||| 1.0 ||| 28 ||| 2.5
||| 3.2 ||| 2.2
| 36
||| 1.1 ||| 2.0
||| 1.8
||| 1.25 ||| 1.4 ||| 1.6

MICROCOPY RESOLUTION TEST CHART

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We consider various physical problems which may be formulated in terms of integral equations of the first kind, including the two-dimensional screen Neumann and Dirichlet problems in acoustics (and crack problems in elasticity). Sharp regularity results for the solutions are available for these problems. We prove the convergence of the p-version for some Galerkin boundary element schemes based on the integral equation formulations. It is shown that the rate of convergence obtained by our method is twice that for the usual h-version.

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On the Convergence of the p-Version of the

Boundary Element Galerkin Method

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Section 1. Introduction

Over the last ten years there has been a spectacular

increase on research and applications of boundary element

techniques.   There has been an explosion of books as well

as a series of Internatio·al Conferences [10], specially

dedicated to boundary element methods (BEM). The state of

the art of asymptotic error estimates of the h-version for

BEM is described in several detailed articles (see for

example [24], [25]).   There the theoretical framework for

both first-kind and second-kind integral equations is the

theory of pseudodifferential operators. As observed in

[21] one has for strongly elliptic pseudodifferential

operators convergence of any Galerkin scheme with conforming

boundary elements; also there holds quasioptimality of the

Galerkin error in the energy norm.

Almost all work on BEM has been performed with the

h-version, where the degree p of the elements is fixed,

usually on low level, typically p = 0,1,2 and the accuracy

is achieved by properly refining the mesh. Only recently

the p-version has been introduced into the BEM [1], [2],

[3], [26]. The p-version fixes the mesh and achieves the

accuracy by increasing the degrees p of the elements
uniformly or selectively. In the finite element method
(FEM) the convergence of the p-version has been thoroughly
investigated for one- and two-dimensional boundary value
problems in a series of papers by Babuška and others [4],
[5], [6], [7], [12]. Meanwhile convergence results have also
been derived for the h-p version of the finite element
method which is a combination of the standard h-version and
the p-version [4], [8], [13], [14].

In this paper, we prove the convergence of the p-version
for some Galerkin boundary element schemes which use first-
kind integral equations. In Section 2 we introduce the
function spaces and corresponding norms used later on. In
Section 3.1 we show that the rate of convergence of the
p-version is an optimal one in the $H^{1/2}$ and $H^{-1/2}$-norms
generalizing known results for H^1 and L^2-norms. In Sections 3.2,
3.3 we approximate singular functions by the p-version in
the $\tilde{H}^{1/2}$ and $\tilde{H}^{-1/2}$-norms and we derive convergence rates
which are twice the rate of the h-version with uniform
mesh. In Section 4 we apply the approximation results of
Section 3 to the Galerkin BEM for several integral equations
which are strongly elliptic pseudodifferential equations.
As examples, we consider the two-dimensional screen Neumann
and Dirichlet problems in acoustics where sharp regularity
results for the solutions are available [22], [23].
Furthermore, we give first-kind boundary integral equations

governing the exterior Dirichlet and Neumann problems of the three-dimensional Helmholtz equation and *we present the* convergence rates for the p-version of the corresponding boundary element Galerkin schemes.

Section 2. Notation

Let $\tilde{\Gamma}$ be a simply connected, bounded, smooth, closed curve in $\mathbb{R}^2$ and Γ be a connected subset of $\tilde{\Gamma}$. By $C^{(k)}(\tilde{\Gamma})$, $0 < k \leq \infty$ (k integer), we denote the space of all functions with continuous derivatives of order up to k on $\tilde{\Gamma}$. The Sobolev spaces $H^s(\tilde{\Gamma})$ are defined for $s \geq 0$ to be the restrictions of $H^{s+1/2}(\mathbb{R}^2)$ to $\tilde{\Gamma}$ and for $s < 0$ by duality,

$$H^s(\tilde{\Gamma}) = (H^{-s}(\tilde{\Gamma}))'$$

with $H^0(\tilde{\Gamma}) = L^2(\tilde{\Gamma})$. These spaces are used to define the corresponding spaces of distributions on Γ, namely, for any real s,

$$\tilde{H}^s(\Gamma) = \{u \in H^s(\tilde{\Gamma}): \text{supp } u \subset \bar{\Gamma}\}$$

$$H^s(\Gamma) = \{u|_\Gamma: u \in H^s(\tilde{\Gamma})\}.$$

The above spaces are normed as follows. For u defined on Γ, let ℓu denote any extension of u on $\tilde{\Gamma}$ and u^* denote the zero extension of u on $\tilde{\Gamma}$. Then

$$\|u\|_{\tilde{H}^s(\Gamma)} = \|u^*\|_{H^s(\tilde{\Gamma})} \tag{2.1}$$

$$\|u\|_{H^s(\Gamma)} = \inf\{\|\ell u\|_{H^s(\tilde{\Gamma})}: \ell u \in H^s(\tilde{\Gamma})\} \tag{2.2}$$

Note that for $s > 1/2$, $s \neq$ integer + 1/2, $\tilde{H}^s(\Gamma)$ is the usual $H_0^s(\Gamma)$ space and for $-1/2 < s < 1/2$, $\tilde{H}^s(\Gamma) = H^s(\Gamma)$. For $s < -1/2$, $s \neq$ integer + 1/2, $\tilde{H}^s(\Gamma) = (H^{-s}(\Gamma))'$. We will be

particularly interested in the cases s = 1/2 and s = -1/2.
For s = 1/2, the space $\tilde{H}^{1/2}(\Gamma)$ is also denoted by $H_{oo}^{1/2}(\Gamma)$,
with the equivalent norm (see [18])

$$\|u\|^2_{1/2(\Gamma)} = \|u\|^2_{H^{1/2}(\Gamma)} + \|(1-x^2)^{-1/2} u\|^2_{H^0(\Gamma)} \tag{2.3}$$

where for nplicity, we have assumed $\Gamma = (-1,+1)$ (the general
case can be treated by affine maps) and x denotes the arc
length. In terms of duality, the following relations hold

$$H^{-1/2}(\Gamma) = (\tilde{H}^{1/2}(\Gamma))', \quad \tilde{H}^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))'$$

Let $\tilde{\Gamma}$ be of length 2π, then $H^s(\tilde{\Gamma})$ may be considered to
be spaces of 2π-periodic functions. For $u \in H^s(\tilde{\Gamma})$, we may
then write

$$u(\xi) - \sum_{j=0}^{\infty} a_j \cos j\xi + \sum_{j=1}^{\infty} b_j \sin j\xi \tag{2.4}$$

so that the $H^s(\tilde{\Gamma})$ norm may be equivalently defined by

$$\|u\|_{H^s(\tilde{\Gamma})} \approx [\sum_{j=0}^{\infty} a_j^2(1+j^2)^s + \sum_{j=1}^{\infty} b_j^2(1+j^2)^s]^{1/2} \tag{2.5}$$

For I a smooth open arc, we will define $P_p(I)$ to be the
set of all algebraic polynomials of degree less than or
equal to p in s, the arc length parameter. $P_p^0(I)$ will denote
the subset of polynomials vanishing at the end points of I.

Let us now subdivide $\tilde{\Gamma}$ into N pieces, $\tilde{\Gamma} = \bigcup_{i=1}^{N} \bar{\Gamma}_i$, such
that Γ_i is a smooth open arc with end points Λ_{i-1}, Λ_i
$(\Lambda_0 = \Lambda_N)$. Then for $p \geq 0$, $S_p(\tilde{\Gamma})$ will denote the set of all
functions u defined on $\tilde{\Gamma}$ such that the restriction $u|_{\Gamma_i}$ to Γ_i

belongs to $P_p(\Gamma_i)$. Moreover, we set for $p \geq 1$,

$$V_p(\tilde{\Gamma}) = S_p(\tilde{\Gamma}) \cap C^{(0)}(\tilde{\Gamma}).$$

We may assume that Γ may be partitioned analogously and define $S_p(\Gamma)$, $V_p(\Gamma)$ as above. Then $S_p^0(\Gamma)$, $V_p^0(\Gamma)$ will denote the subsets of functions that vanish at the end points of Γ.

Note that $S_p(\tilde{\Gamma})$ $(S_p(\Gamma))$ is a subset of $H^{-1/2}(\tilde{\Gamma})$ $(H^{-1/2}(\Gamma))$ while $V_p(\tilde{\Gamma})$ $(V_p(\Gamma))$ is a subset of $H^{1/2}(\tilde{\Gamma})$ $(H^{1/2}(\Gamma))$ and $V_p^0(\Gamma)$ is a subset of $\tilde{H}^{1/2}(\Gamma)$.

So far we have dealt with the one-dimensional case. We will also be interested in a simply connected, bounded, smooth, closed surface $\tilde{\Gamma} \subset \mathbb{R}^3$. The definitions of $H^s(\tilde{\Gamma})$ are analogous to the previous case. We now assume that $\tilde{\Gamma}$ is partitioned into curvilinear quadrilaterals and triangles, i.e., $\tilde{\tilde{\Gamma}} = \overset{n}{\underset{i=1}{\cup}} \bar{\Gamma}_i$. Let Q and T be the reference square and triangle respectively, then $\Gamma_i = F_i(Q)$ or $F_i(T)$, where F_i is a smooth bijective mapping. We assume that the inter-section of any two disjoint Γ_i's is either the empty set or a common vertex or a common side.

By $P_p^1(T)$ we will denote the set of all polynomials of total degree $< p$ on the triangle T. $P_p^2(Q)$ will denote the set of all polynomials of degree $< p$ in each variable on Q. We define

$$S_p(\tilde{\Gamma}) = \{u \mid u \mid_{\Gamma_i}(F_i(\xi)) \in P_p^1(T) \text{ if } \Gamma_i \text{ is a triangle and}$$

$$u \mid_{\Gamma_i}(F_i(\xi)) \in P_p^2(Q) \text{ if } \Gamma_i \text{ is a quadrilateral}\} \qquad (2.6)$$

and $V_p(\hat{\Gamma}) = S_p(\tilde{\Gamma}) \cap C^{(0)}(\hat{\Gamma}).$ \qquad (2.7)

Section 3. Approximation Theorems

In this section, we will be interested in obtaining estimates for the approximation of functions in $H^s(\tilde{\Gamma})$, $\tilde{H}^s(\Gamma)$ and $H^s(\Gamma)$ by piecewise polynomials belonging to the polynomial subspaces introduced in the previous section.

3.1. Approximation of Functions in H^s

We first present some results for the case when u, the function being approximated is known to lie in H^s. These will be used by us in the next section for approximating problems on closed curves and closed surfaces.

In what follows, $\tilde{\Gamma}$ will denote either a closed curve or a closed surface.

Theorem 3.1. Let $\gamma = \tilde{\Gamma}$ or Γ. Let $u \in H^s(\gamma)$, $s > 1/2$. Then for $p = 1,2,\ldots$ there exists $u_p \in V_p(\gamma)$ such that

$$\|u - u_p\|_{H^{1/2}(\gamma)} \leq Cp^{-(s-1/2)} \|u\|_{H^s(\gamma)} \tag{3.1}$$

where the constant C is independent of u and p but depends on s and the partition on γ. Moreover, for $u \in \tilde{H}^s(\Gamma)$,

$$\|u - u_p\|_{\tilde{H}^{1/2}(\Gamma)} \leq Cp^{-(s-1/2)} \log^{1/2} p \|u\|_{\tilde{H}^s(\Gamma)} \tag{3.2}$$

Proof. The estimate (3.1) follows by interpolating the approximation estimates for the p-version obtained in the H^0 and H^1 norm (see [6]). In [7], an alternative proof (for closed curves) using Chebyshev expansions is provided in Theorem 3.2. Moreover, (3.2) is also proved in this theorem, the procedure

being similar to our proof of Theorem 3.3 in Section 3.2.   □

The above theorem provides estimates for the error of the best approximation in the $H^{1/2}$ and $\widetilde{H}^{1/2}$ norms. The next theorem provides estimates in the $H^{-1/2}$ norm. It has been proved in [9] for γ being a closed curve and it is included here for completeness

Theorem 3.2. Let $\gamma = \widetilde{\Gamma}$ *on* Γ, $u \in H^s(\gamma)$, $s > 0$. *Then for* $p = 0,1,2,\ldots$ *there exists* $u_p \in V_p(\gamma)$ *such that*

$$\|u - u_p\|_{\widetilde{H}^{-1/2}(\gamma)} \leq Cp^{-(s+1/2)} \|u\|_{H^s(\gamma)} \qquad (3.3)$$

where C is a constant independent of u and p but depends upon s and the grid on γ.

Proof. Let $u_p \in V_p(\gamma)$ satisfy

$$\int_\gamma u_p \omega \, d\xi = \int_\gamma u \omega \, d\xi \qquad \text{for all } \omega \in V_p(\gamma). \qquad (3.4)$$

Then, with $e = u - u_p$, we have (see [13])

$$\|e\|_{H^0(\gamma)} \leq Cp^{-s} \|u\|_{H^s(\gamma)} \qquad (3.5)$$

Now, for arbitrary $v \in H^1(\gamma)$, we have by (3.4),

$$\frac{\int_\gamma e v \, d\xi}{\|v\|_{H^1(\gamma)}} = \frac{\int_\gamma e(v-y) \, d\xi}{\|v\|_{H^1(\gamma)}} \leq \frac{\|e\|_{H^0(\gamma)} \|v-y\|_{H^0(\gamma)}}{\|v\|_{H^1(\gamma)}}$$

$$\leq Cp^{-1} \|e\|_{H^0(\gamma)}$$

where $y \in V_p(\gamma)$ satisfies

$$\|v - y\|_{H^0(\gamma)} \leq Cp^{-1} \|v\|_{H^1(\gamma)}$$

This yields

$$\|e\|_{(H^1(\gamma))'} \leq Cp^{-(s+1)} \|u\|_{H^s(\gamma)} \tag{3.6}$$

Interpolating (3.5), (3.6) and using the fact that

$$\widetilde{H}^{-1/2}(\gamma) = (H^{1/2}(\gamma))' = (H^0(\gamma), H^1(\gamma))'_{1/2}$$

$$= ((H^1(\gamma))', H^0(\gamma))_{1/2}$$

we obtain (3.4). □

Remark 3.1. For $\gamma = \widetilde{\Gamma}$, we have $\widetilde{H}^k(\Gamma) = H^k(\Gamma)$. For $\gamma = \Gamma$, we have $\|\cdot\|_{H^{-1/2}(\Gamma)} \leq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$. Hence, in either case, (3.3) yields

$$\|u - u_p\|_{H^{-1/2}(\gamma)} \leq Cp^{-(s+1/2)} \|u\|_{H^s(\gamma)} \tag{3.7}$$

Remark 3.2. Since $V_p(\gamma) \subset S_p(\gamma)$, we see that (3.3) and (3.7) also hold for some $u_p \in S_p(\gamma)$.

Remark 3.3. So far we have assumed that $\widetilde{\Gamma}$ and Γ are smooth. The above theorems may also be modified to the case when $\widetilde{\Gamma}$ and Γ are only piecewise smooth.

3.2. $\widetilde{H}^{1/2}$ Approximation of Singular Functions

We are interested here in approximating functions that are defined on the curve Γ and have square root singularities at the end points. For simplicity, we consider a function u defined on $I = [-1, +1]$ by

$$u(x) = (x+1)^{1/2} \chi(x) \tag{3.8}$$

where χ is a C^∞ function satisfying

$$\chi(x) = 1, \quad -1 \le x \le -1/2$$
$$= 0, \quad 1/2 \le x \le 1$$

We consider the approximation of u in the $\tilde{H}^{1/2}(I)$ norm by functions in $P_p(I)$.

Let $\hat{I} = [-\pi, \pi]$. (We may consider $\hat{I}$ to be a closed circle.) Let u be transformed to the periodic function $\hat{u}$ on $\hat{I}$ by the mapping $x = \cos \xi$, i.e., $\hat{u}(\xi) = u(x)$. Then we see that

$$\hat{u}(\xi) = (1+\cos \xi)^{1/2} \chi(\cos \xi) = \sqrt{2} \chi(\cos \xi)(\cos(\xi/2)) \quad (3.9)$$

The following lemma is taken from [7].

Lemma 3.1. $\quad \|u\|_{H^{1/2}(I)} \approx \|\hat{u}\|_{H^{1/2}(\hat{I})} \quad$ *for any* $u \in H^{1/2}(I)$.

The main theorem of this section is the following.

Theorem 3.3. *Let u be defined by* (3.8). *Then for* $p = 1,2,\ldots$ *there exists a polynomial* u_p^0 *in* $P_p(I)$ *such that*

$$u_p^0(\pm 1) = u(\pm 1) \quad (3.10)$$

and

$$\|u-u_p^0\|_{\tilde{H}^{1/2}(I)} < Cp^{-1} \log^{1/2} p \quad (3.11)$$

Proof. We first consider the image $\hat{u}$ of u, which (being even) may be written as

$$\hat{u}(\xi) = \sum_{k=0}^{\infty} a_k \cos k\xi. \quad (3.12)$$

Define

$$u_p^0 := u_p + \bar{u} \tag{3.13}$$

where u_p is defined by

$$\dot{u}_p := \sum_{k=0}^{p} a_k \cos k\xi \tag{3.14}$$

and where $\bar{u}$ is a linear function such that u_p^0 satisfies the condition (3.10), i.e., such that

$$\bar{u}(\pm 1) = (u - u_p)(\pm 1) = (\hat{u} - \dot{u}_p)(\cos^{-1}(\pm 1)) \tag{3.15}$$

We now estimate the coefficients a_k in (3.12). We have

$$a_k = C \int_0^\pi \hat{u} \cos k\xi \, d\xi = C \int_0^\pi \chi(\cos \xi) \cos(\tfrac{\xi}{2}) \cos k\xi \, d\xi$$

$$= C \int_0^\pi \chi(\cos \xi) [\cos((\tfrac{2k+1}{2})\xi) + \cos((\tfrac{2k-1}{2})\xi)] d\xi$$

Here, C may represent different constants. Integrating by parts gives,

$$a_k = C \left\{ [\chi(\cos \xi)(\sin((\tfrac{2k+1}{2})\xi) \cdot \tfrac{1}{2k+1} + \sin((\tfrac{2k-1}{2})\xi) \cdot \tfrac{1}{2k-1})]_0^\pi \right. $$
$$\left. + \int_0^\pi \chi'(\cos \xi) \sin \xi [\sin((\tfrac{2k+1}{2})\xi) \cdot \tfrac{1}{2k+1} + \sin((\tfrac{2k-1}{2})\xi) \tfrac{1}{2k-1}] d\xi \right\}$$

So that with $\chi'(\pm 1) = 0$ further integration by parts yields

$$|a_k| \leq C \left\{ \tfrac{1}{k^2} + | \int_0^\pi \chi'(\cos \xi) \left[\cos((k - \tfrac{1}{2})\xi) \tfrac{1}{2k+1} - \cos((k + \tfrac{3}{2})\xi) \tfrac{1}{2k+1} \right. \right.$$
$$\left. \left. + \cos((k - \tfrac{3}{2})\xi) \tfrac{1}{2k-1} - \cos((k + \tfrac{1}{2})\xi) \tfrac{1}{2k-1} \right] d\xi | \right\}$$

$$\leq C \left\{ \tfrac{1}{k^2} + | \int_0^\pi \chi''(\cos \xi) \sin \xi \left[\frac{\sin(k - \tfrac{1}{2})\xi}{(2k+1)(2k-1)} - \frac{\sin(k + \tfrac{3}{2})\xi}{(2k+1)(2k+3)} \right. \right.$$
$$\left. \left. + \frac{\sin(k - \tfrac{3}{2})\xi}{(2k-1)(2k-3)} - \frac{\sin(k + \tfrac{1}{2})\xi}{(2k-1)(2k+1)} \right] d\xi | \right\}$$

Now since χ'' and all the sine functions are bounded independent of k, we obtain

$$|a_k| \leq \frac{C}{k^2} \tag{3.16}$$

We now estimate $\|u-u_p\|_{H^{1/2}(I)}$. By Lemma 3.1, (2.5), (3.12), and (3.14), we have

$$\|u-u_p\|^2_{H^{1/2}(I)} = \|\hat{u}-\hat{u}_p\|^2_{H^{1/2}(\hat{I})} = C \sum_{k=p+1}^{\infty} a_k^2 (1+k^2)^{1/2}$$

$$\leq C \sum_{p+1}^{\infty} \frac{(1+k^2)^{1/2}}{k^4}$$

which behaves like $\int_{p+1}^{\infty} \frac{C}{x^3} dx = \frac{C}{p^2}$. Hence,

$$\|u-u_p\|_{H^{1/2}(I)} \leq \frac{C}{p} \tag{3.17}$$

Next, we estimate $\|\bar{u}\|_{H^{1/2}(I)}$. Since $\bar{u}$ is linear,

$$\|\bar{u}\|_{H^{1/2}(I)} \leq C\{|u(+1)| + |u(-1)|\} \tag{3.18}$$

Now for any x, by (3.12), (3.14),

$$|(u-u_p)(x)| \leq \sum_{k=p+1}^{\infty} |a_k| \leq \sum_{k=p+1}^{\infty} \frac{C}{k^2} \leq \frac{C}{p} \tag{3.19}$$

Using (3.15), (3.19), and (3.18), we see that

$$\|\bar{u}\|_{H^{1/2}(I)} \leq \frac{C}{p} \tag{3.20}$$

which combined with (3.17) yields

$$\|u-u_p^0\|_{H^{1/2}(I)} \leq \frac{C}{p} \tag{3.21}$$

By (2.3), we know that

$$\|u-u_p^0\|_{\tilde{H}^{1/2}(I)} \leq \|u-u_p^0\|_{H^{1/2}(I)} + \|(1-x^2)^{-1/2}(u-u_p^0)\|_{H^0(I)}$$

$$(3.22)$$

Hence we must bound the second term. We have

$$\int_{-1}^{+1} (1-x^2)^{-1}(u-u_p^0)^2 dx = \left(\int_0^{1/p} + \int_{1/p}^{\pi-\frac{1}{p}} + \int_{\pi-\frac{1}{p}}^{\pi}\right)(\hat{u}-\hat{u}_p^0)^2(\sin \xi)^{-1}d\xi$$

$$(3.23)$$

Now $\frac{1}{\sin \xi}$ is bounded on $[\frac{1}{p}, \pi-\frac{1}{p}]$. Hence, using (3.19)

$$\int_{1/p}^{\pi-\frac{1}{p}} (\hat{u}-\hat{u}_p^0)^2(\sin \xi)^{-1}d\xi \leq \frac{C}{p^2} \int_{1/p}^{\pi-\frac{1}{p}} (\sin \xi)^{-1}d\xi$$

$$\leq \frac{C}{p^2} \log p \qquad (3.24)$$

Also, let [a] denote the integral part of a. Then it may be verified that with

$$\hat{u} := \sum_{j=[\frac{p+2}{2}]}^{\infty} a_{2j} + \sum_{j=[\frac{p+1}{2}]}^{\infty} a_{2j+1} \cos \xi$$

we obtain with (3.13)

$$\hat{u}-\hat{u}_p^0 = \sum_{j=[\frac{p+2}{2}]}^{\infty} a_{2j}(\cos 2j\xi-1) + \sum_{j=[\frac{p+1}{2}]}^{\infty} a_{2j+1}(\cos(2j+1)\xi-\cos \xi)$$

which satisfies

$$(\hat{u}-\hat{u}_p^0)(\cos^{-1}(\pm 1)) = 0$$

as required in (3.10). Hence,

$$(\hat{u}-\hat{u}_p^0)^2 \leq C\{(\sum_{j=[\frac{p+2}{2}]}^{\infty} a_{2j}(\cos 2j\xi-1))^2$$

$$+ (\sum_{j=[\frac{p+1}{2}]}^{\infty} a_{2j+1}(\cos(2j+1)\xi-\cos \xi))^2\}$$

$$\leq C\{(\sum_{j=[\frac{p+2}{2}]}^{\infty} a_{2j} \sin^2 j\xi)^2$$

$$+ (\sum_{j=[\frac{p+1}{2}]}^{\infty} a_{2j+1} \sin(j+1)\xi \sin j\xi)^2\}$$

so that

$$\int_0^{1/p} (\hat{u}-\hat{u}_p^0)^2 (\sin \xi)^{-1}d\xi \leq C\{\int_0^{1/p} (\sum_{j=[\frac{p+2}{2}]}^{\infty} a_{2j}\sin^2 j\xi)^2 (\sin \xi)^{-1}d\xi$$

$$+ \int_0^{1/p} (\sum_{j=[\frac{p+1}{2}]}^{\infty} a_{2j+1}\sin(j+1)\xi \sin j\xi)^2 (\sin \xi)^{-1}d\xi\} \tag{3.25}$$

Now $\sin j\xi \leq j\xi$, so that for any $\varepsilon > 0$,

$$\sin^2 j\xi \leq \sin^\varepsilon j\xi \sin^{2-\varepsilon} j\xi \leq (j\xi)^\varepsilon.1 \leq (j\xi)^\varepsilon$$

Hence, using (3.16), the first term on the right side of (3.25) is bounded by

$$\int_0^{1/p} (\sum_{j=[\frac{p+2}{2}]}^{\infty} \frac{j^\varepsilon}{j^2})^2 \cdot \frac{\xi^{2\varepsilon}}{\sin \xi} d\xi \leq \frac{C}{p^{2(1-\varepsilon)}} \int_0^{1/p} \xi^{2\varepsilon-1}d\xi$$

$$\leq \frac{C}{p^{2(1-\varepsilon)}} \cdot p^{-2\varepsilon} = \frac{C}{p^2}$$

The second term may be similarly bounded, as may the term

$$\int_{\pi -\frac{1}{p}}^{\pi} (\hat{u}-\hat{u}_p^0)^2 (\sin \xi)^{-1}d\xi$$

Using this with (3.23), (3.24) gives

$$\| (1-x^2)^{-1/2} (u-u_p^0) \|_{H^0(I)} \leq \frac{C \log^{1/2} p}{p}$$

which combined with (3.21)-(3.22) yields (3.11).   []

In Section 4 we will use Theorems 3.1 and 3.3 to bound the error made when a function that is smooth in the interior of Γ and behaves like (3.8) at the end points is approximated by functions in $V_p(\Gamma)$.

3.3. $\widetilde{H}^{-1/2}$ Approximation of Singular Functions

In this section, we consider the approximation of functions u defined on $I = [-1,+1]$ of the form

$$u(x) = (x+1)^{-1/2} \chi(x) \tag{3.26}$$

where χ is as before. We are now interested in approximating u in the $\widetilde{H}^{-1/2}(I)$ norm by functions in $P_p(I)$. To this end, we first prove the following lemma.

Lemma 3.2. *Let* $f \in \widetilde{H}^{1/2}(I)$. *Then* $f' \in \widetilde{H}^{-1/2}(I)$ *and*

$$\| f' \|_{\widetilde{H}^{-1/2}(I)} \leq C \| f \|_{\widetilde{H}^{1/2}(I)} \tag{3.27}$$

Proof. Let $\psi \in C_0^\infty(I)$. Define ψ^* to be the extension by 0 of ψ to $\mathbb{R}$. Then it may be easily seen that $\psi^{*'} = \psi'^*$, so that

$$\| \psi' \|_{H^{-1/2}(I)} = \| \psi'^* \|_{H^{-1/2}(\mathbb{R})} = \| \psi^{*'} \|_{H^{-1/2}(\mathbb{R})}$$

$$\leq C \| \psi^* \|_{H^{1/2}(\mathbb{R})} \leq C \| \psi \|_{H^{1/2}(I)} \tag{3.28}$$

since $\psi \in C_0^\infty(I)$. (The inequality (3.28) can be verified taking Fourier transforms, for instance.)

Now, let $f \in \tilde{H}^{1/2}(I)$. We use the following definition, from [11], for the $\tilde{H}^{-1/2}(I)$ norm:

$$\|f'\|_{\tilde{H}^{-1/2}(I)} = \sup_{\psi \in C_0^\infty(I)} \frac{|<f',\psi>_{L^2(I)}|}{\|\psi\|_{H^{1/2}(I)}}$$

Hence with

$$<f',\psi>_{L^2(I)} = - <f,\psi'>_{L^2(I)}$$

We obtain

$$\|f'\|_{\tilde{H}^{-1/2}(I)} = \sup_{\psi \in C_0^\infty(I)} \frac{|<f,\psi'>_{L^2(I)}|}{\|\psi\|_{H^{1/2}(I)}}$$

$$\leq \sup_{\psi \in C_0^\infty(I)} \frac{\|f\|_{\tilde{H}^{1/2}(I)} \|\psi'\|_{H^{-1/2}(I)}}{\|\psi\|_{H^{1/2}(I)}}$$

$$\leq C \|f\|_{\tilde{H}^{1/2}(I)}$$

by (3.28). This proves the lemma. ⬚

With Lemma 3.2, we obtain the following analog to Theorem 3.3.

Theorem 3.4. Let u be defined by (3.26). Then there exists a polynomial u_p in $P_p(I)$ such that

$$\|u-u_p\|_{\tilde{H}^{-1/2}(I)} \leq Cp^{-1}\log^{1/2}p. \tag{3.29}$$

Proof. Let

$$w = \int_{-1}^{x} u \, dx = 2(x+1)^{1/2}\chi(x) - 2\int_{-1}^{x}(x+1)^{1/2}\chi'(x)\,dx = w_1 + w_2$$

By Theorem 3.3, there exists $v_p^1 \in P_p(I)$ satisfying

$$\|w_1 - v_p^1\|_{\tilde{H}^{1/2}(I)} \leq Cp^{-1}\log^{1/2}p.$$

Also, since $\chi(x)$ is smooth, w_2 lies in $H^{2-\epsilon}$ for any $\epsilon > 0$. Applying Theorem 3.1, there exists $v_p^2 \in P_p(I)$ satisfying

$$\|w_2 - v_p^2\|_{\tilde{H}^{1/2}(I)} \leq Cp^{-(2-\epsilon-\frac{1}{2})}\log^{1/2}p \, \|w_2\|_{H^{2-\epsilon}(I)}$$

$$\leq Cp^{-1}\log^{1/2}p.$$

Taking $v_p = v_p^1 + v_p^2$, we have

$$\|w - v_p\|_{\tilde{H}^{1/2}(I)} \leq Cp^{-1}\log^{1/2}p.$$

Finally, using Lemma 3.2 and taking $u_p = v_p'$, we obtain (3.29).
$\square$

Section 4.   The p-Version for Boundary Elements

Before we apply the approximation results of Section 3
to the Galerkin solutions of some integral equations of
the first kind, let us recall some basic facts on the
Galerkin method.   The key to the error analysis of Galerkin's
method is the following result by Hildebrandt and Wienholtz
[15] (see also [11], [21]).

*Lemma 4.1.   Let H be a Hilbert space with dual H' (not
necessarily identified with H) and let A be injective and
continuous from H into H' satisfying a Gårding inequality.
Let u ∈ H denote the solution of*

$$Au = f \qquad (4.1)$$

*where f ∈ H' and let $u_N \in S_N \subset H$ denote the solution of
the Galerkin equations*

$$\langle Au_N, v \rangle = \langle f, v \rangle \qquad \text{for all } v \in S_N \subset H. \qquad (4.2)$$

*Furthermore let for any φ ∈ H there exists $\phi_N \in S_N$ with
$\phi = \lim_{N \to \infty} \phi_N$ in H.   Then for N large enough the Galerkin
equations (4.2) are uniquely solvable and there holds with
a constant C independent of u, u_N and N the error estimate*

$$\|u - u_N\| < C \inf \{\|u - v_N\| : v_N \in S_N\} \qquad (4.3)$$

where $\|\cdot\|$ denotes the norm in H.

Next we list several boundary value problems which can

be reduced to strongly elliptic integral equations, i.e.,
the corresponding integral operators satisfy a Gårding
inequality in appropriate Sobolev spaces. Therefore, due
to Lemma 4.1, the corresponding boundary element Galerkin
methods converge and the quasioptimality (4.3) holds leading
together with the approximation results of Section 3 to
error estimates for the p-version.

The *Neumann screen problem* in acoustics describes the
scattering of a plane wave at a hard obstacle Γ. Here Γ is
given by an oriented open arc being a finite piece of a
smooth curve in $\mathbb{R}^2$. The orientation defines the normal
vector n pointing to the side Γ_2 (see Fig. 1). The opposite
side of Γ will be denoted by Γ_1. The scattering problem
leads to the problem: *Find the pressure amplitude field*
$u \in H^1_{loc}(\Omega_\Gamma)$ *satisfying*

$$(\Delta + k^2)u = 0 \text{ in } \Omega_\Gamma = \mathbb{R}^2 \setminus \bar{\Gamma}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_1} = g_1 \, , \qquad \left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = g_2 \qquad (4.4)$$
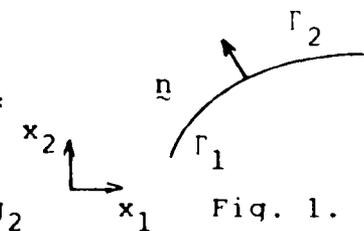
Fig. 1.

Here $k \neq 0$, Im $k \geq 0$ and $g_1, g_2 \in H^{-1/2}(\Gamma)$ are given with
$g := g_1 - g_2 \in \tilde{H}^{-1/2}(\Gamma)$. In addition, we require the
Sommerfeld radiation condition

$$\frac{\partial u}{\partial r} - iku = o(r^{-1/2}) \text{ and } u = O(r^{-1/2}) \text{ as } r = |x| \to \infty. \quad (4.5)$$

From [23] we know that for Im $k \geq 0$, $k \neq 0$ the problem (4.4),
(4.5) has no eigensolutions and furthermore it can be
reduced to a hypersingular integral equation on Γ.

Theorem 4.1 [23]. *Let* g_1, g_2 *and* k *be given as above.* *Then there holds:* (i) $u \in H^1_{loc}(\Omega_\Gamma)$ *solves* (4.4), (4.5) *if and only if the jump* $[u]|_\Gamma \in \tilde{H}^{1/2}(\Gamma)$ *satisfies the integral equation*

$$D[u](z) := -2\int_\Gamma [u](\zeta) \frac{\partial^2}{\partial n_z \partial n_\zeta} \Phi(z,\zeta) ds_\zeta = f(z), \quad z \in \Gamma \quad (4.6)$$

with

$$f(z) := g_1(z) + g_2(z) + 2\int_\Gamma (g_1(\zeta) - g_2(\zeta)) \frac{\partial}{\partial n_z} \Phi(z,\zeta) ds_\zeta \quad (4.7)$$

where

$$\Phi(z,\zeta) := \frac{-i}{4} H_0^{(1)}(k|z-\zeta|) \quad (4.8)$$

and $H_0^{(1)}$ *is the Hankel function of the first kind and order zero.* (ii) *There exists exactly one solution* $\psi \in \tilde{H}^{1/2}(\Gamma)$, $\psi = [u]|_\Gamma$ *of* (4.6).

The proof of the assertion (ii) in [23] hinges on the fact that D is a strongly elliptic pseudodifferential operator of order 1. Therefore there exists a constant $\gamma_1 > 0$ and a compact mapping $C_1: \tilde{H}^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ such that

$$\text{Re}\langle (D+C_1)\psi, \psi \rangle_{L^2(\Gamma)} \geq \gamma_1 \|\psi\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad (4.9)$$

for every $\psi \in \tilde{H}^{1/2}(\Gamma)$. This yields that D is a Fredholm operator of index zero, and the bijectivity of D follows therefore from its injectivity which is guaranteed by the above assumptions on the wave number k. Note that the assumptions on g_1, g_2 imply $f \in H^{-1/2}(\Gamma)$.

Using localization and Mellin transformation Stephan and Wendland derive in [23] the following explicit regularity result for the solution of (4.6) near the end points z_1, z_2 of Γ.

Lemma 4.2 [23]. For $0 \leq \sigma < 1/2$ let $q_j \in H^{1/2+\sigma}(\Gamma)$, $j = 1,2$, be given. Then the solution $[u]|_\Gamma \in \tilde{H}^{1/2}(\Gamma)$ of the integral equation (4.6) has the form

$$[u]|_\Gamma = \sum_{i=1}^{2} \alpha_i \rho_i^{1/2} \chi_i + v_0 \text{ with } v_0 \in \tilde{H}^{3/2+\sigma}(\Gamma), \; \alpha_i \in \mathbb{R}.$$

(4.10)

Here ρ_i denotes the Euclidean distance between $z \in \Gamma$ and the end point z_i of Γ. χ_i is a C^∞-cut-off function with $0 \leq \chi_i \leq 1$ and $\chi_i = 1$ near to z_i, $\chi_i = 0$ at the opposite end point, $i = 1,2$.

The *p-version Galerkin method* for the hypersingular integral equation (4.6) reads: *Find $v_p \in V_p^0(\Gamma)$ such that with $f \in H^{-1/2}(\Gamma)$ given by (4.7) for all $\phi_p \in V_p^0(\Gamma)$ there holds*

$$\langle Dv_p, \phi_p \rangle_{L^2(\Gamma)} = \langle f, \phi_p \rangle_{L^2(\Gamma)}$$

(4.11)

Here $V_p^0(\Gamma)$ denotes the set of continuous, piecewise polynomials of degree $\leq p$ which vanish at the end points of Γ as introduced in Section 2. Note $V_p^0(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$. There holds the following convergence result for the Galerkin scheme (4.11).

Theorem 4.2. *Let p be sufficiently large. Then the Galerkin equations* (4.11) *are uniquely solvable and for the error between the exact solution* $[u]|_\Gamma \in \tilde{H}^{1/2}(\Gamma)$ *of* (4.6) *and the Galerkin solution* $v_p \in v_p^0(\Gamma)$ *we have*

$$\| [u]-v_p \|_{\tilde{H}^{1/2}(\Gamma)} \leq Cp^{-1} \log^{1/2}p \qquad (4.12)$$

where the constant C is independent of p.

Proof. We observe that the operator D in (4.6) fulfills the requirements on A in Lemma 4.1 with $H := \tilde{H}^{1/2}(\Gamma)$, and $H = H^{-1/2}(\Gamma)$ since D satisfies the Gårding inequality (4.9) and D is bijective from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$ due to Theorem 4.2. On the other hand $\{v_p^0(\Gamma)\}$ is a sequence of approximating subspaces of $\tilde{H}^{1/2}(\Gamma)$ as $p \to \infty$ and therefore $v_p^0(\Gamma)$ is a candidate for the subspace S_N in Lemma 4.1 with p instead of N. Thus the convergence of the p-version for the Galerkin procedure (4.11) is an immediate consequence of Lemma 4.1. The rate of convergence in (4.12) follows from the quasioptimality (4.3) together with the regularity result (4.10), where the approximation result (3.11) is used to approximate the singular part in (4.10) and Theorem 3.1 is used to approximate the regular part v_0. $\qquad\qquad\square$

Remark 4.1. (i) The decomposition (4.10) shows that the exact solution $\psi = [u]|_\Gamma$ of the integral equation (4.6) belongs to $H^{1-\epsilon}(\Gamma)$ for any $\epsilon > 0$. Therefore the h-version of the Galerkin procedure for equation (4.6) gives only an estimate of order $O(h^{1/2})$ for the Galerkin error, if a uniform mesh is used. (ii) Application of the estimate (3.2) to the quasioptimality

estimate (4.3) gives with $\psi \in \tilde{H}^{1-\epsilon}(\Gamma)$

$$\|\psi - \psi_p\|_{\tilde{H}^{1/2}(\Gamma)} \leq Cp^{-1/2+\epsilon} \log^{1/2} p \|\psi\|_{\tilde{H}^{1-\epsilon}(\Gamma)} .$$

The better estimate (4.10) follows from Theorem 3.3.

The *Dirichlet screen problem* in acoustics describes the scattering of a plane wave at a soft obstacle Γ. With Γ being an open arc as introduced above the scattering problem becomes: *Find the pressure amplitude field* $u \in H^1_{loc}(\Omega_\Gamma)$ *satisfying*

$$(\Delta+k^2)u = 0 \text{ in } \Omega_\Gamma = \mathbb{R}^2 \setminus \bar{\Gamma}, \quad u = g \text{ on } \Gamma \quad (4.13)$$

together with the radiation condition (4.5) *for given* $g \in H^{1/2}(\Gamma)$ *and* $k \neq 0$, $\text{Im } k \geq 0$.

We know from [22] that with the above restrictions on the wave number k the Dirichlet problem (4.13), (4.5) has no eigensolutions. Furthermore this Dirichlet problem can be reduced to a weakly singular integral equation on Γ [22].

Theorem 4.3 [22]. Let $g \in H^{1/2}(\Gamma)$ *be given and* $k \neq 0$, $\text{Im } k \geq 0$. *Then there holds:* (i) $u \in H^1_{loc}(\Omega_\Gamma)$ *solves* (4.13), (4.5) *if and only if the jump* $[\frac{\partial u}{\partial n}]\big|_\Gamma \in \tilde{H}^{-1/2}(\Gamma)$ *satisfies the integral equation*

$$V[\frac{\partial u}{\partial n}](z) := -2\int_\Gamma [\frac{\partial u}{\partial n}](\zeta) \Phi(z,\zeta) ds_\zeta = 2g(z), \quad z \in \Gamma \quad (4.14)$$

where Φ *is given in* (4.8)

(ii) *There exists exactly one solution* $\psi \in \tilde{H}^{-1/2}(\Gamma)$, $\psi = [\frac{\partial n}{\partial n}]\big|_\Gamma$

of (4.14).

(iii) let $g \in H^{3/2+\sigma}(\Gamma)$, $0 < \sigma < 1/2$, be given. Then with the notation of Lemma 4.2 the solution $[\frac{\partial u}{\partial n}]|_\Gamma \in \tilde{H}^{-1/2}(\Gamma)$ of (4.14) has the form

$$[\frac{\partial u}{\partial n}]|_\Gamma = \sum_{i=1}^{2} \alpha_i \rho_i^{-1/2} \chi_i + \psi_0 \text{ with } \psi_0 \in \tilde{H}^{1/2+\sigma}(\Gamma), \ \alpha_i \in \mathbb{R} \ (4.15)$$

The proof of assertion (ii) in [22] uses that the single layer potential operator V is a strongly elliptic pseudodifferential operator of order -1. Therefore there holds with a constant γ_2 and a compact operator $C_2 : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ the Gårding inequality

$$\text{Re} \langle (V+C_2)\psi, \psi \rangle_{L^2(\Gamma)} \geq \gamma_2 \|\psi\|^2_{\tilde{H}^{-1/2}(\Gamma)} . \qquad (4.16)$$

for any $\psi \in \tilde{H}^{-1/2}(\Gamma)$.

The decomposition (4.15) is obtained in [22] by localizing the weakly singular integral equation (4.14) and applying the Mellin transformation. The explicit form (4.15) of the solution near the end points z_i, $i = 1, 2$, allows us to derive optimal error estimates for the Galerkin solution.

The p-version Galerkin method for the weakly singular integral equation (4.14) reads: Find $\psi_p \in S_p(\Gamma)$ such that with $g \in H^{1/2}(\Gamma)$

$$\langle V\psi_p, \phi_p \rangle = \langle 2g, \phi_p \rangle \text{ for all } \phi_p \in S_p(\Gamma) \qquad (4.17)$$

Here $S_p(\Gamma)$ denotes the set of piecewise polynomials of degree

$\leq$ p subordinate to a partitioning of Γ as introduced in Section 2. Note $S_p(\Gamma) \subset \widetilde{H}^{-1/2}(\Gamma)$.

Theorem 4.4. Let p be sufficiently large. Then the Galerkin equations (4.17) are uniquely solvable and the error between the exact solution ψ of (4.14) and the Galerkin solution $\psi_p \in S_p(\Gamma)$ of (4.17) satisfies

$$\| \psi - \psi_p \|_{\widetilde{H}^{-1/2}(\Gamma)} \cdot Cp^{-1} \log^{1/2} p \qquad (4.18)$$

with a constant C independent of p.

Proof. Due to the Gårding inequality (4.16) application of Lemma 4.1 yields for the choices $\Lambda = V$ and $H = \widetilde{H}^{-1/2}(\Gamma)$ with $H' = H^{1/2}(\Gamma)$ the convergence of the Galerkin scheme (4.17). Note that $\{S_p(\Gamma)\}$ as introduced in Section 2 is a sequence of approximating subspaces for $\widetilde{H}^{-1/2}(\Gamma)$ as $p \to \infty$. The estimate (4.18) follows from the quasioptimality (4.3) together with the regularity result (4.15) where (3.29) is used to approximate the singular part in (4.15) and Theorem 3.2 is used to approximate the regular part ψ_0. $\qquad \Box$

The *exterior Neumann (Dirichlet) problem* in acoustics describes the scattering of a plane wave at a hard (soft) obstacle Ω being a bounded domain in $\mathbb{R}^3$. For simplicity we assume that the boundary $\widetilde{\Gamma}$ of Ω is a closed, smooth, simply connected surface. Then the scattering problem leads to the problem: *Find* $u \in H^1_{loc}(\mathbb{R}^3 \setminus \bar{\Omega})$ *satisfying*

$$(\Lambda + k^2)u = 0 \ in \ \mathbb{R}^3 \setminus \bar{\Omega} \qquad (4.19)$$

$$\frac{\partial u}{\partial n} = g \quad on \ \tilde{\Gamma} \quad (Neumann) \tag{4.20}$$

or

$$u = f \quad on \ \tilde{\Gamma} \quad (Dirichlet) \tag{4.21}$$

for $k \neq 0$, $Im \ k \geq 0$ *together with the radiation condition*

$$\frac{\partial u}{\partial r} - iku = o(r^{-1}), \quad u = O(r^{-1}) \ as \ r = |x| \to \infty. \tag{4.22}$$

Here we make the general assumption:

In the exterior Neumann (Dirichlet) problem in $\mathbb{R}^3 \setminus \bar{\Omega}$
let k^2 *be different from the eigenvalues of the*
interior Dirichlet (Neumann) problem in Ω. (4.23)

The restriction to the three-dimensional case is only for simplicity. Of course we can derive analogous results also for the corresponding 2D problems. Easy modifications of the procedure in [16], [20] lead directly to a boundary integral equation method for the Neumann and the Dirichlet problem. One obtains immediately existence and uniqueness results analogous to Theorems 4.1 and 4.2. Let us first consider again the Neumann problem (4.19), (4.20), (4.22).

Theorem 4.5. Let $g \in H^{-1/2}(\tilde{\Gamma})$ *be given with* $\int_{\tilde{\Gamma}} g \ ds = 0$.
Then there holds with k *as above (i)* $u \in H^1_{loc}(\mathbb{R}^3 \setminus \bar{\Omega})$ *solves*
(4.19), (4.20), (4.22) if and only if $u \in H^{1/2}(\tilde{\Gamma})$ *satisfies*
the integral equation

$$Du(z) := -2\int_{\tilde{\Gamma}} u(\zeta) \frac{\partial^2}{\partial n_z \partial n_\zeta} \Phi(z,\zeta) ds_z = \tilde{g}(z), \quad z \in \Gamma \tag{4.24}$$

where

$$\Phi(z,\zeta) := \frac{e^{ik|z-\zeta|}}{-4\pi|z-\zeta|} \tag{4.25}$$

and

$$\tilde{q}(z) = q(z) - 2\int_{\tilde{\Gamma}} q(\zeta) \frac{\partial}{\partial n_z} \Phi(z,\zeta) ds_\zeta \tag{4.26}$$

(ii) There exists exactly one solution $u \in H^{1/2}(\tilde{\Gamma})$ *of* (4.24).

(iii) Let $q \in H^s(\tilde{\Gamma})$, $s \geq -1/2$ *and* $\tilde{\Gamma}$ *be analytic. Then the solution* u *of* (4.24) *belongs to* $H^{s+1}(\tilde{\Gamma})$.

Correspondingly using the direct approach of [20], [22] one obtains for the Dirichlet problem (4.19), (4.21), (4.22):

Theorem 4.6. Let $f \in H^{1/2}(\tilde{\Gamma})$ *be given. Then with* k *as above there holds: (i)* $u \in H^1_{loc}(\mathbb{R}^3 \setminus \bar{\Omega})$ *solves* (4.19), (4.21), (4.22) *if and only if* $\frac{\partial u}{\partial n} \in H^{-1/2}(\tilde{\Gamma})$ *satisfies the integral equation*

$$V \frac{\partial u}{\partial n}(z) := -2\int_{\tilde{\Gamma}} \frac{\partial u}{\partial n}(\zeta) \Phi(z,\zeta) ds_\zeta = \tilde{f}(z), \quad z \in \Gamma \tag{4.27}$$

with Φ *as in* (4.25) *and*

$$\tilde{f}(z) := f(z) + 2\int_{\tilde{\Gamma}} u(\zeta) \frac{\partial}{\partial n} \Phi(z,\zeta) ds_\zeta \tag{4.28}$$

(ii) There exists exactly one solution $\frac{\partial u}{\partial n} \in H^{-1/2}(\tilde{\Gamma})$ *of* (4.27).

(iii) Let $f \in H^s(\tilde{\Gamma})$, $s > 1/2$, *and* $\tilde{\Gamma}$ *be analytic. Then the solution* $\frac{\partial u}{\partial n}$ *of* (4.27) *belongs to* $H^{s-1}(\tilde{\Gamma})$.

Proofs of Theorems 4.5 and 4.6. For brevity we sketch only the main steps. The equivalence (i) between the boundary

value problems and the integral equations is standard and follows immediately from Green's formula (see [16], [20]). The existence and uniqueness results (ii) of the solution of the respective integral equation are based on the strong ellipticity of the pseudodifferential operators D and V, i.e., there hold with constants $\tilde{\gamma}_1, \tilde{\gamma}_2 > 0$ and compact mappings $\tilde{C}_1: H^{1/2}(\tilde{\Gamma}) \to H^{-1/2}(\tilde{\Gamma})$ and $\tilde{C}_2: H^{-1/2}(\tilde{\Gamma}) \to \overset{\circ}{H}{}^{1/2}(\tilde{\Gamma})$ the Gårding inequalities

$$\mathrm{Re} \langle (D + \tilde{C}_1) v, v \rangle_{L^2(\tilde{\Gamma})} \geq \tilde{\gamma}_1 \| v \|^2_{H^{1/2}(\tilde{\Gamma})} \qquad (4.29)$$

$$\mathrm{Re} \langle (V + \tilde{C}_2) \psi, \psi \rangle_{L^2(\tilde{\Gamma})} \geq \tilde{\gamma}_2 \| \psi \|^2_{H^{-1/2}(\tilde{\Gamma})} \qquad (4.30)$$

for all $v \in H^{1/2}(\tilde{\Gamma})$ and $\psi \in H^{-1/2}(\tilde{\Gamma})$. Hence $D: H^{1/2}(\tilde{\Gamma}) \to H^{-1/2}(\tilde{\Gamma})$ and $V: H^{-1/2}(\tilde{\Gamma}) \to H^{1/2}(\tilde{\Gamma})$ are Fredholm operators of index zero. Under the assumption (4.23) we have for $k \neq 0$, $\mathrm{Im}\, k \geq 0$ that the integral equations have no eigensolutions. Hence the above mappings D and V are bijective yielding assertion (ii). The regularity results (iii) in Theorems 4.5 and 4.6 follow in a standard way from the ellipticity of the pseudodifferential operators D and V (see for example [19], [20]). □

Finally, we consider the Galerkin equations for the integral equations (4.24) and (4.27) and show the convergence of the p-version.

The *Galerkin method (p-version)* for the integral equation (4.24) reads with $V_p(\tilde{\Gamma})$ defined by (2.7):

Find $v_p \in V_p(\tilde{\Gamma})$ *such that with* $\tilde{g} \in H^{-1/2}(\tilde{\Gamma})$ *given by* (4.26)

there holds for all $\phi_p \in V_p(\widetilde{\Gamma})$

$$\langle Dv_p, \phi_p \rangle_{L^2(\widetilde{\Gamma})} = \langle \widetilde{g}, \phi_p \rangle_{L^2(\widetilde{\Gamma})} \tag{4.31}$$

Correspondingly, the *Galerkin method (p-version)* for the integral equation (4.27) reads with $S_p(\widetilde{\Gamma})$ given by (2.6): *Find* $\psi_p \in S_p(\widetilde{\Gamma})$ *such that with* $\widetilde{f} \in H^{1/2}(\widetilde{\Gamma})$ *given by* (4.28) *there holds for all* $\phi_p \in S_p(\widetilde{\Gamma})$

$$\langle V\psi_p, \phi_p \rangle_{L^2(\widetilde{\Gamma})} = \langle \widetilde{f}, \phi_p \rangle_{L^2(\widetilde{\Gamma})}. \tag{4.32}$$

Theorem 4.7. Let p be sufficiently large and $\widetilde{g} \in H^s(\widetilde{\Gamma})$, $s \geq -1/2$. *Then the Galerkin equations* (4.31) *are uniquely solvable. Let* $u \in H^{s+1}(\widetilde{\Gamma})$ *be the exact solution of* (4.24) *and* $v_p \in V_p(\widetilde{\Gamma})$ *be the Galerkin solution then we have for* $s \geq -1/2$

$$\|u - v_p\|_{H^{1/2}(\widetilde{\Gamma})} \leq Cp^{-(s+1/2)} \|\widetilde{g}\|_{H^s(\widetilde{\Gamma})} \tag{4.33}$$

with a constant C independent of u, $\widetilde{g}$ *and p.*

Proof. Obviously, since Gårding's inequality (4.29) holds the assumptions of Lemma 4.1 are satisfied if we choose $A = D$, $H = H^{1/2}(\widetilde{\Gamma})$, $H' = H^{-1/2}(\widetilde{\Gamma})$ and $S_N = V_p(\widetilde{\Gamma}) \subset H^{1/2}(\widetilde{\Gamma})$. Note that $\widetilde{\Gamma}$ is a closed, bounded, analytic surface. Thus for p large enough Lemma 4.1 guarantees the unique solvability of the Galerkin equations (4.31) and the quasioptimal estimate for the Galerkin error

$$\|u - v_p\|_{H^{1/2}(\widetilde{\Gamma})} \leq C \inf\{\|u - w_p\|_{H^{1/2}(\widetilde{\Gamma})} : w_p \in V_p(\widetilde{\Gamma})\} \tag{4.34}$$

From Theorem 4.5 (iii) we know that for $s \geq -1/2$ with a

constant C

$$\|u\|_{H^{s+1}(\widetilde{\Gamma})} \leq C \|g\|_{H^{s}(\widetilde{\Gamma})} . \tag{4.35}$$

Therefore we can apply the approximation result (3.1) of Theorem 3.1 to (4.34) and obtain (4.33) by using (4.35).

Theorem 4.8. Let p be sufficiently large and $\widetilde{f} \in H^{s}(\widetilde{\Gamma})$, $s \geq 1/2$. Then the Galerkin equation (4.32) are uniquely solvable. Let $\frac{\partial u}{\partial n} \in H^{s-1}(\widetilde{\Gamma})$ be the exact solution of (4.27) and $\psi_p \in S_p(\widetilde{\Gamma})$ be the Galerkin solution, then we have for $s \geq 1/2$

$$\left\|\frac{\partial u}{\partial n} - \psi_p\right\|_{H^{-1/2}(\widetilde{\Gamma})} \leq Cp^{-(s-1/2)} \|\widetilde{f}\|_{H^{s}(\widetilde{\Gamma})} \tag{4.36}$$

with a constant C independent of $\frac{\partial u}{\partial n}$, $\widetilde{f}$ and p.

Proof. Again, application of Lemma 4.1 gives the assertion if we take $A = V$, $H = H^{-1/2}(\widetilde{\Gamma})$, $H' = H^{1/2}(\widetilde{\Gamma})$ and $S_N = S_p(\widetilde{\Gamma}) \subset H^{-1/2}(\widetilde{\Gamma})$ since the Gårding inequality (4.30) holds. Note again that $\widetilde{\Gamma}$ is a closed, bounded, analytic surface. From Theorem 4.6 (iii) we know that for $s \geq 1/2$ with a constant C

$$\left\|\frac{\partial u}{\partial n}\right\|_{H^{s-1}(\widetilde{\Gamma})} \leq C \|f\|_{H^{s}(\widetilde{\Gamma})} . \tag{4.37}$$

On the other hand, Lemma 4.1 yields

$$\left\|\frac{\partial u}{\partial n} - \psi_p\right\|_{H^{-1/2}(\widetilde{\Gamma})} \leq C \inf \left\{ \left\|\frac{\partial u}{\partial n} - \phi_p\right\|_{H^{-1/2}(\widetilde{\Gamma})} : \phi \in S_p(\widetilde{\Gamma}) \right\} \tag{4.38}$$

Therefore by applying Theorem 3.2 and Remark 3.1 to (4.38) we

obtain with (4.37) the desired estimate (4.36).   []

Remark 4.2.   Theorems 4.7, 4.8   show that for the p-version, the rate of convergence obtained depends only upon the smoothness of the data. Hence, when $\tilde{f}$ and $\tilde{g}$ are arbitrarily smooth, one obtains arbitrarily high rates of convergence. This is in direct contrast to the h-version, where the rate of convergence depends in addition upon the degree of polynomials used and is therefore not very high even for smooth solutions.

Finally, we remark that results analogous to the above ones can be shown for two-dimensional crack problems in linear elasticity, since those problems can be reduced to first kind integral equations like (4.6) or (4.14) for the *components* of the jumps of the displacement or traction across the crack line Γ. Regularity results analogous to (4.10) and (4.15) hold for the solutions (see [17], [22], [23]). Hence the corresponding Galerkin schemes (4.11) and (4.17) will lead componentwise to error estimates like (4.12) and (4.18) with obvious modifications.

REFERENCES

1. Alarcon, E., Abia, L., Reverter, A.: On the possi-
    bility of adaptive boundary elements, in Accuracy
    Estimates and Adaptive Refinements in Finite Element
    Computations (AFREC), Lisbon, 1984.

2. Alarcon, E., Reverter, A., Molina, J.: Hierarchical
    boundary elements. Computers and Structures 20 (1985)
    151-156.

3. Alarcon, E., Reverter, A.: p-adaptive boundary ele-
    ments. International Journal for Numerical Methods in
    Engineering 23 (1986) 801-829.

4. Babuška, I.: The p and h-p versions of the finite
    elements method. The state of the art. Institute for
    Physical Science and Technology, University of Maryland,
    College Park, MD, Technical Note BN-1156 (1986).

5. Babuška, I., Szabo, B. A., Katz, I. N.: The p-version
    of the finite element method. SIAM J. Numer. Anal. 18
    (1981) 515-545.

6. Babuška, I., Suri, M.: The optimal convergence rate
    of the p-version of the finite element method. Institute
    for Physical Science and Technology, University of
    Maryland, College Park, MD, Technical Note BN-1045
    (1985).

7. Babuška, I., Suri, M.: The treatment of nonhomogeneous
    Dirichlet boundary conditions by the p-version of the
    finite element method. Institute for Physical Science
    and Technology, University of Maryland, College Park,
    MD (1987).

8. Babuška, I., Suri, M.: The h-p version of the finite
    element method with quasiuniform meshes, RAIRO (1987).

9. Babuška, I., Suri, M.: The p-version of the finite
    element method for constraint boundary conditions.
    Institute for Physical Science and Technology, University
    of Maryland, College Park, MD (1987).

10. Brebbia, C. A., editor. Progress in Boundary Element
    Methods, Vols. 1, 2, 3, 4, 5, Springer-Verlag, Berlin,
    1981 ff.

11. Costabel, M., Stephan, E. P.: Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximations. In W. Fiszdon and K. Wilmánski, editors, Mathematical Models and Methods in Mechanics 1981, Banach Center Publications Vol. 15, pages 175-251, PWN-Polish Scientific Publishers, Warsaw 1985.

12. Dorr, M. R.: The approximation theory for the p-version of the finite element method. SIAM J. Numer. Anal. 21 (1984) 1180-1207.

13. Gui, W., Babuška, I.: The h,p and h-p versions of the finite element method in one dimension. Parts 1-3, Institute for Physical Science and Technology, University of Maryland, College Park, MD, Technical Notes BN-1036, 1037, 1038 (1985).

14. Guo, B., Babuška, I.: The h-p version of the finite element method. Part 1: The basic approximation results. Computational Mechanics 1 (1986) 21-41. Part 2: General results and applications. Computational Mechanics 1 (1986) 203-220.

15. Hildebrandt, S., Wienholtz, E.: Constructive proofs of representation theorems in separable Hilbert space, Comm. Pure Appl. Math. 17 (1964) 369-373.

16. Hsiao, G. C., Wendland, W. L.: A finite element method for some integral equations of the first kind, J. Math. Anal. Appl. 58 (1977) 449-481.

17. Hsiao, G. C., Stephan, E. P., Wendland, W. L.: An integral equation formulation for a boundary value problem of elasticity in the domain exterior to an arc. Lecture Notes in Mathematics 1121 (1983) 153-165.

18. Lions, J. L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

19. Petersen, B. E.: Introduction to the Fourier Transform and Pseudodifferential Operators, Pitman, Boston, 1983.

20. Stephan, E. P.: Boundary integral equations for mixed boundary value problems, screen and transmission problems in $\mathbb{R}^3$. Habilitationsschrift, Technische Hochschule Darmstadt, 1984.

21. Stephan, E. P., Wendland, W. L.: Remarks to Galerkin and least squares methods with finite elements for general elliptic problems. Manuscripta Geodaetica 1 (1976) 93-123.

22. Stephan, E. P., Wendland, W. L.: An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems, Appl. Analysis 18 (1984) 183-219.

23. Stephan, E. P., Wendland, W. L.: A hypersingular boundary integral method for two-dimensional screen and crack problems, submitted to Archive of Rational Mechanics (1987).

24. Wendland, W. L.: Boundary element methods and their asymptotic convergence. In P. Filippi, editor, Theoretical Acoustics and Numerical Techniques, CISM Courses 277, pp. 135-216, Springer-Verlag, Wien, New York, 1983.

25. Wendland, W. L.: On some mathematical aspects of boundary element methods for elliptic problems. In J. Whiteman, editor, Mathematics of Finite Elements and Applications V, pp. 193-227, Academic Press, London, 1985.

26. Wendland, W. L.: Splines versus trigonometric poly-nomials. h-versus p-version in 2D boundary integral methods. In D. Griffiths, R. Mitchell eds., Dundee Biennial Conference on Numerical Analysis 1985.

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