

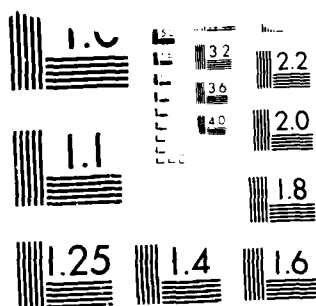
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PROPERTIES OF A MAGNETIC DIPOLE(U) MATERIALS RESEARCH 1/1
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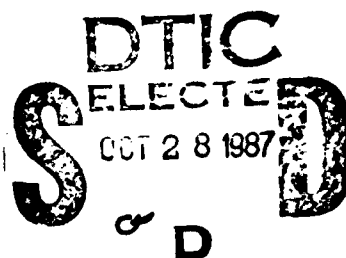
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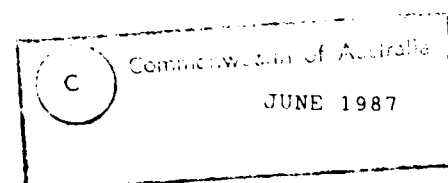
PROPERTIES OF A MAGNETIC DIPOLE

P. J. Ryan



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TECHNICAL NOTE

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PROPERTIES OF A MAGNETIC DIPOLE

P.J. Ryan

ABSTRACT

The properties of a classical magnetic dipole are investigated. The vector potential **A** is derived ab initio and from this the explicit form of the magnetic induction **B** is deduced. It is verified that this dipole magnetic field is both solenoidal ($\nabla \cdot \mathbf{B} = 0$) and irrotational ($\nabla \times \mathbf{B} = 0$) so that Maxwell's equations are satisfied for steady-state conditions. These properties also lead to the existence of a scalar potential ϕ which is explicitly derived.



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PROPERTIES OF A MAGNETIC DIPOLE

1. INTRODUCTION

Magnetic phenomena differ from electric phenomena since there are no free magnetic charges or poles. The basic entity in magnetic phenomena is thus the magnetic dipole - the magnetic analogue of the electric dipole consisting of two opposite charges placed very close together.

Bulk magnetic phenomena such as magnetization are generally described in terms of accumulations of magnetic dipoles. However standard works on electromagnetism (for example refs. [1-3]) usually describe only the simple features of the dipole without verifying all its properties. This note explores the physical and mathematical properties of the classical magnetic dipole. These properties enable the dipole to be considered as one of the fundamental magnetostatic units.

2. THE VECTOR POTENTIAL **A**

The magnetic dipole can be described by an infinitesimally small current loop such as shown in Fig. 1. To calculate the magnetic induction **B** due to this current loop it is customary to first calculate the vector potential **A** defined so that

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (1)$$

The loop of radius a is oriented in the x - y plane with its centre at the origin of the rectangular cartesian coordinate system and the vector potential is to be determined at a point P . For convenience a spherical polar coordinate system is also used with its origin coincident with the cartesian coordinate system (the point O in Fig.1). The following derivation of **A** is similar to that given in standard texts [1-2] but is included here for completeness. Due to the freedom of gauge transformations for the vector

potential, $\nabla \cdot \mathbf{A}$ can be defined as zero (the so-called Coulomb gauge condition). In this case the vector potential $\mathbf{A}$ at point P is given by the volume integral

$$\mathbf{A} = \mu_0 / 4\pi \int \mathbf{J}(\mathbf{r}') / (|\mathbf{r} - \mathbf{r}'|) dV' \quad (2)$$

where $\mathbf{J}$ is the current density, $\mathbf{r}$ is the radial vector defining the point P and $\mathbf{r}'$ is the vector defining the points on the current loop over which the integration is to be carried out. The current density $\mathbf{J}$ has a component only in the ϕ -direction

$$J_\phi = I \delta(\cos\theta') \delta(r' - a) / a \quad (3)$$

and hence so has the vector potential. With no loss of generality and to simplify the calculations the point P is chosen so that $\phi = 0$. Then only the rectangular component $J_y = J_\phi$ is retained. Now $\mathbf{A}$ is given by

$$A_\phi(r, \theta) = \mu_0 I / 4\pi a \int r'^2 dr' d\Omega \cos\phi' \delta(\cos\theta') \delta(r' - a) / (|\mathbf{r} - \mathbf{r}'|) \quad (4)$$

where $d\Omega = \sin\theta' d\theta' d\phi'$. The integration is simplified by the two delta functions: the current density has non-zero value only on the loop where $\cos\theta' = 0$ and $r' = a$. When the integration is carried out over the two delta functions the vector potential reduces to

$$A_\phi(r, \theta) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi' d\phi'}{(a^2 + r^2 - 2ar \cos\phi' \sin\theta)^{1/2}} \quad (5)$$

For points P where $r \gg a$ the denominator may be expanded in a Taylor series as:

$$|\mathbf{r} - \mathbf{r}'|^{-1} = 1/r [1 + (a/r) \sin\theta \cos\phi' + O(a^2/r^2) + \dots] \quad (6)$$

If this expression is substituted into equation (5) and terms of second order and higher in (a/r) neglected, then the vector potential is given by

$$A_\phi(r, \theta) = \frac{\mu_0 I a}{4\pi r} \int_0^{2\pi} \cos\phi' [1 + (a/r) \sin\theta \cos\phi'] d\phi' \quad (7)$$

$$= \frac{\mu_0 I a^2}{4\pi r^2} \int_0^{2\pi} \sin\theta \cos^2\phi' d\phi', \quad (8)$$

$$= \frac{\mu_0 I a^2}{4r^2} \sin\theta, \quad (9)$$

since $\int_0^{2\pi} \cos^2 \phi' d\phi' = \pi$ and $\int_0^{2\pi} \cos \phi' d\phi' = 0$.

Defining the dipole moment $\mathbf{m}$ to be a vector of magnitude $(\pi a^2)I$ normal to the plane of the current loop, $A_\phi(r, \theta)$ can be written as

$$A_\phi(r, \theta) = \frac{\mu_0}{4\pi} m \sin\theta / r^2 \quad (10)$$

Hence the vector potential $\mathbf{A}$ can be expressed as a vector cross product

$$\mathbf{A} = \mu_0 / 4\pi \mathbf{m} \times \mathbf{r} / r^3, \quad (11)$$

since its only non-zero component (in the ϕ -direction) is orthogonal to both the directions of $\mathbf{m}$ and $\mathbf{r}$ and its magnitude is given by $\mu_0 / 4\pi |\mathbf{m}| |\mathbf{r}| / r^3$ times the sine of the angle θ between these two vectors.

Alternatively $\mathbf{A}$ can be expressed as:

$$\mathbf{A} = -\mu_0 / 4\pi \mathbf{m} \times \nabla(1/r) \quad (12)$$

since $\nabla(1/r) = -\mathbf{r}/r^3$ for $r > 0$.

If this expression for $\mathbf{A}$ is correct then it should satisfy the Coulomb gauge condition assumed in its derivation. This follows readily since

$$\nabla \cdot \mathbf{A} = \mu_0 / 4\pi \nabla \cdot (\mathbf{m} \times \mathbf{r} / r^3) \quad (13)$$

$$= \mu_0 / 4\pi [(\nabla \times \mathbf{m}) \cdot \mathbf{r} / r^3 + \mathbf{m} \cdot (\nabla \times \mathbf{r} / r^3)] = 0 \quad (14)$$

because $\nabla \times \mathbf{m}$ vanishes as $\mathbf{m}$ is constant and the curl of the gradient of any function vanishes identically (since $\mathbf{r}/r^3 = -\nabla(1/r)$).

3. COMPONENTS OF MAGNETIC INDUCTION B

The magnetic induction $\mathbf{B}$ due to a magnetic dipole located at the origin can be derived from the relationship

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (15)$$

If the result obtained in section 2 for the vector potential $\mathbf{A}$ is substituted, the magnetic induction at a point $\mathbf{r}$ away from the current source is given by:

$$\mathbf{B} = \mu_0 / 4\pi \nabla \times (\mathbf{m} \times \mathbf{r} / r^3)$$

which can be expanded to

$$\mathbf{B} = \mu_0 / 4\pi [(\mathbf{r}/r^3 \cdot \nabla) \mathbf{m} - (\nabla \cdot \mathbf{m}) \mathbf{r} / r^3 + \mathbf{m}(\nabla \cdot \mathbf{r} / r^3) - (\mathbf{m} \cdot \nabla) \mathbf{r} / r^3] \quad (17)$$

The first two terms are identically zero since $\mathbf{m}$ is constant. The third term is also zero since for any integer n and for $r > 0$ [4]

$$\nabla \cdot (\mathbf{r}^n \mathbf{r}) = r^n (n+3) \quad (18)$$

Thus the only remaining term is

$$\mathbf{B} = - \mu_0 / 4\pi (\mathbf{m} \cdot \nabla) \mathbf{r} / r^3 \quad (19)$$

This expression can be simplified by considering the vector identity:

$$(\mathbf{U} \cdot \nabla) \psi \mathbf{V} = (\mathbf{U} \cdot \nabla \psi) \mathbf{V} + \psi (\mathbf{U} \cdot \nabla) \mathbf{V} \quad (20)$$

where $\mathbf{U}$ and $\mathbf{V}$ are arbitrary differentiable vector functions and ψ is an arbitrary differentiable scalar function. Equation (19) then becomes

$$\mathbf{B} = \frac{-\mu_0}{4\pi} [(\mathbf{m} \cdot \nabla (1/r^3)) \mathbf{r} + \frac{1}{r^3} (\mathbf{m} \cdot \nabla) \mathbf{r}] \quad (21)$$

$$= \frac{-\mu_0}{4\pi} [-3(\mathbf{m} \cdot \mathbf{r}) \mathbf{r} / r^5 + \mathbf{m} / r^3] \quad (22)$$

because $\nabla(1/r^3) = -3\mathbf{r}/r^5$ and $(\mathbf{m} \cdot \nabla)\mathbf{r} = \mathbf{m}$. Thus (22) can be written as

$$\mathbf{B} = \frac{\mu_0}{4\pi} [3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}/r^5 - \mathbf{m}/r^3] \quad (23)$$

In spherical polar coordinates with the origin situated at the location of the dipole the components of $\mathbf{B}$ can be found from (23) by $B_r = \hat{\mathbf{r}} \cdot \mathbf{B}$ and $B_\theta = \hat{\boldsymbol{\theta}} \cdot \mathbf{B}$ where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the radial and zenithal (θ) directions respectively (ref. to Fig. 1). There is no component in the azimuthal (ϕ) direction since $\hat{\boldsymbol{\phi}} \cdot \mathbf{B} = 0$. The two non-zero components may be readily obtained as:

$$B_r = \mu_0/4\pi [2m\cos\theta/r^3] \quad (24)$$

$$B_\theta = \mu_0/4\pi [m\sin\theta/r^3] \quad (25)$$

from the above discussion.

4. DIVERGENCE OF THE MAGNETIC INDUCTION

The expression (23) for $\mathbf{B}$ was derived from the curl of the vector potential $\mathbf{A}$. Hence $\mathbf{B}$ is automatically divergenceless since the divergence of the curl of any vector is zero [4]. However, it is of interest to explicitly verify that the deduced expression for $\mathbf{B}$ is divergenceless. The divergence of the magnetic induction due to a magnetic dipole is given by

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \nabla \cdot [3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}/r^5 - \mathbf{m}/r^3] \quad (26)$$

The divergence of both terms inside the bracket can be calculated from the vector identity [4]

$$\nabla \cdot (\psi \mathbf{V}) = \nabla \psi \cdot \mathbf{V} + \psi \nabla \cdot \mathbf{V} \quad (27)$$

where ψ and $\mathbf{V}$ are defined as in section 3. Now the first term may be expanded to

$$\nabla \cdot (\mathbf{m} \cdot \mathbf{r})\mathbf{r}/r^5 = \mathbf{m} \cdot \mathbf{r}/r^5 + (\mathbf{m} \cdot \mathbf{r})(\nabla \cdot \mathbf{r}/r^5) \quad (28)$$

$$= -\mathbf{m} \cdot \mathbf{r}/r^5 \quad (29)$$

since $\nabla(\mathbf{m} \cdot \mathbf{r}) = \mathbf{m}$ and $\nabla(r/r^5) = -2/r^5$ from equation (18). The divergence of the second term in (26) reduces to

$$\begin{aligned}\nabla \cdot (\mathbf{m}/r^3) &= \nabla(1/r^3) \cdot \mathbf{m} + 1/r^3 (\nabla \cdot \mathbf{m}) \\ &= -3\mathbf{m} \cdot \mathbf{r}/r^5\end{aligned}\quad (30)$$

since $\nabla(1/r^3) = -3\mathbf{r}/r^5$ and $\mathbf{m}$ is a constant. Combining these results the divergence of $\mathbf{B}$ becomes

$$\begin{aligned}\nabla \cdot \mathbf{B} &= \frac{\mu_0}{4\pi} [-3\mathbf{m} \cdot \mathbf{r}/r^5 + 3\mathbf{m} \cdot \mathbf{r}/r^5] \\ &= 0,\end{aligned}\quad (32)$$

so that $\mathbf{B}$ is a solenoidal vector field and thus obeys the fundamental law of magnetostatics.

5. THE CURL OF THE MAGNETIC INDUCTION

From Maxwell's laws $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ assuming steady-state conditions. Hence $\nabla \times \mathbf{B} = 0$ in the region excluding the dipole source. Adapting the same philosophy as in section 4 this result is explicitly verified for the specific form of the dipole's magnetic induction $\mathbf{B}$. The curl of the magnetic induction can be written as

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \nabla \times \{3(\mathbf{m} \cdot \mathbf{r}) \mathbf{r}/r^5 - \mathbf{m}/r^3\} \quad (33)$$

The curl of both terms can be calculated from the vector identity

$$\nabla \times (\psi \mathbf{V}) = \nabla \psi \times \mathbf{V} + \psi (\nabla \times \mathbf{V}), \quad (34)$$

where ψ and $\mathbf{V}$ are as defined in section 3. The first term may be expanded as

$$3 \nabla \times (\mathbf{m} \cdot \mathbf{r}) \mathbf{r}/r^5 = 3 [\nabla(\mathbf{m} \cdot \mathbf{r}) \times \mathbf{r}/r^5 + (\mathbf{m} \cdot \mathbf{r}) \nabla \times (\mathbf{r}/r^5)] \quad (35)$$

$$= 3[\mathbf{m} \times \mathbf{r}/r^5] \quad (36)$$

since $\nabla(\mathbf{m} \cdot \mathbf{r}) = \mathbf{m}$ and $\nabla \times (\mathbf{r}/r^5) = \mathbf{0}$ identically [4]. The second term in equation (33) becomes

$$-\nabla \times (\mathbf{m}/r^3) = -[\nabla(1/r^3) \times \mathbf{m} + 1/r^3(\nabla \times \mathbf{m})] \quad (37)$$

$$= 3 \mathbf{r} \times \mathbf{m}/r^5 \quad (38)$$

since $\nabla(1/r^3) = -3\mathbf{r}/r^5$ for $r > 0$ and $\nabla \times \mathbf{m} = \mathbf{0}$. Combining these results the curl of $\mathbf{B}$ becomes

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} [3(\mathbf{m} \times \mathbf{r})/r^5 + 3(\mathbf{r} \times \mathbf{m})/r^5] \quad (39)$$

$$= \mathbf{0}$$

Hence the magnetic induction vector $\mathbf{B}$ is irrotational as required.

6. SCALAR POTENTIAL OF MAGNETIC DIPOLE

It has been shown that $\mathbf{B}$ is irrotational (i.e. $\nabla \times \mathbf{B} = \mathbf{0}$) for a magnetic dipole. Hence $\mathbf{B}$ can be written as the gradient of a scalar field ϕ .

$$\mathbf{B} = -\nabla\phi \quad (40)$$

Since $\mathbf{B}$ is also solenoidal ($\nabla \cdot \mathbf{B} = 0$) the scalar potential satisfies the Laplacian, $\nabla^2\phi = 0$.

The scalar potential can be deduced from the alternative expression for the vector potential $\mathbf{A}$

$$\mathbf{A} = -\mu_0/4\pi \mathbf{m} \times \nabla(1/r) \quad (41)$$

Thus $\mathbf{B} = \nabla \times \mathbf{A}$ can be expanded as

$$\mathbf{B} = \frac{\mu_0}{4\pi} [\mathbf{m} \cdot \nabla \nabla(1/r) - \mathbf{m} \cdot (\nabla \cdot \nabla(1/r))] \quad (42)$$

$$= \frac{\mu_0}{4\pi} [(\mathbf{m} \cdot \nabla) \nabla(1/r)], \quad (43)$$

since $\nabla^2(1/r) = 0$ for $r > 0$.

Now using the vector identity [4]

$$\begin{aligned} \nabla(\mathbf{m} \cdot \nabla(1/r)) &= (\nabla(1/r) \cdot \nabla) \mathbf{m} + (\mathbf{m} \cdot \nabla) \nabla(1/r) + \nabla(1/r) \times (\nabla \times \mathbf{m}) + \\ &\quad \mathbf{m} \times (\nabla \times \nabla(1/r)) \end{aligned} \quad (44)$$

it can be seen that

$$\nabla(\mathbf{m} \cdot \nabla(1/r)) = (\mathbf{m} \cdot \nabla) \nabla(1/r) \quad (45)$$

since the other terms on the right side vanish because $\mathbf{m}$ is constant and $\nabla \times \nabla(1/r)$ vanishes identically [4]. Thus $\mathbf{B}$ can be expressed as

$$\mathbf{B} = \frac{\mu_0}{4\pi} \nabla [(\mathbf{m} \cdot \nabla(1/r))] \quad (46)$$

so that the scalar potential ϕ is given by

$$\phi = -\mu_0/4\pi \mathbf{m} \cdot \nabla(1/r) \quad (47)$$

It can be readily shown that this potential satisfies the Laplacian since

$$\nabla^2 \phi = -\nabla \cdot \mathbf{B} = 0 \quad (48)$$

7. SUMMARY AND CONCLUSIONS

In this paper the magnetostatic properties of a magnetic dipole have been verified. In summary it has been shown that the magnetic dipole has a vector potential which is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} (\mathbf{m} \times \mathbf{r}/r^3) \quad (49)$$

or

$$\mathbf{A} = \frac{\mu_0}{4\pi} (\mathbf{m} \times \nabla(1/r)) \quad (50)$$

and satisfies the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0 \quad (51)$$

The magnetic induction $\mathbf{B}$ was derived from the curl of the vector potential to be

$$\mathbf{B} = \frac{\mu_0}{4\pi} [3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}/r^5 - \mathbf{m}/r^3] \quad (52)$$

It has also been explicitly verified that Maxwell's equations hold for the dipole field so that

$$\nabla \cdot \mathbf{B} = 0 \quad (53)$$

and

$$\nabla \times \mathbf{B} = 0, \quad (54)$$

in the region excluding the current source.

Further, it was shown that $\mathbf{B}$ can be expressed as the gradient of a scalar potential defined by

$$\phi = \frac{\mu_0}{4\pi} \mathbf{m} \cdot \nabla (1/r) \quad (55)$$

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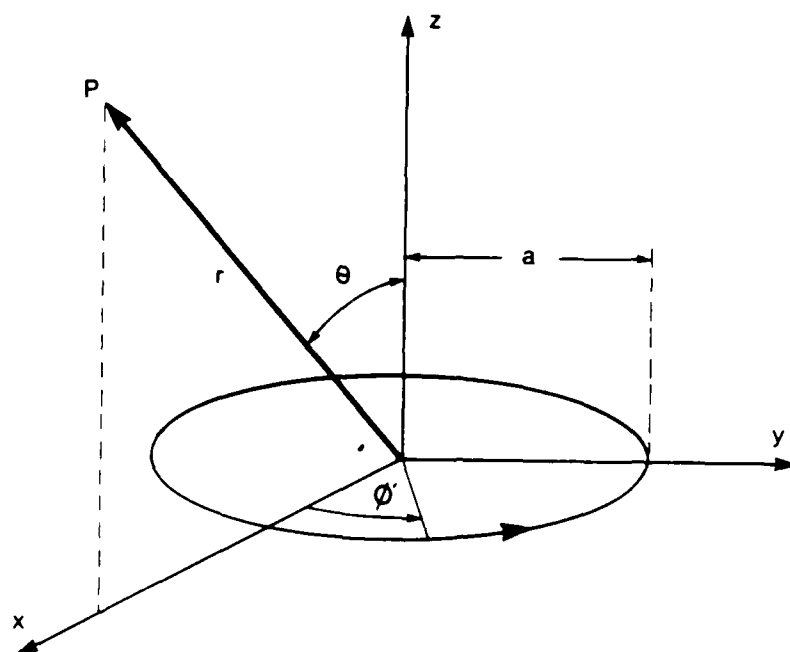


FIGURE 1 The magnetic dipole represented as a circular current loop with both its rectangular cartesian and spherical polar coordinate systems.

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ABSTRACT

The properties of a classical magnetic dipole are investigated. The vector potential $\mathbf{A}$ is derived ab initio and from this the explicit form of the magnetic induction $\mathbf{B}$ is deduced. It is verified that this dipole magnetic field is both solenoidal ($\nabla \cdot \mathbf{B} = 0$) and irrotational ($\nabla \times \mathbf{B} = 0$) so that Maxwell's equations are satisfied for steady-state conditions. These properties also lead to the existence of a scalar potential ϕ which is explicitly derived.

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