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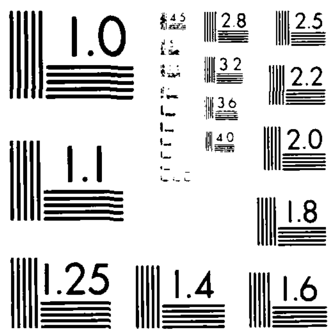


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This is a preprint version of an article written at the request of the International Statistical Review. The article is organized in three main sections. The first of these is a brief overview of Harald Cramer's life and career. The second (and main) section is an account of his work in Probability and Statistics, with historical perspective where possible. The third, final section contains personal comments and recollections from the author's own contacts with Harald Cramer. These are intended to complement the description of the career and scientific contributions of Cramer, with some glimpses of his personal qualities.



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HARALD CRAMÉR 1893 - 1985

by

M.R. Leadbetter

Technical Report No. 192

July 1987

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University of North Carolina



Summary

This is a preprint verion of an article written at the request of the International Statistical Review. The article is organized in three main sections. The first of these is a brief overview of Harald Cramér's life and career. The second (and main) section is an account of his work in Probability and Statistics, with historical perspective where possible. The third, final section contains personal comments and recollections from the author's own contacts with Harald Cramér. These are intended to complement the description of the career and scientific contributions of Cramér, with some glimpses of his personal qualities.





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Harald Cramér 1893-1985

Harald Cramér was born in Stockholm, Sweden on September 25, 1893 and died there on October 5, 1985. For over 60 of his 92 years he made fundamental contributions to probability and statistics, exerting a remarkable and lasting influence on those fields, and an equally remarkable and lasting personal influence on generations of students, colleagues, and friends. In this article I aim to highlight his career and professional contributions, and will attempt to speak to the very warm human qualities of this great scientist.

I have organized the article in three main sections. Section 1 gives a brief overview of Harald Cramér's life and career. Section 2 is an account of his work in Probability and Statistics. In these sections I have drawn on various sources, especially Cramér's own "Personal Recollections" ([21]), a paper which contains important perspectives on the history and development of probability theory (and should be required reading for almost every probabilist).

It has been most helpful to have access to the article [2] written by Gunnar Blom - a former student of Cramér. In particular, Blom's article provides special first hand insights into Cramér's work, personality and methods of operation during the years in which he made his most famous contributions. In Section 2 I have followed a similar historical sequence to that in [21], but concentrating primarily on the contributions of Cramér himself, and have attempted to give some insights into his motivations from the probabilistic activity of the time.

Section 3 contains personal comments from my own contacts with Harald Cramér. While these are somewhat scattered, they are an attempt to complement the description of the career and contributions of Cramér the scientist, with a glimpse of some of his very delightful personal qualities.

A complete bibliography of Cramér's works has been given by Blom and Matern ([3]) and I have therefore listed here only publications referred to specifically in the text. A further complete list of his publications in languages other than Swedish may be found in [2].

It is a pleasure to express my thanks to Gunnar Blom and Ulf Grenander for their reading of my draft manuscript and for suggestions and information they provided. I am also indebted to a number of colleagues and friends for very helpful critical comments.

## 1. HARALD CRAMÉR: AN OVERVIEW OF HIS LIFE AND CAREER.

Harald Cramér, the second son of a banker, was born in Stockholm on September 25, 1893. He spent virtually his entire life in Stockholm, though identifying closely with his family origins in the island of Gotland, where his ancestors had lived for many generations. In 1912 he entered the University of Stockholm, studying chemistry and mathematics. As a research assistant in Biochemistry he made several scientific contributions in that field with H. v.Euler (later a Nobel prize winner) before deciding that his main interest lay in mathematics, and studying with Marcel Riesz who had come to Sweden from Hungary to work at the Mittag Leffler Institute. He graduated in 1917 with a Ph.D., having written a thesis on Dirichlet series, and he published approximately 20 papers in Analytic Number Theory over the next seven years. He notes ([21]) that in the course of this research he became familiar with Fourier Integrals of a type closely related to those which were central to much of his subsequent probabilistic work.

One of the natural employment avenues for young Swedish mathematicians at the time was with insurance companies, and it was on such a career that Cramér embarked in 1918. This was also the year of his marriage to Marta Hansson (of whom more will be said later). They had one daughter, Marie Louise now living in Finland, and two sons, Kim and Tomas, who both live in Stockholm. In 1920 Cramér became an actuary for a life insurance company (see [2] for details of positions held). I recall his great amusement when, on a visit to North Carolina, he saw a notice advertising actuarial opportunities with headlines "ME BE AN ACTUARY? WHAT'S THAT?" His comment was that no young mathematician in Sweden when he grew up would ever have to ask such a question!

Problems of insurance risk led the young Cramér to an interest in probability and statistics, thus sparking his extraordinarily productive career in these fields. This will be described in historical sequence in Sections 2.1-2.5 below.

In 1929 Cramér was appointed to a new chair in "Actuarial Mathematics and Mathematical Statistics" at Stockholm University, set up at the instigation of the Swedish insurance companies. This institute became very active under his leadership, and many of the now celebrated results of Cramér himself and his students and colleagues, emanated from it in the ensuing 20 years. Even in the enforced isolation of the second world war, a high level of activity was maintained, including the preparation of his most famous book "Mathematical Methods of Statistics" which was to profoundly influence postwar statistical education.

For Cramér the period of the war was also a time of service in helping refugees from the European continent. He personally intervened with the authorities on behalf of colleagues and others, to arrange at least temporary asylum in Sweden - a humanitarian activity involving significant personal risk.

In 1950 Cramér was elected President of Stockholm University and in 1958 Chancellor of the entire Swedish university system. This administrative service undoubtedly had a severe impact on his scientific production but by no means halted it - as can be seen from his publications of the period. On his retirement from the Chancellorship in 1961 he again became very active in research. He made several extended visits to the U.S. to participate in research activities and give lectures, accompanied by his wife Marta. Though troubled by increasing deafness, he continued active scientific work until almost the end of his life - his last paper being published at age 89. It was fitting that his last major scientific appearance was at the Stochastic Processes Conference held at Gothenburg in Sweden in 1984. Cramér at 90 gave a masterly, major address to the highly appreciative international audience, just over one year before his death.

## 2. 60 YEARS OF CONTRIBUTION TO PROBABILITY AND STATISTICS.

I shall attempt in this section to set the contributions of Harald Cramér in probability and statistics within their historical context. It is convenient (as in [21]) to group these in decades, heading each with his major activities for that period.

### 2.1 The 1920's: Central Limit Theory

As noted, Cramér was led naturally to an interest in probability through his work in insurance risk. The early sophistication of that interest is striking. For example his first probabilistic paper [4] in 1919 gave a rigorous treatment of the now classical characterization for the Poisson

Process. Yet more significant for his later work were other ideas from insurance. For example the notion of total insurance as a sum of independent individual components, stimulated his subsequent interest in central limit theory, while consideration of economic fluctuations was a precursor to fundamental ideas in the theory of stochastic processes.

Cramér refers to this period as a "decade of preparation" for the explosive development of probability theory which was to follow. Nevertheless significant events occurred -- such as the appearance of Wiener's work on Brownian motion, the blossoming development by Russian probabilists, providing results such as Kolmogorov's Inequality and the three series theorem, and the appearance of Levy's 1925 book [31] containing a systematic account of characteristic functions.

Cramér himself had been using characteristic functions in an extensive study of central limit theory. Motivated by applications to insurance, he investigated the magnitude of the error in approximating the distribution of a standardized sum of i.i.d. random variables by its normal limit. He first improved a treatment of Liapounov [34] to obtain ([5]) a bound for the error of the form $(3\beta_3/\sigma^3)n^{-1/2}$ log n where the original random variables have zero mean, standard deviation $\sigma$, and third absolute moment $\beta_3$. This will be recognized as essentially what is now often called the "Berry-Esseen bound" aside from the factor log n. His main work in this area, however, was to obtain detailed Edgeworth and related expansions for the error (without the factor log n), the principal results being published in 1928 ([6]). In this paper he used his "Condition C", viz $\limsup_{t\to\infty} |\phi(t)| < 1$ for a characteristic function $\phi$. I once asked him if C stood for "Cramér" and received the slightly injured reply that he had introduced also Conditions A and B which, however, had disappeared! It was in this 1928 paper also that Cramér proposed the "goodness of fit" criterion now known as the Cramér-von Mises Statistic.

Anyone with more than minimal interest in central limit theory should read Cramér's description in [21] of the early history of activity in this area, and the perspective it brings not only to his work but that of others such as Liapounov, Chebychev, Lindeberg, Edgeworth and Charlier, and the later definitive account in the 1944 thesis of Esseen ([25]) which, along with that of Berry [1], gave "final form" to results in this area.

In this work Cramér was motivated by actual applications in insurance problems. I conjecture that he always had a primary concern for potential usefulness of results, and was less interested in personally finding marginal improvements to a theorem than in developing new methods or finding new results under conditions which were natural and suited to application. Of course there was never any compromise with rigor, and he had a finely tuned appreciation for mathematical beauty.

## 2.2 The 1930's: Characteristic Functions and Stationary Processes

The chair to which Cramér was appointed in 1929 was the first one in Sweden involving <u>mathematical</u> statistics, (in contrast to the more applied field of "statistics") and he was kept busy developing the new institute thus formed. He continued a vigorous research activity in the area of insurance mathematics and (as described in [2]) undertook some important responsibilities in the development of a new base for insurance premiums in Sweden, also working on a state commission to prepare a new insurance act.

I am grateful to Ulf Grenander for the comment that one of Cramér's major (though less widely known) contributions to insurance mathematics was a treatment of loadings in mortality assumptions. He provided a precise discussion of the bias necessary in premium estimation, clarifying the differences between "life type" and "death type" policies, and developed the so-called "zero point method" which illuminated the problem and was used for many years in setting premium rates.

But above all this he was clearly most excited about the development of mathematical probability theory with its profusion of activity in what he has described as "this heroic period". One cannot read his own description of these years ([21]) without sensing the high excitement, and being substantially infected by it. Obviously Kolmogorov's work on foundations was a "fountainhead" for future development and it had a very important influence on Cramér's own research.

This was also a decade of high activity in Markov processes. Even though Cramér himself never took up a study of the general Markov theory (attributing this to "never feeling quite at home with partial differential equations"), it was carried on in an important way in the Stockholm group by Feller, who spent five years there. Cramér incidentally made strenuous efforts to create a permanent position for Feller, but was unsuccessful, and Feller left in 1939 for the United States. Cramér records that the Stockholm group were very interested in the subclass of Markov processes having independent increments and in particular in the work of Lévy ([32]), and of Khintchine ([30]). Again he was motivated by useful applications and was especially interested in the so-called "Lundberg Risk Process" where claims occur as a Poisson Process, claim amounts being independent with a given distribution function (i.e. a variant of the Compound Poisson Process, with arbitrary type of multiplicity distribution). The ruin problem was studied (ruin occurring if claims up to time t exceed accumulated premiums up to t) in a thesis ([38]) by C.O. Segerdahl, who gave inequalities for the probability of ruin in a given time period. This, of course, is a particular type of "first passage problem" -- other forms of which were to interest Cramér in later years.

The work of Lévy on infinite divisibility (e.g. [32]) also interested Cramér and in particular he proved ([8]) Levy's conjecture that if a normal random variable X can be represented as the sum of independent random variables X = Y + Z, then the "factors" Y and Z must themselves be normal. The proof of this now famous result employed characteristic functions, the use of which had become basic to a great deal of Cramér's work, including the previous investigations in central limit theory. Cramér records that the ideas of characteristic functions go back to Lagrange, Laplace and Cauchy and were used by Liapounov in his first proof of the central limit theorem.

During this decade Cramér and coworkers further developed the subject of characteristic functions. The essential ideas for the celebrated continuity theorem had been given by Lévy in [31] but the definitive form of the result is due to Lévy and Cramér. This result was given in Cramér's Cambridge Tract "Random Variables and Probability Distributions" ([9]) which was written with the earlier encouragement of G.H. Hardy, was published in 1937, and became a widely read reference among probabilists. The statement of the continuity theorem given there contained what Cramér regarded as a "regrettable error" in that convergence of the characteristic functions was assumed only in a finite neighbourhood of t = 0. This error was pointed out by Khintchine, and corrected in later editions. Generalizations of the characteristic function to multidimensional cases were developed in a joint paper [24] with H. Wold, containing the now celebrated "Cramér-Wold device" to reduce certain multivariate distributional problems to univariate cases.

Central limit theory developed dramatically in this decade. Feller, while working in Stockholm, had shown the necessity of the Lindeberg condition for a normal limit. Perhaps the most far reaching development was that initiated by Khintchine who considered row sums $S_n$ of <u>arrays</u> of (uniformly asymptotically negligible) random variables, identifying the possible distributional limits for $S_n - b_n$ with the class of infinitely divisible distributions. It is interesting that Cramér himself had used arrays in considering generalizations of the law of the iterated logarithm in a paper [7] he had published some three years earlier, arising from a correspondence with Cantelli.

Cramér was also concerned with problems of "large deviations" associated with central limit theory, considering in [9], the ratio of the tail of the distribution of a normalized sum, to that of the standard normal distribution. This work foreshadowed a great deal of activity and application in a variety of areas (e.g. by Feller ([26]) in improving the law of the iterated logarithm.

No record of the probabilistic activities of the 30's would be complete without reference to the early development of the theory of stationary processes, originating with the pioneering 1934 paper [29] of Khintchine which had been foreshadowed by the 1930 work of Wiener ([39] -- cf [36]). This clearly stimulated the interest of Cramér and his Stockholm group. For example, Cramér ([10]) generalized Khintchine's spectral representation for the covariance function, to deal with vector processes and "cross spectral" matrices in both continuous and discrete time.

The 1938 Stockholm thesis of H. Wold contained further covariance and spectral properties for stationary sequences, and the original form of the "decomposition" theorem. This result stated that a stationary sequence $[x_n, n=0,\pm1\ldots]$ with finite second moments has the decomposition

(2.2.1) $$x_n = u_n + v_n$$

where $u_n$, $v_n$ are mutually uncorrelated, $v_n$ being deterministic (predictable from the remote past) and $u_n$ purely non-deterministic with moving average representation

(2.2.2) $$u_n = \sum_{i=-\infty}^{n} c_{n-i}\, z_i$$

in terms of uncorrelated random variables $\{z_n\}$ and constants $c_n$. Important generalizations of this result appeared in Cramér's later work.

## 2.3 The 1940's: Stationary Processes; Mathematical Methods of Statistics.

The years of the second world war imposed a period of professional isolation which was keenly felt by Cramér. Nevertheless he maintained a remarkable level of activity at the Stockholm Institute. For example, the thesis by O. Lundberg ([35]) on stochastic models provided an actuarial basis for non-life insurance which was widely used. Cramér made the most of the few available opportunities for international contacts. One of these was a primarily Scandinavian conference which he organized in Stockholm in 1941 -- at which he presented the spectral representation

$$x(t) = \int e^{it\lambda}\, dz(\lambda)$$

of a stationary process with respect to a process $z(\cdot)$ with orthogonal increments. This now classic result provided a very satisfying counterpart to Khintchine's representation for the covariance functions, and was published the following year in [12]. The communication difficulties caused by the war are perhaps illustrated by the fact that Loeve independently obtained this important result, publishing it subsequently in the appendix of [33]. Another mentioned by Cramér concerns the work of Kolmogorov and of Wiener on prediction -- each apparently unaware of the other's results until after the war.

The article [2] by Gunnar Blom contains illuminating firsthand insights into the operation of the Stockholm Institute during those years, and the manner in which Cramér maintained a high level of activity in the group. Undoubtedly the most significant product of this period was the book "Mathematical Methods of Statistics" ([13]), a classic of our field. Cramér had decided to write this at that time as a means of making good use of the period of isolation caused by the war. One wonders whether this timely and profoundly influential work would have even been written if the climate for international research contacts had been more favourable! In any case the decision to write the volume at that time was astute, and it has had immense influence on generations of statisticians.

One can take issue with some minor points (such as the constructive definition of Borel sets) but to find a book which has fewer errors and blemishes would be a difficult assignment. Most of all, the book removed many elements of mystery surrounding statistical theory, exposing its basis as a legitimate branch of mathematics, accessible to anyone with reasonable mathematical training. "Mathematical Methods" incidentally contained the celebrated Cramér-Rao Inequality, obtained independently by Rao in [37].

"Mathematical Methods" is by no means a manual on applied statistics. But it clearly exemplifies the fact that Cramér was substantially motivated in his mathematical work by its potential for application. Indeed it is noted in [2] that he actually attempted to set up an applied statistics section in the Stockholm group -- a plan which did not reach fruition though a less formal applied activity was instituted.

The end of the second world war brought the opportunities for international contact for which Cramér had clearly longed. From conversations with him, and his description in [21], he must have had a high sense of excitement as he seized opportunities to exchange ideas and achievements "pent up" from the previous years. He gave lectures in Paris in the spring of 1946 on stationary processes, and on statistical estimation -- then quite a controversial topic. He records in [21] his mild apprehension at finding R.A. Fisher, whose views he was to some extent opposing, in the audience at one of these lectures, but that a private discussion afterwards had turned out well. Cramér described this incident to me saying that Fisher had praised the lecture but said he had understood very little of it since it was delivered in French! This leads me to mention Cramér's impressive command of languages with the ability to give fluent lectures in English, French and German, as well as his native Swedish. He had some knowledge of Russian, and one of the last projects of his life when in his late 80's, was to study classical Greek.

The fall of 1946 was spent at Princeton, the spring of '47 at Yale and the summer at Berkeley -- visits in which Cramér met or renewed acquaintance with leading probabilists and statisticians and also greatly influenced a number of students and younger workers who would later make their own substantial contributions to the field.

In the remaining years of the decade Cramér's Institute was very active in the stochastic process area. The activity included fundamental work of Karhunen on stationary processes, the pathbreaking thesis of Grenander [27] involving statistical inference for stochastic processes, and the work of Cramér himself on general spectral representations for some classes of non stationary processes ([14]).

## 2.4 The 1950's: Administration, Collective Risk, Towards Multiplicity Theory.

Cramér's administrative responsibilities as President of Stockholm University from 1950, and system-wide Chancellor from 1958 to 1961 clearly limited his time for scientific work, but he continued to make significant contributions in this period. His monograph "Collective Risk Theory" ([15]) was published in 1955 and treated insurance risk within a stochastic process framework; in particular his estimates of ruin probabilities have been widely used. He also wrote a number of expository and general articles in the areas of probability, statistics, and actuarial mathematics. In 1958 he began work on important questions regarding the generalization of representations for stationary (vector) processes, to include non-stationary cases. This will be described in the next section under "Multiplicity Theory", as his published works belong to that period.

## 2.5 The 1960's And Beyond: Multiplicity Theory, Extremes And Excursions By Stochastic Processes

From 1960 on Cramér made notable contributions in the two above listed areas of stochastic process theory. His work on multiplicity theory had begun in 1958 with attempts to generalize available representations for stationary vector processes to non-stationary cases. Concluding that spectral theory would lead to only limited results, he concentrated on time domain analysis, generalizing the representations (2.2.1) (2.2.2) obtained by Wold, in important and interesting ways. More specifically (using the notation and statement of [21]), if $(x_n, n=0,\pm1\ldots)$ is a q-dimensional vector process, $x_n = (x_{n1},\ldots x_{nq})$, where the components $x_{nj}$ have zero means and finite second moments, then the generalization of the Wold decomposition holds, viz. $x_n=u_n+v_n$ where $u_n$, $v_n$ are now (orthogonal) vector processes, $v_n$ being deterministic and $u_n$ purely non-deterministic. Further $u_n$ has the representation

$$u_n = \sum_{i=-\infty}^{n} c_{ni} z_i \tag{2.5.1}$$

where the "innovations" $z_i = (z_{i1} \dots z_{ir_i})$ are vectors of order $r_i \leq q$, $z_{ij}$ being orthogonal for all i,j and $c_{ni}$ being a $q \times r_i$ matrix. It is worth emphasizing that stationarity is not assumed and this is reflected in the fact that the representation (2.5.1) for $u_n$ is not now of moving average type, i.e. even in the univariate case $c_{ni}$ is not necessarily of the form $c_{n-i}$. Cramér also investigated corresponding representations for a continuous parameter vector process $x(t)=(x_1(t) \dots x_q(t))$, showing that the nondeterministic part $u(t)$ can be written as

$$u(t) = \int_{-\infty}^{t} G(t,v)\, dz(v)$$

where $z(v)$ is a vector of some fixed order N (the "multiplicity") which may in the continuous case exceed q and indeed may be $+\infty$.

These representations provide immediate solutions to the linear least squares prediction problem, and were studied by Cramér in approximately ten papers from 1960, (e.g. [16], [17], [18]) the last being [22], published in 1982 -- graphically demonstrating his activity in what would normally be years of retirement. (A more detailed summary of this work and references to other authors may be found in [21]).

On retiring from the University Chancellorship in 1961, Cramér accepted the invitation of Gertrude Cox to visit North Carolina and participate in work being done at the Research Triangle Institute (RTI) under contract to NASA, on the reliability of spacecraft missions. In this connection he made three visits of some months each, in 1962, 1963 and 1965. One of the applied problems considered involved the drift of gyroscopic guidance systems, which Cramér modeled by stationary normal processes, instigating his interest in maxima of such processes, and in their crossings of high levels.

He wrote seven papers (cf. [19,20]) on these and related topics in the period 1961-71. In particular he obtained the asymptotic distribution of the maximum of a stationary normal process and discussed limiting forms for the (Palm) distributions of times between high level upcrossings and downcrossings, in a rigorous framework. In joint work we obtained formulae for the moments of the numbers of high level upcrossings, and, together with R.J. Serfling, discussed relationships between these moments and various Palm distributions (e.g. of the time from an upcrossing to the $n^{th}$ subsequent downcrossing).

These and related topics form the heart of the joint book [23] which was published in 1967. As well as bringing this work together and indicating its application, a major purpose of the book was to provide an account of basic theory for stationary and related processes, which would be readily accessible to a reader with a reasonable background in probability theory. From a personal viewpoint it was naturally a great experience to work with Cramér on such a project. While he generously shared the writing in a rather evenly split way, I attempted to follow his structure and format, and it was inspirational to observe his artistry with words and especially to see his first hand descriptions of areas of development (such as spectral and real time representations) in which he had been so intimately involved.

HARALD CRAMÉR: SOME PERSONAL OBSERVATIONS.

When, in 1961, I began work in North Carolina at the Research Triangle Institute, I felt well acquainted with Cramér through "Mathematical Methods" though I never expected to meet this almost legendary figure. It was with open disbelief that the RTI statisticians heard Gertrude Cox saying "How would you like to work with Harald Cramér? I have persuaded him to come here". I add that the early development of Statistics, especially in North Carolina, owed a very great deal to the imaginative efforts of Gertrude Cox whose aim was always to obtain the most outstanding people possible in any endeavour in which she was involved. Her ability to recognize and seize opportunities (as in this case) accounted for much of her extraordinary success.

In any case, some months later Harald Cramér walked up the steps of the Institute, presumably oblivious of the faces peering from windows to catch sight of him. He had on the way stopped briefly at Princeton "to find out from Will Feller what of importance had been going on in probability during my term as Chancellor". One suspects that in spite of the demands of administration he had never really lost contact!

Cramér quickly confirmed the perception of his personality one obtained from "Mathematical Methods" and gave an immediate impression of graciousness and smoothness in dealing with people - making it evident why his Swedish colleagues had selected him for important administrative tasks. A minor illustration comes to mind. Early in his visit he requested use of a typewriter only to be told politely "We don't provide typewriters - we have secretaries". To this Cramér evidently responded innocently with "And will the secretaries be able to take Swedish?" A typewriter was provided without delay.

Harald Cramér represented the formal Swedish academic tradition

-inspirational and genial towards his students, but in an acceptably distant kind of way. While vestiges of that formality were apparent, he related to the RTI staff in a very relaxed fashion as if recognizing cultural differences, and wishing to be sensitive to them.

His work style appeared to be relaxed, but very productive. He took up problems of maxima and level crossings by normal processes, in order to model for example the drift of gyroscopes in spacecraft and to determine the probability that this would remain within limits needed for a successful mission. While specific actual numerical applications were indeed made to this problem, one has to admit that the immediate impact on the space program was not too substantial! However the scientific endeavours thus instigated have led to wide further development of theory and indeed significant application. To me, one of the most charming of these is the use made by D.G. Kendall ([28]) of the asymptotic distribution of extremes of a normal stationary process in significance tests for the existence of units of measurement in the building of prehistoric structures such as Stonehenge. Even in these "retirement" years Cramér's power as a mathematician was impressive and one could only wonder how he must have appeared to colleagues and students when working at full speed in his younger days. Occasionally he would give an informal talk on results he was obtaining. Their "cleanness" and beauty prompted one of our engineers to say "I think he must only work on problems which he knows will have a nice solution!"

On visits to North Carolina, as elsewhere, Cramér was much in demand as a seminar speaker, and he acceded to such requests wherever possible, but always after careful consideration. On one occasion he received an invitation by 'phone and gave a noncommittal reply. On putting the 'phone down he said

"You know, I have a policy never to make agreements over the telephone!" This seemed a good lesson for those of us who recklessly agree to whatever is proposed - especially if it is far enough ahead in the future! Cramér's conversation was always tinged with a quiet but very real humour - usually showing a splendid appreciation for the nuances of English. He loved a good story - especially when it involved an amusing incident from real life, but even if it were apocryphal. When they concerned real people the stories were always kindly. He hardly ever expressed negative feelings and the rare occasion on which this did happen would come as quite a surprise. Some of his anecdotes are recorded on a tape which we persuaded him to make during one of his later visits.

No picture of Harald Cramér would be complete without mention of his wife, Marta. She was gracious, intellectual, and talented, providing continual support in and enhancement of the various roles which Harald was called upon to assume - whether it be the demands of long hours spent on research, or the public spotlight of an administrative leader. She combined a keen sense of history with a strong interest in preservation and would proudly guide visitors around the "Old City"; one of her favourite projects was the redecoration of certain rooms at Stockholm University in authentic period style and furnishings.

One of the most lasting impressions of both Marta and Harald Cramér was their deep enjoyment of their family and love of occasions in which family and grandchildren would visit them. I was privileged to be present at such an occasion when Marta prepared a banquet indeed, for an assembly of much of the family. My fond visions are of Marta in the kitchen surrounded by a seemingly infinite number of pots, pans and ingredients of all types which went into the preparation of the sumptuous meal, the closeness of the family gathering, and the sight of Harald subsequently loading the dishwasher - a "packing problem" which he insisted no one else was quite as well equipped to handle as he.

On their visits to North Carolina they treated us to a warm family relationship which our children greatly enjoyed. A favourite memory - recorded with a delightful photograph - is that of Harald reading from the "Jungle Book" about the adventures of Rikki-Tikki-Tavi to our two wide-eyed young sons at bedtime, in a motel room during a trip to the North Carolina coast. I recall also the Cramérs' arrival from Sweden on one of their trips, when we met them at the Raleigh railway station and found they had almost an entire railroad "trolley" full of baggage. On noticing my surprise Harald commented "The only thing Marta and I have significant disagreements about is the amount of luggage to bring!" During these trips Marta vigilantly watched over her husband's health - making sure that his activities did not exceed his strength.

The letter we received from Harald following Marta's death in 1973 began "My beloved Marta has left me..." and spoke worlds for the extraordinary partnership which they had enjoyed in their 55 years together.

In the last decade or so of his life Harald lived in an upstairs apartment in a small house near to that of his son Tomas and his family, in a park bordering the Stockholm harbour. The apartment though smaller, reflected the atmosphere of their wonderful Djursholm house which had supported so many family and academic traditions. It was very pleasant to visit him in his library - carefully set out and organized as it had been in the previous home - and to eat a meal which he had prepared, perhaps from a recipe he had recently found in the newspaper.

All too often some "law of total maturity" seems to dictate that great intellectuals must be less agreeable or have some other weakness in their makeup. This was not the case with Harald Cramér as these glimpses into his later life show. He was a scholar of remarkable ability and achievement, with an extraordinary influence on his field. He was a gentleman in the finest sense, arising from the very best of a strong Swedish tradition. But beyond this, he and Marta will be remembered by those fortunate enough to have known them personally, for their constant and universal graciousness, their strong family relationships, and as warm and true friends.

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