\|) = b$ and $\Pr(X_n \neq Y_n) = \frac{1}{n}$ which shows (2.6).

Further suppose $a \geq b$. Without loss of generality we may assume that $u = \bar{0}$. Then consider the random pair $(\tilde{X}_n, \tilde{Y}_n)$ with the following joint distribution:

$$\Pr\{\tilde{X}_n = \bar{0}, \tilde{Y}_n = \bar{0}\} = 1 - a, \Pr\{\tilde{X}_n = \frac{1}{n}\bar{e}, \tilde{Y}_n = \bar{0}\} = a - b, \Pr\{\tilde{X}_n = \frac{1}{n}\bar{e}, \tilde{Y}_n = \frac{1}{n}\bar{e}\} = b.$$

Obviously, $(\tilde{X}_n, \tilde{Y}_n)$ satisfies the side conditions

$$\Pr(\tilde{X}_n \neq \bar{0}) = a, \Pr(\tilde{Y}_n \neq \bar{0}) = b \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{E} h(\|X_n - Y_n\|) = 0,$$

which proves (2.7).

The proofs of iii) and iv) are quite analogous to those of i) and ii) respectively. QED

Note that if $A(h, g)$ and $B(g)$ hold we have constructed an *optimal* pair (X^*, Y^*) (see (2.13)), i.e. (X^*, Y^*) realizes the infimum in (2.1). However, if $D(h, g)$ holds and $a \neq b$ then optimal pairs do not exist, because $\mathbb{E}h(\|X - Y\|) = 0$ implies $a = b$.

Corollary 1. For any $a \geq 0, b \geq 0, a + b \geq 0, p \geq 0, q \geq 0,$

$$(2.15) \quad I(p, q; a, b) = \begin{cases} |a^{1/q} - b^{1/q}|^p & \text{if } p \geq q \geq 1, \\ |a - b|^{p/q} & \text{if } p \geq q, 0 < q < 1, \\ 0 & \text{if } 0 \leq p < q \text{ or } q = 0, p > 0, \\ |a - b| & \text{if } p = q = 0. \end{cases}$$

$$(2.16) \quad S(p, q; a, b) = \begin{cases} (a^{1/q} + b^{1/q})^p & \text{if } 0 \leq p \leq q, q \geq 1, \\ (a + b)^{p/q} & \text{if } 0 \leq p \leq q < 1, q \neq 0, \\ \infty & \text{if } p > q \geq 0, \\ \min(a + b, 1) & \text{if } p = q = 0. \end{cases}$$

Remark. Obviously, if $q = 0$ in (2.15) or (2.16) the values of I and S make sense for $a, b \in [0, 1]$.

3. Moment problems with two fixed pairs of marginal moments

The main part of this section is devoted to the explicit description of the following bounds:

$$(3.1) \quad I(h, g_1, g_2; a_1, b_1, a_2, b_2) := \inf \mathbb{E} h(\|X - Y\|),$$

$$(3.2) \quad S(h, g_1, g_2; a_1, b_1, a_2, b_2) := \sup \mathbb{E} h(\|X - Y\|),$$

where $h, g_1, g_2 \in M$, and the infimum in (3.1) and the supremum in (3.2) are taken over the set of all pairs of rv's $X, Y \in X(U)$ satisfying the moment conditions

$$(3.3) \quad \mathbb{E} g_i(\|X\|) = a_i, \quad \mathbb{E} g_i(\|Y\|) = g_i, \quad i = 1, 2, ..$$

In particular, if $h(t) = t^p$, $g_i(t) = t^{q_i}$, $i = 1, 2$ ($p \geq 0$, $q_2 > q_1 \geq 0$), we write

$$(3.4) \quad I(p, q_1, q_2; a_1, b_1, a_2, b_2) := I(h, g_1, g_2; a_1, b_1, a_2, b_2),$$

$$(3.5) \quad S(p, q_1, q_2; a_1, b_1, a_2, b_2) := S(h, g_1, g_2; a_1, b_1, a_2, b_2).$$

The moment problem with two pairs of marginal conditions is considerably more complicated and in the present section, our results are not as complete as in the previous one.

Theorem 2. *Let the conditions $A(g_2, g_1)$ and $G(g_2, g_1)$ hold. Let $a_i \geq 0$, $b_i \geq 0$, $i = 1, 2$, $a_1 + a_2 > 0$, $b_1 + b_2 > 0$ and*

$$(3.6) \quad g_1^{-1}(a_1) \leq g_2^{-1}(a_2), \quad g_1^{-1}(b_1) \leq g_2^{-1}(b_2).$$

i) *If $A(h, g_1)$, $B(g_1)$ and $D(h, g_2)$ are fulfilled, then*

$$(3.7) \quad I(h, g_1, g_2; a_1, b_1, a_2, b_2) = I(h, g_1; a_1, b_1) = h(|g_1^{-1}(a_1) - g_1^{-1}(b_1)|).$$

ii) Let $D(h, g_2)$ be fulfilled. If $F(h, g)$ holds or if $B(g)$ and $E(h, g)$ hold then

$$(3.8) \quad S(h, g_1, g_2; a_1, b_1, a_2, b_2) = S(h, g_1; a_1, b_1) = h(g_1^{-1}(a_1) + g_1^{-1}(b_1)).$$

iii) If $G(h, g_2)$ is fulfilled and $g_1^{-1}(a_1) \neq g_2^{-1}(a_2)$ or $g_1^{-1}(b_1) \neq g_2^{-1}(b_2)$, then

$$(3.9) \quad S(h, g_1, g_2; a_1, b_1, a_2, b_2) = S(h, g_1; a_1, b_1) = \infty.$$

Proof: By Theorem 1 i) we have

$$(3.10) \quad I(h, g_1, g_2; a_1, b_1, a_2, b_2) \geq I(h, g_1; a_1, b_1) = \phi(a_1, b_1).$$

Further we shall define an appropriate sequence of rv's (X_t, Y_t) that satisfy the side conditions (3.3) and $\lim_{t \rightarrow \infty} \mathbb{E} h(\|X_t - Y_t\|) = \phi(a_1, b_1)$. Let $f(x) = g_2 \circ g_1^{-1}(x)$. Then, by the Jensen's inequality and $A(g_2, g_1)$,

$$(3.11) \quad f(a_1) = f(\mathbb{E} g_1(\|X\|)) \leq \mathbb{E} f \circ g_1(\|X\|) = a_2$$

as well as $f(b_1) \leq b_2$. Moreover $\lim_{t \rightarrow \infty} f(t)/t = \infty$ by $G(g_1, g_2)$.

Case 1. Suppose that $f(a_1) < a_2$, $f(b_1) < b_2$.

Claim. If the functions $f \in M$ and the reals c_1, c_2 possess the properties

$$(3.12) \quad f(c_1) < c_2, \quad \lim_{t \rightarrow \infty} f(t)/t = \infty$$

then there exist a positive t_0 and a function $k(t)$ ($t \geq t_0$) such that the following relations hold for any $t \geq t_0$:

$$(3.13) \quad 0 < k(t) < c_1,$$

$$(3.14) \quad tf(c_1 - k(t)) + k(t)f(c_1 + t) = c_2(k(t) + t),$$

$$(3.15) \quad \frac{k(t)}{k(t)+t} \leq \frac{c_2}{f(c_1+t)},$$

and

$$(3.16) \quad \lim_{t \rightarrow \infty} k(t) = 0.$$

Proof of the claim: Let us take such t_0 that $f(c_1+t)/(c_1+t) > c_2/c_1$, $t \geq t_0$ and consider the equation

$$F(t,x) = c_2,$$

where $F(t,x) := f(c_1-x)t/(x+t) + f(c_1+t)x/(x+t)$. For each $t \geq t_0$ we have $F(t,c_1) > c_2$, $F(t,0) = f(c_1) < c_2$. Hence, for each $t \geq t_0$ there exists such $x = k(t)$, that $k(t) \in (0, c_1)$ and $F(t, k(t)) = c_2$, which provides (3.13) and (3.14). Further (3.14) implies (3.15) as well as (3.13) and (3.15) imply (3.16). The claim is established.

From the claim, we see that there exist $t_0 > 0$ and functions $\ell(t)$ and $m(t)$ ($t \geq t_0$) such that for all $t > t_0$ we have

$$(3.17) \quad 0 < \ell(t) < a_1, \quad 0 < m(t) < b_1,$$

$$(3.18) \quad tf(a_1 - \ell(t)) + \ell(t)f(a_1 + t) = a_2(\ell(t) + t),$$

$$(3.19) \quad tf(b_1 - m(t)) + m(t)f(b_1 + t) = b_2(m(t) + t),$$

$$(3.20) \quad \lim_{t \rightarrow \infty} \ell(t) = 0, \quad \lim_{t \rightarrow \infty} m(t) = 0.$$

Using (3.17)-(3.20) and the conditions $A(h, g_1)$, $D(h, g_2)$ and $G(g_2, g_1)$, one can easily check that the rv's (X_t, Y_t) ($t > t_0$) determined by the equalities

$$\Pr\{X_t = x_i(t), Y_t = y_j(t)\} = p_{ij}(t), \quad i, j = 1, 2,$$

where

$$x_1(t) := g_1^{-1}(a_1 - \ell(t))\bar{e}, \quad x_2(t) := g_1^{-1}(a_1 + t)\bar{e},$$

$$y_1(t) := g_1^{-1}(b_1 - m(t))\bar{e}, \quad y_2(t) := g_1^{-1}(b_1 + t)\bar{e},$$

$$p_{11}(t) := \min\{t/(\ell(t) + t), t/(m(t) + t)\},$$

$$p_{12}(t) := t/(\ell(t) + t) - p_{11}(t),$$

$$p_{21}(t) := t/(m(t) + t) - p_{11}(t),$$

$$p_{22}(t) := \min\{\ell(t)/(\ell(t) + t), m(t)/(m(t) + t)\}$$

possess all the desired properties.

Case 2. Suppose $f(a_1) = a_2$ (i.e. $g_1^{-1}(a_1) = g_2^{-1}(a_2)$), $f(b_1) < b_2$.

Then we determine (X_t, Y_t) by the equalities

$$\Pr\{X_t = g_1^{-1}(a_1), Y_t = y_1(t)\} = t/(m(t) + t),$$

$$\Pr\{X_t = g_1^{-1}(a_1), Y_t = y_2(t)\} = m(t)/(m(t) + t).$$

Case 3. The cases $(f(a_1) < a_2, f(b_1) = b_2)$, $(f(a_1) = a_2, f(b_1) = b_2)$ are considered in the same way as in Case 2.

ii) and iii) are proved by the analogous arguments.

QED

Corollary 2. Let $a_i \geq 0, b_i \geq 0, a_1 + a_2 > 0, b_1 + b_2 > 0$

$$a_1^{1/q_1} \leq a_2^{1/q_2}, \quad b_1^{1/q_1} \leq b_2^{1/q_2}.$$

i) If $1 \leq q_1 \leq p < q_2$, then

$$(3.21) \quad I(p, q_1, q_2; a_1, b_1, a_2, b_2) = I(p, q_1; a_1, b_1) = (a_1^{1/q_1} - b_1^{1/q_1})^p.$$

ii) If $0 < p \leq q_1$, $1 \leq q_1 < q_2$ then

$$(3.22) \quad S(p, q_1, q_2; a_1, b_1, a_2, b_2) = S(p, q_1; a_1, b_1) = (a_1^{1/q_1} + b_1^{1/q_1})^p.$$

iii) If $0 < q_1 < q_2 < p$ and $a_1^{1/q_1} = a_2^{1/q_2}$ or $b_1^{1/q_1} = b_2^{1/q_2}$ then

$$S(p, q_1, q_2; a_1, b_1, a_2, b_2) = S(p, q_1; a_1, b_1) = \infty.$$

Corollary 2 describes situations in which the "additional moment information" $a_2 = \mathbb{E} \|X\|^{q_2}$, $b_2 = \mathbb{E} \|Y\|^{q_2}$ does not affect the bounds

$$I(p, q_1, q_2; a_1, b_1, a_2, b_2) = I(p, q_1, a_1, a_2), \quad S(p, q_1, q_2; a_1, b_1, a_2, b_2) = S(p, q_1, a_1, a_2),$$

(and likewise Theorem 2).

We conclude this section by giving the explicit solution of the following moment problem: *determine the extremal values*

$$(3.23) \quad \tilde{I} := \tilde{I}(p, q_1, q_2, a_1, a_2) := \inf\{\mathbb{E} \|X\|^p : \mathbb{E} \|X\|^{q_1} = a_1, \mathbb{E} \|X\|^{q_2} = a_2\},$$

$$(3.24) \quad \tilde{S} := \tilde{S}(p, q_1, q_2, a_1, a_2) := \sup\{\mathbb{E} \|X\|^p : \mathbb{E} \|X\|^{q_1} = a_1, \mathbb{E} \|X\|^{q_2} = a_2\}$$

for all $p \geq 0$, $0 \leq q_1 \leq q_2$.

Theorem 3. Let $p \geq 0$, $0 \leq q_1 < q_2$. Then

$$\tilde{I} = \begin{cases} \frac{a_1^{(q_2-p)/(q_2-q_1)}}{a_2^{(q_1-p)/(q_2-q_1)}} & \text{if } 0 \leq p \leq q_1 < q_2, \\ \frac{a_2^{(p-q_1)/(q_2-q_1)}}{a_1^{(p-q_2)/(q_2-q_1)}} & \text{if } 0 \leq q_1 < q_2 \leq p, \\ a_1^{p/q} & \text{if } 0 < q_1 \leq p < q_2 \text{ or } a_1^{1/q_1} = a_2^{1/q_2}, 0 < q_1 < q_2, \\ a_i & \text{if } p = q_i \quad i = 1 \text{ or } 2 \\ 0 & \text{if } 0 = q_1 < p < q_2 \end{cases}$$

and

$$\tilde{S} = \begin{cases} \frac{a_1 \frac{(q_2 - p)/(q_2 - q_1)}{a_2 \frac{(q_1 - p)/(q_2 - q_1)}}{a_1^{p/q_1}} & \text{if } 0 \leq q_1 \leq p \leq q_2, q_1 < q_2 \\ a_1^{p/q_1} & \text{if } 0 < p \leq q_1 < q_2 \text{ or } 0 = p < q_1 < q_2 \\ \text{or } a_1^{1/q_1} = a_2^{1/q_2}, 0 < q_1 < q_2, \\ a_i & \text{if } p = q_i, i = 1 \text{ or } 2, \\ \infty & \text{if } 0 \leq q_1 < q_2 < p, a_1^{1/q_1} \neq a_2^{1/q_2} \\ 1 & \text{if } p = 0, 0 < q_1 < q_2 \text{ or } a_1^{1/q_1} = a_2^{1/q_2}. \end{cases}$$

Proof: If $a_1^{1/q_1} = a_2^{1/q_2}$ then it is easy to see that $\|X\| = a_1^{1/q_1}$ with probability 1. Hence,

$$(3.25) \quad \tilde{I} = \tilde{S} = a_1^{p/q_1} \text{ if } p > 0, 0 < q_1 < q_2, a_1^{1/q_1} = a_2^{1/q_2}.$$

Further, suppose that $a_1^{1/q_1} \neq a_2^{1/q_2}$. For any $r > 1$ and any real-valued ξ and η ,

$$(3.26) \quad \mathbb{E}|\xi\eta| \leq (\mathbb{E}|\xi|^r)^{1/r} (\mathbb{E}|\eta|^{\frac{r}{r-1}})^{\frac{r-1}{r}}$$

by the Hölder's inequality. Let $0 < r_1 < r_2 < r_3$, $r = (r_3 - r_1)/(r_3 - r_2)$,

$$\xi = \|X\|^{r_1(r_3 - r_2)/(r_3 - r_1)}, \eta = \|X\|^{r_3(r_2 - r_1)/(r_3 - r_2)}. \text{ Then}$$

by (3.26),

$$(3.27) \quad \mathbb{E}\|X\|^{r_2} \leq (\mathbb{E}\|X\|^{r_1})^{r_1(r_3 - r_2)/(r_3 - r_1)} (\mathbb{E}\|X\|^{r_3})^{r_3(r_2 - r_1)/(r_3 - r_2)}.$$

Taking $r_1 = p$, $r_2 = q_1$, $r_3 = q_2$ in (3.37) we obtain

$$(3.28) \quad \mathbb{E} \|X\|^{q_1} \leq (\mathbb{E} \|X\|^p)^{(q_2 - q_1)/(q_2 - p)} (\mathbb{E} \|X\|^{q_2})^{(q_1 - p)/(q_2 - p)}$$

which implies for $0 < q_1 < q_2$,

$$(3.29) \quad \tilde{I} \geq a_1^{(q_2 - p)/(q_2 - q_1)} a_2^{(p - q_1)/(q_2 - q_1)}.$$

Taking $r_1 = q_1$, $r_2 = q_2$, $r_3 = p$ in (3.7) we obtain

$$(3.30) \quad \tilde{I} \geq a_2^{(p - q_1)/(q_2 - q_1)} a_1^{(p - q_2)/(q_2 - q_1)} \quad \text{for } q_1 < q_2 < p.$$

Finally, putting $r_1 = q_1$, $r_2 = p$, $r_3 = q_2$ in (3.27) we have

$$(3.31) \quad \tilde{S} \leq a_1^{(q_2 - p)/(q_2 - q_1)} a_2^{(p - q_1)/(q_2 - q_1)} \quad \text{for } q_1 < p < q_2.$$

The "optimal" rv X^* (for all inequalities (3.29)-(3.31) and $p \neq 0$) is given by

$$\Pr\{X^* = a_2^{1/(q_2 - q_1)} a_1^{-1/(q_2 - q_1)} e\} = a_1^{q_2/(q_2 - q_1)} a_2^{-q_1/(q_2 - q_1)} := b,$$

$$\Pr\{X^* = 0\} = 1 - b.$$

Thus the equalities

$$(3.32) \quad \tilde{I} = \begin{cases} a_1^{(q_2 - p)/(q_2 - q_1)} a_2^{-(q_1 - p)/(q_2 - q_1)} & \text{if } 0 < p < q_1 < q_2 \\ a_2^{(p - q_1)/(q_2 - q_1)} a_1^{-(p - q_2)/(q_2 - q_1)} & \text{if } 0 < q_1 < q_2 < p, \end{cases}$$

$$(3.33) \quad \tilde{S} = a_1^{(q_2 - p)/(q_2 - q_1)} a_2^{(p - q_1)/(q_2 - q_1)} \quad \text{if } 0 < q_1 < p < q_2$$

and the inequality

$$(3.34) \quad \tilde{I} \leq a_1^{q_2/(q_2 - q_1)} a_2^{-q_1/(q_2 - q_1)} \quad \text{if } p = 0, q_1 < q_2$$

are claimed.

Further, Ljapunov's inequality implies

$$(3.35) \quad \tilde{I} \geq a_1^{p/q_1} \quad \text{for } q_1 < p$$

and

$$(3.36) \quad \tilde{S} \leq a_1^{p/q_1} \quad \text{for } p < q_1.$$

Thus, it is sufficient to determine a sequence of rv's $\{X_t\}_{t \geq t_0}$ such that

$$(3.37) \quad \mathbb{E} \|X_t\|^{q_1} = a_1, \quad \mathbb{E} \|X_t\|^{q_2} = a_2, \quad \lim_{t \rightarrow \infty} \mathbb{E} \|X_t\|^p = a_1^{p/q_1}.$$

Now we can use the claim in the proof of Theorem 2 with $f(t) = t^{q_2/q_1}$, $c_i = a_i$ ($i = 1, 2$) and define the sequence $\{X_t\}_{t \geq t_0}$ by

$$(3.38) \quad \Pr\{X_t = (a_1 - k(t))^{1/q_1} e\} = \frac{t}{k(t) + t}$$

$$(3.39) \quad \Pr\{X_t = (a_1 + t)^{1/q_1} e\} = \frac{k(t)}{k(t) + t},$$

where $k(t)$ satisfies the relationships (3.13)-(3.16). By (3.35)-

(3.37), it follows that

$$(3.40) \quad \tilde{I} = a_1^{p/q_1} \quad \text{if } 0 < q_1 < p < q_2,$$

$$(3.41) \quad \tilde{S} = a_1^{p/q_1} \quad \text{if } 0 < p < q_1 < q_2.$$

If $p > q_2$ then the sequence $\{X_t\}$, $t \geq 0$, possesses the property

$$\lim_{t \rightarrow \infty} \mathbb{E} \|X_t\|^p = \infty. \quad \text{So,}$$

$$(3.42) \quad \tilde{S} = \infty \quad \text{if } 0 < q_1 < q_2 < p.$$

Putting $r = r_2/r_1$, $\xi = \|X\|^{r_1}$, $\eta = I\{X \neq \bar{0}\}$ in (3.6) we obtain

$$(3.43) \quad \mathbb{E} \|X\|^{r_1} \leq (\mathbb{E} \|X\|^{r_2})^{r_1/r_2} (\Pr\{X \neq \bar{0}\})^{(r_2 - r_1)/r_2}$$

Taking $r_1 = q_1$, $r_2 = q_2$ we have

$$(3.44) \quad \tilde{I} \geq a_1^{q_2/(q_2 - q_1)} a_2^{-q_1/(q_2 - q_1)} \quad \text{for } 0 = p < q_1 < q_2.$$

By (3.34) and (3.43),

$$(3.45) \quad \tilde{I} = a_1^{q_2/(q_2 - q_1)} a_2^{-q_1/(q_2 - q_1)} \quad \text{for } 0 = p < q_1 < q_2.$$

Analogously, letting $r_1 = p$, $r_2 = q_2$ in (3.43) we have

$$(3.46) \quad \tilde{S} \leq a_1^{(q_2 - p)/q_2} a_2^{p/q_2} \quad \text{if } 0 = q_1 < q_2 < p.$$

Also (3.3) with $r_1 = q_2$, $r_2 = p$ implies

$$(3.47) \quad \tilde{I} \geq a_2^{p/q_2} a_1^{(p - q_2)/q_2} \quad \text{if } 0 = q_1 < q_2 < p.$$

The "optimal" rv X^* for the last two cases is given by

$$\Pr\{X^* = (\frac{a_2}{a_1})^{1/q_2}\} = a_1, \quad \Pr\{X^* = 0\} = 1 - a.$$

Thus,

$$(3.48) \quad \tilde{S} = a_1^{(q_2 - p)/q_2} a_2^{p/q_2}, \quad \tilde{I} = a_2^{p/q_2} a_1^{(p - q_2)/q_2} \quad \text{if } 0 = q_1 < q_2 < p.$$

Further, the sequence $\{X_n\}_{n>1/a_1}$ defined by

$$\Pr\{X_n = (na_2 - 1)^{1/q_2} e\} = 1/n,$$

$$\Pr\{X_n = (na_2 - 1)^{-1/q_2} e\} = a_1 - 1/n,$$

$$\Pr\{X_n = \bar{0}\} = 1 - a_1$$

possesses the properties:

$$\Pr\{X_n \neq \bar{0}\} = a_1, \quad \mathbb{E} \|X_n\|^{q_2} = a_2,$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \|X_n\|^p = \infty \quad \text{if } q_2 < p,$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \|X_n\|^p = 0 \quad \text{if } p < q_2.$$

Hence,

$$(3.49) \quad \tilde{S} = \infty \quad \text{if } 0 = q_1 < q_2 < p,$$

$$(3.50) \quad \tilde{I} = 0 \quad \text{if } 0 = q_1 < p < q_2.$$

Finally, if $p = q_i$, then

$$(3.51) \quad \tilde{I} = \tilde{S} = a_i, \quad i = 1, 2.$$

Summarizing (3.32), (3.32), (3.40), (3.41), (3.42), (3.45), (3.48)-
(3.51) we obtain the desired explicit representations for \tilde{I} and \tilde{S} .

4. General remarks

Combining Theorems A and 1 we obtain the following precise estimates of the extremal functionals $\hat{L}_H(P,Q)$ ($P,Q \in \mathcal{P}(U)$) and $\check{L}_H(P,Q)$ (see Theorem A) in terms of the moments

$$(4.1) \quad a = \int_U g(x)P(dx), \quad b = \int_U g(x)Q(dx).$$

Theorem 4. *Let $(U, \|\cdot\|)$ be a separable normed space and $H \in \mathcal{H}$ (see (1.7)).*

i) *If $A(H,g)$ and $B(g)$ hold, then*

$$(4.2) \quad \hat{L}_H(P,Q) \geq H(|g^{-1}(a) - g^{-1}(b)|).$$

ii) *If $B(g)$ and $E(H,g)$ hold, then*

$$(4.3) \quad \check{L}_H(P,Q) \leq H(g^{-1}(a) + g^{-1}(b)).$$

Moreover there exist $P_i, Q_i \in \mathcal{P}(U)$, $i = 1, 2$ with

$$a = \int_U g(x)P_i(dx), \quad b = \int_U g(x)Q_i(dx)$$

such that

$$\hat{L}_H(P_1, Q_1) = H(|g^{-1}(a) - g^{-1}(b)|)$$

and

$$\check{L}_H(P_2, Q_2) = H(g^{-1}(a) + g^{-1}(b)).$$

The following theorem is an extension for $p = q = 1$ of Theorem 1 to a non-normed space U such as the Skorokhod space $D[0,1]$, which is of special interest in probability and statistics (see Billingsley (1968)).

Theorem 5. Let (U, d) be a separable metric space, $X = X(U)$ the space of all U -valued rv's, $u \in U$, $a \geq 0$, $b \geq 0$. Assume that there exists $z \in U$ such that $d(z, u) \geq \max(a, b)$. Then

$$(4.4) \quad \min\{\mathbb{E}d(X, Y) : X, Y \in X, \mathbb{E}d(X, u) = a, \mathbb{E}d(Y, u) = b\} = |a - b|$$

and

$$(4.5) \quad \max\{\mathbb{E}d(X, Y) : X, Y \in X, \mathbb{E}d(X, u) = a, \mathbb{E}d(Y, u) = b\} = a + b.$$

Proof: Let $a \leq b$, $\gamma = d(z, u)$. By the triangle inequality the minimum in (4.4) is greater than $b - a$. On the other hand if $\Pr(X = u, Y = u) = 1 - b/\gamma$, $\Pr(X = u, Y = z) = (b - a)/\gamma$, $\Pr(Z = z, Y = u) = 0$, $\Pr(X = z, Y = z) = a/\gamma$ then $\mathbb{E}d(X, u) = a$, $\mathbb{E}d(Y, u) = b$, $\mathbb{E}d(X, Y) = b - a$, which proves (4.4). Analogously, one shows (4.5). QED

We conclude by stating explicitly the following open problems.

(i) Find the explicit expression of $I(a, q_1, q_2; a_1, b_1, a_2, b_2)$ and $S(p, q_1, q_2; a_1, b_1, a_2, b_2)$ for all $p \geq 0$, $q_2 > 0$, $q_1 \geq 0$ (see (3.4), Corollary 2 and Theorem 3).

One could start with the following one-dimensional version of i). Let $g_i : [0, \infty) \rightarrow \mathbb{R}$ ($i = 1, 2$) and $h : \mathbb{R} \rightarrow \mathbb{R}$ be given continuous functions with h symmetric and strictly increasing on $[0, \infty)$. Let further X and Y be nonnegative random variables having given moments

$$a_i = \mathbb{E}g_i(X), \quad b_i = \mathbb{E}g_i(Y), \quad i = 1, 2.$$

The problem is to evaluate

$$(4.5) \quad I = \inf \mathbb{E}h(X - Y), \quad S = \sup \mathbb{E}h(X + Y).$$

If desired one could think of $X = X(t)$ and $Y = Y(t)$ as functions on the unit interval (with Lebesgue measure), see Karlin and Studden (1966), Chapter 3 and Rogosinski (1958).

The 5 moments a_1, a_2, b_1, b_2 and $\mathbb{E}h(X \pm Y)$ depend only on the joint distribution of the pair (X, Y) and the extremal values in (4.5) are realized by a probability measure supported by 6 points. (See Rogosinski (1958), Theorem 1; Karlin and Studden (1966), Chapter 3; Kemperman (1985)). Thus the problem can also be formulated as the nonlinear programming problem to find

$$I = \inf \sum_{j=1}^6 p_j h(u_j - v_j), \quad S = \sup \sum_{j=1}^6 p_j h(u_j + v_j),$$

subject to

$$p_j \geq 0, \quad \sum_{j=1}^6 p_j = 1, \quad u_j \geq 0, \quad v_j \geq 0, \quad j = 1, \dots, 6,$$

$$\sum_{j=1}^6 p_j g_i(u_j) = a_i, \quad \sum_{j=1}^6 p_j g_i(v_j) = b_i, \quad i = 1, 2, \dots$$

Such a problem becomes simpler when the functions g_i and h on \mathbb{R}_+ are convex (see, for example, Karlin and Studden (1966), Chapter XIV).

(ii) *Find the explicit solutions of moment problems with one fixed pair of marginal moments for rv's with values in a separable metric space U . (See Theorem 5).*

Note that in the case when U is a normed space, the moment problem was easily reduced to the one-dimensional moment problem ($U = \mathbb{R}$). This is no longer possible for general (non-normed) spaces U rendering the moment problem (ii) quite different for that considered in Sections 2 and 3.

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