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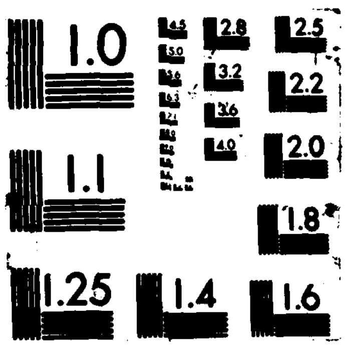
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Research Report CCS 555

AN EXTENDED COBB-DOUGLAS FORM FOR USE IN PRODUCTION ECONOMICS

by

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A.P. Schinnar *

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ABSTRACT

Sub alpha

The Cobb-Douglas function is widely used in production economics in the following form: $A = AL^{\alpha}K^{\beta}$, where α and β along with A are positive constants that relate the inputs L (=Labor) and K (=Capital) to the amount of output Q. A variety of supposed alternatives and generalizations have been suggested in place of the Cobb-Douglas form for use in production economics. These alternatives and generalizations are here shown to be representable in an extended Cobb-Douglas form in which A, α and β are functions of L and K rather than constants. This extension is then formally related to other general forms, such as the minimum discrimination information statistic, and used to explain the successful uses of the Cobb-Douglas function for empirical applications in many different countries and contexts.

KEY WORDS

Production functions

Aggregation

Homogeneous functions

Information statistic

1.0 Introduction

The Cobb-Douglas function, which made its first appearance in [14][1/] may be written in the form

$$(1.1) \qquad Q = AL^{\alpha}K^{\beta}$$

where α and β along with A are constants which relate the variables, e.g., the amounts of inputs L (= Labor) and K (= Capital) to the amount of output Q. This has been the most extensively used form of function for studying production behavior in empirical economics and econometrics.

In this paper we shall build on what has already been accomplished by a straightforward extension to

$$(1.2) \qquad Q = A(L,K)\ L^{\alpha(L,K)}\ K^{\beta(L,K)}$$

in which A, α and 3 are now functions of the inputs with

$$(1.3) \qquad \begin{gathered} \alpha + \beta = 1 \\ \alpha,\ \beta \geq 0\ . \end{gathered}$$

This formulation, as we shall see, makes easy contact with a variety of other disciplines from which most of the supposedly more general alternatives to production theory may also be accommodated. Hence nothing need be lost. Consistent also with empirical results from a wide variety of studies, it leads to certain new relations in the mathematical theory of homogeneous functions, and it provides access to more general results, such as are exhibited in [8] and [11], from which still further developments may be effected in mathematics, economics and statistics, and information theory

[1/] Tintner et. al. [38] notes a use of this kind of function in economic theory by Wicksell [42] in 1893 but the use of this function as one of the earliest (and most successful) examples of anything like a modern econometric study must certainly be dated from the paper by C. H. Cobb and P. H. Douglas in [14]. See also [19] which must still count as a model for economic-econometric investigations and which must rank very high in the order of such models.

as well.

After first delineating some of these possibilities, this paper will introduce a series of specializing assumptions to tie these more general developments into known results and relations in empirical and theoretical economics. These topics will be treated first from the standpoint of optimal production at the level of the individual firm -- what is sometimes called "intra-firm" analysis[1/] -- and then extended to aggregations of firms via efficiency frontiers and expansion paths at the level of the individual firm. The specializing assumptions needed to attain them will be clarified en route to these results, and then they will be interpreted and related to some of the already extant findings in empirical studies of production.

In the concluding section, a return will then be made to the natural extension, to more than two factors, for the general formulation, given above -- which we shall call an "Extended Cobb-Douglas Form" -- and suggestions will then be supplied on how the results given in this paper can be extended to developments in other fields besides the ones studied here.

[1/] See [4], [5] and [6].

2.0 A New Representation for Homogeneous Functions

To initiate our analysis we now let x represent a vector of variables x_i, i=1,...,n, and consider the expression

$$(2.1)\qquad hf(x) = \sum_{i=1}^{n} f_i x_i ,$$

which, by Euler's theorem, always holds when the function f(x) is homogeneous of degree h with continuous partial derivatives $f_i \equiv \frac{\partial f(x)}{\partial x_i}$. We also consider

$$(2.2)\qquad \prod_{i=1}^{n} \left(\frac{f_i x_i}{\delta_i(x)}\right)^{\delta_i(x)}$$

together with

$$(2.3)\qquad \sum_{i=1}^{n} \delta_i(x) = 1$$

and specify, for any i, that

$$(2.4)\qquad \delta_i^{\delta_i} = \delta_i^{-\delta_i} = 1$$

if $\delta_i = 0$.[1/]

If we can now choose

$$(2.5)\qquad \delta_i = \frac{f_i x_i}{hf(x)} = \frac{f_i x_i}{\sum_{i=1}^{n} f_i x_i}$$

with hf(x) > 0 to satisfy (2.3) then, by direct substitution in (2.2), we can obtain

$$(2.6)\qquad \prod_{i=1}^{n} \left(\frac{f_i x_i}{f_i x_i / hf(x)}\right)^{f_i x_i / hf(x)} = \left(hf(x)\right)^{\sum_{i=1}^{n} f_i x_i / hf(x)} = hf(x)$$

1/ Cf. Duffin, Peterson and Zener [17], p. 79. This same convention is also used in information theory. Cf., e.g., Theil [37].

This is the same $hf(x)$ that holds for (2.1). Hence via the derivation from (2.2) we have obtained an apparently new result for homogeneous functions which we may summarize in a way that makes it immediately pertinent to our analysis via the following:

<u>Theorem 1</u>: Let $hf(x) > 0$ where $f(x)$ is a function which is homogeneous of degree h, and let $f(x)$ have continuous partial derivatives, then

$$f(x) = A(x) \prod_{i=1}^{n} (x_i)^{\delta_i(x)} \tag{2.7}$$

for all x where

$$A(x) = \frac{1}{h} \prod_{i=1}^{n} \left(\frac{f_i}{\delta_i(x)} \right)^{\delta_i(x)} \tag{2.8}$$

with

$$\delta_i(x) = \frac{f_i x_i}{hf(x)} = \frac{f_i x_i}{\sum_{i=1}^{n} f_i x_i} \tag{2.9}$$

and

$$\sum_{i=1}^{n} \delta_i(x) = 1. \tag{2.10}$$

In the next section we shall supply illustrative examples to show how other production functions (which are homogeneous) may be represented in this extended Cobb-Douglas format. For the present, however, we observe that the form on the right of (2.7) as specified by (2.8) - (2.9), may be used to express <u>any</u> homogeneous function, and it also leads to a variety of important interpretations and uses.[1/]

[1/] Including the zero functional which is satisfied by choosing $A(x) \equiv 0$ and adding an extra $\delta_i(x) \equiv 1$ with all other $\delta_i(x) = 0$ as in (2.7). There is no real trouble for $h = 0$ either since we need merely restate (2.7) in terms of $hf(x)$ to handle such cases. We do not undertake to develop this further, however, since such functions are not ordinarily of interest in the economic theory of production.

Hereafter we shall refer to this as the "Extended Cobb-Douglas Form". We now exhibit one interpretation of the extended Cobb-Douglas function via

<u>Theorem 2</u>: For a production function $f(x)$, homogeneous of degree h, and with continuous partial derivatives, the $h\delta_i(x)$ in the Extended Cobb-Douglas format are the partial output elasticities of $f(x)$.

<u>Proof</u>: By definition, the ith partial output elasticity is

(2.11)
$$w_i(x) \equiv \frac{Ef(x)}{Ex_i} \equiv \frac{\partial \ln f(x)}{\partial \ln x_i}$$
$$= \frac{x_i}{f(x)} \frac{\partial f(x)}{\partial x_i} = \frac{x_i f_i}{f(x)}$$
$$= h\delta_i(x), \text{ by } (2.9).$$

Q.E.D.

We here observe that Ey/Ez represents the partial elasticity of y with respect to z in the "elasticity calculus" provided by R. G. D. Allen.[1/] Then we also observe that we have the following:

<u>Corollary 2.1</u>: The total output elasticity $\sum_i w_i(x)$ equals the degree of homogeneity.

<u>Proof</u>:

(2.12)
$$\sum_i w_i(x) = \sum_i h\delta_i(x) = h$$
since $\sum_i \delta_i(x) = 1$.

Q.E.D.

1/ See [1].

We further remark that these $\delta_i(x)$ are required to be non-negative, at least in economic theory, which in turn implies that the f_i, which are the marginal productivities, are also non-negative when $x_i \geq 0$, for all i. Our representation in Theorem 1 thus carries us at least part of the way toward explaining the success of the ordinary Cobb-Douglas functions in econometric investigations. These functions have the $\delta_i(x) = \delta_i$, as constants, which is to be expected when the partial output elasticities and the f_i do not vary much in the range of observations. This is to say that in such cases the ordinary Cobb-Douglas functions should give a good approximation to the "true" production function whenever the latter is homogeneous.1/

Parts of Professor Douglas' originally motivated search for "Laws of Production" or what might better be called "Laws of Production and Distribution" may now come into better view. See [14] and [19]. In terms of the preceding development, we may now phrase the research question as follows: For f(x) allowed to range over an entire class of admissible functions to select some best fitting function by reference to specified criteria which will thereby determine the $\delta_i(x) \geq 0$, $\sum \delta_i(x) = 1$ -- i.e., the factor proportions (and payments).

We shall return to this topic later in this paper. Here, however, we want to turn to the interpretation of A(x). From the start this term has been regarded as a sort of catch-all -- "A repository of other things." See [14] and [18]. As (2.8) makes clear, however, it contains a great deal of information and may be used as part of a distributional measure in a way that relates it to modern information theory.

1/The situation we have exhibited in Theorem 1 is not restricted to economics. Indeed it also helps to explain the success of techniques like dimensional analysis in physics and engineering. The fact that equivalent expressions for a physical quantity must agree dimensionally in powers of length, time, mass, etc., may also be regarded as generalizations of the concept of homogeneity.

In fact, applying Theorem 2 to (2.8), we observe that

$$A(x) = \prod_{i=1}^{n} \left(\frac{f_i}{w_i(x)} \right)^{\delta_i(x)} . \tag{2.13}$$

Hence A(x) denotes the geometric mean of the ratios of the marginal productivities of the factors to their respective output elasticities for all factors. Alternatively we can return to (2.8) and write

$$\ln [hA(x)] = \sum_{i=1}^{n} \delta_i(x) \ \ln \left[\frac{f_i}{\delta_i(x)} \right] \tag{2.14}$$

where "ln" is the Napierian logarithm. Subtracting $\ln \sum_{j=1}^{n} f_j$ from each side we obtain

$$\ln \left[\frac{hA(x)}{\sum_j f_j} \right] = - \sum_{i=1}^{n} \delta_i(x) \ \ln \left[\frac{\delta_i(x)}{f_i / \sum_j f_j} \right] . \tag{2.15}$$

One can thus interpret the logarithm of the ratio of hA(x) to $\sum_j f_j$ as the negative of the mean information for discrimination in favor of the hypothesis given by the "δ distribution" against the hypothesis given by the "$f_i / \sum_j f_j$ distribution."[1/]

This kind of "distributional" interpretation will be used later, as in our discussion of aggregation, but for the present it is of interest because of interpretations that have also been employed to explain some of the Cobb-Douglas properties and findings. E.g., we might especially note the use of Pareto distributions, as discussed in Houthakker [25] and Tintner et. al. [38] because of the fact that such distributions are also homogeneous.[2/]

1/ Cf. [29] p. 5.

2/ [illegible]

We next relate these developments, i.e., all of the developments in this section, to the classical arithmetic-geometric mean inequality, viz.,

$$\sum_{i=1}^{n} \delta_i \mu_i \geq \prod_{i=1}^{n} \mu_i^{\delta_i} , \tag{2.16}$$

where the μ_i are non-negative[1] but are otherwise arbitrary. The δ_i, which are "weights", must satisfy

$$\sum_{i=1}^{n} \delta_i = 1, \ \delta_i \geq 0, \text{ all } i \tag{2.17}$$

but are otherwise also arbitrary. As is well known, classically,[2] the necessary and sufficient conditions for equality in (2.16) are

$$\mu_1 = \mu_2 = \ldots = \mu_n \tag{2.18}$$

Now for f(x) homogeneous with continuous partial derivatives and f_i, $x_i \geq 0$, all i, we can set $\delta_i \mu_i = f_i x_i \geq 0$ and, using Euler's theorem, obtain

$$hf(x) = \sum_{i=1}^{n} f_i x_i \geq \prod_{i=1}^{n} \left(\frac{f_i x_i}{\delta_i} \right)^{\delta_i} \tag{2.19}$$

which is in the standard form of the geometric-arithmetic mean inequality as used in geometric programming.[3] In this form the necessary and sufficient conditions for equality become

$$\frac{f_1 x_1}{\delta_1} = \frac{f_2 x_2}{\delta_2} = \ldots = \frac{f_n x_n}{\delta_n} , \tag{2.20}$$

and this term-by-term relationship may be employed, as we shall later see, in identifying the $\delta_i(x)$ which are optimal in the sense of cost minimizing choices.

1/ Strictly speaking they should all be positive. See [17] pp. 4 ff. This will normally be the case in economic applications, however, and so we need not concern ourselves on this point here.

2/ See [23].

3/ See [17], pp. 4 ff.

3.0 Some Illustrative Applications

Before proceeding further, we provide some simple illustrations. Drawing on commonly employed production functions (which are also homogeneous)[1/] and using only two factors of production, L = "Labor" and K = "Capital", we apply theorem 1 to three such examples as follows:

(i) Linear Case: We write this as

$$Q = a_1K + a_2L, \tag{3.1}$$

a production function which is homogeneous of degree h=1 and has infinite elasticity of substitution between the indicated factors. To apply Theorem 1 we need

$$\frac{\partial Q}{\partial K} = a_1 = f_1$$

$$\frac{\partial Q}{\partial L} = a_2 = f_2$$

and

$$\delta = \frac{a_1K}{a_1K + a_2L}$$

$$1-\delta = \frac{a_2L}{a_1K + a_2L}$$

for use in

$$\hat{Q} = AK^{\delta} L^{1-\delta} \tag{3.2}$$

where

$$A = \frac{1}{h}\left(\frac{a_1}{\delta}\right)^{\delta} \left(\frac{a_2}{1-\delta}\right)^{1-\delta} .$$

[1/] Cf., the tabulation given in Intriligator [26], p. 187.

By direct substitution in (3.2) we then obtain

$$\hat{Q} = \left(\frac{a_1K + a_2L}{K}\right)^{\delta} \left(\frac{a_1K + a_2L}{L}\right)^{1-\delta} K^{\delta} L^{1-\delta}$$

$$= a_1K + a_2L,$$

which, as wanted, is the same Q as in (3.1).

(ii) Ordinary Cobb-Douglas with output elasticities constant, viz.,

(3.3) $$Q = BK^{\alpha}L^{\beta}$$

where $\alpha + \beta = h$ gives the degree of homogeneity -- not necessarily 1. Again we need

$$\frac{\partial Q}{\partial K} = \alpha \frac{Q}{K} = f_1$$

$$\frac{\partial Q}{\partial L} = \beta \frac{Q}{L} = f_2$$

for use in

(3.4) $$\hat{Q} = AK^{\delta}L^{1-\delta}$$

with

$$\delta = \frac{\alpha}{h}$$

$$1-\delta = \frac{\beta}{h}$$

and

$$A = \frac{1}{h} \left(\frac{\alpha \frac{Q}{K}}{\delta}\right)^{\delta} \left(\frac{\beta \frac{Q}{L}}{1-\delta}\right)^{1-\delta} .$$

Direct substitution in (3.4) then produces

$$\hat{Q} = \frac{1}{h}\left(\frac{\frac{\alpha Q}{K}}{\frac{\alpha}{h}}\right)^{\delta} \left(\frac{\frac{\beta Q}{L}}{\frac{\beta}{h}}\right)^{1-\delta} K^{\delta}L^{1-\delta}$$

$$= BK^{\alpha}L^{\beta} ,$$

which is (3.3). Q.E.D.

(iii) <u>Constant Elasticity of Substitution, CES, Production Function:</u>

This is usually written

(3.5) $$Q = B\left[\alpha K^{-\sigma} + (1-\alpha)L^{-\sigma}\right]^{-h/\sigma}$$

with the constants $B > 0$, $0 < \alpha < 1$ and $h > 0$, the degree of homogeneity. The elasticity of substitution, a constant, is then given by $1/(1+\sigma)$.[1/] Differentiating the above expression partially for K and L we obtain

$$\frac{\partial Q}{\partial K} = \alpha hB\left[\alpha K^{-\sigma} + (1-\alpha)L^{-\sigma}\right]^{-\frac{h}{\sigma}-1} K^{-\sigma-1}$$

$$\frac{\partial Q}{\partial L} = (1-\alpha)hB\left[\alpha K^{-\sigma} + (1-\alpha)L^{-\sigma}\right]^{-\frac{h}{\sigma}-1} L^{-\sigma-1},$$

for use in

(3.6) $$\hat{Q} = AK^{\delta}L^{1-\delta}$$

[1/] The preceding two cases are obtainable from this one by means of specializing assumptions on this elasticity. See, e.g., [2]. See also [24], pp. 86-88.

where

$$\delta = \frac{\alpha K^{-\sigma}}{\alpha K^{-\sigma} + (1-\alpha)L^{-\sigma}}$$

$$1-\delta = \frac{(1-\alpha)L^{-\sigma}}{\alpha K^{-\sigma} + (1-\alpha)L^{-\sigma}}$$

and

$$A = \frac{1}{h}\left(\frac{hB\left[\alpha K^{-\sigma} + (1-\alpha)L^{-\sigma}\right]^{-\frac{h}{\sigma}}}{K}\right)^{\delta}\left(\frac{hB\left[\alpha K^{-\sigma} + (1-\alpha)L^{-\sigma}\right]^{-\frac{h}{\sigma}}}{L}\right)^{1-\delta}.$$

Via direct substitution in (3.6), the extended Cobb-Douglas format,

$$\hat{Q} = B\left[\alpha K^{-\sigma} + (1-\alpha)L^{-\sigma}\right]^{-\frac{h}{\sigma}},$$

which is the CES function represented by (3.5).

Evidently, we can get these functions and many more, in fact any homogeneous function with continuous partial derivatives, via Theorem 1. We do not pursue this further, however, so that we can begin making more pointed interpretations via a series of specializing assumptions in the sections that follow.

4.0 Efficiency Frontiers and Expansion Paths

We utilize this section to cover certain preliminaries as follows. Let r_i, a positive constant, represent the price of the ith factor of production

$$
\begin{aligned}
&\min \sum_{i=1}^{n} r_i x_i \\
(4.1) \qquad &\text{subject to } f(x) \geq q^o \\
&\qquad\qquad\quad x \geq 0
\end{aligned}
$$

where q^o, also a constant, greater than zero, represents a stipulated minimal amount which is to be produced via the production function $f(x)$. I.e., the problem is to minimize the cost of producing at least the stipulated amount, q^o, by selecting the factor quantities $x_i \geq 0$, $i=1,\ldots,n$ under the given price structure.

We shall assume $f(x)$ is concave as well as positively homogeneous with continuous partial derivatives[1/] in the interior of the set $\chi = \{x: f(x) \geq 0,\ x \geq 0\}$. We shall also assume the "Slater conditions"[2/] for χ, i.e., we shall assume that there is some $x' > 0$ such that $f(x') > 0$. Because of our homogeneity assumption, this means that there is $x'' > 0$ such that $f(x'') > q^o$.

Many of the production functions employed in economic analysis, e.g., the ordinary Cobb-Douglas or the CES functions, have partial derivatives which become infinite at any $x_i = 0$. They also often have the property that if $f(\hat{x})>0$ one must have $\hat{x}>0$. Thus, for these cases (4.1) reduces to the problem

---

[1/] Cf., e.g., Henderson and Quandt [24], p. 61.

[2/] Cf. [12].

$$\min \sum_{i=1}^{n} r_i x_i$$

(4.2)

$$\text{subject to } f(x) \geq q^0 > 0$$

in which one is sure that an optimum $x^* > 0$ -- i.e., the optimum is taken on in the interior of χ. Therefore, the Kuhn-Tucker conditions on the existence of x^* and λ^* reduces to equalities rather than inequalities. Thereby, i.e., whenever the optimum occurs at an interior point, regardless of the condition which assures it (e.g., $f(x) > 0$ implies $x > 0$) one has

(4.3)
$$r_i = \lambda^* f_i^*$$
$$\lambda^*[f(x^*) - q^0] = 0$$
$$\lambda^* \geq 0,$$

are necessary and sufficient for the optimality of x^* in (4.2).

We note further that since $r_i > 0$, all i, that $\lambda^* > 0$ and

(4.4)
$$f(x^*) = q^0$$

Indeed,

(4.5)
$$\min. \sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{n} r_i x_i^* = \lambda^* \sum_{i=1}^{n} f_i^* x_i^* = \lambda^* h f(x^*)$$
$$= \lambda^* h\, q^0$$

where f_i^* is the partial derivative of $f(x)$ evaluated at x^*. Thus

(4.6)
$$\lambda^* = \frac{\sum_{i=1}^{n} r_i x_i^*}{h\, q^0} = \frac{r_i}{f_i^*} \text{ , all } i,$$

For the kinds of analyses we are about to undertake, it is usually assumed that $f(x)$ is concave, as well as homogeneous, and that for some x' we have $f(x') > 0$. To clarify the consequences concerning the degree of homogeneity we now prove,

<u>Theorem 3</u>: If (i) $f(x)$ is positively homogeneous (ii) concave and (iii) for some x', $f(x') > 0$, then $0 < h \leq 1$.

<u>Proof</u>: First note that (i) implies $f(0) = 0$. By concavity,

$$f(\gamma x' + (1-\gamma)0) \geq \gamma f(x') + (1-\gamma)f(0) \tag{4.7}$$

for $0 < \gamma < 1$. Hence, $f(\gamma x') \geq \gamma f(x')$. Now since $f(x)$ is homogeneous of degree h this becomes

$$\gamma^h f(x') \geq \gamma f(x'). \tag{4.8}$$

Since $f(x') > 0$, this gives

$$\gamma^{h-1} \geq 1 \tag{4.9}$$

and this is possible only if $h \leq 1$. Q.E.D.

Before proceeding to employ these results in our subsequent development we might observe that they can be extended by relaxing some of the above assumptions such as, e.g., concavity. For example, if $f(x)$ is <u>not</u> concave but $\tilde{f}(x) \equiv T(f(x))$ is, where $T(\mu)$ is any homogeneous strictly isotone function of μ, our results will apply to $\tilde{f}(x)$ and can be translated immediately back to $f(x)$. We do not pursue this here, however, because we wish to make easy contact with known empirical and theoretical results in economics and therefore reserve this more extended treatment for a subsequent paper.

Hereafter we shall assume that our $f(x)$ are positively homogeneous and concave and that $f(x) > 0$ implies $x > 0$. We shall call a function satisfying these conditions a "proper production function" and proceed to use this concept in the development of conditions for optimal expansion or contraction of production in the following: First we substitute from (4.3) in Theorem 1 to obtain

<u>Theorem 4</u>: Let $f(x)$ be any proper production function we then have on the expansion path,

$$f(x^*) = A^* \prod_{i=1}^{n} (x_i^*)^{\delta_i^*} \tag{4.10}$$

where

$$A^* = \frac{1}{h\lambda^*} \prod_{i=1}^{n} \left(\frac{r_i}{\delta_i^*}\right)^{\delta_i^*} \tag{4.11}$$

and

$$\delta_i^* = \frac{r_i x_i^*}{\sum_{j=1}^{n} r_j x_i^*}, \text{ all } i \tag{4.12}$$

with δ_i^*, A^* and $f(x^*)$ denoting respectively δ, A and $f(x)$ evaluated at x^*. I.e., $f(x^*)$ is a point on the efficiency frontier of $f(x)$.

This theorem, which is easily proved, by substitution from (4.3) in theorem 1, offers additional important interpretations to the components of $f(x)$ when a firm produces optimally. First $f(x^*)$ characterizes the minimal cost of production of the stipulated output level q^o.[1] Also, since the constants $r_i > 0$ are the same for each firm, e.g., for an economy in competitive equilibrium, we also obtain a characterization of the production process which will later be used to effect an aggregation in which the production functions though proper, possibly differ in form from one firm to another.

These δ_i^* when interpreted as in (2.20) then denote a "cost minimizing distribution" for the cost of each factor in the production process, and this obtains when the firm is on its efficiency frontier -- or expansion path -- while A* clearly reflects the way in which the optimal marginal productivities are to be combined and then replaced by their corresponding unit prices.

We now draw some easy consequences for subsequent use via

Corollary 4.1: If $f(x)$ is any proper production function then at the efficiency frontier,

$$f(x^*) = \frac{\sum_{i=1}^{n} r_i x_i^*}{h\lambda^*} \tag{4.13}$$

where

$$\lambda^* = r_i / f_i^* . \tag{4.14}$$

Proof of Corollary 4.1 follows directly from the arithmetic-geometric mean inequality.[2] This being fairly obvious we proceed to

[1] Alternatively, $f(x^*)$ may be viewed as the maximal level output for a given budget.

[2] Alternatively, we can obtain (4.13) directly from the Kuhn-Tucker conditions and Euler's theorem as in (2.1) ff.

<u>Theorem 5</u>: Let $f(x)$ be any proper production function, as above, and let x^o, x^* be input vectors employed to attain positive outputs q^o and q^* at minimum total cost,

Then

$$\frac{r_i}{r_k} = \frac{\delta_i^o}{x_i^o} \Big/ \frac{\delta_k^o}{x_k^o} = \frac{\delta_i^*}{x_i^*} \Big/ \frac{\delta_k^*}{x_k^*} \quad , \tag{4.15}$$

if and only if the choices x^o, x^* are both on the firm's cost-minimising expansion path.

<u>Proof</u>: Via the Kuhn-Tucker conditions in (4.3),

$$\frac{r_i}{r_k} = \frac{f_i^o}{f_k^o} = \frac{f_i^*}{f_k^*} \tag{4.16}$$

and so, also,

$$f_i^o r_k - f_k^o r_i = 0 \ . \tag{4.17}$$

Via Theorem 1, this gives

$$\frac{\delta_i^o}{x_i^o} r_k - \frac{\delta_k^o}{x_k^o} r_i = 0 \tag{4.18}$$

by means of (4.12) and suitable cancellations. In the same way we also obtain

$$\frac{\delta_i^*}{x_i^*} r_k - \frac{\delta_k^*}{x_k^*} r_i = 0 \tag{4.19}$$

if x^* is on the cost minimizing expansion path.[1/] Q.E.D.

[1/] In the customary terminology of economics, x^o and x^* are both on the locus formed from conditions of tangency between isocosts and isoquants. See Intriligator [26], p. 193.

This now leads us naturally to consider the characterization in terms of partial derivatives of production functions which may be represented in Extended-Cobb Douglas form with their δ_i constant:

<u>Theorem 6</u>: The δ_i are constant (output) elasticities for all x if and only if

$$x_i \left[\left(\frac{f_i}{f(x)}\right)^2 - \frac{f_{ii}}{f(x)}\right] = \frac{f_i}{f(x)} \tag{4.20}$$

and

$$x_i \left[f_{ik} - \frac{f_i f_k}{f(x)}\right] = 0 \tag{4.21}$$

on $\{x : h f(x) > 0\} \neq \emptyset$

and[1] assuming the second partials, viz.,

$$f_{ik} = \frac{\partial^2 f(x)}{\partial x_i \partial x_k}$$

all exist for $i,k=1,\ldots,n$.

<u>Proof</u>: Referring to (2.9) in Theorem 1 we are in the situation where $\delta_i(x) = f_i x_i / h f(x)$. We have $\delta_i(x) =$ constant if and only if its first partial derivatives vanish for all x. Taking the derivatives with respect to x_i we have

---

[1] The symbol $\emptyset$ refers to the empty set.

(4.22) $$\frac{\partial \delta_i}{\partial x_i} = \frac{1}{h} \left[\frac{f_{ii} x_i}{f(x)} + \frac{f_i}{f(x)} - x_i \left(\frac{f_i}{f(x)} \right)^2 \right] = 0$$

or

(4.23) $$x_i \left[\left(\frac{f_i}{f(x)} \right)^2 - \frac{f_{ii}}{f(x)} \right] = \frac{f_i}{f(x)} \quad .$$

Taking the partial derivative with respect to x_k

(4.24) $$\frac{\partial \delta_i}{\partial x_k} = \frac{f_{ik} x_i}{hf(x)} - \frac{f_i f_k x_i}{h[f(x)]^2} = 0$$

or

(4.25) $$x_i \left[f_{ik} - \frac{f_i f_k}{f(x)} \right] = 0$$

which proves the theorem.

From Theorem 5 it now follows that if δ_i is constant, all i, then the expansion path will be linear. In fact we will then have $A(x^*)$ = constant -- cf. Theorem 4 -- which is the ordinary Cobb-Douglas function and the corresponding $f(x^*)$ in Theorem 2 is homogeneous of degree 1. It follows that the δ_i^* are output elasticities by Theorem 2 and, we may observe, λ^* then becomes the "long run cost of production". All of the other usual conditions and properties of these functions are then also present. See Henderson and Quandt [24], p. 85.[1/]

1/ See also Nerlove [33].

5.0 Aggregation

We now want to go from preceding results for an individual enterprise to an aggregate of firms. We want to do this, however, not merely from the standpoint of "pure" aggregation theory, but in a way that will enable us to obtain insight into results from ordinarily employed statistical practices in which, e.g., the data are all given in the form of sums of pertinent inputs and outputs. Therefore, we return to (4.3) and write

$$r_i = \lambda_j^* f_i^{*j} \tag{5.1}$$

where $r_i > 0$, $i=1,\ldots,n$, is the price of the ith factor, the same for each of $j=1,\ldots,m$ firms. For each such firm $x_{ij}^* > 0$ implies the above equality if the star is to indicate an optimal solution and so also $f_i^{*j} > 0$ for cases of interest in economics.

Here the term "optimal" refers to cost minimization, which means, as usual, that the price of an added unit of factor must equal its contribution to the marginal cost of production for any factor used in a positive amount, e.g., for conditions of competitive equilibrium.

This assumes, as already noted, that r_i is the same for all firms but, of course, this need not be true for λ_j^* or f_i^{*j} provided the equality indicated in (5.1) obtains for each of the $j=1,\ldots,m$ firms. Although we will hereafter assume that the firms all have proper production functions, we will also allow them to vary from firm to firm. Thus we may multiply both sides of (5.1) by x_{ij}^* and sum over all i and j to obtain

$$\sum_{j=1}^{m}\sum_{i=1}^{n} r_i x_{ij}^* = \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i^* f_i^{*j} x_{ij}^* \tag{5.2}$$

or

(5.3)
$$\sum_{i=1}^{n} r_i \sum_{j=1}^{m} x_{ij}^* = \sum_{j=1}^{m} \lambda_j^* \sum_{i=1}^{n} f_i^{*j} x_{ij}^*$$

$$= \sum_{j=1}^{m} \lambda_j^* h_j f_j(x_j^*)$$

where $f_j(x_j^*)$ denotes the production function of firm j with degree of homogeneity $h_j \leq 1$ and input vector $x_j^* = (x_{1j}^*, \ldots, x_{nj}^*)$.

Now let

(5.4)
$$\bar{x}_i^* = \sum_{j=1}^{m} x_{ij}^*$$

denote the total amount of factor i used by these j=1,...,m firms. We can then apply the arithmetic-geometric mean inequality to obtain

(5.5)
$$\sum_{i=1}^{n} r_i \bar{x}_i^* \geq \prod_{i=1}^{n} \left(\frac{r_i \bar{x}_i^*}{\delta_i}\right)^{\delta_i}$$

where $\delta_i \geq 0$, $\sum_{i=1}^{n} \delta_i = 1$. We obtain equality when

(5.6)
$$\delta_i^* = \frac{r_i \bar{x}_i^*}{\sum_{i=1}^{n} r_i \bar{x}_i^*} ,$$

i.e., when the cost minimizing choices of $\bar{x}$ are employed in the δ_i. See (2.18) & (2.20

Letting $\underline{\lambda}^*$, $\underline{h}$ denote the vectors $\{\lambda_j^*\}$, $\{h_j\}$ and X* a matrix with x_j^* for its jth row, j=1,...,m we then have

<u>Theorem 7</u>: Let $f_j(x_j)$ j=1,...,m be any proper production function, then at the expansion path of the economy

(5.8)
$$F(X, \underline{h}, \underline{\lambda}^*) = A^* \prod_{i=1}^{n} (\bar{x}_i)^{\delta_i^*}$$

where

$$(5.9) \qquad F(X^*,\underline{h},\underline{\lambda}^*) = \sum_{j=1}^{m} \lambda_j^* h_j \; f_j(x_j^*) = \sum_{i=1}^{n} r_i \bar{x}_i^*$$

$$(5.10) \qquad A^* = \prod_{i=1}^{n} \left(\frac{r_i}{\delta_i^*}\right)^{\delta_i^*}$$

and

$$(5.11) \qquad \delta_i^* = \frac{r_i \bar{x}_i^*}{\sum_{i=1}^{n} r_i \bar{x}_i^*} \quad i=1,\ldots,n.$$

In these economy- or industry-wide models, we may expect $\delta_i^* > 0$, all i so that also, as in Theorem 1, $\sum_{i=1}^{n} \delta_i^* = 1$. Theorems 5 and 6 also continue to apply, of course, which means that the expansion path for this economy or industry is

$$(5.12) \qquad \delta_i^* r_k \bar{x}_k^* - \delta_k^* r_i \bar{x}_i^* = 0,$$

and $F(\underline{x}^*,\underline{h},\underline{\lambda}^*)$ is of degree one at every point on this path. Furthermore, if Theorem 6 also applies -- i.e., δ_i^* is constant for every i -- this expansion path is linear and (5.8) reduces to the ordinary Cobb-Douglas function.

Dropping the assumption of constant elasticities we can achieve simplicity (for purposes of insight and understanding) in another way by reverting once more to the two-factor case as in Section 4. From Theorem 7 we can characterize the expansion path for an economy or industry via

$$(5.13) \qquad F(\underline{L}^*,\underline{K}^*,\underline{h},\underline{\lambda}^*) = A^* K^{*\delta^*} L^{*1-\delta^*}$$

where

$$\underline{L}^* = \{L_j^*\},\ \underline{K}^* = \{K_j^*\},\ \underline{\lambda}^* = \{\lambda_j^*\},\ \underline{h} = \{h_j\}$$

(5.14)

$$K^* = \sum_{j=1}^{m} K_j^*, \quad L^* = \sum_{j=1}^{m} L_j^*$$

and

$$A^* = \left(\frac{\rho}{\delta^*}\right)^{\delta^*} \left(\frac{\omega}{1-\delta^*}\right)^{1-\delta^*} \tag{5.15}$$

with

$$\delta^* = \frac{\rho K^*}{\rho K^* + \omega L^*} \quad , \quad 1-\delta^* = \frac{\omega L^*}{\rho K^* + \omega L^*} \tag{5.16}$$

where we are using ρ to represent the price per unit "capital", and ω the price per unit "labor".

In fact, substitution of the indicated expressions for A^*, δ^* and $1-\delta^*$ on this expansion path produces

$$F(\underline{L}^*,\underline{K}^*,\underline{h},\underline{\lambda}^*) = \rho K^* + \omega L^* \tag{5.17}$$

directly from (5.13). This says that total output is equal to total value added. Moreover, δ^* and $1-\delta^*$ as in (5.16) equals the share of each of these factors in value added.

The latter coincides, we might explicitly note, with A. A. Walters' observation that in most (cross-section) studies the coefficients of labor (i.e., its exponents) in the Cobb-Douglas functions utilized, are "a close approximation to the share of labour in value added"[1/] and that "the sum of the factor coefficients is at or near unity." See the table below. Unlike Walters, however, we do not conclude that this is the result of "an accounting convention that all income must be allocated to one factor or another."[2/] We conclude rather that this suggests (in study after study) that the firms are on or close to an optimal expansion path. The term "optimal", as throughout this paper, continues to refer to cost minimization. Hence, we do not draw any inference as to the presence or absence of perfect competition or the presence of exploitation, etc.[3/] since such cost minimization is compatible with fixed

[1/] See [39], pp. 329-330. See also p. 37 in [41].

[2/] See also Levy and Simon [31].

[3/] See Bronfenbrenner [5] and [4] and Reder [34].

INTER-INDUSTRY PRODUCTION FUNCTIONS[1/]
CROSS SECTION ESTIMATES

Year	Country	Labour α_1	Capital α_2	Sum $\alpha_1+\alpha_2$	Reference
1889	U.S.A.	.51	.43	.94	Douglas (1948)
1899		.62	.33	.95	
1904		.65	.31	.96	
1909		.63	.34	.97	
1914		.61	.37	.98	
1919		.76	.25	1.01	
1912	Australia	.52	.47	.99	Gunn and Douglas (1941)
1922-3		.53	.49	1.02	
1926-7		.59	.34	.93	
1934-5		.64	.36	1.00	
1936-7		.49	.49	.98	
1910-11	Victoria	.74	.25	.99	Gunn and Douglas (1940)
1923-4		.62	.31	.92	
1927-8		.59	.27	.86	
1933-4	N. S. Wales	.65	.34	.99	
1937-8	S. Africa 1.	.66	.32	.98	Browne (1943)
	2.	.65	.37	1.02	
1923	Canada	.48	.48	.96	Daly and Douglas (1943)
1927		.46	.52	.98	
1935		.50	.52	1.02	
1937		.43	.58	1.01	
1938-9	N. Zealand	.46	.51	.97	Williams (1945)
1924	U.K. (industry)	.72	.18	.90	Lomax (1950)
1930		.75	.13	.88	
1946	India	.66	.31	.97	Tewari (1954)
1947		.57	.50	1.07	Dutt (1955)
1951		.59	.40	.99	Murti and Sastry (1957)
1909	U.S.A. (industry)	.74	.32	1.06	Marschak and Andrews (1944)

[1/] Source: A. A. Walters [41]. We are indebted to A. A. Walters and to Econometrica, the journal of the Econometric Society, for permission to reproduce this table.

prices and production quotas, e.g., such as might be imposed by a central authority.

We have also left unattended the issues revolving around a choice of production functions[1/] by the underlying firms and plants. Although the notation in (2.8) and (2.9) ff. properly suggest that A(x) and the $\delta_i(x)$ are all to be estimated, with x,. and Q serving as the relevant input and output data, we do not want to enter into questions of statistical estimation and identification in this paper. On the other hand, we may make at least one suggestion which flows naturally and easily from the above analysis by observing that a choice of $\hat{\delta}_i \neq \delta_i^*$ for any i must produce

$$F(\delta^*) > F(\hat{\delta}) \tag{5.18}$$

where F is defined as in (5.13), since, by Theorem 1 and its corollaries, equality holds for each firm if and only if $\hat{\delta}_i = \delta_i^*$ all i and this is evidently reflected in Theorem 4 as well.[2/]

In view of the importance of these δ_i choices we may single them out in a way that also makes contact with other important recent conceptual and methodological developments by introducing

$$\Delta = -\sum_{i=1}^{n} \hat{\delta}_i \log \frac{\delta_i^*}{\hat{\delta}_i} = \sum_{i=1}^{n} \hat{\delta}_i \log \frac{\hat{\delta}_i}{\delta_i^*} \tag{5.19}$$

In modern statistical information theory, this is called the "information discrimination function". Cf., e.g., Kullback [30]. Here it might better be called the "efficiency discrimination function" because, via the indicated inequality, $\Delta = 0$, if and only if $\hat{\delta}_i = \delta_i^* \geq 0$ for every i.

1/ I.e., we do not inquire as to whether the functions themselves also result from optimal choices.

2/ Cf. also Klein [28] for a comparable discussion of the conditions for aggregation of Cobb-Douglas functions.

6.0 Conclusion

We now return to the issue of the generality of our Extended Cobb-Douglas function, which has, we think, been exhibited in a variety of ways. Thus, for instance, we showed how other functions, often used in economics, can be represented in this form. Nevertheless, our representation theorem -- i.e., Theorem 1 -- is essentially that of a Cobb-Douglas form. Indeed, the relationship of the ordinary Cobb-Douglas form to our general representation may be likened to developments in other disciplines, too, where practitioners are confronted with "actual" applications. A case in point is the use of "linear lumped constant" circuit representation in electrical engineering and control theory. An analyst will ordinarily use this approach to a valid approximation with essential simplifications to obtain the major properties of a complex system. Analogously our representation permits us to go directly to a suitable (ordinary) Cobb-Douglas function by specifying the δ_i and A as constants. Similarly the "lumping" of actual elements occurs through aggregation (e.g., across firms) and classification (across factors or products).

In fact we have now exhibited the Extended Cobb-Douglas Function as canonical for production theory. Concomitantly we have provided analytical developments and interpretations which relate this canonical format to standard economic constructs. Finally, we have also shown how to interpret a wide range of empirical results in ways which are consistent with these constructs.

In this paper we have restricted our attention to the simplest and most basic case of static production with a single output to be produced with a single function[1/] -- one to a firm or plant -- from factors which are acquired at

[1/] Cf. [10] for possibilities which include varying functional forms over time.

fixed prices per unit. The representation in Theorem 1 is evidently more general than this, however, and admits of numerous other extensions and applications. In preparation therefore we have taken some pains to make contact with other new and important disciplines such as geometric programming and its related theory. This sets the stage for other developments as well. It seems best, however, to reserve these other developments for treatment in their own right in separate papers.

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The Cobb-Douglas function is widely used in production economics in the following form: $A=AL^{\alpha}K^{\beta}$, where α and β along with A are positive constants that relate the inputs L(=Labor) and K (=Capital) to the amount of output Q. A variety of supposed alternatives and generalizations have been suggested in place of the Cobb-Douglas form for use in production economics. These alternatives and generalizations are here shown to be representable in an extended Cobb-Douglas form in which A, α and β are functions of L and K

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rather than constants. This extension is then formally related to other general forms, such as the minimum discrimination information statistic, and used to explain the successful uses of the Cobb-Douglas function for empirical applications in many different countries and contexts.

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