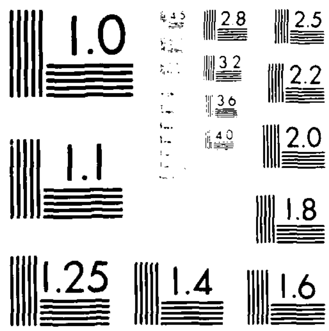
AD-A186 433 STRONG REPRESENTATION OF WEAK CONVERGENCE(U) NORTH CAROLINA UNIV AT CHAPEL HILL DEPT OF STATISTICS Z D BAI ET AL JUN 87 TR-186 AFOSR-TR-87-1354 1/1

UNCLASSIFIED F49620-85-C-0144 F/G 12/3 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A186 433

DTIC FILE COPY (2)

DTIC ELECTE OCT 0 2 1987 S D

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY: NA	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; Distribution Unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: NA	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 186	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 87-1354
6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina; 6b. OFFICE SYMBOL (If applicable):	7a. NAME OF MONITORING ORGANIZATION: AFOSR/NM
6c. ADDRESS (City, State and ZIP Code): Statistics Dept. Phillips Hall 039-A Chapel Hill, NC 27514	7b. ADDRESS (City, State and ZIP Code): Building 410 Bolling AFB, DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620 85 C 0144
8c. ADDRESS (City, State and ZIP Code): Building 410, Bolling AFB, DC 20332-6448	10. SOURCE OF FUNDING NOS.: PROGRAM ELEMENT NO. 6.1102F; PROJECT NO. 2304; TASK NO. A5; WORK UNIT NO.
11. TITLE (Include Security Classification): Strong Representation of Weak Convergence	

12. PERSONAL AUTHOR(S): Bai, Z.D., Liang, W.Q. and Vervaat, W.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
preprint	FROM 9/86 TO 9/87	June 1987	5

16. SUPPLEMENTARY NOTATION

17. COSATI CODES		
FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Keywords and phrases: Skorohod's representation theorem strong representation of weak convergence.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The following result is proved. If S_n is a separable metric space for $n\le\infty$, $\phi_n: S_n\to S_\infty$ is measurable for $n<\infty$, X_n is an S_n-valued random variable for $n\le\infty$ and $\phi_n(X_n)\to_d X_\infty$ in S_∞, then there exist S_n-valued random variables X_n^* such that $X_n =_d X_n^*$ for $n\le\infty$ and $\phi_n(X_n^*)\to X_\infty^*$ wp1. Conditions on S_n and ϕ_n are presented that allow a construction in the context of Polish spaces.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: ~~Peggy Ravitch~~ Maj Crowley	22b. TELEPHONE NUMBER (Include Area Code): ~~919-962-2307~~ 202 767 5025; 22c. OFFICE SYMBOL: AFOSR/NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE. UNCLASSIFIED

AFOSR-TR- 87-1354

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

STRONG REPRESENTATION OF WEAK CONVERGENCE

by

Z.D. Bai

W.Q. Liang

and

W. Vervaat

Technical Report No. 186

June 1987

STRONG REPRESENTATION OF WEAK CONVERGENCE

Z.D. BAI*

Center for Multivariate Analysis, 515 Thackeray Hall, University of Pittsburgh, Pittsburgh, PA 15260, USA

W.Q. LIANG*

Center for Multivariate Analysis, 515 Thackeray Hall, University of Pittsburgh, Pittsburgh, PA 15260, USA

Wim VERVAAT**

Department of Mathematics, Catholic University, Toernooiveld 5, 6525 ED Nijmegen, The Netherlands

The following result is proved. If S_n is a separable metric space for $n \leq \infty$, $\varphi_n: S_n \to S_\infty$ is measurable for $n < \infty$, X_n is an S_n-valued random variable for $n \leq \infty$ and $\varphi_n(X_n) \to_d X_\infty$ in S_∞, then there exist S_n-valued random variables X_n^* such that $X_n^* =_d X_n$ for $n \leq \infty$ and $\varphi_n(X_n^*) \to X_\infty^*$ wp1. Conditions on S_n and φ_n are presented that allow a construction in the context of Polish spaces.

AMS 1980 Subject Classification: Primary 60B10

Skorohod's representation theorem • strong representation of weak convergence

* This research has been supported by the Air Force Office of Scientific Research Grant No. F49620-85-C-0008.

** Research completed during a visit to the Center for Stochastic Processes, University of North Carolina and supported by the Air Force Office of Scientific Research Grant No. F49620-85-C-0144.

In this paper we prove the following variant of Skorohod's representation theorem for weak convergence. Equality in distribution is denoted by $=_d$, convergence in distribution by $\to_d$.

Theorem 1. *Let S_n for $n = 1, 2, \dots, \infty$ be a separable metric space and let φ_n for $n = 1, 2, \dots$ be a measurable function from S_n into S_∞. If X_n is an S_n-valued random variable for $n = 1, 2, \dots, \infty$ and $\varphi_n(X_n) \to_d X_\infty$ in S_∞, then there exist S_n-valued random variables X_n^* for $n = 1, 2, \dots, \infty$ defined on one probability space and such that $X_n^* =_d X_n$ in S_n for $n = 1, 2, \dots, \infty$ and $\varphi_n(X_n^*) \to X_\infty^*$ wp1 in S_∞.*

When $S_n = S$ (separable) and $q_n = \mathrm{id}_S$ for all n, then the above theorem specializes to Dudley's (1968) variant of Skorohod's representation theorem. In Skorohod's (1956) original version S was required to be complete as well. See Wichura (1970) and Blackwell & Dubins (1983) for further extensions. Our proof of the present theorem amounts to the construction of a special metric space T to which Dudley's theorem can be applied.

Theorem 1 turns out to be useful in many instances. It is applied in Bai (1984), Bai & Yin (1986) and Yin (1984). The need for these applications led the first two authors to the present research.

Here is the simplest example of a theorem that can be proved by Theorem 1, but not by the theorem of Skorohod-Dudley in its original form. It is Theorem 4.1 of Billingsley (1968), restricted to separable metric spaces.

Theorem 2. *If S is a separable metric space with metric ρ, (X_n, Y_n) are S^2-valued random variables for $n = 1, 2, \ldots$ and X is an S-valued random variable such that $X_n \to_d X$ in S and $\rho(X_n, Y_n) \to_d 0$ in* R, *then $Y_n \to_d X$ in S.*

Proof. By Billingsley (1968, Th.4.4) we have $(X_n, \rho(X_n, Y_n)) \to_d (X, 0)$ in $S \times \mathrm{R}$. Apply Theorem 1 with $S_\infty = S \times \mathrm{R}$, $S_n = S^2$, X_n replaced by (X_n, Y_n) and $q_n(x, y) = (x, \rho(x, y))$, all for $n < \infty$. □

Proof of Theorem 1. All statements involving n are supposed to hold for $n = 1, 2, \ldots, \infty$ unless restricted explicitly; limit statements without explicit tendency hold as $n \to \infty$. Let T be the disjoint union of all S_n. Let $s : T \to \{1, 2, \ldots, \infty\}$ be defined by $s(x) = n$ if $x \in S_n$. Set $q_\infty := \mathrm{id}_S$ and define $q : T \to S_\infty$ by $q(x) := q_{s(x)}(x)$. Let ρ_n be the metric of S_n. Let ε_n be positive for $n < \infty$, decreasing to 0 as $n \to \infty$, and set $\varepsilon_\infty := 0$. We now define what is going to be the metric on T:

$$\delta(x, y) := \rho_\infty(q(x), q(y)) + \begin{cases} \varepsilon_{s(x)} \wedge \rho_{s(x)}(x, y) & \text{if } s(x) = s(y), \\ \varepsilon_{s(x)} \vee \varepsilon_{s(y)} & \text{if } s(x) \neq s(y). \end{cases} \tag{1}$$

Let us first verify that δ is indeed a metric. Obviously, $\delta(x, y) = \delta(y, x)$ and $\delta(x, x) = 0$. If $\delta(x, y) = 0$, then $s(x) = s(y)$ and $\rho_{s(x)}(x, y) = 0$, so $x = y$. The triangle inequality can be verified separately for both terms on the right-hand side of (1). We note the following properties of δ-convergence:

(2) $\delta = \rho_\infty$ on $S_\infty \times S_\infty$.

(3) If $x, x_k \in S_n$, then $\delta(x, x_k) \to 0$ as $k \to \infty$ iff $\rho_n(x, x_k) \to 0$ and $\rho_\infty(q_n(x), q_n(x_k)) \to 0$ as $k \to \infty$. It follows that the ρ_n-topology in S_n is coarser than or equal to the trace in S_n of the δ-topology in T, which is homeomorphic via $S_n \ni x \mapsto (x, q_n(x))$ to the trace in the graph of q_n of the product topology in $S_n \times S_\infty$. The last topology is separable, and so is the trace of the δ-topology in each S_n. Consequently, δ is separable.

(4) S_n is δ-open for $n < \infty$.

(5) If $x_n \in S_n$ for each $n < \infty$, then $x_n \to x$ iff $x \in S_\infty$ and $q_n(x_n) \to x$ in S_∞.

Having established that T with δ is a separable metric space, we may apply Dudley's theorem to T-valued random variables. However, there is one more barrier to take. We want to identify S_n-valued random variables with T-valued random variables having range in S_n. In the first appearance random variables must be $\mathcal{S}_n$-measurable, where $\mathcal{S}_n$ is the Borel field in S_n generated by ρ_n. In the second appearance they must be $\mathcal{T}_n$-measurable, where $\mathcal{T}_n$ is the trace in S_n of $\mathcal{T}$, the Borel field in T generated by δ. So we must prove $\mathcal{S}_n = \mathcal{T}_n$.

From the second clause in (3) it follows that $\mathcal{S}_n \subset \mathcal{T}_n$. For the converse inclusion we must do a little more. First note that $\mathcal{T}$ is already generated by the open δ-balls in T, since δ is separable. This can be phrased equivalently by stating that $\mathcal{T}$ is the smallest σ-field in T which makes the functions $\delta(x,\cdot)$ measurable for all $x \in T$. Consequently, $\mathcal{T}_n$ is the smallest σ-field in S_n which makes the functions $\delta(x,\cdot)$, restricted to S_n, measurable for all $x \in T$. First suppose $n = \infty$. Then $\delta(x,y) = \rho_\infty(\varphi_\infty(x),y) + \varepsilon_{s(x)}$ for $x \in T$, $y \in S_\infty$, which, as a function of y, is obviously $\mathcal{S}_\infty$-measurable. So $\mathcal{T}_\infty \subset \mathcal{S}_\infty$. Considering (1) for $x \in T$ as a function of $y \in S_n$ we observe that $\rho_n(x,\cdot)$ is $\mathcal{S}_n$-measurable, and that $\rho_\infty(\varphi_\infty(x),\varphi_n(\cdot))$ is $\mathcal{S}_n$-measurable as composition of the $\mathcal{S}_n/\mathcal{S}_\infty$-measurable function φ_n and the $\mathcal{S}_\infty$-measurable function $\rho_\infty(\varphi_\infty(x),\cdot)$.

We now write down the scheme of implications that proves the theorem. We are given S_n-valued random variables X_n such that

(6) $\quad \varphi_n(X_n) \to_d X_\infty \quad$ in S_∞.

The major point, to be proved below, is that this implies

(7) $\quad X_n \to_d X_\infty \quad$ in T.

By Dudley's theorem there are T-valued random variables Y_n, defined on one probability space, such that $Y_n =_d X_n$ in T and $Y_n \to Y_\infty$ wp1 in T. By (4) we have $S_n \in \mathcal{T}$ for each n, so there is a measurable function $j_n: T \to S_n$ such that the restriction of j_n to S_n is the identity map on S_n: take j_n to be the identity map on S_n and constant on $T \setminus S_n$. Set $X_n^* := j_n Y_n$. Then X_n^* has range in S_n, and $X_n^* = Y_n$ wp1 in T, since $\mathsf{P}[Y_n \in S_n] = \mathsf{P}[X_n \in S_n] = 1$. From $X_n^* = Y_n$ wp1 and $Y_n =_d X_n$ in T it follows that $X_n^* =_d X_n$ in T. As X_n^* has range in S_n, this implies $X_n^* =_d X_n$ in S_n. From $X_n^* = Y_n$ wp1 in T and $Y_n \to Y_\infty$ wp1 in T we obtain $X_n^* \to X_\infty^*$ wp1 in T. As X_n^* has range in S_n, this implies by (5)

$$\varphi_n(X_n^*) \to X_\infty^* \text{ wp1} \quad \text{in } S_\infty.$$

We have arrived at all conclusions of the theorem.

It remains to prove the implication (6) $\Rightarrow$ (7). We will interpret (6) and (7) by convergence of probability distributions on continuity sets, so we must compare the boundaries under ρ_n and δ. By (2) we have for $A \subset T$:

(8) $\quad \partial_{\rho_n}(A \cap S_n) = \partial_{\delta|S_n}(A \cap S_n) \subset \partial_\delta(A)$.

Let $B(x,\varepsilon) := \{y \in T : \delta(x,y) < \varepsilon\}$ and set

(9) $\quad \mathcal{V} := \{B(x,\varepsilon) : \varepsilon < \varepsilon_{s(x)} \text{ if } s(x) < \infty,\ \mathsf{P}[X_\infty \in \partial_\delta B(x,\varepsilon)] = 0\}$,

and let $\mathcal{U}$ consist of the unions of finitely many elements of $\mathcal{V}$. By Billingsley (1968, Corollary 2 on p 15) it is sufficient for (7) that

(10) $\quad \mathsf{P}[X_n \in A] \to \mathsf{P}[X_\infty \in A] \quad$ for $A \in \mathcal{U}$.

If $B(x,\varepsilon) \in \mathcal{V}$ with $s(x) < \infty$, then $B(x,\varepsilon) \subset S_{s(x)}$, so $\mathsf{P}[X_n \in B(x,\varepsilon)] = 0$ unless $n = s(x)$. If $B(x,\varepsilon) \in \mathcal{V}$ with $x \in S_\infty$, then

$$[X_n \in B(x,\varepsilon)] = [\varrho_\infty(x, \varphi_n(X_n)) < \varepsilon - \varepsilon_n] = [\varphi_n(X_n) \in B(x, \varepsilon - \varepsilon_n) \cap S_\infty],$$

so

$$(11) \qquad [\varphi_n(X_n) \in B(x,\varepsilon) \cap S_\infty] \subset \liminf [X_n \in B(x,\varepsilon)] \subset \limsup [X_n \in B(x,\varepsilon)]$$
$$\subset [\varphi_n(X_n) \in \overline{B(x,\varepsilon) \cap S_\infty}].$$

By (6), (8) and (9) the outmost sides of (11) have equal probabilities. Combining the previous observations for separate $B(x,\varepsilon) \in \mathcal{V}$ we arrive at (11) with $A \in \mathcal{U}$ instead of $B(x,\varepsilon) \in \mathcal{V}$, again with equal probabilities for the outmost sides. This proves (10), hence (7). The proof of the theorem is complete.

Remarks. In general the space T is not complete under δ, even if all S_n are under ϱ_n. To see this, consider the case that all x_n $(n < \infty)$ lie in S_m for one fixed m. Then (x_n) is δ-Cauchy iff $((x_n, \varphi_m(x_n)))_{n<\infty}$ is $\varrho_m \times \varrho_\infty$-Cauchy. If the latter holds, then $((x_n, \varphi_m(x_n)))$ converges in $S_m \times S_\infty$, but not necessarily in graph φ_m, unless the latter is closed. This combined with the observation that δ-Cauchy sequences $(x_n)_{n<\infty}$ with $x_n \in S_n$ converge if S_∞ is ϱ_∞-complete leads us to the following result.

Theorem 3. *Let S_n be separable and ϱ_n-complete for each n. Then T is δ-complete iff* graph φ_n *is closed in $S_n \times S_\infty$ for each n.*

It is well-known that graph φ_n is closed if φ_n is continuous, and that φ_n is continuous if graph φ_n is closed and S_∞ is compact. Using the fact that a subset of a Polish space is Polish iff it is G_δ (Dugundji (1966, Th.XIV.8.3)), we arrive at the following variation on Theorem 3.

Theorem 4. *Let S_n be Polish for each n. Then T is Polish iff* graph φ_n *is G_δ in $S_n \times S_\infty$ for each n.*

For results on real functions with G_δ graphs, see van Rooij & Schikhof (1982, Exerc. 11.Y,Z). Functions of the first class of Baire (pointwise limits of continuous functions) have G_δ graphs. F_σ graphs are also G_δ.

Acknowledgment. Thomas M. Liggett has contributed to our discussions about this proof.

References

[1] Bai, Z.D. (1984). A note on the joint distribution of the eigenvalues of noncentral multivariate F matrix. Center for Multivariate Analysis, University of Pittsburgh. Technical Report **84-49**.

[2] Bai, Z.D., & Y.Q. Yin (1986). Limiting behavior of the norm of products of random matrices and two problems of Geman and Hwang. *Probab. Theory Rel. Fields* **73** 555-570.

[3] Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley.

[4] Blackwell, D. & L.E. Dubins (1983). An extension of Skorohod's almost sure representation theorem. *Proc. Amer. Math. Soc.* **89** 691-692.

[5] Dudley, R.M. (1968). Distances of probability measures and random variables. *Ann. Math. Statist.* **39** 1563-1572.

[6] Dugundji, J. (1966). *Topology*. Allyn and Bacon.

[7] Skorohod, A.V. (1956). Limit theorems for stochastic processes. *Theory Probab. Appl.* **1.3** 261-290.

[8] van Rooij, A.C.M., & W.H. Schikhof (1982). *A Second Course on Real Functions*. Cambridge University Press.

[9] Wichura, M.J. (1970). On the construction of almost uniformly convergent random variables with given weakly convergent image laws. *Ann. Math. Statist.* **41** 284-291.

[10] Yin, Y.Q. (1984). Limiting sepetral distributions for a class of random matrices. Center for Multivariate Analysis, University of Pittsburgh. Technical Report **84-50**.

END

DATE

FILMED

DEC.

1987