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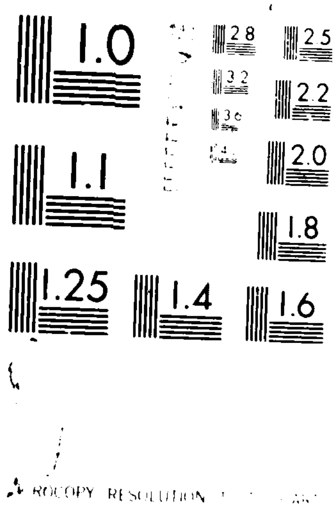
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Suppose that $X_1, \ldots, X_n$ are samples drawn from a population with density function f, and $f_n(x) = f_n(x; X_1, \ldots, X_n)$ is an estimate of f(x). Denote by $m_{nr} = \int |f_n(x) - f(x)|^r dx$ and $M_{nr} = E(m_{nr})$ the Integrated r-th Order Error and Mean Integrated r-th Order Error of f_n for some $r \geq 1$ (when r = 2, they are the familiar

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF INTEGRATED AND MEAN-INTEGRATED r-th ORDER ERROR OF HISTOGRAM DENSITY ESTIMATES*

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ABSTRACT

Suppose that $X_1, \ldots, X_n$ are samples drawn from a population with density function f, and $f_n(x) = f_n(x;X_1, \ldots, X_n)$ is an estimate of f(x). Denote by $m_{nr} = \int |f_n(x) - f(x)|^r dx$ and $M_{nr} = E(m_{nr})$ the Integrated r-th Order Error and Mean Integrated r-th Order Error of f_n for some $r \geq 1$ (when r = 2, they are the familiar and widely studied ISE and MISE). In this paper the same necessary and sufficient condition for $\lim_{n\to\infty} M_{nr} = 0$ and $\lim_{n\to\infty} m_{nr} = 0$ a.s. is obtained when $f_n(x)$ is the ordinary histogram estimator.

AMS 1980 Subject Classification : Primary 62G05.

Key Words and Phrases : Consistency, Density Estimation, Histogram.

Suppose that $X_1, \ldots, X_n$ are iid samples drawn from a d-dimensional population with density function f. Let $f_n(x) = f_n(x; X_1, \ldots, X_n)$ be an estimator of f(x). The Integrated Square Error (ISE) and Mean Integrated Square Error (MISE) of f_n are defined by

$$ISE = \int |f_n(x) - f(x)|^2 dx,$$

$$MISE = E\int |f_n(x) - f(x)|^2 dx$$

respectively. They are important and widely used criteria in evaluating the performance of an estimator f_n. Quite a lot of works appeared in the statistical literature dealing with the asymptotic properties of them, for various types of estimators, such as kernel estimator, orthogonal series estimator, nearest neighbor estimator etc. For example, a much discussed problem is to find the conditions under which the assertions

$$\lim_{n\to\infty} MISE = 0, \quad \lim_{n\to\infty} ISE = 0, a.s. \tag{1}$$

are true. Various conditions have been proposed in the literature which are far from necessary.

In this paper we obtain the necessary and sufficient conditions of (1) for the histogram estimator. In fact, we have done more. For given constant $r \geq 1$, define the integrated r-th order error m_{nr} and mean integrated r-th order error M_{nr} by

$$m_{nr} = \int |f_n(x) - f(x)|^r dx, \quad M_{nr} = E(m_{nr})$$

We obtain the necessary and sufficient conditions of $M_{nr} \to 0$ and $m_{nr} \to 0$, a.s. In the case of r=1, the problem has been considered by Abou-Jaoude([1], [2]) (see also [4], pp. 19-23).

A d-dimensional histogram is defined as follows. Choose $\mathbf{a}_n = (a_{1n}, \ldots, a_{dn}) \in R^d$, $\tilde{\mathbf{h}}_n = (h_{1n}, \ldots, h_{dn}) \in R^d$ with $h_{in} > 0$, $i=1, \ldots, d$. Denote by k the number of those $X_1, \ldots, X_n$ falling into the set $A_n(c_1, \ldots, c_d) = \{x=(x^{(1)}, \ldots, x^{(d)}): a_{in} + c_i h_{in} \leq x^{(i)} < a_{in} + (c_i + 1)h_{in}, i=1, \ldots, d\}$

Define

$$f_n(x) = k/(nh_{1n} \cdots h_{dn}), \quad \text{when } x \in A_n(c_1, \ldots, c_d),$$
$$c_1, \ldots, c_d = 0, \pm 1, \pm 2, \ldots \tag{2}$$

which is a histogram estimator of $f(x)$.

Theorem. Suppose that

$$\int f^r(x)\,dx < \infty, \quad \text{for some } r \geq 1 \tag{3}$$

$$\lim_{n\to\infty} \max_{1\leq i\leq d} h_{in} = 0 \tag{4'}$$

$$\lim_{n\to\infty} nh_{1n} \cdots h_{dn} = \infty \tag{5'}$$

Then for the histogram f_n defined by (2) we have

$$\lim_{n\to\infty} m_{nr} = 0, \text{a.s.} \tag{6}$$

$$\lim_{n\to\infty} M_{nr} = 0, \tag{7}$$

Conversely, if (6) or (7) is true, then (3), (5') are true. Further, if $f(x)$ does not have a form

$$f(x) = \sum_{i_1,\cdots,i_d=-\infty}^{\infty} b_{i_1\cdots i_d} I_{A(i_1,\ldots,i_d)}(x)$$

Then (4') is also true. Here $b_{i_1 \ldots i_d}$ is a constant, $A(i_1, \ldots, i_d) = \{x = (x^{(1)}, \ldots, x^{(d)})' : a_{j0} + i_j h_{j0} \leq x^{(j)} < a_{j0} + (i_j + 1)h_{j0}, \ j = 1, \ldots, d\}$, and $I_B(x)$ denotes the indicator of B.

Proof. For simplicity of writing, we shall give only the proof for the case of $d=1$, as the general case involves no essential change. For $d=1$ and writing h_n for $\tilde{h}_n$, the conditions (4') and (5') reduce to

$$\lim_{n\to\infty} h_n = 0 \tag{4}$$

$$\lim_{n\to\infty} nh_n = \infty \tag{5}$$

In the sequel we shall often write h for h_n, a for a_n. Note that we may assume $|a| \le h$ without loss of generality. Write

$$\Delta_\ell = \Delta_{n\ell} = [a + \ell h, a + (\ell + 1)h),$$
$$I_\ell(x) = I_{\Delta_\ell}(x),$$
$$p_\ell = \int_{\Delta_\ell} f(x)\,dx$$

Then $Ef_n(x) = p_\ell/h$, when $x \in \Delta_\ell$. The symbol "C" will be used to denote any constant not depending on n (but may depend on r).

The proof will be divided into four parts.

(A). Sufficiency of (3)–(5) for (7)

First consider the case of $r > 1$. In order to prove (7), we need only to show that

$$\int |f_n(x) - Ef_n(x)|^r dx = n^{-r} h^{-r+1} \sum_\ell \left| \sum_{i=1}^{n} (I_\ell(X_i) - p_\ell) \right|^r \to 0 \tag{8}$$

$$\int |f(x) - Ef_n(x)|^r dx \to 0 \tag{9}$$

as $n \to \infty$. (9) is easy to prove : Suppose first that f is continuous on $(-\infty,\infty)$, then for any fixed constant $A > 0$,

$$\int_{-A}^{A} |f(x) - Ef_n(x)|^r dx \to 0, \text{ as } n\to\infty$$

On the other hand, by Hölder inequality, for any fixed constant $B > 0$ we have

$$\sum \{\int_{\Delta_\ell} (Ef_n(x))^r dx : \Delta_\ell \subset [-B,B]^c\}$$
$$= \sum \{h^{-r+1} (\int_{\Delta_\ell} f(x)\,dx)^r : \Delta_\ell \subset [-B,B]^c\}$$
$$\le \sum \{\int_{\Delta_\ell} f^r(x)\,dx : \Delta_\ell \subset [-B,B]^c\}$$

$$\leq \int_{[-B,B]^c} f^r(x)\,dx \tag{10}$$

Hence

$$\sum \{ \int_{\Delta_\ell} |f(x) - Ef_n(x)|^r dx : \Delta_\ell \subset [-B,B]^c \}$$

$$\leq 2^r \{ \int_{[-B,B]^c} f^r(x)\,dx + \sum (\int_{\Delta_\ell} (Ef_n(x))^r dx : \Delta_\ell \subset [-B,B]^c \}$$

$$\leq 2^{r+1} \int_{[-B,B]^c} f^r(x)\,dx$$

Summing up the above arguments and noticing (3), we obtain (9). For the general case, choose a function g such that $\int |f(x) - g(x)|^r dx \leq$ some given $\varepsilon > 0$, define $g_n(x) = h^{-1} \int_{\Delta_\ell} g(x)dx$ for $x \in \Delta_\ell$, $\ell = 0, \pm 1, \ldots$, an argument similar to that leading to (10) gives

$$\int |Ef_n(x) - g_n(x)|^r dx \leq \int |f(x) - g(x)|^r dx \leq \varepsilon$$

From this and the fact that (9) is true when f is continuous, (9) follows easily.

For a proof of (8), put $Y_{i\ell} = I_\ell(X_i) - p_\ell$, $S_{j\ell} = \sum_{i=1}^{j} (I_\ell(X_i) - p_\ell)$, $T_{nr} = \sum_\ell |S_{n\ell}|^r$. If $1 < r \leq 2$, then from the inequality $|1 + a|^r \leq 1 + ra + C|a|^r$ (a: real), we have

$$|S_{n\ell}|^r \leq |S_{n-1,\ell}|^r + C|Y_{n\ell}|^r + r|S_{n-1,\ell}|^{r-2} S_{n-1,\ell} Y_{n\ell}$$

Therefore,

$$E|S_{n\ell}|^r \leq E|S_{n-1,\ell}|^r + CE|Y_{n\ell}|^r \leq C\sum_{i=1}^{n} E|Y_{i\ell}|^r \leq 2Cnp_\ell$$

which in turn implies

$$n^{-r}h^{-r+1} \sup_{1\leq j\leq n} ET_{jr} \leq n^{-r}h^{-r+1} \sum_\ell 2Cnp_\ell = 2C(nh)^{-r+1} \to 0 \tag{11}$$

Suppose that (11) is true for $r \in (1, m]$, we proceed to show that it is true for $r \in (m, m+1]$. Since when $r > 2$ we have the inequality

$$|1 + a|^r \leq 1 + ra + Ca^2(1 + |a|^{r-2}) \text{ (a: real)}$$

It follows that

$$|S_{n\ell}|^r \le |S_{n-1,\ell}|^r + r|S_{n-1,\ell}|^{-2}S_{n-1,\ell}Y_{n\ell}$$
$$+ C|S_{n-1,\ell}|^{r-2}Y_{n\ell}^2 + C|Y_{n\ell}|^r \tag{12}$$

which in turn implies

$$E|S_{n\ell}|^r \le E|S_{n-1,\ell}|^r + CE|S_{n-1,\ell}|^{r-2}EY_{n\ell}^2 + CE|Y_{n\ell}|^r$$

Noticing that $EY^2_{n\ell} \le p_\ell$, $E|Y_n|^r \le p_\ell$, we have

$$E|S_{\tilde{n}\ell}|^r \le E|S_{\tilde{n}-1,\ell}|^r + Cp_\ell(E|S_{\tilde{n}-1,\ell}|^{r-2}+1), \quad \tilde{n} = n, n-1, \ldots$$

Therefore

$$E|S_{n\ell}|^r \le Cp_\ell \sum_{j=1}^{n-1} E|S_{j\ell}|^{r-2} + np_\ell C \tag{13}$$

$$n^{-r}h^{-r+1} \sup_{1\le j\le n} ET_{jr}$$
$$\le C(nh)^{-r+1} \max_{1\le j\le n} \sum_\ell p_\ell E|S_{j\ell}|^{r-2} + C(nh)^{-r+1} \tag{14}$$

Since f is a probability density and $\int f^r(x)dx < \infty$, $r > 2$, we have $\int f^{r-1}(x)dx < \infty$. Hence when $r > 2$

$$\sum_\ell h^{-r+2} p_\ell^{r-1} \le \int f^{r-1}(x)\,dx = C < \infty \tag{15}$$

By induction hypothesis

$$n^{-r+1}h^{-r+2} \max_{1\le j\le n} ET_{j,r-1} \to 0, \quad \text{as } n \to \infty \tag{16}$$

from (14)-(16), we have

$$n^{-r}h^{-r+1} \max_{1\le j\le n} ET_{jr}$$
$$\le C(nh)^{-r+1} \left(\sum_\ell p_\ell^{r-1}\right)^{1/(r-1)} \left(\max_{1\le j\le n} \sum_\ell E|S_{j\ell}|^{r-1}\right)^{(r-2)/(r-1)}$$
$$+ C(nh)^{-r+1}$$

$$\le C(nh)^{-r+1} (h^{(r-2)/(r-1)}) (n^{r-2}h^{(r-2)^2/(r-1)})$$
$$(n^{-r+1}h^{-r+2} \max_{1\le j\le n} ET_{j,r-1})^{(r-2)/(r-1)} + C(nh)^{-r+1}$$
$$= C(nh)^{-1} (n^{-r+1}h^{-r+2} \max_{1\le j\le n} ET_{j,r-1})^{(r-2)/(r-1)} + C(nh)^{-r+1} \to 0$$

as $n \to \infty$. This shows that (11) is true for $r \in (m, m+1]$, concluding the proof of (11) hence (7), for $r>1$. When $r = 1$, (7) is a consequence of (6) for $r = 1$, which we are now going to prove.

(B) Sufficiency of (3)-(5) for (6)

Again first consider the case of $r>1$. Define T_{nr} as before, then

$$I_{n\ell} \equiv \int |f_n(x) - f(x)|^r dx = n^{-r} h^{-r+1} T_{nr}$$

Hence we need only to show that

$$\lim_{n\to\infty} I_{nr} = 0, \quad \text{a. s.} \tag{17}$$

Define

$$\xi_{j\ell 1} = Y_{j\ell}, \qquad \eta_{j\ell 1} = |S_{j\ell}|^{r-1} \operatorname{sign}(S_{j\ell})$$

$$\xi_{j\ell 2} = Y_{j\ell}^2 - p_\ell(1-p_\ell), \qquad \eta_{j\ell 2} = |S_{j\ell}|^{r-1}$$

and proceed to show that for any given $\varepsilon > 0$

$$J \equiv P(n^{-r} h^{-r+1} \max_{2\le k\le n} |\sum_{j=2}^{k} \sum_{\ell} \xi_{j\ell i} \eta_{j-1,\ell i}| \ge \varepsilon) \le Cn^{-2}, \quad i = 1,2 \tag{18}$$

In the following we shall write $\xi_{j\ell}$, $\eta_{j\ell}$ for $\xi_{j\ell i}$, $\eta_{j\ell i}$. Since $\{\sum_{j=2}^{k} \xi_{j\ell}\eta_{j-1,\ell}$, $k = 2, 3, \ldots\}$ is a martingale sequence, we have

$$J \le \varepsilon^{-4} n^{-4r} h^{-4r+4} CE(\sum_{j=2}^{n} \sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell})^4 \tag{19}$$

From an inequality of Rosenthal (see [5], p 23),

$$\begin{aligned} &E(\sum_{j=2}^{n} \sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell})^4 \\ &\le C(E\{\sum_{j=2}^{n} E[(\sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell})^2 \mid F_{j-1}]\}^2 \\ &+ \sum_{j=2}^{n} E(\sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell})^4) \end{aligned} \tag{20}$$

Here F_j is the σ-field generated by $X_1, \ldots, X_j$. Since $\sum_{\ell} |\xi_{j\ell}| \le 3$, by Jensen's inequality we have

$$\sum_{j=2}^{n} E\Big(\sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell}\Big)^4$$
$$\le C\sum_{j=2}^{n} E\sum_{\ell} |\xi_{j\ell}|\eta_{j-1,\ell}^4$$
$$\le C\sum_{j=2}^{n} p_\ell E\eta_{j-1,\ell}^4$$

It can be shown easily by using (13) that

$$E\eta_{j-1,\ell}^4 \le C(1+(np_\ell)^{2r-2}) \le C(1+(np_\ell)^{2r-1})$$

Noticing that $\sum_{\ell} p_\ell^r h^{-r+1} \le \int |f(x)|^r dx < \infty$ and $nh \to \infty$, we have

$$\sum_{j=2}^{n} E\Big(\sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell}\Big)^4$$
$$\le Cn\Big(\sum_{\ell} p_\ell + n^{2r-1}\sum_{\ell} p_\ell^{2r}\Big)$$
$$\le Cn\Big(1+n^{2r-1}h^{2r-2}\big(\sum_{\ell} p_\ell^r h^{-r+1}\big)^2\Big)$$
$$\le Cn^2 (nh)^{2r-2} \tag{21}$$

$$E\Big(\sum_{j=2}^{n} E\Big[\Big(\sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell}\Big)^2 \Big| F_{j-1}\Big]\Big)^2$$
$$\le n\sum_{j=2}^{n} E\Big(E\Big[\Big(\sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell}\Big)^2 \Big| F_{j-1}\Big]\Big)^2$$
$$\le 2n\sum_{j=2}^{n} E\Big(\sum_{\ell} p_\ell \eta_{j-1,\ell}^2\Big)^2$$
$$+ 2n\sum_{j=2}^{n} E\Big(\sum_{\ell \ne m} E(\xi_{j\ell}\xi_{jm}\eta_{j-1,\ell}\eta_{j-1,m} | F_{j-1})\Big)^2$$
$$\le Cn\sum_{j=2}^{n} E\Big(\sum_{\ell} p_\ell \eta_{j-1,\ell}^2\Big)^2 + Cn\sum_{j=2}^{n} E\Big(\sum_{\ell} p_\ell |\eta_{j-1,\ell}|\Big)^4$$
$$\le Cn\sum_{j=2}^{n} E\Big(\sum_{\ell} p_\ell \eta_{j-1,\ell}^2\Big)^2 \tag{22}$$

The last step is valid in view of Jensen's inequality and $\sum_{\ell} p_\ell = 1$. Therefore, from

$$E\eta_{j-1,\ell}^2\eta_{j-1,m}^2 \le (E\eta_{j-1,\ell}^4 E\eta_{j-1,m}^4)^{1/2},$$

and $\quad E\eta_{j-1,\ell}^4 \le C(1+(np_\ell)^{2r-2})$

We obtain

$$E\left(\sum_{j=2}^{n} E[(\sum_{\ell} \xi_{j\ell}\eta_{j-1,\ell})^2 | F_{j-1}]\right)^2$$

$$\leq Cn \sum_{j=2}^{n} \sum_{\ell} p_\ell^{\,2} E\eta_{j-1,\ell}^{\,4} + Cn \sum_{j=2}^{n} \sum_{\ell \neq m} p_\ell p_m (E\eta_{j-1,\ell}^{\,4} E\eta_{j-1,m}^{\,4})^{1/2}$$

$$\leq Cn \sum_{j=2}^{n} \sum_{\ell} p_\ell^{\,2}(1+(np_\ell)^{2r-2}) + Cn \sum_{j=2}^{n} (\sum_{\ell} p_\ell (1+(np_\ell)^{r-1})^2$$

$$\leq Cn^2(1 + (nh)^{2r-2}(\sum_{\ell} p_\ell^{\,r} h^{-r+1})^2) + Cn^2(1 + (nh)^{r-1} \sum_{\ell} p_\ell^{\,r} h^{-r+1})^2$$

$$\leq Cn^2 (nh)^{2r-2} \tag{23}$$

From (19)-(21) and (23), (18) follows.

Now to prove that for $r > 1$ and given $\varepsilon > 0$, we have

$$P(n^{-r}h^{-r+1} \max_{1 \leq k \leq n} T_{kr} \geq \varepsilon) \leq C/n^2 \tag{24}$$

First suppose that $1 < r \leq 2$. We have

$$|S_{n\ell}|^r \leq |S_{n-1,\ell}|^r + C|Y_{n\ell}|^r + r|S_{n-1,\ell}|^{r-1} \operatorname{sign}(S_{n-1,\ell}) Y_{n\ell}$$

Hence

$$|S_{n\ell}|^r \leq C \sum_{j=1}^{n} |Y_{j\ell}|^r + r \sum_{j=2}^{n} Y_{j\ell}\eta_{j-1,\ell 1}$$

and

$$n^{-r}h^{-r+1} \max_{1 \leq k \leq n} T_{kr}$$

$$\leq Cn^{-r}h^{-r+1} \sum_{j=1}^{n} \sum_{\ell} |Y_{j\ell}|^r$$

$$+ n^{-r}h^{-r+1} r \max_{2 \leq k \leq n} \left| \sum_{j=2}^{k} \sum_{\ell} Y_{j\ell}\eta_{j-1,\ell 1} \right| \tag{25}$$

Since

$$n^{-r}h^{-r+1} \sum_{j=1}^{n} \sum_{\ell} |Y_{j\ell}|^r \leq 2(nh)^{-r+1} \to 0$$

From (18) and (25), (24) follows.

Suppose that (24) is true for $r \leq m$, and proceed to show that it is true for $r \in (m, m+1]$. Since $r > 2$, we have

$$|S_{n\ell}|^r \le |S_{n-1,\ell}|^r + r|S_{n-1,\ell}|^{r-2} S_{n-1,\ell} Y_{n\ell}$$
$$+ C|S_{n-1,\ell}|^{r-2} Y_{n\ell}^2 + C|Y_{n\ell}|^r$$

Hence

$$n^{-r} h^{-r+1} \max_{1\le k\le n} T_{kr} \le Cn^{-r} h^{-r+1} \sum_{j=1}^{n} \sum_{\ell} |Y_{j\ell}|^r$$
$$+ rn^{-r} h^{-r+1} \max_{2\le k\le n} \left| \sum_{j=2}^{k} \sum_{\ell} Y_{j\ell} \eta_{j-1,\ell 1} \right|$$
$$+ Cn^{-r} h^{-r+1} \max_{2\le k\le n} \left| \sum_{j=2}^{k} \sum_{\ell} \xi_{j\ell 2} \tilde{\eta}_{j-1,\ell} \right|$$
$$+ C(nh)^{-r+1} \max_{2\le j\le n} \sum_{\ell} p_\ell |S_{j-1,\ell}|^{r-2} \equiv \sum_{i=1}^{4} H_i \tag{26}$$

where $\tilde{\eta}_{j-1,\ell} = |S_{j-1,\ell}|^{r-2}$. Observe that H_2/r is the expression in (18), nhH_3/C is also of the form in (18) with r replaced by r − 1. Further, since r > 2 and $\int f^r(x)dx<\infty$, we have $\int f^{r-1}(x)dx<\infty$. Therefore, the inequality (18) can be applied to both H_2 and nhH_3. Further, $nh \to \infty$, so we obtain

$$P(H_2+H_3 \ge \varepsilon) \le C/n^2 \tag{27}$$

for arbitrarily given $\varepsilon > 0$. Also,

$$n^{-r} h^{-r+1} \sum_{j=1}^{n} \sum_{\ell} |Y_{j\ell}|^r \le 2(nh)^{-r+1} \to 0 \tag{28}$$

Since $\int f^{r-1}(x)dx < \infty$, by induction hypothesis, we have

$$P(n^{-r+1} h^{-r+2} \max_{1\le k\le n} T_{k,r-1} \ge \varepsilon) \le C/n^2 \tag{29}$$

By Hölder inequality and the fact $\sum_{\ell} p_\ell^{r-1} \le Ch^{r-2}$,

$$(nh)^{-r+1} \max_{2\le j\le n} \sum_{\ell} p_\ell |S_{j-1,1}|^{r-2}$$
$$\le C(nh)^{-1} (n^{-r+1} h^{-r+2} \max_{1\le k\le n} T_{k,r-1})^{(r-2)/(r-1)} \tag{30}$$

(29), (30) together give

$$P(H_4 \geq \varepsilon) \leq C/n^2 \tag{31}$$

for arbitrarily given $\varepsilon > 0$. Summing up (26)–(28) and (31) we see that (24) is true for $r \in (m, m+1]$, concluding the proof of (24) hence (17). Thus we have proved (6) for the case of $r>1$.

Now consider the case of $r = 1$. Since $f_n(x)$ as a function of x in $(-\infty, \infty)$ is a probability density, in order to prove (6), it suffices to show that for any fixed positive integer N, it is true that

$$\lim_{n\to\infty}\int_I |f_n(x)-f(x)|dx = 0, \quad \text{a. s.} \tag{32}$$

where $I = \bigcup_{\ell=-NH}^{NH} \Delta_\ell$, $H = [h^{-1}]$, the integer part of h^{-1}.

To prove (32), it suffices to verify the following two assertions:

$$\lim_{n\to\infty}\int_I |f(x) - Ef_n(x)|dx = 0 \tag{33}$$

$$\lim_{n\to\infty}\int_I |f_n(x) - Ef_n(x)|dx = 0, \quad \text{a. s.} \tag{34}$$

The second assertion follows directly from Lemma 3 of Devroye [3] if we note the fact that $NH/n \leq N/nh \to 0$. The first assertion can be verified by using a continuous function g on $[a-N-1, a+N+1]$ such that the integral

$$\int_{a-N-1}^{a+N+1} |f(x) - g(x)|dx$$

does not exceed given $\varepsilon > 0$. Trivial details are omitted.

(C). Necessity of (3), (4)

Since

$$\int f_n^r(x)dx = n^{-r}h^{-r+1}\sum_\ell \left|\sum_{i=1}^n (I_\ell(X_i) - p_\ell)\right|^r$$

$$\leq n^{-r}h^{-r+1}\left(\sum_\ell \left|\sum_{i=1}^n (I_\ell(X_i) - p_\ell)\right|\right)^r \leq 2^r/h^{r-1} < \infty$$

Therefore, if (6) or (7) is true, then (3) is true.

Suppose that f(x) does not have a form

$$\sum_{\ell=-\infty}^{\infty} C_\ell I_{[a_0+\ell h_0,\, a_0+(\ell+1)h_0)}(x) \tag{35}$$

for some a_0, $h_0 > 0$ and $\{C_\ell\}$. We want to prove that if (6) or (7) is true, then $h_n \to 0$. Suppose in the contraty that $h_n \not\searrow 0$. Then there exists subsequence $\{h_{n_i},\ i = 1, 2, \ldots\}$ such that $h_{n_i} \to h_0 > 0$ as $i \to \infty$. h_0 must be finite, otherwise we would have $f_n(x) \to 0$ uniformly in $\{X_1, X_2, \ldots\}$ and x, and this contradicts obviously with any one of (6) or (7). Since $|a_{n_i})| \le h_{n_i}$, without losing generality we may assume that $a_{n_i} \to a_0$, also finite. From these facts it follows by the law of large numbers that if we define

$$g(x) = h_0^{-1} \int_{a_0+\ell h_0}^{a_0+(\ell+1)h_0} f(t)\,dt,$$

$$x \in [a_0+\ell h_0,\ a_0+(\ell+1)h_0),\quad \ell = 0, \pm 1, \pm 2, \ldots$$

Then we have

$$\lim_{n\to\infty} \int_I |f_n(x) - g(x)|^r dx = 0, \quad \text{a. s.} \tag{36}$$

for any bounded interval I. Since at least one of (6) and (7) is true, (36) implies that f = g almost everywhere on $(-\infty, \infty)$ (Lebesgue measure). Hence f can be expressed in the form (35), contradicting the assumption.

(D). Necessity of (5).

Suppose in the contratry that (5) is not true, then there exists subsequence $\{h_{n_i}\}$ such that $h_{n_i} \to 0$, $n_i h_{n_i} \to u < \infty$. We restrict ourselves to the discussion of the subsequence. For simplicity of writing and without losing generality, we may assume that $h_n \to 0$ and

$$\lim_{n\to\infty} nh_n = u < \infty \tag{37}$$

Since $h \to 0$, we have

$$\lim_{n\to\infty} \int |f(x) - Ef_n(x)|dx = 0 \tag{38}$$

In fact, as pointed out earlier, (33) is true for any bounded interval I. Since f(x) and $Ef_n(x)$ are probability density functions on $(-\infty, \infty)$, this fact implies (38). Now it is easily seen that if at least one of (6) and (7) is true, then

$$\lim_{n\to\infty} E\int |f_n(x) - Ef_n(x)|dx = 0 \tag{39}$$

In fact, if (7) is true, then for any bounded interval I we have by Hölder inequality that

$$E\int_I |f_n(x) - f(x)|dx \to 0, \quad \text{as } n \to \infty \tag{40}$$

Since f(x), $f_n(x)$ are probability density functions, it is easily seen that

$$\int |f_n(x) - f(x)|dx \le 2\int_I |f_n(x) - f(x)|dx + 2[1-\int_I f(x)dx] \tag{41}$$

From (40), (41), it follows that

$$\lim_{n\to\infty} E\int |f_n(x) - f(x)|dx = 0 \tag{42}$$

Now (39) follows from (38) and (42) If (6) is true, then by Hölder inequality we have

$$\lim_{n\to\infty} \int_I |f(x) - f_n(x)|dx = 0, \quad \text{a. s.} \tag{43}$$

for any bounded interval I. From (43) and the dominated convergence theorem, we again have (40) and hence (39).

First we assume that u=0. Write $A_n = \bigcup_{i=1}^{n} [X_i - h, X_i + h]$. By the definition, $f_n(x) = 0$ for $x \bar{\varepsilon} A_n$. So we have

$$E\int_{A_n} f_n(x)dx = 1.$$

Denote by $\lambda(A_n)$ the Lebesgue measure of A_n. Then

$$\lambda(A_n) \le 2nh \to 0, \quad \text{as } n \to \infty,$$

which implies that

$$\lim_{n\to\infty} E\int_{A_n} f(x)dx = 0.$$

Thus

$$E\int |f_n(x) - f(x)|dx \ge E\int_{A_n} |f_n(x) - f(x)|dx$$

$$\geq E\int_{A_n} f_n(x)\,dx - E\int_{A_n} f(x)\,dx \to 1, \quad \text{as } n \to \infty.$$

From this and (38), it follows that

$$\lim_{n\to\infty} E\int |f_n(x) - Ef_n(x)|\,dx = 1, \tag{44}$$

which contradicts (39).

Now we assume $0 < u < \infty$ By (39), there exist a sequence of positive constants, say C_n, such that $C_n \to \infty$, $C_n/n \to 0$ and

$$C_n E\int |f_n(x) - Ef_n(x)|\,dx \to 0 \tag{45}$$

Write $k = [n/C_n]$. Then

$$\begin{aligned} &C_n E\int |f_n(x) - Ef_n(x)|\,dx \\ &= C_n (nh)^{-1} \sum_{\ell} E\left|\sum_{i=1}^{n} (I_\ell(X_i) - p_\ell)\right| \\ &\geq (kh)^{-1} \sum_{\ell} E\left|E\left\{\sum_{i=1}^{n} (I_\ell(X_i) - p_\ell) \,\middle|\, X_1, \ldots, X_k\right\}\right| \\ &= (kh)^{-1} \sum_{\ell} E\left|\sum_{i=1}^{k} (I_\ell(X_i) - p_\ell)\right|. \end{aligned} \tag{46}$$

Since $h \to 0$ and $kh \leq nh/C_n \to 0$, by (44) we have

$$\lim_{k\to\infty} (kh)^{-1} \sum_{\ell} E\left|\sum_{i=1}^{k} (I_\ell(X_i) - p_1)\right| = 1.$$

On the other hand, (45) and (46) together imply

$$(kh)^{-1} \sum_{\ell} E\left|\sum_{i=1}^{k} (I_\ell(X_i) - p_\ell))\right| \to 0, \quad \text{as } n \to \infty.$$

Thus a contradiction is reached, and the proof of the Theorem is completed

Remark. The method of proof in (D) can easily be adopted to the case of the kernel estimate.

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