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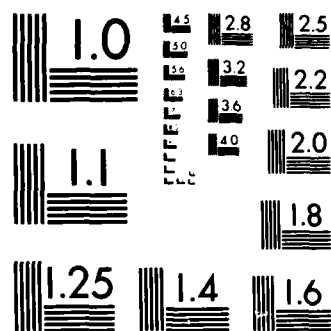
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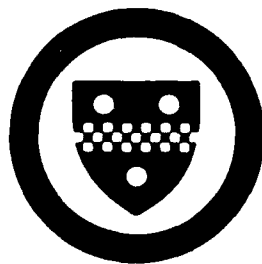
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LINEAR PREDICTION ESTIMATES OF THE NUMBER
OF SIGNALS AND FREQUENCIES OF MULTIPLE SINUSOIDS*

Z.D. Bai, P.R. Krishnaiah and L.C. Zhao

Center for Multivariate Analysis
University of Pittsburgh

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Center for Multivariate Analysis
 Fifth Floor, Thackeray Hall
 University of Pittsburgh
 Pittsburgh, PA 15260



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1. INTRODUCTION

In a companion paper, the authors (see Bai, Krishnaiah and Zhao (1986)) considered the problem of estimation of the number of signals and the frequencies of these signals under a signal processing model with multiple sinusoids. The number of signals was estimated by using an information theoretic criterion. They have also established the strong consistency of their estimates. In this paper, we establish the rates of convergence of the above estimates of the number of signals and frequencies.

2. PRELIMINARIES AND STATEMENT OF PROBLEM

Consider the model

$$y(n) = \sum_{j=1}^t a_j \exp(i\omega_j n) + w(n), \quad n = 1, 2, \dots, N \quad (2.1)$$

where $i = \sqrt{-1}$, $\{a_j\}$ is a set of complex amplitudes, $\{\omega_j\}$ is a set of frequencies and $\{w(n)\}$ is the noise sequence of independent and identically distributed (i.i.d.) complex random variables with mean zero and $E|w(n)|^2 = \sigma^2 < \infty$. We assume that the frequencies $\omega_j \in (0, 2\pi)$ are different from each other. Also, $y(n)$ is complex valued spatial sample observed at n -th array element. We will now describe the method of estimation of t_0 , the number of signals, and ω_j 's, the frequencies, considered in our earlier paper (Bai, Krishnaiah and Zhao(1986)). To determine t_0 , it is assumed a priori that $t_0 \leq T < \infty$. Let

$$S_t = \min\left\{\frac{1}{(N-t)} \sum_{n=t+1}^N \left| \sum_{\ell=0}^t b_{\ell}^{(t)} \overline{y(n-\ell)} \right|^2\right\} \quad (2.2)$$

for $t = 0, 1, \dots, T$, where the coefficients $b_{\ell}^{(t)}$ are subject to the restriction $\sum_{\ell=0}^t |b_{\ell}^{(t)}|^2 = 1$. Also, let

$$R_t = S_t + tC_N \quad (2.3)$$

where C_N satisfies the following restrictions:

$$(i) \quad \lim_{N \rightarrow \infty} C_N = 0$$

$$(ii) \quad \lim_{N \rightarrow \infty} \{\sqrt{N} C_N / \sqrt{\log \log N}\} = \infty. \quad (2.4)$$

The proposed estimate of t_0 is given by $\hat{t}_0$ where $\hat{t}_0$ is given by

$$R_{\hat{t}_0} = \min\{R_0, R_1, \dots, R_T\} \quad (2.5)$$

Now, let $\hat{b} = (\hat{b}_0, \dots, \hat{b}_{\hat{t}_0})'$ be such that

$$S_{\hat{t}_0} = \frac{1}{(N - \hat{t}_0)} \sum_{n=\hat{t}_0+1}^N \left| \sum_{\ell=0}^{\hat{t}_0} \hat{b}_{\ell} \overline{y(n-\ell)} \right|^2 \quad (2.6)$$

and let $\hat{\rho}_j \exp(i\hat{\omega}_j)$ be the roots of

$$\hat{H}(z) = \sum_{j=0}^{\hat{t}_0} \hat{b}_j z^j \quad (2.7)$$

where $\hat{\rho}_j \geq 0$ and $\hat{\omega}_j \in [0, 2\pi)$ for $j = 1, \dots, \hat{t}_0$. Then $\omega_1, \dots, \omega_{t_0}$ were estimated with $\hat{\omega}_1, \dots, \hat{\omega}_{\hat{t}_0}$ respectively. The strong consistency of $\hat{t}_0$ and $\hat{\omega}_j$'s was also established. In this paper, we are interested in establishing the rates of convergence of the above estimates.

Throughout this paper, A^* and A^- respectively denote transpose of the conjugate of A and general inverse of A .

3. CONVERGENCE RATE OF THE ESTIMATE OF THE NUMBER OF SIGNALS

In this section, we establish the rate of convergence of $\hat{t}_0$. Let

$$\hat{\gamma}_{\ell m}^{(t)} = \frac{1}{N-t} \sum_{n=t+1}^N y(n-\ell) \overline{y(n-m)}, \quad \ell, m=0,1,\dots,t, \quad (3.1)$$

and $\hat{\Gamma}^{(t)} = (\hat{\gamma}_{\ell m}^{(t)})$. Also, let $\hat{b}_t = (\hat{b}_0^{(t)}, \hat{b}_1^{(t)}, \dots, \hat{b}_t^{(t)})'$ be such that

$$S_t = \frac{1}{N-t} \sum_{n=t+1}^N \left| \sum_{\ell=0}^t \hat{b}_\ell^{(t)} \overline{y(n-\ell)} \right|^2. \quad (3.2)$$

Here we note that S_t is the smallest eigenvalue of the matrix $\hat{\Gamma}^{(t)}$.

Denote by $\delta_{\ell m}$ the Kronecker delta. Write

$$\gamma_{\ell m} = \sum_{j=1}^t |a_j|^2 e^{i(m-\ell)\omega_j} + \sigma^2 \delta_{\ell m}, \quad (3.3)$$

and $\Gamma^{(t)} = (\gamma_{\ell m})$ for $0 \leq \ell, m \leq t$. We can write $\hat{\gamma}_{\ell m}^{(t)}$ as

$$\hat{\gamma}_{\ell m}^{(t)} = \gamma_{\ell m} + J_1 + J_2 + J_3 + J_4 \triangleq \gamma_{\ell m} + \Delta_{N\ell m}^{(t)}, \quad (3.4)$$

$$\begin{aligned} J_1 &= \sum_{\substack{j,k=1 \\ j \neq k}}^t a_j \overline{a_k} e^{i(m\omega_k - \ell\omega_j)} \frac{1}{N-t} \sum_{n=t+1}^N e^{in(\omega_j - \omega_k)} \\ J_2 &= \sum_{j=1}^t a_j e^{i(m-\ell)\omega_j} \frac{1}{N-t} \sum_{n=t+1}^N e^{i(n-m)\omega_j} \overline{w(n-m)} \\ J_3 &= \sum_{j=1}^t \overline{a_j} e^{i(m-\ell)\omega_j} \frac{1}{N-t} e^{-i(n-\ell)\omega_j} w(n-\ell) \\ J_4 &= \frac{1}{N-t} \sum_{n=t+1}^N [w(n-\ell) \overline{w(n-m)} - \delta_{\ell m} \sigma^2]. \end{aligned} \quad (3.5)$$

Write the eigenvalues of $\Gamma^{(t)}$ as $\lambda_0^{(t)} \geq \dots \geq \lambda_t^{(t)} > 0$. From the structure of $\Gamma^{(t)}$, we have, for any $t \geq 1$,

$$\lambda_0^{(t)} \geq \lambda_0^{(t-1)} \geq \lambda_1^{(t)} \geq \lambda_1^{(t-1)} \geq \dots \geq \lambda_{t-1}^{(t-1)} \geq \lambda_{(t)}^{(t)} \geq \sigma^2$$

and for $t \geq t_0$

$$\lambda_{t_0}^{(t)} = \lambda_{t_0+1}^{(t)} = \dots = \lambda_t^{(t)} = \sigma^2$$

$$\lambda_{t_0-1}^{(t)} \geq \lambda_{t_0-1}^{(t_0-1)} > \sigma^2. \quad (3.6)$$

Now, let $\Delta = \lambda_{t_0-1}^{(t_0-1)} - \sigma^2$. Also, denote the eigenvalues of $\hat{\Gamma}(t)$ by

$$\hat{\lambda}_0^{(t)} \geq \hat{\lambda}_1^{(t)} \geq \dots \geq \hat{\lambda}_t^{(t)}. \quad \text{Then by Lemma 2.1 in Bai, Krishnaiah and Zhao (1985),}$$

we have

$$\sum_{\ell=0}^t (\hat{\lambda}_\ell^{(t)} - \lambda_\ell^{(t)})^2 \leq \text{tr}(\hat{\Gamma}^{(t)} - \Gamma^{(t)})^2. \quad (3.7)$$

Also, since $\hat{\lambda}_t^{(t)} = S_t$, we have

$$|S_t - \lambda_t^{(t)}| \leq \sqrt{\text{tr}(\hat{\Gamma}^{(t)} - \Gamma^{(t)})^2} \quad t = 0, 1, 2, \dots, T. \quad (3.8)$$

If $t > t_0$, then

$$\text{tr}(\hat{\Gamma}^{(t)} - \Gamma^{(t)})^2 < \frac{1}{4} C_N^2 \quad (3.9)$$

and

$$\text{tr}(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)})^2 < \frac{1}{4} C_N^2 \quad (3.10)$$

implies that

$$\begin{aligned} R_t - R_{t_0} &= S_t - S_{t_0} + (t - t_0)C_N \\ &\geq C_N - [\sqrt{\text{tr}(\Gamma^{(t)} - \hat{\Gamma}^{(t)})^2} + \sqrt{\text{tr}(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)})^2}] \\ &> 0. \end{aligned} \quad (3.11)$$

Here we use the fact that $\lambda_t^{(t)} = \lambda_{t_0}^{(t_0)} = \sigma^2$. Hence $\hat{t}_0 \neq t$. If $t < t_0$, then

$$\text{tr}(\hat{\Gamma}^{(t)} - \hat{\Gamma}^{(t)})^2 < \frac{1}{4} C_N^2 \quad (3.12)$$

and (3.10) implies that

$$\begin{aligned} R_t - R_{t_0} &= S_t - S_{t_0} - (t_0 - t)C_N \\ &\geq \lambda_t^{(t)} - \sigma^2 - \{\sqrt{\text{tr}(\hat{\Gamma}^{(t)} - \hat{\Gamma}^{(t)})^2} + \sqrt{\text{tr}(\hat{\Gamma}^{(t_0)} - \hat{\Gamma}^{(t_0)})^2} + (t_0 - t)C_N\} \\ &\geq \Delta - (t_0 + 2)C_N > 0 \end{aligned}$$

provided that $C_N < \Delta/(t_0 + 2)$. Hence $\hat{t}_0 \neq t$. Since $C_N \rightarrow 0$ for large N , we have $C_N < \Delta/(t_0 + 2)$. Therefore

$$P(\hat{t}_0 \neq t_0) \leq P\left(\bigcup_{t=0}^T \{\text{tr}(\hat{\Gamma}^{(t)} - \hat{\Gamma}^{(t)})^2 \geq \frac{1}{4} C_N^2\}\right). \quad (3.13)$$

Theorem 3.1. If we choose C_N satisfying the conditions

$$\lim_{N \rightarrow \infty} C_N = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \sqrt{N} C_N = \infty$$

then $E|W(n)|^{2\eta} < \infty$, $\eta > 1$ implies

$$P(\hat{t}_0 \neq t_0) = O(N(N C_N)^{-\eta}), \text{ as } N \rightarrow \infty.$$

Also, $E \exp\{h|W(n)|^2\} < \infty$ for some $h > 0$, implies

$$P(\hat{t}_0 \neq t_0) = O(e^{-\delta N C_N^2}), \text{ for some } \delta > 0.$$

Proof. Both conclusions of this theorem follow from (3.13) and the expressions given in (3.5) and the well-known results of limit theorems concerning sums of independent random variables.

4. CONVERGENCE RATE OF FREQUENCY ESTIMATES

In this section, we establish the rate of convergence of the frequency estimates $\hat{\omega}_1, \dots, \hat{\omega}_{t_0}$.

Let $\underline{b}_{t_0}$ be the eigenvector of $\hat{\Gamma}^{(t_0)}$ corresponding to the smallest eigenvalue σ^2 . Also, let

$$\begin{aligned}\hat{\underline{b}}_{t_0} &= \beta [I - (\hat{\Gamma}^{(t_0)} - S_{t_0} I)^* (\hat{\Gamma}^{(t)} - S_{t_0} I) (\hat{\Gamma}^{(t_0)} - S_{t_0} I)^* - (\hat{\Gamma}^{(t_0)} - S_{t_0} I)] \underline{b}_{t_0} \\ &= \beta [I - (\hat{\Gamma}^{(t_0)} - S_{t_0} I) + (\hat{\Gamma}^{(t_0)} - S_{t_0} I)] \underline{b}_{t_0}\end{aligned}\quad (4.1)$$

where $\beta = \beta_N$ is a positive constant which normalizes $\hat{\underline{b}}_{t_0}$ to unit length. Note that $\hat{\Gamma}^{(t_0)} \rightarrow \Gamma^{(t_0)}$ and $S_{t_0} \rightarrow \sigma^2$, with probability one. When N is large enough, we know that

$$[I - (\hat{\Gamma}^{(t_0)} - S_{t_0} I) + (\hat{\Gamma}^{(t_0)} - S_{t_0} I)] \underline{b}_{t_0} \neq 0.$$

Hence $\hat{\underline{b}}_{t_0}$ is well-defined. It is easy to verify that $\hat{\underline{b}}_{t_0}$ is the eigenvector of $\hat{\Gamma}^{(t_0)}$ corresponding to the smallest eigenvalue S_{t_0} .

By the triangle inequality and (4.1), we have

$$|\hat{\underline{b}}_{t_0} - \underline{b}_{t_0}| \leq 2 |(\hat{\Gamma}^{(t_0)} - S_{t_0} I) + (\hat{\Gamma}^{(t_0)} - S_{t_0} I) \underline{b}_{t_0}|. \quad (4.2)$$

Since $(\Gamma^{(t_0)} - \sigma^2 I) \underline{b}_{t_0} = 0$ and $|\underline{b}_{t_0}| = 1$, we have

$$\begin{aligned}& |(\hat{\Gamma}^{(t_0)} - S_{t_0} I) \underline{b}_{t_0}| \\ &= |(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)} - (S_{t_0} - \sigma^2) I) \underline{b}_{t_0}| \\ &\leq [\text{tr}(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)})^2]^{\frac{1}{2}} + |S_{t_0} - \sigma^2|\end{aligned}$$

Hence

$$|\hat{b}_{t_0} - b_{t_0}| \leq 2|\hat{\lambda}_{t_0-1}^{(t_0)} - s_{t_0}|^{-1} \{ [\text{tr}(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)})^2]^{\frac{1}{2}} + |s_{t_0} - \sigma^2| \}. \quad (4.3)$$

Now, let $\Delta = \lambda_{t_0-1}^{(t_0)} - \sigma^2 > 0$. Also, let $n > 0$ be a small number such that $n < \Delta/4(t_0+1)$. Then, for $t = t_0$, using (3.7) $|\hat{\gamma}_{j\ell} - \gamma_{j\ell}| \leq n$, $j, \ell = 0, 1, \dots, t_0$, implies

$$[\text{tr}(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)})^2]^{\frac{1}{2}} \leq (t_0+1)n,$$

$$|s_{t_0} - \sigma^2| \leq (t_0+1)n,$$

and

$$\begin{aligned} \hat{\lambda}_{t_0-1}^{(t_0)} - s_{t_0} &\geq \lambda_{t_0-1}^{(t_0)} - \sigma^2 - 2(\text{tr}(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)})^2)^{\frac{1}{2}} \\ &\geq \Delta - 2(t_0+1)n > \frac{\Delta}{2}. \end{aligned} \quad (4.4)$$

Thus, by (4.2) - (4.4),

$$|\hat{b}_{t_0} - b_{t_0}| < 8(t_0+1)\Delta^{-1}n. \quad (4.5)$$

For any $\varepsilon > 0$, let $n = \varepsilon\Delta/[8(t_0+1)]$. Then $|\hat{\gamma}_{\ell m} - \gamma_{\ell m}| \leq n$, $\ell, m = 0, 1, \dots, t_0$ implies that

$$|\hat{b}_{t_0} - b_{t_0}| < \varepsilon. \quad (4.6)$$

Therefore

$$P(|\hat{b}_{t_0} - b_{t_0}| \geq \varepsilon) \leq \sum_{\ell=0}^{t_0} \sum_{m=0}^{t_0} P(|\hat{\gamma}_{\ell m} - \gamma_{\ell m}| \geq n). \quad (4.7)$$

Now, define $|b_1 - b_2|$ as usual if b_1, b_2 have common dimensionality and ∞ otherwise. Also, $\hat{b} = (\hat{b}_0, \hat{b}_1, \dots, \hat{b}_{t_0})'$. From (4.7) we have

$$P(|\hat{b} - b_{t_0}| \geq \varepsilon) \leq P(|\hat{b}_{t_0} - b_{t_0}| \geq \varepsilon) + P(\hat{t}_0 \neq t_0) \quad (4.8)$$

From (4.7), (4.8) and what has been proved in Section 3 we obtain the following theorem.

Theorem 4.1. Let C_N be chosen satisfying the conditions

$$(i) \lim_{N \rightarrow \infty} C_N = 0 \quad (ii) \lim_{N \rightarrow \infty} \sqrt{N} C_N / \sqrt{\log \log N} = \infty.$$

Then $E|w(n)|^{2\eta} < \infty$, $\eta > 1$ implies

$$P(|\hat{b} - b_{t_0}| \geq \epsilon) = O(N(NC_N)^{-\eta}). \quad (4.9)$$

Also,

$$E \exp\{h|w(n)|^2\} < \infty \text{ for some } h > 0$$

implies

$$P(|\hat{b} - b_{t_0}| > \epsilon) = O(\exp\{-bNC_N^2\}) \text{ for some } b > 0 \quad (4.10)$$

Since the roots of the polynomial are continuous functions of coefficients of the polynomial, we have

Theorem 4.2. Suppose $\hat{\rho}_j e^{i\hat{\omega}_j}$, $j = 0, 1, \dots, \hat{t}_0$ are the roots of $\sum_{\ell=0}^{\hat{t}_0} \hat{b}_\ell z^\ell$. $\hat{\omega}_j \in [0, 2\pi)$ and $\hat{\omega}_j$'s are arranged in increasing order. Also, $\exp(i\omega_j)$, $j=0, 1, \dots, t_0$ are ranked in the increasing order of ω_j . In addition, let $\hat{z} = (\hat{\rho}_0 e^{i\hat{\omega}_0}, \hat{\rho}_1 e^{i\hat{\omega}_1}, \dots, \hat{\rho}_{\hat{t}_0} e^{i\hat{\omega}_{\hat{t}_0}})$ and $z = (e^{i\omega_0}, \dots, e^{i\omega_{t_0}})$, we choose C_N satisfying the conditions

$$(i) \lim_{N \rightarrow \infty} C_N = 0, \quad (ii) \lim_{N \rightarrow \infty} \sqrt{N} C_N / \sqrt{\log \log N} = \infty.$$

Then $E|w(n)|^{2\eta} < \infty$, $\eta > 1$ implies

$$P(|\hat{z} - z| \geq \epsilon) = O(N(C_N N)^{-\eta}). \quad (4.11)$$

Also,

$$E \exp\{h|w(n)|^2\} < \infty \text{ for some } h > 0 \quad (4.12)$$

implies

$$P(|\hat{z} - z| \geq \epsilon) = O(\exp\{-bNC_N^2\}) \text{ for some } b > 0 \quad (4.13)$$

We note that (4.12) and (4.13) are respectively equivalent to the statements that $P(\max_{1 \leq \ell \leq \hat{t}_0} |\hat{\rho}_\ell - 1| \geq \epsilon)$ is of order in (4.9) and

$P(\max_{1 \leq \ell \leq \hat{t}_0 \wedge t_0} |\hat{\omega}_\ell - \omega_\ell| \geq \epsilon)$ is of order in (4.10) where $a \wedge b$ denotes the minimum of

a and b .

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