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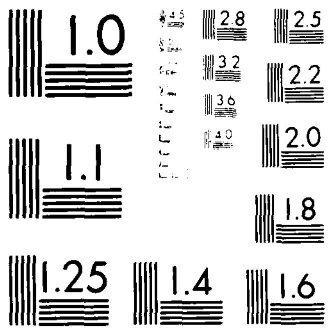

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for all x and y from a normed space.

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CENTER FOR STOCHASTIC PROCESSES

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On Hypercontractivity of α-stable Random Variables, $0 < \alpha < 2$

Jerzy Szulga

Technical Report No. 196

July 1987

ON HYPERCONTRACTIVITY OF α-STABLE RANDOM VARIABLES, $0 < \alpha < 2$.[*)]

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Let θ be a symmetric α-stable random variable, $0 < \alpha < 2$. We show that for any p and q, $0 < q < p < \alpha$, there is a constant $s = s(p,q,\alpha)$ such that

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1. Introduction

The classical Nelson inequality [9]

(1.1) $$((|x+s_{p,q}y|^p + |x-s_{p,q}y|^p)/2)^{1/p} \le ((|x+y|^p + |x-y|^p)/2)^{1/p},$$

where $1 < q \le p < \infty$, $x,y \in \mathbb{R}$, and $s_{p,q} = ((q-1)/(p-1))^{1/2}$, and its extensions play a fundamental role in the theory of hypercontractive operators [1,4,12], convolution inequalities [11], logarithmic Sobolev inequalities [4], stochastic Ising models [5] and related subjects in harmonic analysis, statistical mechanics and quantum physics, to name just a few areas of great importance nowadays.

Some more sophisicated counterparts of Nelson's inequality were recently applied in the intensively developing theory of multiple stochastic integrals, random multilinear forms and stochastic chaoses, topics originated with Wiener [13] in the late 30's. Using some properties of hypercontractive operators, Borell [2] showed that all p^{th} norms, $p \ge 2$, of Hilbert space valued polynomial chaoses in independent random variables are comparable, generalizing *a fortiori* classical results of Marcinkiewicz, Paley and Zygmund [8,10].

Furthermore, this concept became a basic part of a construction of a stable multiple stochastic integral due to Krakowiak and Szulga [7]. A notion of hypercontractive random variables was introduced for this purpose, having also an intrinsic interest.

Following the key feature of the Nelson inequality, a random variable θ is called *hypercontractive* with indices $p,q > 0$ in a normed space X if there exists a constant s such that

(1.2) $$(E(||x + s\theta y||^p)^{1/p} \le (E||x + \theta y||^q)^{1/q}$$

for all $x,y \in X$; $\theta \in HC(p,q,X;\ s)$ in short.

For example, Borell's extension of (1.1) ([2]) says that for a Bernoulli random variable σ taking the value +1 or -1 with probability $\frac{1}{2}$, $\sigma \in HC(p,q,X;\ s_{p,q})$ in any normed space X, where $1 < q < p < \infty$.

By a central limit theorem argument, a symmetric Gaussian random variable $\gamma \in HC(p,q,X;\ s_{p,q})$ with the same parameters p,q and $s_{p,q}$ (cf. also Nelson [9]).

The number $s_{p,q}$ is the best possible constant for τ and σ. Moreover, q must be greater than 1. This is obvious in case of a Bernoulli random variable because, for $t > 0$,

$$E|1 + t\sigma|^q \quad \begin{cases} \leq 1 & \text{if } q \leq 1, \\ > 1 & \text{if } q \geq 1. \end{cases}$$

In case of a Gaussian random variable γ we have, for $0 < q \leq 1 < p$ and $|t| < 1$,

$$E|1 + t\gamma|^q \leq 1 + q(q-1)/2\ E\tau^2 I\ \{|\gamma| \leq 1\} + \text{const}\ e^{-1/t},$$

$$E|1 + t\gamma|^p \geq 1 + p(p-1)/2\ E\gamma^2 I\{|\gamma| \leq 1\}t^2.$$

The latter inequalities are due to S. Kwapien.

An α-stable symmetric random variable is hypercontractive in an arbitrary normed space if $1 < q < p < \alpha \leq 2$ (Krakowiak and Szulga [6]). Although this is an attribute of a more general class of probability distributions (*ibidem*) integrability and limitation to parameters $q > 1$ seemed like the prerequisite conditions.

We show in this paper that α-stable symmetric random variables are hypercontractive for all α,p,q, with $0 < q < p < \alpha < 2$, in any normed space.

2. Properties of hypercontractive random variables

For a $p > 0$ and a Banach space X we denote by $L_p(X)$ the Banach (Fréchet, if $p < 1$) space of X-valued random variables θ such that $||\theta||_p = (E||\theta||^p)^{1/p} < \infty$. A function $F=F_k\colon N^k \to X$, $k \geq 1$, is called *tetrahedral* if F vanishes outside the "tetrahedron" $D_k = \{\underline{i}_k = (i_1,\ldots,i_k)\colon 1 \leq i_1 < \ldots < i_k\}$. We define a homogeneous polynomial of degree k on R^N by

$$\langle F_k;\ (\underline{t})_k\rangle = \sum_{\underline{i}_k \in D_k} F(\underline{i}_k)\quad t_{i_1}\ldots t_{i_k},\ \underline{t} = (t_i) \in \mathbb{R}^N,$$

(by convention, $F_0 \in X$).

The class of hypercontractive random variables is closed under certain algebraic operations and under weak limits of its distributions. The latter statement follows immediately from the definition of weak convergence of probability measures (see also proof of Proposition 2.2 in [6] for more details). The precise meaning of the first statement is contained in the following result (Krakowiak and Szulga [6]):

<u>Theorem 2.1.</u> *Let* $0 < q \leq p < \infty$, $\underline{\theta} = (\theta_i)$ *be a sequence of independent p-integrable random variables, and* $F_0, \ldots, F_k$ *be finitely supported tetrahedral functions with values in* X. *If* $\theta_j \in HC(p,q,X;s,)$ *for each* $j=1,2,\ldots$ *then*

$$(2.1) \qquad \left|\left| \sum_{k-1}^{n} \langle F_k ; (\underline{s\theta})^k \rangle \right|\right|_p \leq \left|\left| \sum_{k=1}^{n} \langle F_k ; (\underline{\theta})^k \rangle \right|\right|_q ,$$

where $\underline{s\theta} = (s_i \theta_i)$.

In other words, the hypercontractivity is a hereditary feature under forming polynomials of several variables (not necessarily homogeneous). In particular, using linear forms, we obtain immediately the following statements:

COROLLARY 2.2 *If* (θ_i) *is a sequence of i.i.d. random variables and each* $\theta_i \in HC(p,q,X;s)$ *then the space of all linear combinations of* θ_i*'s with* X*-valued coefficients consists of hypercontractive random variables, i.e.*

$$\{\sum_i x_i \theta_i : (x_i) \in X^N\} \in HC(p,q,X;s).$$

COROLLARY 2.3 *If* $\theta \in HC(p,q,X;s)$ *and* θ *belongs to the domain of normal attraction of a random variable* ξ *then* $\xi \in HC(p,q,X;s)$.

Therefore, by the central limit theorem argument, we obtain the following corollary

COROLLARY 2.4 (observed by Dr. Rama-Murthy). *If* $\theta \in HC(p,q,X;s)$ *for some* $s \neq 0$ *and* $E\theta^2 < \infty$ *then* $q > 1$.

Notice that if θ is a hypercontractive random variable then necessarily $|s| < 1$ and $E||x + y\theta||^q \geq 1$.

By homogeneity of the norms appearing in the definition of hypercontractivity it suffices to consider only vectors x with $||x|| = 1$. Except a limited set of random variables, we may also assume that $||y|| \leq 1$; the precise meaning of this remark is explained below.

LEMMA 2.5 *Suppose that*

$$\inf\{E||x + y\theta||^q;\ ||x|| = 1,\ ||y|| \geq 1\} > 1$$

and (1.2) *is satisfied for all* y *with* $||y|| \leq 1$. *Then* $\theta \in HC(p,q,X;s)$ *for some* $s \neq 0$.

PROOF. Assume that $||x|| = 1$. Let $t = ||y|| \geq 1$ and $q > 0$. Since

$$\lim_{t\to\infty} (||x + y^\theta||_q - 1)/t = ||\theta||_q$$

there is a positive constant c such that

$$||1+t\theta||_q \geq 1 + ct\ \ ||\theta||_q.$$

Let $r = \min(1,p)$. We solve the following inequality with respect to s, uniformly for $t \geq 1$,

$$(2.2)\qquad ||x + sy\theta||_p/||x + y\theta||^q \leq (1+s^r t^t||\theta||_p^r)^{1/r}/(1+ct||\theta||_q) \leq 1.$$

Define s_1 by

$$s_1 = \inf_{t\geq 1} ((1 + ct||\theta||_q)^r - 1)^{1/r}/(t||\theta||_p)$$
$$= ((1 + c||\theta||)^r - 1)^{1/r}||\theta||^p > 0$$

and check that all s, $|s| \leq s_1$, satisfy (2.2), which completes the proof. □

LEMMA 2.6. *For any normed space* X, *any* $q < 1$ *and a symmetric random variable* $\theta \in L^q$ *we have*

$$(2.3)\qquad \delta_q(t) \overset{df}{=} \inf\{E||x + y\theta||^q:\ ||x|| = 1,\ ||y|| = t\} = E|1+t\theta|^q.$$

PROOF. The inequality "$\leq$" is obvious.

Let $x,y \in X$ with $||x|| = ||y|| = 1$ and x^* and y^* be functionals on X with norm 1 such that $\langle x^*,x\rangle = \langle y^*,y\rangle = 1$. Put $a = \langle x^*,y\rangle$ and $b = \langle y^*,x\rangle$. Then

$$\delta_q(t) \geq \inf\ \{E\max(|1 + a\theta|^q,\ |b + \theta|^q:\ 0\leq a\leq 1,\ |b|\leq 1\}$$

Since, for a Bernoulli random variable σ independent of θ,

$$E||x + y\theta||^q = E_\theta E_\sigma ||x + \sigma|\theta|y||^q,$$

it is enough to prove that for every $t \geq 0$

$$\inf\{E\max(|1 + a\sigma t|^q, |b + \sigma t|^q): 0 \leq a \leq 1, |b| \leq 1\} \geq E|1 + t\sigma|^q.$$

Observe that, for $t \leq 1$ and $0 \leq a \leq 1$,

$$E|1 + at\sigma|^q \geq E|1 + t\sigma|^q \geq \inf_{|b| \leq 1} E|b + t\sigma|^q.$$

Therefore, for $t \leq 1$,

$$\begin{aligned}
&\inf_a \inf_b E\max(|1 + at\sigma|^q, |b + t\sigma|^q) \\
&\geq \inf_a \inf_b \max(E|1 + at\sigma|^q, E|b + t\sigma|^q) \\
&\geq \max(\inf_a E|1 + at\sigma|^q, \inf_b E|b + t\sigma|^q) \\
&\geq \max(E|1 + t\sigma|^q, \inf_b E|b + t\sigma|^q) \\
&= E|1 + t\sigma|^q.
\end{aligned}$$

Finally, for $t > 1$, we have

$$\inf_a E|1 + at\sigma|^q = 2^{q-1}.$$

Also, for a fixed $t > 1$, the function $b \to E|t + \sigma b|^q$ is decreasing for $b \in [0,1]$, hence

$$\inf_b E|b + t\sigma|^q \geq E|1 + t\sigma|^q \geq q^{q-1}. \quad \square$$

REMARK. In the case $q \geq 1$ the quantity $\delta_q(t)$, defined in (2.3), depends essentially on the geometry of the normed space X (cf. e.g. [14]).

If a random variable θ is integrable and $q \geq 1$ then the necessary and sufficient condition for hypercontractivity of θ is the finiteness of the function

$$V(s) = \sup_{t>0} f_p(st)/\{(1 + f_q(t) + \mu(t))^{p/q} - 1\}, \tag{2.4}$$

where

$$f_p(t) = E|t\theta|^p I\{|t\theta| > 2\},$$

$$\mu(t) = \inf\{E||x + t\,\theta y||^2 - 1)I\{|t\theta| \leq 2\}: ||x||=||y||=1\}$$

(Krakowiak and Szulga [6]). Note that $\mu(t)$ may be replaced by any function $\mu'(t)$ with the same asymptotic behavior at 0. For example, if X is a Hilbert space then one may choose

$$\mu'(t) = E|t\theta|^2 I\{|t\theta| \leq 2\}$$

This criterion enables us to give an example of a non-hypercontractive random variable for certain parameters p and q, $1 < q < p < 2$.

EXAMPLE. Let θ be a symmetric random variable such that

$$N(x) = P(|\theta| > x) = x^{-\alpha} (\log x)^c, \quad x \geq e,$$

where $\alpha > 1$ and $c > 1$. Writing $f(t) \approx g(t)$ whenever $\lim_{t \to 0} f(t)/g(t) = \alpha_c$, $0 < |c| < \infty$, we derive the following asymptotic behavior of f_p and μ by a routine computation

$$f_p(t) \approx \begin{cases} N(1/t) & \text{if } p < \alpha, \\ N(1/t)\log(1/t) & \text{if } p = \alpha, \end{cases}$$

$$p(t) \approx \begin{cases} n(1/t) & \text{if } \alpha < 2, \\ t^2 & \text{if } \alpha \geq 2. \end{cases}$$

Therefore $V(s) = \infty$ for any $s \neq 0$ and for all p,q such that $1 < q < p = \alpha < 2$. In other words, for these parameters,

$$\theta \notin HC(p,q,R;s).$$

Unfortunately, the use of the criterion (2.4) is restricted to integrable random variables and to moments $p,q > 1$.

In the following sections a straightforward computation proves the hypercontractivity of all α-stable symmetric random variables, $0 < \alpha < 2$, in an arbitrary normed space, and, a *fortiori*, of all random variables whose distributions are weak limits of random α-stable multilinear forms (cf. discussion at the beginning of this section).

3. Hypercontractivity on the real line

Let ξ be a symmetric α-stable random variable. The aim of this section is to evaluate the absolute p^{th} moment of the transformed α-stable random variable $x + y\xi$. The exact integral formula

$$E|x + y\xi|^p = 2\Gamma(1+p)\sin(\pi p/2) \int_0^\infty (1-\cos v \exp(-|vy|^\alpha))v^{-p-1}dv$$

(cf. e.g. Zolotarev [15], p. 63) is too complex for our purposes, so we prefer to switch from ξ to an appropriate random variable from its domain of normal attraction.

THEOREM 3.1 *Let* $0 < q < p < \alpha$. *Then every α-stable symmetric random variable* $\xi \in HC(p,q,\mathbb{R};s)$ *for some* $s = s(p,q,\alpha) > 0$.

PROOF. By virtue of Corollary 2.3 it is enough to prove that $\theta \in HC(p,q,\mathbb{R};s)$ for some $s > 0$, where θ is the random variable belonging to the domain of normal attraction of ξ with the density

$$f(x) = \begin{cases} 0 & \text{if } |x| < 1, \\ \alpha/2\ |x|^{-1-\alpha} & \text{if } |x| \geq 1. \end{cases} \tag{3.1}$$

Since L_p-norms are homogeneous and θ is symmetric, it suffices to evaluate the function $t \to E|1 + t\theta|^p$, $t \geq 0$. Define

$$\Phi_p(u) = (|1 + u|^p + |1 - u|^p)/2.$$

We check that

$$\begin{aligned} E|1 + t\theta|^p &= \alpha t^\alpha \int_t^\infty \Phi_p(u)u^{-1-\alpha}\, du \\ &= 1 + \alpha t^\alpha \int_t^\infty (\Phi_p(u) - 1)u^{-1-\alpha}\, du \\ &= 1 + a_{p,\alpha}\, t^\alpha + g_{p,\alpha}(t), \end{aligned} \tag{3.2}$$

where

$$g_{p,\alpha}(t) = \alpha t^\alpha \int_0^t (1-\Phi_p(u))u^{-1-\alpha}\, du,$$

$$a_{p,\alpha} = \alpha\int_0^\infty (\Phi_p(u) - 1)u^{-1-\alpha}\, du$$

$$= \alpha\int_0^1 \{(\Phi_p(u) - 1)u^{-1-\alpha} + (\Phi_p(u)u^{-p} - 1)u^{-1+\alpha}\}du$$

$$= p/(\alpha - p)(1 - h(p,\alpha)).$$

The two-parameter function h is given by the formula

$$h(p,\alpha) = \alpha(\alpha - p)/p \int_0^1 (1-\Phi_p(u))(u^{-1-\alpha} + u^{-1-p+\alpha})du.$$

We claim that $h(p,\alpha) < 1$ for $0 < p < \alpha$. If $p > 1$ then $h(p,\alpha) < 0$ because $\Phi_p(u) > 1$ for $p > 1$ and $u < 1$; if $p=1$ then $h(p,\alpha) = 0$ because $\Phi_1(u) = 0$ for $u \le 1$.

For the remaining p's, observe that the function $p \to h(p,\alpha)$ is decreasing on the interval $(0,\min(1,\alpha))$ and

$$\lim_{p\to 0} h(p,\alpha) = 1 - (\alpha\pi/2)\text{cotan }(\alpha\pi/2).$$

Indeed, integrating the series

$$1 - \Phi_p(u) = - \sum_{k=1}^{\infty} \binom{p}{2k} u^{2k}, \quad 0 \le u < 1,$$

term by term, we obtain the following representation of the function h

$$h(p,\alpha) = \alpha(\alpha-p) \sum_{k=1}^{\infty} (1-p)\ldots(2k-1-p)((2k-\alpha)^{-1} + (2k+\alpha-p)^{-1}/(2k)!.$$

A routine calculation shows that $p \to h(p,\alpha)$ is a decreasing function if $0 < p < \min(1,\alpha)$ and that

$$\lim_{p\to 0} h(p,\alpha) = \sum_{k=1}^{\infty} 2\alpha^2/(4k^2-\alpha^2) = 1 - (\alpha\pi/2)\text{cotan}(\alpha\pi/2) < 1$$

(cf. e.g. Gradshteyn and Ryzhik [3]).

From the latter statement and from (3.2) we infer immediately that

(3.3) $$E|1 + t\theta|^p > 1 \quad \text{for all } t > 0.$$

Moreover,

(3.4) $$\lim_{t\to 0} g_{p,\alpha}(t)/t^2 = p(p-2)\alpha/(4-2\alpha).$$

Therefore there are positive constants $a = a_{p,\alpha}$ and $b = b_{p,\alpha}$ such that for all $t, 0 \le t \le 1$, we have

(3.5) $$1 + at^{\alpha} \le E|1 + t\theta|^p \le 1 + bt^{\alpha}.$$

The latter estimate yields the inequality

$$||1 + st\theta||_p \le ||1 + t\theta||_q$$

valid for all $t, 0 \le t \le 1$, and all s, $|s| \le s_0 \overset{df}{=} (b/a)^{1/\alpha}$.

We complete the proof by choosing $s = \min(s_0,s_1)$, where s_1 appears in Lemma (2.5).

In the following section we extend the above result to an arbitrary normed space.

4. Hypercontractivity in normed spaces.

The proof of the main result is based on some elementary inequalitis.

LEMMA 4.1. *Let* $c > 0$ *and* $p > 0$. *Define*

$$f_p(v) = |(1 + v)^p - pv|, \ -1 \le v \le c.$$

Then

$$f_p(t) = \begin{cases} (1 + v)^p - 1 - pv & \text{if } p \ge 1, \\ pv - (1 + v)^p + 1 & \text{if } p \le 1, \end{cases}$$

and there are constants $d_p, D_p > 0$ *such that for all* v, $-1 \le v \le c$,

$$d \le f_p(v)/v^2 \le D_p.$$

We omit the routine proof.

For fixed $x, y \in X$, $||x|| = 1$, $||y|| \le c$ and a symmetric random variable η we define a stochastic process $V(s)$, $s \in [0,1]$,

$$V(s) = ||x + s\eta y|| - 1$$

with values in $[-1,c]$.

LEMMA 4.2 *For a symmetric integrable random variable* η *the following contractions hold true*

$$V(s) \le s\, V(1); \tag{4.1}$$

$$EV^2(s) \le s^2\, EV^2(1) + 4s\, EV(1). \tag{4.2}$$

PROOF. (4.1) follows from the convexity of the norm. Since

$$\begin{aligned} E||x + s\eta y||^2 &= E||s(x + \eta y) + (1-s)x||^2 \\ &\le E(s||x + \eta y|| + 1-s)^2 \\ &= E(sV(1) + 1)^2 \\ &= s^2 EV^2(1) + 2sEV(1) + 1, \end{aligned}$$

and

$$E||x + s\eta y|| = E||x - s\eta y||$$
$$= E||(1+s)x - s(x + \eta y)||$$
$$\geq (1+s) - sE||x + \eta y||$$
$$= 1 - sEV(1)$$

then

$$EV^2(s) = E(||x + s\eta y||^2 - 2||x + s\eta y|| + 1)$$
$$\leq (s^2EV^2(1) + 2sEV(1) + 1) - (2 - 2sEV(1)) + 1$$
$$= s^2EV^2(1) + 4s\ EV(1). \quad \square$$

LEMMA 4.3 *For every* $a > c$,

(4.3) $$(a - 1)^q \geq 1 + ((c - 1)^q - 1)/c^q\ a^q$$

if $c > 2$, *and*

(4.4) $$(1 + a)^p \leq 1 + (1 + \frac{1}{c})^{(p-1)^+} a^p.$$

PROOF. Omitted. $\square$

LEMMA 4.4 *There exists a constant* $c > 2$ *such that for every* $x,y \in X$, *with* $||x||$ $= 1$ *and* $||y|| = t \leq 1$,

$$E(||x + \theta y||^q - 1)1\{|t\theta| \leq c\} > k_c t^\alpha.$$

PROOF. From Lemma 2.2 and Theorem 3.1 we infer that

$$E(||x + \theta y||^q - 1)1\{|t\theta| \leq c\}$$
$$\geq E(|1 + \theta t|^q - 1)1\{|t\theta| \leq c\}$$
$$= \alpha t^a \int_t^c (E_\sigma|1 + \sigma u|^q - 1)\ u^{-\alpha-1}\ du$$
$$> \alpha t^\alpha \int_0^c (E_\sigma|1 + \sigma u|^q - 1)\ u^{-\alpha-1}\ du$$
$$= k_c t^\alpha. \quad \square$$

LEMMA 4.5. *Let* θ *have the density* (3.1). *Then*

$$E|t\theta|^2\ 1\{|t\theta| \leq c\} \leq \alpha\ c^{2-\alpha}/(2-\alpha)t^\alpha;$$
$$E|t\theta|^p\ 1\{|t\theta| > c\} = \alpha c^{p-\alpha}/(\alpha-p)t^\alpha.$$

PROOF. Omitted. $\square$

Now we extend the main result of the previous section to an arbitrary normed space.

THEOREM 4.5. *For every* $p, q, 0 < q < p < \alpha < 2$, *there is* $s > 0$ *such that every symmetric* α*-stable random variable* $\xi \in HC(p,q,X;s)$, *where* X *is an arbitrary normed space.*

PROOF. It suffices to show that the random variable θ with density (3.1) (see the proof of Theorem 3.1) is hypercontractive. We may and do assume that $q < 1$ and $x, y \in X$ with $||x|| = 1$, $||y|| = t \le 1$.

We set up directly a convenient decomposition

$$E||x + s\theta y||^p =$$

$$1 + E(||x + s\theta y||^p - 1)1\{|t\theta| \le c\} + E(||x + s\theta y||^p - 1)1\{|t\theta| > c\}$$

$$= 1 + A_p(t) + B_p(t),$$

where c results from Lemma 4.4.

In order to find the needed upper estimate of $A_p(t)$ we apply Lemma 4.1, using notation and contractions from Lemma 4.2 with $\eta = \sigma\theta 1\{|t\theta| \le c\}$, as follows

$$\begin{aligned} A_p(t) &= E(1 + V(s))^p - 1 \\ &= E(pV(s) + sgn(p-1)\ f_p(V(s)) \\ &\le ps\ EV(1) + (p-1)^+ D_p\ EV^2(s) \\ &\le ps\ EV(1) + (p-1)^+ D_p(s^2 EV^2(1) + 4sEV(1)) \\ &= s(p + 4(p-1)^+)EV(1) + (p-1)^+ D_p s^2 EV^2(1) \end{aligned}$$

$$= s(p + 4(p-1)^+)/q\ (E(1 + V(1))^q - 1 + f_q(V(1)) + (p-1)^+\ D_p s^2 EV^2(1)$$

$$\le s(p + 4(p-1)^+)/q\ (E(1 + V(1))^q - 1) + s\ K_{p,q} EV^2(1)$$

$$\le sk_{p,q}\ E(||x + \theta y||^q - 1)1\{|t\theta| \le c\} + sK_{p,q}\ E|t\theta|^2 1\ \{|t\theta| \le c\},$$

where $k_{p,q}$ and $K_{p,q}$ are suitable positive constants.

By Lemma 4.3 and 4.5 there exists a constant $L_{p,q}$ such that

$$B_p(t) \le s^p\ L_{p,q}\ E(||x + \theta y||^q - 1)1\{|t\theta| > c\},$$

$$K_{p,q}\ E|t\theta|^2 1\{|t\theta| \le c\} \le L_{p,q}\ E(||x + \theta y||^q - 1)1\{|t\theta| > c\}.$$

We complete the proof by selecting a number s such that

$$s^{\min(p,1)}\, L_{p,q} \le p/(2q).$$

This yields the concluding inequality

$$(E||x + s\theta y||^p)^{1/p} \le (1 + p/q\ (E||x + \theta y||^q - 1))^{1/p}$$
$$\le E||x + \theta y||^q)^{1/q}. \qquad \square$$

COROLLARY 4.6. *Let* M *be a symmetric* α*-stable measure,* $0 < \alpha < 2$ *independently scattered on the interval* $[0,T]$, *and let* f_j, *be function on* $[0,T]^j$ *with values in a Banach space* X, *integrable with respect to the product random measures* M^j, $j=1,\ldots,n$ *(as defined in* [7]*). Put*

$$\theta = \sum_{j=1}^{n} \int_{[0,T]^j} f_j dM^j. \tag{4.5}$$

Then θ *is a hypercontractive random variable in* X *with parameters* p,q $(q < p < \alpha)$ *and* s^n, *where* $s = s(p,q,\alpha)$ *appears in Theorem* 4.5. *Note that an* α*-stable polynomial*

$$\theta = \sum_{j=1}^{n} \sum_{k_1,\ldots,k_j} a_j(k_1,\ldots,k_j)\xi_{k_1}\cdots\xi_{k_j}$$

is a special case of (4.5).

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