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NEW METHODS FOR NUMERICAL SOLUTION OF ONE CLASS OF
STRONGLY NONLINEAR PAR. (U) EMORY UNIV ATLANTA GA DEPT
OF MATHEMATICS AND COMPUTER SCIENC. V I OLIKER ET AL.

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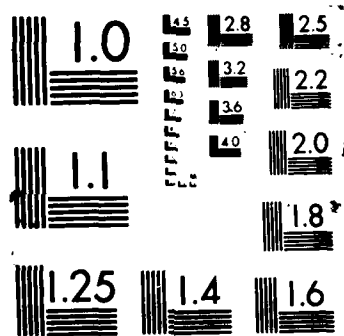
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AD-A186 166

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V.I. Oliker and P. Waltman

equation" in dynamic meteorology has the form (1) [H]. In a more complicated form, an equation of this type appears in the von Karman system of equations for elasticity and also in an inverse problem of geometric optics [W]; see below part B. It also turned out that recent progress in the study of fully nonlinear equations became possible after important properties of the equation (1) were discovered; [GT], ch. 17. For quasilinear elliptic and parabolic equations the use of equations of the form (1) is crucial in obtaining C^0 and Hölder estimates (see [GT], ch. 17, [K], and further references there).

In spite of the increasing number of papers in this area the theory of the problem (1), (2) and its generalizations is far from being complete. Researchers in the USA, USSR, Germany, England and other countries at present are actively pursuing this direction.

The proposers were fortunate to start their research at a relatively early stage of all these developments. The current project funded by the AFOSR Grant 84-0285 covers two major areas:

A. Investigation of numerical methods for solving problem (1), (2) and its generalizations.

B. Investigation of solubility of a Monge-Ampère equation arising in shaped antenna design.

We describe now briefly the results obtained so far, current activities, and proposed directions for future studies.

Part A. Numerical Methods. The type of the operator M depends on the function on which it is evaluated. For that reason, one usually seeks a solution of (1), (2) in the class of functions on which M has a fixed type, for example, elliptic or hyperbolic. In the case under consideration, the requirement $f > 0$ in Ω forces any function satisfying (1) to be an "elliptic" solution, that is, in the class where M is elliptic.

In view of the important practical applications several heuristic approaches were suggested for numerical solution of some modified forms of (1), (2), [Ar], [Sh], [Sh1]. Though no rigorous analysis of these methods exists, one may note that they all are local methods based on a finite difference approximation and linearization. Because of the strong nonlinearity of M this approach might be successful only in a neighborhood of the true solution and therefore, if a priori a good initial approximation is not available, these methods will not produce, generally, a sequence converging to the true solution.

We investigated this problem in detail and obtained the following results:

- A special discretization scheme for (1), (2) was suggested different from standard finite element or finite difference schemes. It can be shown that latter ones in known forms will not work here;
- an iterative method has been developed for solving the discretized version of (1), (2);
- the question of finding an initial approximation in our scheme is completely and effectively resolved; it is just a routine step of the iteration process;
- the iterations are selfcorrecting;
- global convergence is established;
- our algorithm is suitable for a parallel computer;
- a computer code has been written and tested.

The experience gained in testing our procedure on different types of examples, including ones with large gradients, shows that its most effective use will be in combination with some fast Newton-type scheme. More precisely, a particular criteria exists for checking when the current approximation can be used as the beginning step for a converging Newton-type iterative procedure. In this combined scheme we proved convergence (quadratic) of the Newton iterates, but the above mentioned criteria involves some heuristic arguments and more work needs to be done here.

The computer code for the method is quite sophisticated; it involves, as a step, construction of a convex hull of sets of points in R^3 . There are here different approaches and the effectiveness of the algorithm depends on it. We have been currently testing various schemes and the code presently is a substantial improvement over its original version of 1984-85. A full time employed graduate student in Computer Science with a Ph.D. in Mathematics has been helping us with this part of work.

The results have been submitted for publication (see the subsection on publications and presentations).

In the third year of the funding period we intend to study the already mentioned before "combined schemes", two sided approximations, and extensions to equations of the form

$$\begin{aligned} \frac{\partial^2 z}{\partial^2 x} \frac{\partial^2 z}{\partial^2 y} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 + a(x, y) \frac{\partial^2 z}{\partial x^2} + 2b(x, y) \frac{\partial^2 z}{\partial x \partial y} \\ + c(x, y) \frac{\partial^2 z}{\partial y^2} + g(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0. \end{aligned}$$

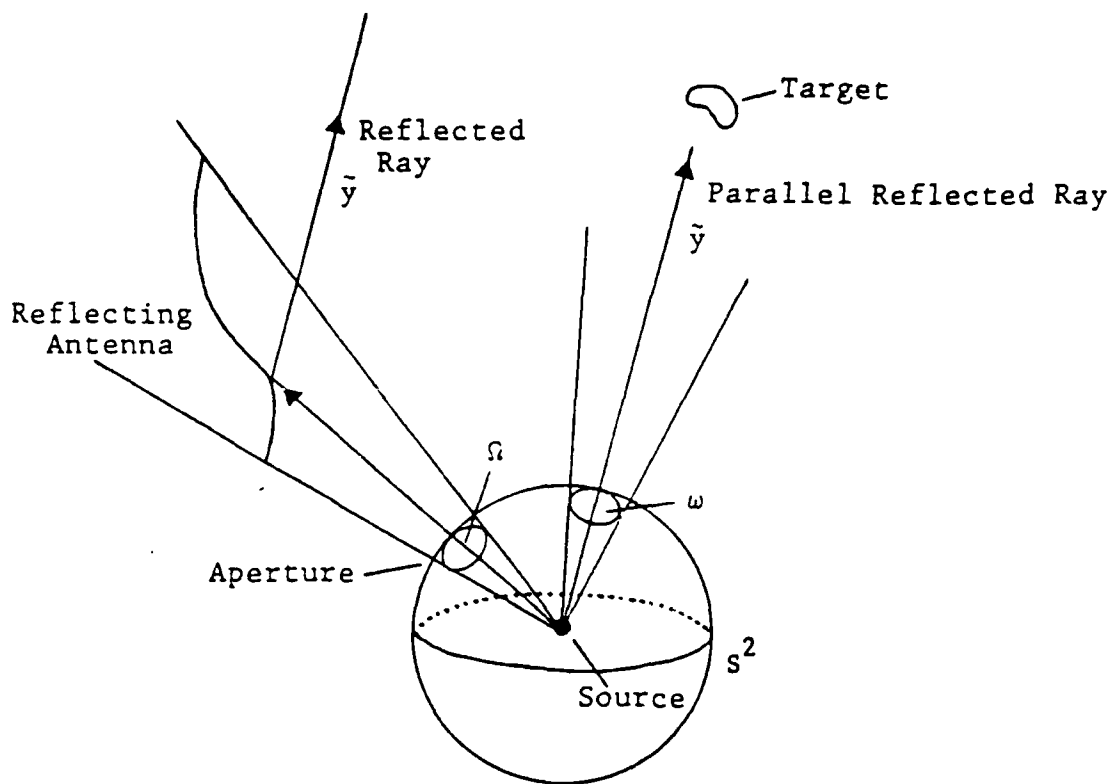
We have already initiated work in these directions and are quite sure of its successful completion. The last equation includes as a special case the balance equation of the atmospheric dynamics. If time and resources permit we want to tailor the code we have for a parallel machine. At the University of Georgia in Athens a Cyber 205 is available and we have made some contacts with the program. We have also established contact with Professor D. Hoffman from the University of Massachusetts at Amherst who agreed in principle to help us to analyze our work with computer graphics. His experience in this area is well known [H] and we hope, it will be valuable for us.

Part B. Applications to Shaped Antenna Design. In a practical problem of shaped antenna design it is required to determine a reflecting surface such that for a given point-source of light the reflected rays cover a prescribed region of the far sphere and the density of the distribution of reflected rays is a prescribed in advance function of the direction. It is assumed that the power density of the source as well as the aperture of the incident ray cone are known, and the reflection process obeys the laws of geometric optics (see Figure 1 on next page). In this form the problem was posed by Westcott and Norris [WN] and later it has been considered in Brickell, Marder, and Westcott [BMW], Norris and Westcott [NW], Westcott [W]. Further references can be found in [W]. The research of these investigators has been supported by Plessey Radar, Ltd. for many years.

The problem admits a precise mathematical formulation and in this form it reduces to solving the equation

$$\frac{4p^2 \det[\nabla_{ij} p + (p - \frac{|\nabla p|^2 + p^2}{2p}) e_{ij}]}{(|\nabla p|^2 + p^2)^2 \det(e_{ij})} = \rho \text{ in } \omega, \quad (3)$$

$$\text{nonlinear boundary condition on } \partial\omega, \quad (4)$$



OBJECT: To Illuminate Target with Prescribed Intensity
(Through Given Aperture)

Equivalent to Determining the Surface and Position of Reflector

with respect to the unknown function $p(>0)$ naturally associated with the problem; here (e_{ij}) is the matrix of the first fundamental form e of the unit sphere S^2 , ∇ the gradient in the metric e , ∇_{ij} - second covariant derivatives in e . The condition (4) is somewhat complicated to be presented here without considerable expansion. In [NW], and also in [W], p. 40, the authors state that the question of existence of solutions is open. Conditions for uniqueness were given by Marder [M]. In [NW], [W] results of numerical studies and applications are presented. Still there are no rigorous convergence results, and even the linearized version of the problem has not been investigated. In 1957 J. Keller [K] obtained some results pertaining to the radially symmetric case without satisfying a particular boundary condition.

With the support of AFOSR we started our investigation of the problem with the radially symmetric (r.s.) case, that is, when the incidence ray cone Ω and the far field domain ω are circular, the prescribed density of reflected rays is a function of the azimuthal angle only, and the reflecting surface is sought as a surface of revolution. The following results have been obtained:

- The problem splits naturally into two parts. In the first part one finds a class of surfaces for which the reflected directions cover the prescribed domain $\bar{\omega}$. Those surfaces can be conveniently parametrized by points of $\bar{\omega}$. In the second part one seeks in the above class a particular surface for which the density of the reflected rays is a prescribed function in ω , and which projects onto the given domain $\bar{\Omega}$;
- it is shown that for any function $p \in C^2(\bar{\omega})$, $p > 0$ in $\bar{\omega}$, the surface F defined by the map

$$\begin{aligned}
 -r &= \nabla p + (p - \rho)y, \quad y \in \bar{\omega}, \\
 \rho &= (p^2 + |\nabla p|^2)/2p = |r|, |\nabla p|^2 = \langle \nabla p, \nabla p \rangle,
 \end{aligned}
 \tag{5}$$

satisfies the requirement of being a reflector;

- simple and verifiable necessary and sufficient conditions for solubility of (3), (4). It was shown that radially symmetric solutions can be constructed explicitly whenever all of the parameters are appropriate.

In a separate study that just has been completed we proved that

- nonradially symmetric solutions to the problem for general distribution densities which are close (in $C^{0,6}$ - norm) to r.s. solutions can be found under the natural restriction corresponding to the conservation law. The result appears to be quite delicate, because, in general, the problem is unstable and even a small perturbation of the data may produce a situation where a solution does not exist.

During the past year we have also experimented with numerical methods for solving (3), (4) for ϕ sufficiently close to a r.s. symmetric density. In r.s. case the equation is singular at the endpoint corresponding to axis of revolution, and as a result straight forward linearizations about r.s. solutions do not seem to work very well, though in [W] some success was claimed.

During the third year we intend to continue investigation of the following topics:

- Solubility of (3), (4) with the right hand side subject only to the conservation of energy requirement (a necessary condition in this problem);

- numerical solution of (3), (4) for ϕ close to r.s. data. With our latest work now complete we have much better insights into the problem and anticipate definite progress here.
- Application of numerical methods developed in Part A to the problem (3), (4).
- Extension of the techniques used in the solution of the r.s. case to more general, nonlinear eigenvalue problem for ordinary differential equations.

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Publications and Presentations

The results obtained in Part A were presented in

- a paper "On the numerical solution of the equation

$$\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = f$$

and its discretizations, I". It has been submitted to the SIAM Journal of Numerical Analysis and is currently under revision;

- a lecture at a special session of the Southeastern-Atlantic Regional Conference on Differential Equation, Atlanta, October, 1985
- a plenary lecture was delivered on the subject by one of the principal investigators at the South Eastern Regional SIAM Conference, March, 1986.

The results in Part B were presented in

- a paper "Radially Symmetric Solutions of a Monge-Ampère Equation arising in a reflector mapping problem", pp 1-19; accepted for publication in Proceedings of UAB Conference on differential equations and mathematical physics, Springer;
- a paper "Near radially symmeytric solution of an inverse problem in geometric optics", pp 1-23, to be submitted;
- a report "On the Monge-Ampère equation arising in the reflector mapping problem", Institute for Mathematics and its Applications, University of Minnesota, preprint series #198, pp. 1-43 (this is an expanded version of the first paper mentioned in this part);

- an hour lecture given at the Technical University in Berlin, West Germany;
- a 30-minute presentation at the conference on Differential Equations and Mathematical Physics, University of Alabama, Birmingham.

END

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