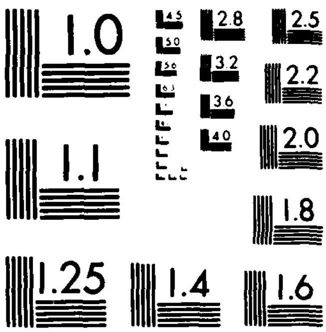
AD-A186 031 ON THE DIRECTION OF ARRIVAL ESTIMATION(U) PITTSBURGH UNIV PA CENTER FOR MULTIVARIATE ANALYSIS Z D BAI ET AL. JUN 87 TR-87-12 AFOSR-TR-87-1110 1/1

UNCLASSIFIED F49620-85-C-0008 F/G 17/11 NL



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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 87-1110	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On the direction of arrival estimation		5. TYPE OF REPORT & PERIOD COVERED Journal ~~technical~~ - June 1987
		6. PERFORMING ORG. REPORT NUMBER 87-12
7. AUTHOR(s) Z.D. Bai, P.R. Krishnaiah & L.C. Zhao		8. CONTRACT OR GRANT NUMBER(s) N00014-85-K-0292 F49620-85-C-0008
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Multivariate Analysis University of Pittsburgh 515 Thackeray Hall, Pittsburgh, PA 15260		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304 A5
11. CONTROLLING OFFICE NAME AND ADDRESS nm Air Force Office of Scientific Research Bldg 410 BAFB DC 20332-6448		12. REPORT DATE June 1987
		13. NUMBER OF PAGES 17
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Same as 11		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DTIC ELECTE OCT 14 1987 S D E

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Direction of arrival, signal processing

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The estimation of arrival direction is an important task in signal processing and has recently received considerable attention in the literature. In this paper, the authors proposed a method to estimate the direction of arrival and proved the strong consistency of the estimates for both cases in presence of white noise and colored noise.

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AFOSR-TR- 87-1110

ON THE DIRECTION OF ARRIVAL ESTIMATION*

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June 1987

Technical Report 87-12

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*This work is supported by Contract N00014-85-K-0292 of the Office of Naval Research and contract F49620-85-C-0008 of the Air Force Office of Scientific Research. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

ABSTRACT

The estimation of arrival direction is an important task in signal processing and has recently received considerable attention in the literature. In this paper, the authors proposed a method to estimate the direction of arrival and proved the strong consistency of the estimates for both cases in presence of white noise and colored noise.

AMS 1980 Subject classifications: Primary 62F12; Secondary 62H12.

Keywords and phrases: Direction of arrival, signal processing.

1. INTRODUCTION

Since the work of Schmidt (1981) and that of Bienvenu (1979), which in turn were extensions of Pisarenko (1973), the eigenstructure methods for direction of arrival (DOA) have been developed rapidly in the past few years, and have attracted considerable interest. When the additive sensor noise is spatially white, Wax, Shan and Kailath (1984) proposed a method for estimating the DOA. This method is based on the fact that the DOA vectors are orthogonal to those eigenvectors of the true covariance matrix of observations associated with the smallest eigenvalue. In some cases, the noise is not spatially white and its covariance is unknown and in this case the algorithm of Wax, Shan and Kailath is no longer appliable. In these cases, Paulraj and Kailath (1986) proposed a method to estimate the DOA based on the difference of two covariance matrices. Their method relies on the fact that the DOA vectors are orthogonal to the eigenvectors of the difference matrix associated with the zero eigenvalue. Both methods of Shan-Wax-Kailath and Paulraj-Kailath are based upon finding the infimum of a Hermitian form with constrained variables.

However, though simulation results strongly supported the above two methods for estimating the DOA, it is not an easy task to find the solutions for the infimum of the constrained Hermitian form. In the present paper, we investigate the estimation of DOA for both cases where the noise is white or colored. In the argorithm for estimating the DOA, we only need to solve a polynomial equation whose degree is just the number of signals. Also, we shall prove that this estimate is strongly consistent under minor moment restrictions. In another paper (in preparation) we shall investigate the asymptotic normality of these estimates.

The organization of this paper is as follows: In Section 2, we shall describe the algorithm for estimating DOA when the noise is spatially white and prove the strong consistency of these estimates. In Section 3, we shall briefly describe the procedure for finding the estimate of number of signals by using information theoretical criteria and the estimate of DOA by the proposed method when the noise is colored. We only point out that these estimates are also strongly consistent and omit the details, because the proofs are almost the same as the proof for the strong consistency of signal number estimate (see Zhao, Krishnaiah and Bai (1986 a,b) and the proof given in Section 2 for the strong consistency of estimtes of DOA.

2. ESTIMATE OF DOA IN THE PRESENCE OF SPATIALLY WHITE NOISE

Consider the model

$$\underset{\sim}{x}(t) = A\underset{\sim}{s}(t) + \underset{\sim}{n}(t), \quad t = 1,2,\ldots,N, \tag{2.1}$$

where $\underset{\sim}{x}(t)$: $p\times 1$, the observations received by p sensors, $\underset{\sim}{s}(t)$: $q\times 1$, the signal vector emitted by q sources, $q < p$, $\underset{\sim}{n}(t)$ is the white noise vector, $A = (\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_q)$ and $\underset{\sim}{a}_k = (1, e^{-j\omega_0\tau_k},\ldots,e^{-j\omega_0(p-1)\tau_k})^T$, called the direction-frequency vector associated with the k^{th} signal $j = \sqrt{-1}$, ω_0 the center frequency of signals and $\tau_k = \frac{\Delta}{c}\sin\theta_k$, Δ the spacing between sensors, c the speed of propagation and θ_k the direction of k^{th} signal. Since ω_0 is known, we can assume ω_0 in the sequel.

It is usual to assume that

$$\left.\begin{array}{ll} \text{(i)} & \{\underset{\sim}{s}(t)\} \text{ are independent and identically distributed (i.i.d.), } \{\underset{\sim}{n}(t)\} \\ & \text{are i.i.d., and independent of } \{\underset{\sim}{s}(t)\} \\ \text{(ii)} & E\underset{\sim}{s}(t) = \underset{\sim}{0}, E\underset{\sim}{n}(t) = \underset{\sim}{0},\ E\underset{\sim}{s}(t)\underset{\sim}{s}^*(t) = \Psi > 0, \\ & E\underset{\sim}{n}(t)\underset{\sim}{n}^*(t) = \sigma^2 I_p \text{ with } \sigma^2 \text{ unknown,} \\ \text{(III)} & \tau_k\text{'s are distinct,} \end{array}\right\} \tag{2.2}$$

where * denotes complex conjugate transpose.

Under the model (2.1), our problem is to find an estimate of τ_k's based on the sample covariance matrix

$$\hat{\Sigma} = \frac{1}{N}\sum_{i=1}^{N} \underset{\sim}{x}(i)\underset{\sim}{x}^*(i).$$

The covariance matrix of $\underset{\sim}{x}(t)$ is given by

$$\Sigma = A\Psi A^* + \sigma^2 I_p$$

Denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ and $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_p$ the eigenvalues of Σ and $\hat{\Sigma}$ respectively. Also, let $\underset{\sim}{e}_1,\ldots,\underset{\sim}{e}_p$ and $\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p$ denote the eigenvectors associated with these λ's and δ's respectively. Without loss of generality, we assume that $\underset{\sim}{u}$'s are of unit length and orthogonal of each other, and the same is true for $\underset{\sim}{e}$'s. If the number of sources is q, then we have

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q > \lambda_{q+1} = \cdots = \lambda_p (=\sigma^2).$$

The key steps of Wax - Shan - Kailath algorithm are as follows. First, determine the number of sources q. Next, find the so-called noise subspace as the span of the eigenvectors corresponding to the minimal (noise) eigenvalues σ^2 of Σ. The subspace spanned by the direction vectors of the impinging signal wavefronts, which is called signal subspace, can be obtained as the orthogonal complement of the noise subspace. For determining the DOA's, they plotted the inverse Hermitian form that measures the orthogonality between the direction vectors and the noise subspace, i.e.,

$$H_\theta^{(\Sigma)} = [\underset{\sim}{a}_\theta^* E_n^{(\Sigma)} E_n^{(\Sigma)*} \underset{\sim}{a}_\theta]^{-1} \tag{2.3}$$

where $\underset{\sim}{a}_\theta = (1,e^{-j\theta},\ldots,e^{-j(p-1)\theta})^T$ and $E_n^{(\Sigma)}$ is an $p\times(p-q)$ matrix whose columns are the eigenvectors associated with the minimum eigenvalues of Σ. They pointed out that, "Ideally, $\underset{\sim}{a}_\theta \perp E_n^{(\Sigma)}$, for $\theta = \theta_k$, and hence $H_\theta^{(\Sigma)}$ should become very large at these θ_k, enabling us to pick out the source directions." In other words, they might extract these θ_k's by seeking for the extreme points of $H_\theta^{(\hat{\Sigma})-1}$, a polynomial of $e^{-j\theta}$ with degree $2(p-1)$.

But there are two problems: (1) We do not know the number of the extreme points. (2) No method is proposed to extract the desired q θ_k's from these extreme points.

Now we introduce a new method as follows:

Since the information theoretic criterion (ITC) gives strongly consistent estimate of the number of signals, we can assume that q is known throughout this section. (refer to Zhao, Krishnaiah and Bai [1986a]).

Write

$$W_N = (\underset{\sim}{u}_{q+1}, \ldots, \underset{\sim}{u}_p). \tag{2.4}$$

By a knowledge of linear algebra, there exists an unitary matrix $0_N:(p-q)\times(p-q)$, such that

$$\begin{aligned} &W_n 0_N = (\hat{\underset{\sim}{u}}_{q+1}, \ldots, \hat{\underset{\sim}{u}}_p) = (\hat{u}_{ik})\ _{1\le i\le p,\ q+1\le k\le p} \\ &\text{with } \hat{u}_{ik} = 0, \text{ for } k = q+1,\ldots,p-1 \text{ and } i = k+1,\ldots,p, \\ &\qquad \hat{u}_{ii} \geq 0, \text{ for } i = q+1,\ldots,p. \end{aligned} \tag{2.5}$$

Also, if all $\hat{u}_{ii} > 0$, then 0_N is uniquely determined.

Let $z_k = \hat{\rho}_k \exp(j\hat{\tau}_k)$, $k = 1,2,\ldots,q$ be roots of

$$B(z) = \sum_{k=1}^{q+1} \underset{\sim}{u}_{k,q+1} z^{k-1} \tag{2.6}$$

where $\hat{\rho}_k \geq 0$ and $\hat{\tau}_k \in [0,2\pi)$. Then we take $\hat{\tau}_k$, $k = 1,2,\ldots,q$ as the estimates of τ_k's.

<u>Remark 2.1</u>. Sometimes, $\hat{u}_{q+1,q+1}$ may be zero. In such a case, there may be less than q roots for B(z), and we can not get q estimates of τ_k's. However, in the large sample case, we can prove that with probability one, $\hat{u}_{q+1,q+1} > 0$ for large N.

<u>Remark 2.2</u>. Using the Schmidt orthogonalization procedure, we can seek for 0_N and $(\hat{u}_{ik})$.

Remark 2.3. Using our method, we do not bother about answering the two problems mentioned above.

In the sequel, we will establish the strong consistency of $\hat{\tau}_k$'s. Before doing that, we introduce the following lemma.

Lemma 2.1. Let $A = (a_{ik})$ and $B = (b_{ik})$ are two Hermitian p×p matrices with spectrum decompositions

$$A = \sum_{i=1}^{p} \delta_i \underset{\sim}{u}_i \underset{\sim}{u}_i^*, \quad \delta_1 \geq \delta_2 \geq \cdots \geq \delta_p,$$

and

$$B = \sum_{i=1}^{p} \lambda_i \underset{\sim}{v}_i \underset{\sim}{v}_i^*, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p,$$

where δ's and λ's are eigenvalues of A and B respectively, $\underset{\sim}{u}$'s and $\underset{\sim}{v}$'s are orthogonal unit eigenvectors associated with δ's and λ's respectively. Further, we assume that

$$\lambda_{n_{h-1}+1} = \lambda_{n_h} = \tilde{\lambda}_h, \; n_0 = 0 < n_1 < \cdots < n_s = p, \; h = 1,2,\ldots,s,$$

$$\tilde{\lambda}_1 > \tilde{\lambda}_2 > \cdots > \tilde{\lambda}_s,$$

and that

$$|a_{ik} - b_{ik}| < \alpha, \quad i, k = 1,\ldots,p.$$

Then there is a constant M independent of α, such that

(i) $|\delta_i - \lambda_i| < M\alpha, \quad i = 1,2,\ldots,p$

(ii) $\sum_{i=n_{h-1}+1}^{n_h} \underset{\sim}{u}_i \underset{\sim}{u}_i^* = \sum_{i=n_{h-1}+1}^{n_h} \underset{\sim}{v}_i \underset{\sim}{v}_i^* + C^{(h)}$ with

$$C^{(h)} = (C^{(h)}_{\ell k}), \; |C^{(h)}_{\ell k}| \le M\alpha, \; \ell, \; k = 1,2,\ldots,p, \; h = 1,2,\ldots,s.$$

Proof. By Von-Neumann's inequality, one can easily obtain

$$\sum_{i=1}^{p} (\delta_i - \lambda_i)^2 \le \operatorname{tr}(A-B)^2,$$

which implies (i) with $M = p$.

For simplicity, we denote by $D = 0(\alpha)$ the fact that $|d_{ik}| \le M\alpha$, $i=1,2,\ldots,m$, $k = 1,2,\ldots,n$ for any $m\times n$ matrix $D = (d_{ik})$. To prove (ii), without loss of generality, we can assume

$$A = \sum_{h=1}^{s} \tilde{\lambda}_h \sum_{i\in L_h} \underset{\sim}{u}_i \underset{\sim}{u}_i^*, \quad B = \sum_{h=1}^{s} \tilde{\lambda}_h \sum_{i\in L_h} \underset{\sim}{v}_i \underset{\sim}{v}_i^*,$$

where $L_h = \{n_{h-1}+1,\ldots,n_h\}$. When $s = 1$, (ii) is trivial. Now we assume (ii) is true for $s = t-1$, and proceed to prove (ii) for $s = t$. When $s = t$,

$$\sum_{h=1}^{t-1} (\tilde{\lambda}_h - \tilde{\lambda}_t) \sum_{i\in L_h} \underset{\sim}{u}_i \underset{\sim}{u}_i^* = \sum_{h=1}^{t-1} (\tilde{\lambda}_h - \tilde{\lambda}_t) \sum_{i\in L_h} \underset{\sim}{v}_i \underset{\sim}{v}_i^* + 0(\alpha). \tag{2.7}$$

Multiply from right hand by $\underset{\sim}{v}_k$, $k\in L_t$ in the two hand sides of (2.7), we get

$$\sum_{h=1}^{t-1} (\tilde{\lambda}_h - \tilde{\lambda}_t) \sum_{i\in L_h} \underset{\sim}{u}_i (\underset{\sim}{u}_i^* \underset{\sim}{v}_k) = 0(\alpha)$$

which implies that

$$\underset{\sim}{u}_i^* \underset{\sim}{v}_k = 0(\alpha), \quad i\notin L_s, \; k\in L_s,$$

Thus, we have

$$U_1^* V_2 = 0(\alpha), \quad V_1^* U_2 = 0(\alpha), \tag{2.8}$$

where

$$U_1 = (\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_{n_{t-1}}), \quad U_2 = (\underset{\sim}{u}_{n_{t-1}+1},\ldots,\underset{\sim}{u}_{n_t}), \ n_t = p,$$

$$V_1 = (\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_{n_{t-1}}), \quad V_2 = (\underset{\sim}{v}_{n_{t-1}+1},\ldots,\underset{\sim}{v}_{n_t}).$$

Put $U_2 = V_1G_1 + V_2G_2$, where G_1: $n_{t-1}\times(p-n_{t-1})$, G_2: $(p-n_{t-1})\times(p-n_{t-1})$. By (2.8),

$$\begin{aligned} V_2^*U_2U_2^*V_2 &= V_2^*(I_p-U_1U_1^*)V_2 = V_2^*V_2 + 0(\alpha^2) \\ &= I_{p-n_{t-1}} + 0(\alpha) \end{aligned} \tag{2.9}$$

By (2.8) and (2.9), we get

$$0(\alpha) = V_1^*U_2 = G_1 + V_1^*V_2G_2 = G_1,$$

which implies that

$$U_2 = V_2G_2 + 0(\alpha). \tag{2.10}$$

By (2.9) and (2.10),

$$\begin{aligned} G_2G_2^* &= V_2^*V_2G_2G_2^*V_2^*V_2 = V_2^*U_2U_2^*V_2 + 0(\alpha) \\ &= I_{p-n_{t-1}} + 0(\alpha) \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), it follows that

$$\sum_{i\in L_t} \underset{\sim}{u}_i\underset{\sim}{u}_i^* = U_2U_2^* = V_2V_2^* + 0(\alpha) = \sum_{i\in L_t} \underset{\sim}{v}_i\underset{\sim}{v}_i^* + 0(\alpha), \tag{2.12}$$

and that

$$\sum_{h=1}^{t-2} \tilde{\lambda}_h \sum_{i \in L_h} \underset{\sim}{u}_i \underset{\sim}{u}_i^* + \tilde{\lambda}_{t-1} \sum_{i \in L_{t-1}+L_t} \underset{\sim}{u}_i \underset{\sim}{u}_i^*$$

$$= \sum_{h=1}^{t-2} \tilde{\lambda}_h \sum_{i \in L_h} \underset{\sim}{v}_i \underset{\sim}{v}_i^* + \tilde{\lambda}_{t-1} \sum_{i \in L_{t-1}+L_t} \underset{\sim}{v}_i \underset{\sim}{v}_i^* + O(\alpha)$$

By the induction assumption,

$$\sum_{i \in L_h} \underset{\sim}{u}_i \underset{\sim}{u}_i^* = \sum_{i \in L_h} \underset{\sim}{v}_i \underset{\sim}{v}_i^* + O(\alpha), \quad h = 1,\ldots,t-2, \tag{2.13}$$

and

$$\sum_{i \in L_{t-1}} \underset{\sim}{u}_i \underset{\sim}{u}_i^* = \sum_{i \in L_{t-1}+L_t} \underset{\sim}{v}_i \underset{\sim}{v}_i^* + O(\alpha). \tag{2.14}$$

Thus, (ii) is true for $s = t$ by (2.12) - (2.14). Lemma 2.1 is proved.

We have the following:

<u>THEOREM 2.1</u> Suppose the 4^{th} moments of $\underset{\sim}{s}(t)$ and $\underset{\sim}{n}(t)$ are finite. Then the estimates $\hat{\tau}_k$'s are strongly consistent.

Proof. Let $\underset{\sim}{b} = (b_1, b_2, \ldots, b_{q+1}, 0, \ldots, 0)^T$ be the p×1 vector whose elements $b_1, b_2, \ldots, b_{q+1}$ are the coefficients of the polynomial $b_{q+1} \prod_{k=1}^{q} (z - e^{j\tau_k}) \triangleq f(z)$ with restrictions $\sum_{k=1}^{q+1} |b_k|^2 = 1$ and $b_{q+1} > 0$.

Let $\underset{\sim}{\eta}_{q+1} = \underset{\sim}{b}$, $\underset{\sim}{\eta}_{q+2} = (0, b_1, b_2, \ldots, b_{q+1}, 0, \ldots, 0)^T, \ldots, \underset{\sim}{\eta}_p = (0, \ldots, 0, b_1, \ldots, b_{q+1})^T$ be all p×1 vectors. From $a_k^* \underset{\sim}{\eta}_{q+\ell} = e^{j(\ell-1)\tau_k} f(e^{j\tau_k}) = 0$, $k = 1,2,\ldots,q$, $\ell = 1,\ldots,p-q$, it follows that $\underset{\sim}{\eta}_{q+1}, \ldots, \underset{\sim}{\eta}_p$ are all eigenvectors of $A\Psi A^*$ associated with zero eigenvalue. Since they are linearly independent, they span the eigensubspace of $A\Psi A^*$ associated with zero eigenvalue. Let V_2 denote this subspace and let $P_2(\hat{\underset{\sim}{u}}_{q+1}) = \sum_{k=q+1}^{p} \beta_k^{(N)} \underset{\sim}{\eta}_k$ denote the projection of $\underset{\sim}{u}_{q+1}$ on v_2. By the strong law of large numbers, we have

$$\hat{\Sigma} \to \Sigma = A\Psi A^* + \sigma^2 I_p, \quad \text{a.s.} \quad \text{as } N \to \infty$$

by Lemma 2.1, it is easy to see that

$$\hat{\underset{\sim}{u}}_{q+1} = P_2(\hat{\underset{\sim}{u}}_{q+1}) + o(1) \quad \text{a.s.} \quad \text{as } N \to \infty \tag{2.14}$$

Since $\underset{\sim}{u}_{q+1,\ell} = 0$ for $\ell = q+2,\ldots,p$, we see that the last p-q-1 components of $P_2(\hat{\underset{\sim}{u}}_{q+1}) = \sum_{k=q+1}^{p} \beta_k^{(N)} \underset{\sim}{\eta}_k$ tend to zero almost surely. From this and the expressions of $\underset{\sim}{\eta}_k$, $k = q+1,\ldots,p$, noting that b_{q+1} are positive constants, we get

$$\lim_{N\to\infty} \beta_k^{(N)} = 0 \quad \text{a.s.} \quad \text{for } k = q+2,\ldots,p$$

and

$$\lim_{N\to\infty} \beta_{q+1}^{(N)} = 1 \quad \text{a.s.},$$

which implies that

$$\hat{\underset{\sim}{u}}_{q+1} \to \underset{\sim}{\eta}_{q+1} = \underset{\sim}{b}, \quad \text{a.s.} \quad \text{as } N \to \infty. \tag{2.15}$$

By the definition of $\underset{\sim}{b}$, we know that $e^{j\tau_k}$, $k = 1,2,\ldots,q$, are the roots of the polynomial equation

$$\sum_{k=1}^{q+1} b_k z^{k-1} = 0. \tag{2.16}$$

Hence, after suitable rearrangement,

$$\hat{\rho}_k e^{j\hat{\tau}_k} \to e^{j\tau_k}, \quad \text{a.s.}, \quad k = 1,2,\ldots,q.$$

and consequently,

$$\hat{\rho}_k \to 1 \quad \text{a.s.}$$

$$\hat{\tau}_k \to \tau_k, \quad k = 1,2,\ldots,q, \quad \text{a.s.} \quad \text{as } N \to \infty \tag{2.17}$$

which proves the theorem.

Remark 2.4. If q is known, to ensure the strong consistency of $\hat{\tau}_k$'s, we only need to assume the second moments of $\underset{\sim}{s}(t)$'s and $\underset{\sim}{n}(t)$'s. But in ITC procedure, to guarantee the strong consistency of the estimate of the signal number, we assumed the 4^{th} moments of $\underset{\sim}{s}(t)$ and $\underset{\sim}{n}(t)$ exist. (Refer to Zhao, Krishnaiah and Bai [1986a]). Therefore in this theorem we still assume the 4^{th} moments exist, so that the conclusion of Theorem 2.1 is still true by using the ITC estimate of signal number $\hat{q}$ instead of q when q is unknown.

3. ESTIMATE OF DOA IN THE PRESENCE OF COLORED NOISE

In the section 2, we obtain an estimate of DOA's when the additive sensor noise is spatially white. When the sensor noise is colored, the case is more complicated. For this case, Paulraj and Kailath (1986) proposed a solution to the DOA estimation problem. Their technique is applicable to situations where it is possible to obtain two estimates of the array covariance in which the unknown noise field remains invariant while the signal field undergoes some change. This method is based on computing the difference of the two measured covariances, thus subtracting out the unknown noise covariance and leaving only the difference matrix of the two signal covariance.

Assume that there are two estimates of the array covariance with the array being displaced between the measurements. This displacement could be of several types. Examples of spatial displacements are rotations, translations, or a combination of the two. Displacements can also be of a temporal nature with the noise statistics being long-term stationary, while those of the signals are only short-term stationary. For the details, refer to Paulraj and Kailath (1986). Here we assume that the noise covariance matrix is invariant across the two measurements while the signal covariance matrix and the DOA's change in some manner between the measurements. Thus we have the following model:

$$\underset{\sim}{x}^{(\ell)}(t) = A^{(\ell)}\underset{\sim}{s}^{(\ell)}(t) + \underset{\sim}{n}^{(\ell)}(t), \quad t = 1,2,\ldots,N, \ \ell = 1,2 \tag{3.1}$$

where $\underset{\sim}{x}^{(\ell)}(t)$: $p\times 1$, the observations received by p sensors for the ℓ^{th} measurements, $\underset{\sim}{s}^{(\ell)}(t)$:$q_1\times 1$, the signal vector emitted by q_1 sources, $\ell = 1,2$, $\underset{\sim}{n}^{(\ell)}(t)$ is the colored noise for the ℓ^{th} measurements, $A^{(\ell)} = (\underset{\sim}{a}_1^{(\ell)},\ldots,\underset{\sim}{a}_{q_1}^{(\ell)})$ and $\underset{\sim}{a}_k^{(\ell)} = (1,e^{-j\tau_k^{(\ell)}},\ldots,e^{-j(p-1)\tau_k^{(\ell)}})^T$, $k = 1,2,\ldots,q_1$ and $\ell = 1,2$.

It is usual to assume that

(i) For each ℓ, $\ell = 1,2$, $\{\underset{\sim}{s}^{(\ell)}(t)\}$ iid., $\{\underset{\sim}{n}^{(\ell)}(t)\}$ iid., and independent of $\{\underset{\sim}{s}^{(\ell)}(t)\}$,

$$\left.\begin{aligned}
&\text{(ii) } E\underset{\sim}{s}^{(\ell)}(t) = \underset{\sim}{0},\ E\underset{\sim}{n}^{(\ell)}(t) = \underset{\sim}{0},\\
&\quad E\underset{\sim}{s}^{(\ell)}(t)\underset{\sim}{s}^{(\ell)*}(t) = \Psi^{(\ell)} > 0,\\
&\quad E\underset{\sim}{n}^{(\ell)}(t)\underset{\sim}{n}^{(\ell)*}(t) = \Sigma_0 > 0, \quad \ell = 1,2,\ t = 1,2,\ldots,N,
\end{aligned}\right\} \tag{3.2}$$

where $\Psi^{(1)}$, $\Psi^{(2)}$ and Σ_0 are all unknown.

The covariance matrix of $x^{(\ell)}(t)$ is given by

$$\Sigma^{(\ell)} = A^{(\ell)}\Psi^{(\ell)}A^{(\ell)*} + \Sigma_0, \quad \ell = 1,2.$$

For the translational invariance model, we know that $A^{(1)} = A^{(2)}$, and

$$\Sigma^{(1)} - \Sigma^{(2)} = A^{(1)}(\Psi^{(1)} - \Psi^{(2)})A^{(1)*}, \tag{3.3}$$

where we assume that $\Psi^{(1)} - \Psi^{(2)}$ is of rank q_1 and $q_1 < p$. Also, we assume that $\tau_k^{(1)}$'s are distinguished. This means that $A^{(1)}$ is of full rank (i.e., $= q_1$).

For other invariance models, we have

$$\Sigma^{(1)} - \Sigma^{(2)} = (A^{(1)},A^{(2)})\begin{pmatrix}\Psi^{(1)} & 0\\ 0 & -\Psi^{(2)}\end{pmatrix}(A^{(1)},A^{(2)})^*, \tag{3.4}$$

were we assume that $\tau_k^{(1)}$'s and $\tau_k^{(2)}$'s are all distinguished and $2q_1 < p$. In this case, $(A^{(1)},A^{(2)})$ is full rank (i.e., $= 2q_1$).

For the model (3.3), we write $A = A^{(1)}$, $\tilde{\Psi} = \Psi^{(1)} - \Psi^{(2)}$, and $q = q_1$. For the model (3.4), we write $A = (A^{(1)},A^{(2)})$, $\tilde{\Psi} = \begin{pmatrix}\Psi^{(1)} & 0\\ 0 & -\Psi^{(2)}\end{pmatrix}$ and $q = 2q_1$.

Put

$$\hat{\Sigma}_\ell = \frac{1}{N}\sum_{i=1}^{N} x^{(\ell)}(i)x^{(\ell)*}(i), \quad \ell = 1,2.$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q > \lambda_{q+1} = \cdots = \lambda_p = 0$ and $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_p$ denote the eigenvalues of $(\Sigma_1-\Sigma_2)^2$ and $(\hat{\Sigma}_1-\hat{\Sigma}_2)^2$ respectively. Take C_N satisfying

$$\frac{C_N}{N} \to 0 \quad \text{and} \quad \frac{C_N}{\log\log N} \to \infty \quad \text{as } N \to \infty. \tag{3.5}$$

Write

$$I(k,C_N) = N \sum_{i=k+1}^{p} \delta_i + k\, C_N, \tag{3.6}$$

and define $\hat{q}$ as follows

$$I(\hat{q},C_N) = \min\{I(0,C_N),\ldots,I(p-1,C_N)\}. \tag{3.7}$$

We have the following

<u>THEOREM 3.1</u>. Suppose that (3.2) holds, A is of rank q and the 4^{th} moments of $\underset{\sim}{n}^{(\ell)}(t)$ exist for $\ell = 1,2$. Then $\hat{q}$ is a strongly consistent estimate of q.

Proof. By the law of the iterated logarithm, for $\ell = 1,2$,

$$\hat{\Sigma}_\ell = \Sigma_\ell + O\left(\sqrt{\tfrac{1}{N}\log\log N}\right) \quad \text{a.s.} \quad \text{as } N \to \infty, \tag{3.8}$$

Using Lemma 2.1, we have

$$\lim_{N\to\infty} \delta_i = \lambda_i \quad \text{a.s.}, \quad i = 1,2,\ldots,p. \tag{3.9}$$

Since the matrix $\Sigma_1 - \Sigma_2$ is of rank q, there exists a $p\times(p-q)$ matrix $Q_0^T Q_0 = I_{p-q}$ and $Q_0^T(\Sigma_1-\Sigma_2) = 0$. It is well known that

$$\sum_{i=q+1}^{p} \delta_i = \min_{Q^TQ=I_{p-q}} \operatorname{tr} Q^T(\hat{\Sigma}_1-\hat{\Sigma}_2)^2 Q. \tag{3.10}$$

By (3.8),

$$Q_0^T(\hat{\Sigma}_1-\hat{\Sigma}_2) = Q_0^T(\hat{\Sigma}_1-\hat{\Sigma}_2-(\Sigma_1-\Sigma_2)) = O(\sqrt{\tfrac{1}{N}\log\log N}), \quad \text{a.s.} \tag{3.11}$$

By (3.10) and (3.11),

$$0 \le \sum_{i=q+1}^{p} \delta_i \le \operatorname{tr} Q_0^T(\hat{\Sigma}_1-\hat{\Sigma}_2)^2 Q_0 = O(\tfrac{1}{N}\log\log N), \text{ a.s.} \tag{3.12}$$

Using (3.9) and (3.12), noticing that $\lambda_q > 0$, we can easily prove that, with probability one for N large,

$$I(q,C_N) < I(k,C_N), \quad k \neq q, \ k \le p-1,$$

which implies that

$$\hat{q} = q.$$

Theorem 3.1 is proved.

In the sequel, we assume that q is known. Write $\Psi = \tilde{\Psi}A^*A\tilde{\Psi}$, then $(\Sigma_1-\Sigma_2)^2$ can be rewritten as

$$(\Sigma_1-\Sigma_2)^2 = A\Psi A^*.$$

Note that A is of the form $A = (\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_q)$ with

$$\underset{\sim}{a}_k = (1,e^{-j\tau_k},\ldots,e^{-j(p-1)\tau_k})^T, \quad k = 1,\ldots,q,$$

where τ_k's are distinguished. So the problem of estimating the DOA's reduces the case of section 2. Let $\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p$ denote the eigenvectors of $(\hat{\Sigma}_1-\hat{\Sigma}_2)^2$ associated with $\delta_1,\ldots,\delta_p$. Based on $W_N = (\underset{\sim}{u}_{q+1},\ldots,\underset{\sim}{u}_p)$, we can use the method proposed by us in the section 2, and take $\hat{\tau}_k$, $k = 1,2,\ldots,q$, as the estimates of τ_k's. In the same way, we have

<u>THEOREM 3.2</u>. Under the conditions of Theorem 3.1, $\hat{\tau}_k$'s are strongly consistent estimates of τ_k's.

Remark 2.4 also applies to this case.

END

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