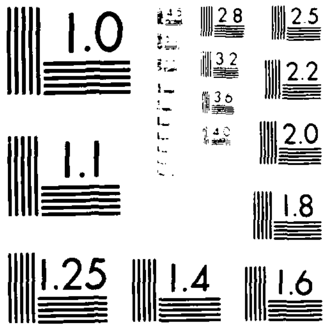
AD-A186 432 LOCAL PROPERTIES OF INDEX-ALPHA STABLE FIELDS(U) NORTH CAROLINA UNIV AT CHAPEL HILL DEPT OF STATISTICS J P NOLAN DEC 86 TR-171 AFOSR-TR-87-1059 1/1

UNCLASSIFIED F49620-85-C-0144 F/G 12/3 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

DTIC FILE COPY

UNCLASSIFIED

AD-A186 432

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY: NA	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for Public Release; Distribution Unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: NA	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 171	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 87-1059
6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: AFOSR/NM
6c. ADDRESS (City, State and ZIP Code): Center for Stochastic Processes, Statistics Department, Philiips Hall 039-A, Chapel Hill, NC 27514	7b. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620 85 C 0144
8c. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, DC	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
6.1102F	2304	A5	

11. TITLE (Include Security Classification): Local properties of index-β stable fields

12. PERSONAL AUTHOR(S): Nolan, John P.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
~~technical~~ preprint	FROM 10/1/86 TO 9/30/87	December 1986	19

16. SUPPLEMENTARY NOTATION

17. COSATI CODES		
FIELD	GROUP	SUB. GR.
XXXXX	XXXXXXXXX	XXXX

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Keywords: Stable processes, random fields, Hausdorff dimension, local times.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

We examine the paths of the stable fields that are the analogs of index-β Gaussian fields. We find Hölder conditions on their paths and find the Hausdorff dimension of the image, graph, and level sets when we have local nondeterminism, generalizing the Gaussian results.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: ~~Peggy Ravitch~~	22b. TELEPHONE NUMBER (Include Area Code): ~~919-962-2307~~; 22c. OFFICE SYMBOL: AFOSR/NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE. UNCLASSIFIED

AFOSR-TR- 87-1059

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

COPY INSPECTED 7

LOCAL PROPERTIES OF INDEX-β STABLE FIELDS

by

John P. Nolan

Technical Report No. 171

December 1986

LOCAL PROPERTIES OF INDEX-β STABLE FIELDS

by

John P. Nolan
University of North Florida
and
Center for Stochastic Processes
University of North Carolina, Chapel Hill

Abstract: We examine the paths of the stable fields that are the analogs of index-β Gaussian fields. We find Hölder conditions on their paths and find the Hausdorff dimension of the image, graph, and level sets when we have local nondeterminism, generalizing the Gaussian results.

AMS 1980 Subject Classifications: 60G17, 60G60.

Keywords and Phrases: Stable processes, random fields, Hausdorff dimension, local times.

This research has been supported by the Air Force Office of Scientific Research Grant No. F49620 85 C 0144.

1. INTRODUCTION

Our objective is to study the sample paths of the stable analogs of the index-β Gaussian fields. One example of the latter is a Gaussian process with $\mathrm{Var}(X(t)-X(s)) \approx |t-s|^{2\beta}$ for some $0 < \beta \le 1$. The work of Cuzick (1978), Adler (1981), Pitt (1978) and Geman and Horowitz (1980) have resulted in detailed knowledge of the sample paths of these Gaussian fields. A good reference for these results is Chapter 8 of Adler's book, where one can find any Gaussian result we don't explicitly reference.

In Section 2 we define our terms and give some consequences of local nondeterminism. Section 3 is concerned with Hölder conditions for the sample paths of (N,1) stable fields. Briefly, the stable result does not follow the Gaussian one and we give a surprising example of how $||\cdot||_p^p$ is a poor replacement for $\mathrm{Var}(\cdot)$. We describe what we can for harmonizable, subgaussian and moving average processes. Finally, in Section 4, we examine (N,d) stable fields. We allow the indices of stability to be different for different components. We find the Hausdorff dimension of the image, graph and level sets for classes of stable fields, as well as show their trajectories are Jarnik functions. Perhaps surprising, $||\cdot||_p^p$ is an adequate tool for these erraticism results and there is no dependence on the index of stability.

2. PRELIMINARIES

Points in $\mathbb{R}^n$ will be denoted by $x = (x^1, \ldots, x^n)$, the usual inner product by $\langle x,y \rangle = \Sigma x^i y^i$ and the Euclidean norm by $|x| = \langle x,x \rangle^{1/2}$. The notation $A \approx_{C(a_1,a_2,\ldots)} B$ will mean that there is a positive constant C depending on the parameters $a_1, a_2, \ldots$ such that $C^{-1} \le A/B \le C$. For $s,t \in \mathbb{R}^n$, $s \le t$ means $s^i \le t^i$ for all $i = 1, \ldots, n$ in which case $[s,t]$ will mean the n-dimensional rectangle $\Pi_{i=1}^n [s^i, t^i]$. Lebesgue measure on $\mathbb{R}^n$ will be denoted by Leb_n.

If X is a symmetric p-stable r.v., $0 < p \le 2$, then we let

$$(2.1) \qquad \|X\|_p = [-\log E\exp(iX)]^{1/p}.$$

This is a norm (p quasi-norm if $0 < p < 1$) on the space of symmetric p-stable random variables. Of course $\|X\|_2^2 = \mathrm{Var}(X)$ in the Gaussian case, so one is tempted to think of $\|X\|_p^p$ as a generalization of the variance. It is known that for any $0 < q < p$, there is a $C(p,q) > 0$ such that

$$(2.2) \qquad E|X|^q = C(p,q) \|X\|_p^q$$

for every symmetric p-stable r.v. X.

Let $0 < p \le 2$ and $T \subset \mathbb{R}^N$, then a real valued random field $X = \{X(t) : t \in T\}$ is called an $(N,1,p)$ stable field if every finite linear combination $\Sigma_{j=1}^m a_j X(t_j)$ is a symmetric p-stable r.v. Then by (2.1)

$$(2.3) \qquad E\exp(i \sum_{j=1}^m a_j X(t_j)) = \exp(-\|\sum_{j=1}^m a_j X(t_j)\|_p^p),$$

so $\|\Sigma a_j X(t_j)\|_p$ completely determines the distribution of $(X(t_1), \ldots, X(t_m))$. If $X(t) = \int f(t) dW$ is the L^p stochastic integral representation for $X(t)$, then $\|\Sigma a_j X(t_j)\|_p = \|\Sigma a_j f(t_j)\|_{L^p}$. Throughout we assume that $\|X(t) - X(s)\|_p \to 0$ as $t \to s$, which is equivalent to X being continuous in probability.

Points $t_1, \ldots, t_m \in T$ are ordered if $t_1 < t_2 < \ldots < t_m$ when $T \subset \mathbb{R}$; if $T \subset \mathbb{R}^N$ $(N > 1)$ we call them *ordered* if for every $j = 2, \ldots, m$, $|t_j - t_{j-1}| \le |t_j - t_k|$ for every $k = 1, \ldots, j-1$, i.e. t_{j-1} is closest to t_j among $t_1, \ldots, t_{j-1}$. In Nolan (1986) we used this definition of order and $\|\cdot\|$ to define local non-determinism (LND) for symmetric stable fields. A curious consequence of LND is the following.

Lemma 2.1. Let $0 < p \le 2$ and X and Y be $(N,1,p)$ stable fields on T. Assume both are LND, $\|X(t)\|_p \approx_{C_1} \|Y(t)\|_p$ for all $t \in T$ and $\|X(t) - X(s)\|_p \approx_{C_2} \|Y(t) - Y(s)\|_p$ for all $|t - s| < \delta_1$ where δ_1 is some positive number. Then locally X and Y have equivalent norms, i.e. there is a $\delta_2 > 0$ such that for any $m \ge 2$,

$$\left\|\sum_{j=1}^{m} u_j X(t_j)\right\|_p \approx_{C(m,p)} \left\|\sum_{j=1}^{m} u_j Y(t_j)\right\|_p$$

for all $u_1, \ldots, u_m \in \mathbb{R}$ and all $t_1, \ldots, t_m \in T$ with $|t_i - t_j| < \delta_2$ for all i and j.

Proof: We will assume $t_1, \ldots, t_m$ are ordered. This is no loss of generality as there is always a permutation π of $\{1, \ldots, m\}$ with $t_{\pi(1)}, \ldots, t_{\pi(m)}$ ordered and the following proof works on this rearrangement. Letting

$$v_j = \sum_{k=j}^{m} u_k, \text{ then } v_{j+1} - v_j = u_j \text{ for } j = 1, \ldots, m-1$$

so

$$\sum u_j X(t_j) = v_1 X(t_1) + \sum_{j=2}^{m} v_j (X(t_j) - X(t_{j-1})).$$

By LND,

$$(2.4) \qquad \|\sum u_j X(t_j)\|_p \approx_{C(m,p)} \|v_1 X(t_1)\|_p + \sum_{j=2}^{m} \|v_j (X(t_j) - X(t_{j-1}))\|_p$$

when $t_1, \ldots, t_m$ are close. The same argument works for Y, so the assumptions on the p-norms of $X(t)$, $Y(t)$, $X(t) - X(s)$ and $Y(t) - Y(s)$ give the result. □

In view of (2.3), one is tempted to conclude that if X and Y satisfy this theorem, then they will have the same local properties. This is true in the Gaussian case, but not necessarily true when $0 < p < 2$, as we shall see in the next section.

The following consequence of LND is the crucial one for local time applications. Since it was not explicitly stated in Nolan (1986) we present it here.

Lemma 2.2. Let X be a LND $(N,1,p)$, $0 < p \le 2$, stable field on compact T with joint density $p(\bar{t};\bar{x}) = p(t_1, \ldots, t_m; x_1, \ldots, x_m)$ of $(X(t_1), \ldots, X(t_m))$. Then there is a $\delta > 0$ such that if $t_1, \ldots, t_m$ are ordered, distinct, and $|t_i - t_j| < \delta$ for all i,j,

$$p(\bar{t};\bar{x}) \le K_1(m,p) J(\bar{t})$$

and for any $0 < \gamma \le 1$,

$$|p(\bar{t};\bar{x}) - p(\bar{t};\bar{y})| \le K_2(m,p) \prod_{j=1}^{m} |x_j - y_j| J(\bar{t})^{1+2\gamma}$$

where

$$J(\bar{t}) = [\|X(t_1)\|_p \prod_{j=2}^{m} \|X(t_j) - X(t_{j-1})\|_p]^{-1}.$$

Proof: The inversion formula for characteristic functions shows

$$p(\bar{t};\bar{x}) \le (2\pi)^{-m} \int_{\mathbb{R}^m} |E\exp(i \sum_{j=1}^{m} u_j X(t_j))| d\bar{u}.$$

Letting $v_j = \Sigma_{k=j}^{m} u_j$ as in (2.4), LND shows that there is some $\delta > 0$ such that $|t_i - t_j| < \delta$ for all i,j, implies that the integrand above is bounded by

$$\exp(-C(\|v_1 X(t_1)\|_p + \sum_{j=2}^{m} \|v_j (X(t_j) - X(t_{j-1}))\|_p)^p).$$

Let $w_1 = \|X(t_1)\|_p v_1$ and $w_j = \|X(t_j) - X(t_{j-1})\|_p v_j$ for $j = 2,\ldots, m$. (Recall that these norms are positive as part of our definition of LND.) Some calculation shows that $J(\bar{t})$ is precisely the Jacobian of the transformation $(u_1,\ldots, u_m) \to (w_1,\ldots, w_m)$, yielding

$$p(\bar{t};\bar{x}) \le (2\pi)^{-m} \int_{\mathbb{R}^m} \exp(-C(\sum_{j=1}^{m} |w_j|)^p) J(\bar{t}) d\bar{w}$$

$$= K_1(m,p) J(\bar{t}).$$

For the second part, the inversion formula yields

$$|p(\bar{t};\bar{x}) - p(\bar{t};\bar{y})| \le (2\pi)^{-m} \int |1 - \exp(-i \sum_{j=1}^{m} (x_j - y_j) u_j)| \times |E\exp(i \sum_{j=1}^{m} u_j X(t_j))| d\bar{u}.$$

For any $0 < \gamma \le 1$, the first term inside the integral is $\le \pi_{j=1}^{m} |x_j - y_j|^{\gamma} |u_j|^{\gamma}$. Letting v_j be as above, $u_m = v_m$ and $u_j = v_j - v_{j+1}$ for $j = 1,\ldots, m-1$ so we can replace the u_j's with

v_j's and expand to get

$$\prod_{j=1}^{m} |u_j|^{\gamma} \leq \sum_{\{\theta\}} \prod_{j=1}^{m} |v_j|^{\theta_j \gamma}, \quad \{\theta\} \subset \{0,1,2\}^m.$$

Using LND as above,

$$|p(\bar{t};\bar{x}) - p(\bar{t};\bar{y})| \leq (2\pi)^{-m} \prod_{j=1}^{m} |x_j - y_j|^{\gamma}$$

$$\times \int_{\mathbb{R}^m} \sum_{\{\theta\}} \left|\frac{w_1}{\|X(t_1)\|_p}\right|^{\theta_1 \gamma} \prod_{j=2}^{m} \left|\frac{w_j}{\|X(t_j) - X(t_{j-1})\|_p}\right|^{\theta_j \gamma}$$

$$\times \exp(-C(\sum |w_j|)^p) J(\bar{t}) d\bar{w}.$$

Since $t \to \|X(t)\|_p$ is continuous on compact T, $\|X(t)\|_p$ and $\|X(t) - X(s)\|_p$ are bounded, implying that for $\theta = 0, 1$ or 2,

$$\|X(t)\|_p^{-\theta\gamma} \leq \text{constant} \|X(t)\|_p^{-2\gamma},$$

$$\|X(t) - X(s)\|_p^{-\theta\gamma} \leq \text{constant} \|X(t) - X(s)\|_p^{-2\gamma}$$

and we can combine these terms with $J(\bar{t})$ to get

$$|p(\bar{t};\bar{x}) - p(\bar{t};\bar{y})| \leq \text{const.} J(\bar{t})^{1+2\gamma} \times \int_{\mathbb{R}^m} \sum_{\{\theta\}} \prod_{j=1}^{m} |w_j|^{\theta_j \gamma} \exp(-C(\sum |w_j|)^p) d\bar{w}$$

$$\times \prod_{j=1}^{m} |x_j - y_j|^{\gamma}$$

$$= K_2(m,p) J(\bar{t})^{1+2\gamma} \prod_{j=1}^{m} |x_j - y_j|^{\gamma}.$$

□

3. REGULARITY FOR (N,1) FIELDS

We will examine $(N,1,p)$ stable fields X similar to the index-β Gaussian fields studied in Chapter 8 of Adler (1981). Specifically, for all t in the interior of T and some $0 < \beta \le \max(1,p^{-1})$, define the two conditions:

(3.1) $$\|X(t+h) - X(t)\|_p = o(|h|^\alpha) \quad \text{as } |h| \downarrow 0 \text{ for all } 0 < \alpha < \beta.$$

(3.2) $$|h|^\alpha = o(\|X(t+h) - X(t)\|_p) \quad \text{as } |h| \downarrow 0 \text{ for all } \alpha > \beta.$$

Note that we may have $\beta > 1$ when $p < 1$ because $\|\cdot\|_p^p$, not $\|\cdot\|_p$, is subadditive. If both (3.1) and (3.2) hold, we call X an *index*-β $(N,1,p)$ stable field. In this section we will examine when sample paths of X satisfy a uniform stochastic Hölder condition of order α on T, i.e. there is an a.s. finite, positive r.v. $C(\omega)$ such that whenever $|h|$ is small and $t, t+h \in T$

(3.3) $$|X(t+h) - X(t)| \le C(\omega)|h|^\alpha.$$

In the Gaussian case (3.1) implies (3.3) for every $\alpha < \beta$ and (3.2) implies (3.3) fails for every $\alpha > \beta$. The stable Lévy process with $\beta = p^{-1}$ shows that the stable result cannot be as simple.

Theorem 3.1. Let X be an $(N,1,p)$ stable field on compact $T \in \mathbb{R}^N$ and $0 < p < 2$.

(i) If $N = 1$, $p > 1$ and (3.1) holds for some $\beta > p^{-1}$, then (3.3) is valid for every $\alpha < \beta$. When $N = 1$, no other values of p and β are sufficient for (3.1) to imply continuous paths.

(ii) For $N \geq 1$, (3.2) implies (3.3) fails for every $\alpha > \beta$.

Proof: (i) By compactness of T is suffices to consider $T = [0, h_0]$ where h_0 is small. Using (3.1) and (2.2) we have for any $0 < q < p$, any $\alpha < \beta$

$$E|X(t+h) - X(t)|^q \leq K|h|^{\alpha q}$$

for $|h|$ small. If $p > 1$ and $\beta > p^{-1}$ then choosing $\alpha \in (p^{-1}, \beta)$ and $q \in (\alpha^{-1}, p)$, gives $\alpha q > 1$ and Kolmogorov's classic result guarantees continuous sample paths. However we need something stronger to get the desired modulus of continuity. Theorem 1.1 of Pisier (1983), with $d(t,s) = |t-s|^\alpha$ and $\psi(u) = |u|^q$ gives the desired modulus of continuity.

The second part of (i) comes from example (d) below.

(ii) As in the Gaussian case, if $\alpha > \beta$, then (3.2) shows $(X(h) - X(0))/|h|^\alpha$ is a.s. unbounded as $|h| \downarrow 0$, so (3.3) cannot hold. $\square$

Note that the proof of part (i) is a moment argument and applies to q^{th} moment processes regardless of whether they are stable or not. Also, this proof fails when $N > 1$. We know of no general result when $N \geq 1$, though the value $\beta = p^{-1}$ is always a lower bound as example (c) below shows. In the next section we will strengthen (ii) using local times.

We now give examples to illustrate the possibilities for specific classes of (N,1,p) stable fields. We will mention when these examples are LND, both for an interesting reason here and for use in the next section. Any unreferenced statements below come from Nolan (1986).

(a) Harmonizable fields. Let $N \geq 1$, $0 < p \leq 2$, μ a finite Borel measure on $\mathbb{R}^N$, W the complex p-stable noise generated by μ and define

(3.4) $$X(t) = \mathrm{Re} \int_{\mathbb{R}^N} \exp(i\langle t,\lambda\rangle)dW(\lambda)$$

for $t \in [0,2\pi]^N$. This gives a stationary (N,1,p) stable field, but does not exhaust that class, e.g. Cambanis and Soltani (1983). Sufficient conditions for (3.1) and (3.2) to hold are respectively

(3.5) $$\limsup_{|\lambda|\to\infty} |\lambda|^{N+\alpha p}\mu(\lambda + Q) < \infty \quad \text{for all } \alpha < \beta,$$

(3.6) $$\liminf_{|\lambda|\to\infty} |\lambda|^{N+\alpha p}\mu(\lambda + Q) > 0 \quad \text{for all } \alpha > \beta,$$

where Q is any bounded cube $[-a,a]^N$ in $\mathbb{R}^N$. If both hold, then X is an index-β (N,1,p) stable field. If both hold and $p \geq 1$, then X is LND. Taking $Q = [-\frac{1}{2},\frac{1}{2}]^N$, this includes random p-stable Fourier series $X(t) = \mathrm{Re}(\Sigma a_n \exp(int)\theta_n)$, e.g. $|a_n| \approx |n|^{-(1+\beta p)}$ for large $|n|$ implies X(t) is an index-β p-stable process.

When $p \geq 1$, Marcus and Pisier (1984) give necessary and sufficient conditions for (3.4) to be continuous. Since (3.5) implies $\tau(t,s) = \|X(t) - X(s)\|_p \leq \text{constant}|t-s|^\alpha$ for $\alpha < \beta$, their logarithmic metric entropy is finite, giving continuity. Even more, using Theorem 1.6 of Marcus and Pisier (1984a), we get (3.3) for every $\alpha < \beta$. Using their notation $\|\exp(iu\cdot)\|_\tau \leq \text{constant}|u|^\alpha$, making

$$\int \|\exp(iu\cdot)\|_\tau^p d\mu(u) < \infty.$$

Furthermore, $J_q(\tau,\delta) \le$ constant $\delta^{1-\varepsilon}$ and $\phi_q(\tau,\delta) \le$ constant $\delta^{1-\varepsilon}$ for small $\varepsilon > 0$. For $|t-s| \le h$, $\delta = \tau(t,s) \le h$, and their result shows

$$|Y(t) - X(s)| \le C(\omega)|h|^{\alpha(1-\varepsilon)}$$

When $0 < p < 1$, the finiteness of μ insures the continuity of X, see Marcus and Woyczynski (1977).

(b) Subgaussian fields. Let $0 < p < 2$, A a positive (p/2)-stable r.v. and Y(t) an (N,1) Gaussian field. Then $X(t) = A^{1/2}Y(t)$ is a subgaussian (N,1,p) stable field. It satisfies (3.1) and/or (3.2) if and only if Y satisfies the respective condition. It is LND if and only if Y is. Since the sample paths of X are simply multiples of those of Y, (3.1) implies (3.3) for every $\alpha < \beta$ as in the Gaussian case.

(c) Multiparameter Lévy stable fields. Let $0 < p \le 2$, $T = [0,1]^N$ and $X(t) = W([0,t])$ where W is the p-stable noise generated by Lebesgue measure on T. When $N = 1$, $\|X(t+h) - X(t)\|_p = |h|^{1/p}$, so X is an index-(1/p) stable process. It is also LND. In contrast, when $N > 1$ and T is compact we have (3.1) for $\beta = p^{-1}$, but (3.2) fails for every β and these fields are not LND. To see the claims about (3.1) and (3.2), we note that

$$\|X(t+h) - X(t)\|_p^p = \mathrm{Leb}_N([0,t+h] \,\Delta\, [0,t]).$$

Taking any component of h to be zero, this is zero, so (3.2) cannot hold. The proof of (3.1) is in the following elementary argument.

For any $M > 0$, and any $t, t+h \in [0,M]^N$, $\mathrm{Leb}_N([0,t+h] \Delta [0,t]) \le M^{N-1} N^{1/2} |h|$. Let $a, b \in \mathbb{R}^N$ have coordinates $a^i = \min(t^i, t^i + h^i)$ and $b^i = \max(t^i, t^i + h^i)$. Then $[0,t+h] \cup [0,t] \subset [0,b]$ and $[0,t+h] \cap [0,t] = [0,a]$, so $[0,t+h] \Delta [0,t] \subset [0,b] \Delta [0,a]$. Now the last term is equal to $\cup_{i=1}^N Q_i$, where

$$Q_i = \left(\prod_{j<i} [0,b^j]\right) \times [a^i, b^i] \times \left(\prod_{j>i} [0,b^j]\right).$$

Hence,

$$\mathrm{Leb}_N([0,t+h] \Delta [0,t]) \le \sum_{i=1}^N \mathrm{Leb}_N(Q_i) = \sum_{i=1}^N \left(\prod_{j \ne i} |b^j|\right) \cdot |b^i - a^i|$$

$$\le \sum M^{N-1} |h^i| \le M^{N-1} N^{1/2} \left(\sum |h^i|^2\right)^{1/2}.$$

Of course these fields are discontinuous when $0 < p < 2$, explaining the critical value of $\beta = p^{-1}$ in Theorem 3.1. We note that Ehm (1981) has derived some of the results in the next section for these fields without LND by using a direct approach to the integrals involved in the proof of our Lemma 2.2.

(d) <u>A class of moving average processes</u>. Let $0 < p \le 2$, $0 < \beta \le 1$ and set

$$X(t) = \int_{-\infty}^{t} |t - \lambda|^{\beta - p^{-1}} e^{-|t-\lambda|} dW(\lambda) \tag{3.7}$$

for $t \in \mathbb{R}$, where W is the symmetric p-stable Lévy process. For every value of p and β, this is an index-β and LND p-stable process. When $p > 1$ and $\beta > p^{-1}$, then Theorem 3.1 (i) above shows (3.3) holds for every $\alpha < \beta$. However, in all other cases ($p \le 1$ or $\beta \le p^{-1}$), the kernel in (3.7) is discontinuous and hence, by Theorem 5.1 of Rosinski (1985), $X(t)$ cannot have continuous sample paths.

Looking a bit further at this example leads to an unexpected result. Let X be one of these discontinuous moving average processes. Take the same p and β and get a subgaussian process Z using (a) and (b) above that is index-β and LND. Then Lemma 2.1 shows

$$\left\| \sum_{j=1}^{m} u_j X(t_j) \right\|_p \sim C(m,p) \left\| \sum_{j=1}^{m} u_j Z(t_j) \right\|_p$$

locally. In view of (2.3) it is surprising that X is discontinuous, while Z is continuous! Perhaps the lesson here is that for regularity results, $\|\cdot\|_p^p$ fails to express what Var($\cdot$) does in the Gaussian case and that a Banach space approach like Rosinski (1985), or Marcus and Pisier (1984, 1984a) is necessary.

4. REGULARITY AND ERRATICISM FOR (N,d) STABLE FIELDS

We will now consider stable fields having state space $\mathbb{R}^d$, i.e. $X = \{X(t) = (X^1(t), \ldots, X^d(t)) : t \in T \subset \mathbb{R}^N\}$. Each component $X^i(t)$ will be an $(N,1,p_i)$ symmetric stable field. We allow components to have different stability indices. This will be abbreviated as an $(N,d,\bar{p})$ stable field, where $\bar{p} = (p_1, \ldots, p_d)$. For simplicity we will assume $T = [0,1]^N$ and that X has stationary increments, although this is not strictly necessary for most of these results.

Since (3.1) fails to imply uniform stochastic Hölder conditions on the sample paths in general, we will replace (3.1) and (3.2) by

(4.1) X satisfies a uniform stochastic Hölder condition of every order $\bar{\alpha} < \bar{\beta}$, i.e. $\bar{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\bar{\beta} = (\beta_1, \ldots, \beta_d)$ and component X^i satisfies (3.3) for every $\alpha_i < \beta_i$.

(4.2) For each $\bar{\alpha} > \bar{\beta}$, we have simultaneously for all components $i = 1, \ldots, d$,

$$|h|^{\alpha_i} = o(\|X^i(t+h) - X^i(t)\|_{p_i}) \quad \text{as } |h| \downarrow 0.$$

The results of the last section apply to each component separately, but to study all the components together we need to rule out degeneracy caused by too much dependence between components. For example, if a field X has one component a scalar multiple of another, then the image of X can be quite different from when the components are independent. For Gaussian fields, Cuzick (1978) gave such a condition in terms of the covariance. We alter slightly our earlier definition in Nolan

(1986). An (N,d) random field has *characteristic function locally approximately independent components*, if for all $m \geq 1$ there is a $\delta = \delta(m) > 0$ and a $C = C(d,m) > 0$ such that for all $u_1, \ldots, u_m \in \mathbb{R}^d$, and all $t_1, \ldots, t_m \in T$ with $|t_i - t_j| < \delta$ for all i and j,

$$(4.3) \qquad \prod_{i=1}^{d} |E\exp(iC^{-1} \sum_{j=1}^{m} u_j^i X^i(t_j))| \leq |E\exp(i \sum_{j=1}^{m} \langle u_j, X(t_j)\rangle)|$$
$$\leq \prod_{i=1}^{d} |E\exp(iC \sum_{j=1}^{m} u_j^i X^i(t_j))|.$$

Clearly (4.3) holds if the components of X are independent. If the indices $p_1, p_2, \ldots, p_d$ are all the same, then the techniques in the above paper give an equivalent condition in terms of the common $\|\cdot\|_{p_i}$ norm.

We can start our analysis by looking at the Hausdorff dimension of the image and graph of X, denoted by ImX and GrX. This result generalizes Cuzick's (1978) Theorem 1.

Theorem 4.1. Let X be an $(N,d,\bar{p})$ stable field on $[0,1]^N$ with stationary increments that satisfies (4.1), (4.2) and (4.3) for some $\bar{\beta}$ with coordinates arranged so that $0 < \beta_1 \leq \beta_2 \leq \ldots \leq \beta_d \leq 1$.

$$(4.4) \qquad \dim(\mathrm{Im}X) = \begin{cases} d & \text{if } N \geq \sum_{i=1}^{d} \beta_i, \\ d + (N - \sum_{i=1}^{d} \beta_i)/\beta_d & \text{if } N < \sum_{i=1}^{d} \beta_i, \end{cases}$$

$$(4.5) \qquad \dim(\mathrm{Gr}X) = \begin{cases} d + N - \sum_{i=1}^{d} \beta_i & \text{if } N \geq \sum_{i=1}^{d} \beta_i, \\ d + (N - \sum_{i=1}^{d} \beta_i)/\beta_d & \text{if } N < \sum_{i=1}^{d} \beta_i. \end{cases}$$

Proof: We make minor adjustments to Cuzick's proof. For both ImX and GrX, (4.1) and real variable arguments show that the right hand sides above are upper bounds for the respective dimensions. The lower bound for dim(ImX) come from standard capacity arguments if we can show $\int_{[-1,1]^N} E|X(t) - X(0)|^{-\lambda} dt < \infty$ for all $\lambda <$ right hand side of (4.4). As he does, substitute $Y^i(t) = (X^i(t) - X^i(0))/\|X^i(t) - X^i(0)\|_{p_i}$. Our (4.3) plays the role of Cuzick's condition (1A) and guarantees that the joint density of $(Y^1(t), \ldots, Y^d(t))$ is bounded above by a constant independent of t. That density is, using (4.3) and $\|Y^i(t)\|_{p_i} = 1$,

$$p_Y(t; y^1, \ldots, y^d) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp(-i \sum_{i=1}^{d} y^i u^i) E\exp(i \sum_{i=1}^{d} u^i Y^i(t)) d\bar{u}$$

$$\leq \text{constant} \int_{\mathbb{R}^d} \prod_{i=1}^{d} |E\exp(i\alpha u^i Y^i(t))| d\bar{u}$$

$$= \text{constant} \int_{\mathbb{R}^d} \prod_{i=1}^{d} \exp(-\|\alpha u^i Y^i(t)\|_{p_i}^{p_i}) d\bar{u}$$

$$= \int_{\mathbb{R}^d} \exp(-\sum_{i=1}^{d} |\alpha u^i|^{p_i}) d\bar{u} < \infty$$

The rest of the proof follows Cuzick. □

It is worth noting that the result does not depend on the indices $p_1, \ldots, p_d$. As in the Gaussian case, the sum $\Sigma_{i=1}^d \beta_i$ is the critical value. If this is less than N, then dim(ImX) < d a.s., so $\mathrm{Leb}_d(\mathrm{Im}X) = 0$ a.s. and almost every point in $\mathbb{R}^d$ is not hit by X. If the sum is N or more than we can ask about hitting points and existence of local times. The latter was done in Nolan (1986) for stable fields, we state it here for these fields. We will say an $(N,d,\bar{p})$ stable field is LND if the components are individually LND and (4.3) holds.

Theorem 4.2. Let X be an $(N,d,\bar{p})$ stable field on $T=[0,1]^N$ with stationary increments that is LND and satisfies (4.2) for some $\bar{\beta}$. If $N>\Sigma_{i=1}^{d}\beta_i$, then X has a jointly continuous local time $\alpha(x,t)$ that for any compact $U\subset \mathbb{R}^d$ is

(i) Hölder continuous in $x\in U$ for any order $0<\gamma<\min(1,((N/\Sigma_{i=1}^{d}\beta_i)-1)/2)$, i.e. there is an a.s. finite positive r.v. $C_1(\omega)$ with

$$|\alpha(x,B)-\alpha(y,B)| \le C_1(\omega)|x-y|^\gamma$$

for all $x,y\in U$ and all rectangles $B\subset T$ with rational vertices;

(ii) Hölder continuous in t for any order $0<\delta<1-(\Sigma_{i=1}^{d}\beta_i/N)$, i.e. there is an a.s. finite positive r.v. $C_2(\omega)$ with

$$\alpha(x,B) \le C_2(\omega)(\mathrm{Leb}_N(B))^\delta$$

for all $x\in U$ and all rectangles $B\subset T$ of sufficiently small edge length.

Proof: The (4.3) part of LND lets us generalize Lemma 2.2 to

$$p(\bar{t};\bar{x}) \le K_1(m,p,d)\bar{J}(\bar{t})$$

and

$$|p(\bar{t};\bar{x})-p(\bar{t};\bar{y})| \le K_2(m,p,d)\prod_{j=1}^{m}|x_j-y_j|^\gamma \bar{J}(\bar{t})^{1+2\gamma}$$

where $0<\gamma<1$, $m\ge 1$, $\bar{t}=(t_1,\ldots,t_m)\in T^m$, $t_1,\ldots,t_m$ are ordered, $\bar{x}=(x_1,\ldots,x_m)$, $\bar{y}=(y_1,\ldots,y_m)\in(\mathbb{R}^d)^m$ and

$$\bar{J}(\bar{t}) = \prod_{i=1}^{d} J^i(\bar{t}),$$

where J^i is the Jacobian term for component X^i as in Lemma 2.2. The rest of the proof is as in sections 25-30 of Geman and Horowitz (1980). The only essential change is to use

$$V_{m,\gamma}(B) = \int_{B^m} \bar{J}(A(\bar{t}))^{1+2\gamma} d\bar{t}$$

(where $A : T^m \to T^m$ rearranges $t_1, \ldots, t_m$ so that they are ordered) instead of their $V_{m,\gamma}(B)$. □

As in (30.7) of Geman and Horowitz (1980), we can strengthen Theorem 3.1 (ii) with LND. This can be applied separately to the components of an $(N,d,\bar{p})$ field even when they do not satisfy (4.3).

Corollary 4.3. Let X be an (N,1,p) stable field on $T = [0,1]^N$ that is LND and satisfies (3.2) for some $\beta < 1$. Then a.s. the sample paths of X are Jarnik(α) for every $\alpha > \beta$, i.e. for every $t \in T$,

$$\operatorname*{ap-lim}_{s \to t} \frac{|X(t) - X(s)|}{|t - s|^{\alpha}} = +\infty \quad \text{a.s.}$$

The fact that this holds at every t means much more than Theorem 3.1 (ii)-- it guarantees that the paths are uniformly erratic.

Let X be as in Theorem 4.2 and assume (4.1) holds. Continuing our discussion after Theorem 4.1, a natural question is how big is the level set $\{t \in T : X(t) = x\}$. Consider the open set $\mathcal{O} = \mathcal{O}(\omega) = \{x \in \mathbb{R}^d : \alpha(x,T) > 0\}$. Adler (1981) shows that for each ω, $\mathcal{O}$ and ImX are essentially the same: his Theorem 8.6.1 shows $\mathcal{O} \subset$ closure(ImX) and his Lemma 8.7.2 shows that the complement of $\mathcal{O}$ is nowhere dense in ImX. His Theorem 8.8.4 can be extended to the stable case.

Corollary 4.4. Assume X is as in Theorem 4.2 and (4.1) holds. Then a.s.

$$\dim X^{-1}(x) = N - \sum_{i=1}^{d} \beta_i$$

for all $x \in \mathcal{O}$.

Finally, we comment on recent results of Monrad and Pitt (1986). Assuming $\bar{\beta} = (\beta,\ldots,\beta)$ has all components the same, then Gaussian fields similar to those here satisfy a uniform dimension result that strengthens Corollary 4.4: $\dim X^{-1}(F) = N - \beta d + \beta \dim F$ for every closed set $F \subset \mathcal{O}$. The stable fields can be dealt with in the same way if we assume (4.1). They also show that (4.4) can be strengthened: if $N \leq \beta d$, then $\dim X(E) = (\dim E)/\beta$ a.s. for every closed set $E \subset T$. We do not see immediately how to generalize this when $p < 2$. Their "strongly LND" can be defined (the limit in the definition of LND in Nolan (1986) is independent of m), but the constant in (2.4), and hence in Lemma 2.2, depends on m when $p < 2$.

References

1. Adler, R.J. (1981) The Geometry of Random Fields, John Wiley and Sons, New York.

2. Cambanis, S. and Soltani, A.R. (1983) "Prediction of stable processes: spectral and moving average representations." Z. Wahrsch. verw. Geb. 66, 593-612.

3. Cuzick, J. (1978) "Some local properties of Gaussian vector fields." Ann. Prob. 6, 984-994.

4. Ehm, W. (1981) "Sample function properties of multiparameter stable processes." Z. Wahrsch. verw. Geb. 56, 195-228.

5. Geman, D. and Horowitz, J. (1980) "Occupation densities." Ann. Prob. 8, 1-67.

6. Marcus, M.B. and Pisier, G. (1984) "Characterization of almost surely continuous p-stable random Fourier series and strongly stationary processes." Acta. Math. 152, 245-301.

7. Marcus, M.B. and Pisier, G. (1984a) "Some results on the continuity of stable processes and the domain of attraction of continuous stable processes." Ann. Inst. Henri Poincaré 20(1984), 177-199.

8. Marcus, M. and Woyczynski, W. (1979) "Stable measures and central limit theorem in spaces of stable type." TAMS 251, 71-102.

9. Monrad, D. and Pitt, L.D. (1986) "Local nondeterminism and Hausdorff dimension." To appear in Seminar on Stochastic Processes.

10. Nolan, J.P. (1986) "Local nondeterminism and local times, I: stable processes" and "II: stable fields" to appear.

11. Pisier, G. (1983) "Some applications of the metric entropy condition to harmonic analysis." Banach Spaces, Harmonic Analysis and Probability, Proceedings 1980-81, Springer Lecture Notes 995, 123-154.

12. Pitt, L.D. (1978) "Local times for Gaussian vector fields." Indiana Univ. Math. J. 27, 309-330.

13. Rosinski, J. (1985) "On stochastic integral representation of stable processes with sample paths in Banach spaces." UNC Center for Stochastic Processes Tech. Rept. No. 88.

END

DATE

FILMED

DEC.

1987