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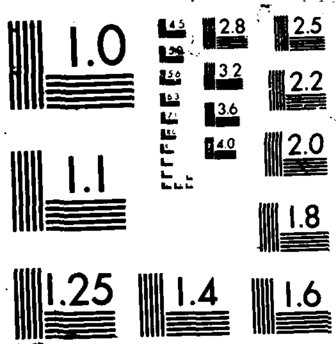
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A. HEDAYAT AND J. STUFKEN
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ABSTRACT

In this paper we show that Bush's bound for the maximum number of constraints in an orthogonal array of index unity is uniformly better than Rao's bound. In addition it is shown, using an argument similar to that needed in the proof of the above result, that Noda's characterization of parameters in orthogonal arrays of strength 4 achieving equality in Rao's bound, leads easily to a similar characterization in arrays of strength 5. [1]

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1. Introduction. Orthogonal arrays were introduced by Rao (1947a) because of their statistical properties when used as fractional factorial designs. See also Rao (1946, 1947b). Rao's papers have been a source of inspiration for many other researchers, who continued to study these arrays and various generalizations. It is mainly due to the work by Taguchi and his colleagues that orthogonal arrays have gained a renewed interest in industrial experimentation for product improvement during recent years, also among researchers in this country.

For the sake of completeness we start with a formal definition of an orthogonal array.

Definition. A $k \times N$ array with entries from $S = \{0, 1, \ldots, s-1\}$ is called an orthogonal array of strength t and index λ if in any t rows of the array each (ordered) t-tuple based on S appears λ times.

We will denote such an array by $OA(N, k, s, t)$. Observe that $N = \lambda s^t$. Often N is called the number of runs, k the number of constraints, and s the number of levels.

One of the more interesting problems, both from a mathematical and statistical point of view, is the following. For given values of N, s and t, what is the maximum number of constraints that can be accomodated in an orthogonal array? Various results on upper bounds for this number are known. Some are for very general settings, others for very specific ones. For some it is known that equality can be achieved, for others this is an open problem. In this paper we will compare two such upper bounds, one by Rao (1947a), the other by Bush (1952). Rao's result may be stated as follows.

Theorem 1. *In an $OA(\lambda s^t, k, s, t)$ the following inequalities hold.*
If $t = 2u$,

$$\lambda s^t \geq 1 + \binom{k}{1}(s-1) + \binom{k}{2}(s-1)^2 + \ldots + \binom{k}{u}(s-1)^u.$$

If $t = 2u + 1$,

$$\lambda s^t \geq 1 + \binom{k}{1}(s-1) + \binom{k}{2}(s-1)^2 + \ldots + \binom{k}{u}(s-1)^u + \binom{k-1}{u}(s-1)^{u+1}.$$

The upper bound for k, implicitly given by these inequalities, is the most general result on the maximum number of constraints in the literature. Improvements have been obtained for special cases of the parameters N, s and t. Bush (1952) considered the special, but still important case of $N = s^t$, arrays of index unity. He proved the following result.

Theorem 2. *In an $OA(s^t, k, s, t)$ the following inequalities hold:*

$$\begin{cases} k \leq t+1 & \text{if } s \leq t, \\ k \leq s+t-1 & \text{if } s > t \text{ and } s \text{ is even , or if } t = 2, \\ k \leq s+t-2 & \text{if } s > t \geq 3 \text{ and } s \text{ is odd.} \end{cases}$$

Improvements on this result for special cases have been obtained, for example by Kounias and Petros (1975). Other improvements result from studying the case $t = 2$, a problem that leads its own life. We refer the reader to Dénes and Keedwell (1974) for more on this.

Numerical examples seem to indicate that for orthogonal arrays of index unity the bound for the maximum number of constraints from Theorem 2 is sharper than that from Theorem 1 (Bush (1952)). Personal communications (Bush (1986) and Raghavarao (1986)) made the authors aware that there is apparently no analytical proof of this observation. The main objective of this paper is to provide such a proof, and use the technique in the proof to extend a result by Noda (1979).

2. Main results. The following result provides a comparison of the bounds in Theorems 1 and 2.

Theorem 3. *For orthogonal arrays of index unity, Bush's bound on the maximum number of constraints is uniformly better than Rao's bound.*

Proof. It is well known and easy to show that if $s \leq t$ the maximum number of constraints equals $t+1$. So we can restrict ourselves to the case $s > t$. To prove the theorem it suffices to show that if we use $k = s+t-1$ in the inequalities of Theorem 1, we obtain valid inequalities. We distinguish between $t = 2u$ and $t = 2u+1$.

Case 1. $t = 2u$. We like to show that $s^{2u} \geq 1 + \binom{s+2u-1}{1}(s-1) + \ldots + \binom{s+2u-1}{u}(s-1)^u$. First notice that

$$\binom{u}{i}(s+1)^i \geq \binom{s+2u-1}{i}, \quad i = 0, 1, \ldots, u.$$

This is obvious for $i = 0$. For $1 \le i \le u-1$ it follows if we can show that

$$(u-j)(s+1) \ge s + 2u - j - 1, \quad j = 0, \dots, u-2,$$

or

$$(u-j-1)s \ge u-1, \quad j = 0, \dots, u-2.$$

This is true since $j \le u-2$ and $s > 2u$. Finally, for $i = u$ the above inequality follows since

$$u!(s+1)^u = u(s+1).(u-1)(s+1)\dots 2(s+1).(s+1) \ge$$

$$u(s+1).(s+2u-2)\dots(s+u+1)(s+1) \ge$$

$$(s+2u-1)(s+2u-2)\dots(s+u+1)(s+u).$$

The proof for Case 1 is now completed as follows.

$$s^{2u} = (s^2-1+1)^u = \sum_{i=0}^{u} \binom{u}{i}(s^2-1)^i = \sum_{i=0}^{u} \binom{u}{i}(s+1)^i(s-1)^i$$

$$\ge \sum_{i=0}^{u} \binom{s+2u-1}{i}(s-1)^i.$$

Case 2. $t = 2u+1$. Here we like to show that

$$s^{2u+1} \ge 1 + \binom{s+2u}{1}(s-1) + \dots + \binom{s+2u}{u}(s-1)^u + \binom{s+2u-1}{u}(s-1)^{u+1}.$$

As in Case 1 we have that $(s+1)^u \ge \binom{s+2u-1}{u}$. Observe further that

$$\binom{u}{i-1}(s+1)^{i-1} + \binom{u}{i}(s+1)^i \ge \binom{s+2u-1}{i-1} + \binom{s+2u-1}{i} = \binom{s+2u}{i}.$$

Hence,

$$s^{2u+1} = s\sum_{i=0}^{u} \binom{u}{i}(s+1)^i(s-1)^i$$

$$= \sum_{i=0}^{u} \binom{u}{i}(s+1)^i(s-1)^{i+1} + \sum_{i=0}^{u} \binom{u}{i}(s+1)^i(s-1)^i$$

$$= 1 + \sum_{i=1}^{u}\left[\binom{u}{i-1}(s+1)^{i-1} + \binom{u}{i}(s+1)^i\right](s-1)^i + (s+1)^u(s-1)^{u+1}$$

$$\ge 1 + \sum_{i=1}^{u} \binom{s+2u}{i}(s-1)^i + \binom{s+2u-1}{u}(s-1)^{u+1}.$$

This concludes the proof of Case 2 and establishes the result. □

Similar as in the proof for Case 2 it can be shown that

$$s\left[1+\binom{k}{1}(s-1)+\ldots+\binom{k}{u}(s-1)^u\right]=\sum_{i=0}^{u}\binom{k+1}{i}(s-1)^i+\binom{k}{u}(s-1)^{u+1}.$$

Hence

$$\lambda s^{2u}\geq\sum_{i=0}^{u}\binom{k}{i}(s-1)^i$$

if and only if

$$\lambda s^{2u+1}\geq\sum_{i=0}^{u}\binom{k+1}{i}(s-1)^i+\binom{k}{u}(s-1)^{u+1}.$$

Equality can only occur simultaneously. Together with the simple fact that the existence of an $OA(N,k,s,t)$ implies the existence of an $OA(N/s,k-1,s,t-1)$, we are now ready to prove the following result.

Theorem 4. *The only values of N,k and s for which an $OA(N,k,s,5)$ with equality in Rao's inequality can exist, are:*

(i) $(N,k,s)=(32,6,2)$, or

(ii) $(N,k,s)=(729,12,3)$, or

(iii) $(N,k,s)=(27a^2(9a^2-1),\frac{3}{5}(3a^2+2),6)$, *where* $a\equiv 0(\bmod 3)$, $a\equiv\pm 1(\bmod 5)$ *and* $a\equiv 5(\bmod 16)$.

Proof. An $OA(N,k,s,5)$ implies the existence of an $OA(N/s,k-1,s,4)$. By our previous observation, equality in Rao's inequality for the first array occurs if and only if it ocurs for the second. The result follows now by the characterization of those cases where equality is possible for arrays of strength 4 by Noda (1979). □

The arrays corresponding to (i) and (ii) in Theorem 4 are known to exist, see for example Gulati (1971). Finally we point out that an argument as in the proof of Theorem 4 fails for the case $t=6$.

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