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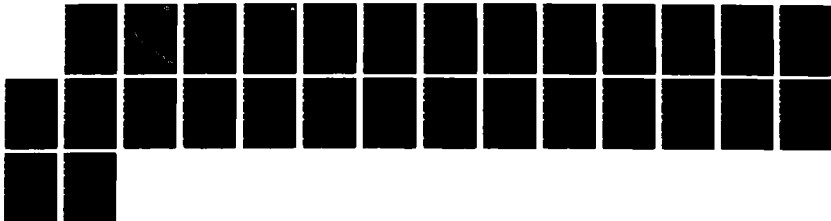
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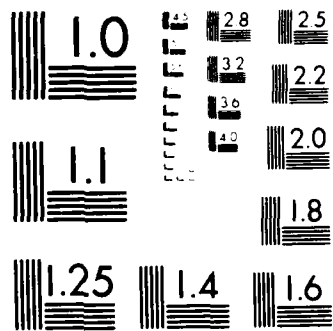
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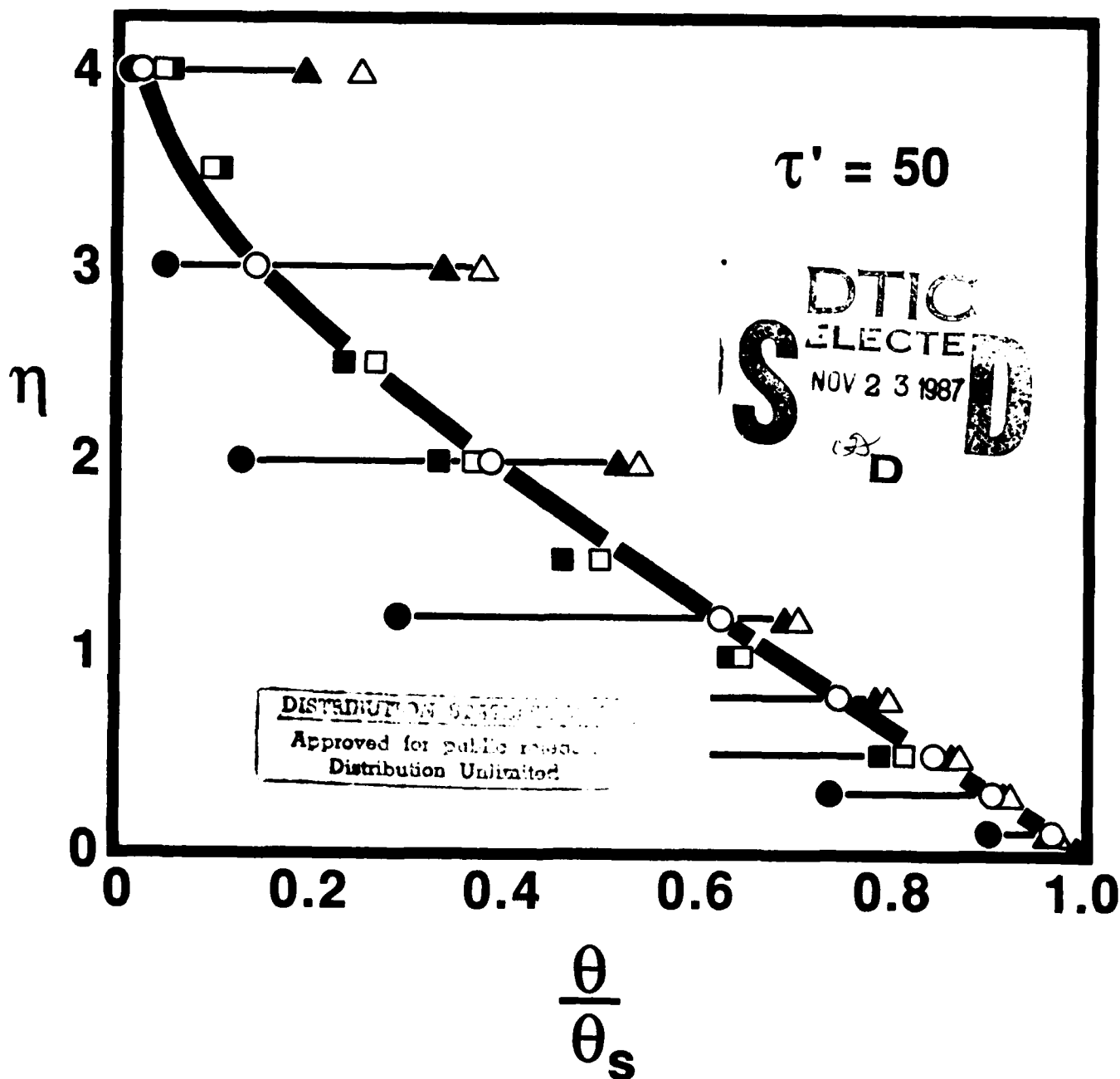
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## Approximate solutions of heat conduction in a medium with variable properties



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# CRREL Report 87-19

September 1987

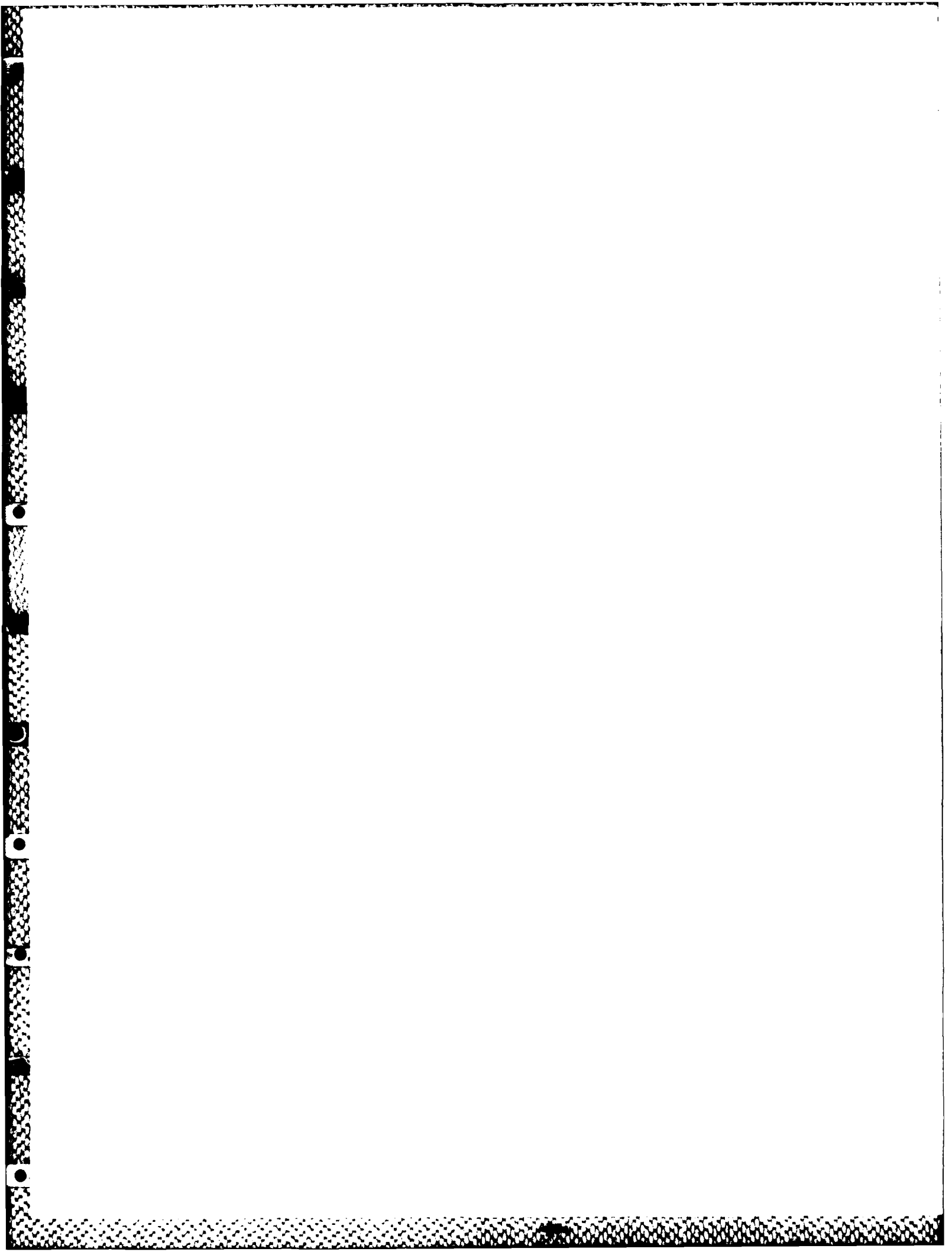


## *Approximate solutions of heat conduction in a medium with variable properties*

Yin-Chao Yen

Prepared for  
OFFICE OF THE CHIEF OF ENGINEERS

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## NOMENCLATURE

$c_p$	heat capacity
$E_i$	exponential integral
$F(y)$	defined by eq 4
$H$	dimensionless heat flux defined by eq 23
$k$	thermal conductivity
$\ell$	$1/\beta$ , characteristic length
$q$	heat flux
$R$	defined by eq 33
$t$	time
$T$	temperature
$V$	a variable defined by eq 10
$x$	coordinate in direction of heat flow
$y$	defined by eq 2
$\rho$	density
$\alpha$	thermal diffusivity
$\theta$	dimensionless temperature defined as $T/T_r$
$\eta$	defined as $y/\ell$
$\tau$	dimensionless time defined as $\alpha_0 t/\ell^2$
$\tau'$	dimensionless time defined by eq 32
$\delta$	thermal boundary thickness
$\epsilon$	dimensionless thermal boundary thickness defined as $\delta/\ell$
$\beta$	coefficient in the expression $k(x) = k_0(1 + \beta x)$
$\gamma$	dimensionless distance defined as $x/\ell$ or $x\beta$ ; or Euler's constant

### *Subscripts*

0,r	reference state
s	surface

# Approximate Solutions of Heat Conduction in a Medium with Variable Properties

YIN-CHAO YEN

## INTRODUCTION

A number of books deal with heat conduction in various geometries, in one or multiple dimensions, and with constant or variable properties that are either a function of temperature or position. One classic work that deals exclusively with conduction is Carslaw and Jaeger (1959), in which all the problems are solved analytically. But even in these simple problems, the solutions are often expressed in terms of integrals and series, so that considerable computational effort is needed to obtain the numerical results.

The work reported here represents an effort in obtaining a simple but reasonably accurate method for the prediction of temperature distribution in a medium with variable properties. The approximate heat balance integral method (HBIM) introduced by Goodman (1958, 1964), which has shown its applicability to a number of heat conduction problems, is extended to the present case of a medium with variable properties. The concept of the integral method is based on the same principles involved in the momentum integral method of the boundary layer in fluid mechanics initially proposed by Pohlhausen (1921) and von Karman (1921). The essence of this method is that at any time  $t$  the thermal boundary layer will be limited to a boundary layer thickness  $\delta(t)$ . Beyond this depth, the temperature will remain at its initial temperature and there will be no heat transfer beyond this point. This solution method is analogous to the momentum integral method in that the basic equations are satisfied on the average over the volume of thickness  $\delta(t)$  rather than at each point.

## MATHEMATICAL ANALYSIS

### Constant surface temperature

Consider the case of a semi-infinite solid medium initially at uniform temperature (let it be zero for convenience in the mathematical treatment) with its physical properties given as a function of position. For  $t > 0$ , the surface is maintained at a temperature of  $T_s$ , and the unidirectional heat conduction equation is given as

$$\frac{\partial}{\partial x} \left[ k(x) \frac{\partial T}{\partial x} \right] = c_p(x) \rho(x) \frac{\partial T}{\partial t} \quad (1)$$

with initial and boundary conditions as

$$T = 0, \quad \text{for } x > 0 \quad \text{and} \quad t \leq 0$$

$$T = T_s, \quad \text{for } x = 0 \quad \text{and} \quad t \geq 0$$

where  $x$  = coordinate in the direction of heat flow

$k$  = thermal conductivity

$c_p$  = heat capacity

$\rho$  = density

$t$  = time.

A new dependent variable  $y$  is defined as

$$y = \int_0^x \frac{k_0}{k(x')} dx' \quad (2)$$

where  $x'$  is the dummy variable and  $k_0$  is the thermal conductivity at some arbitrary reference point (it is chosen in such a manner that the subsequent mathematical treatment will be simplified). By substituting eq 2 into eq 1 and simplifying, eq 1 becomes

$$\frac{\partial^2 T}{\partial y^2} = \frac{F(y)}{\alpha_0} \frac{\partial T}{\partial t} \quad (3)$$

where

$$F(y) = \frac{k(x)c_p(x)\rho(x)}{k_0 c_{p0} \rho_0} \quad (4)$$

and

$$\alpha_0 = \frac{k_0}{c_{p0} \rho_0} \quad (5)$$

By rewriting eq 3 in dimensionless form, this becomes

$$\frac{\partial^2 \theta}{\partial \eta^2} = F(\eta) \frac{\partial \theta}{\partial \tau} \quad (6)$$

where

$$\theta = \frac{T}{T_r} \quad (7)$$

$$\eta = \frac{y}{\ell} \quad (8)$$

$$\tau = \frac{\alpha_0 t}{\ell^2} \quad (9)$$

$T_r$  and  $\ell$  are the reference temperature and the characteristic length of the medium under study, respectively. A new dependent variable  $V$  is introduced and defined by

$$V = \int_0^y F(y) \theta(y) dy \quad (10)$$

By using eq 10 to evaluate the derivatives of  $\theta$  with  $\eta$  and  $\tau$ , eq 6 can be expressed in terms of  $V$  by

$$\frac{\partial}{\partial \eta} \left\{ \frac{1}{F(\eta)} \left[ \frac{\partial^2 V}{\partial \eta^2} - \frac{F'(\eta)}{F(\eta)} \frac{\partial V}{\partial \eta} \right] \right\} = \frac{\partial^2 V}{\partial \tau \partial \eta} \quad (11)$$

where  $F' = dF/d\eta$ . The corresponding initial and boundary conditions are

$$V = 0, \quad \eta = 0$$

$$V = 0, \quad \text{for } \eta \leq 0, \quad \text{and } \tau \leq 0$$

$$\frac{\partial V}{\partial \eta} = F(0)\theta_s, \quad \text{for } \eta = 0 \quad \text{and } \tau \geq 0$$

where  $\theta_s = T_s/T_r$ . Integrating eq 11 from 0 to  $\epsilon$  (which is defined as  $\delta/l$  where  $\delta$  is thermal boundary thickness), it becomes

$$\frac{1}{F(\epsilon)} \left[ \frac{\partial^2 V}{\partial \eta^2} \Big|_{\epsilon} - \frac{F'(\epsilon)}{F(\epsilon)} \frac{\partial V}{\partial \eta} \Big|_{\epsilon} \right] - \frac{1}{F(0)} \left[ \frac{\partial^2 V}{\partial \eta^2} \Big|_0 - \frac{F'(0)}{F(0)} \frac{\partial V}{\partial \eta} \Big|_0 \right] = \frac{d}{d\tau} [V(\epsilon) - V(0)]. \quad (12)$$

Since the effect of surface temperature at any given time is confined to  $\eta < \epsilon$ , it is expected that the following conditions will be satisfied to smooth the temperature profile:

$$\theta = 0, \quad \eta = \epsilon \quad (13a)$$

$$\frac{\partial \theta}{\partial \eta} = 0, \quad \eta = \epsilon \quad (13b)$$

$$\frac{\partial^2 \theta}{\partial \eta^2} = 0, \quad \eta = \epsilon. \quad (13c)$$

The corresponding requirement on  $V$  can be found as

$$V = 0, \quad \eta = 0 \quad (14a)$$

$$\frac{\partial V}{\partial \eta} = F(0)\theta_s, \quad \eta = 0 \quad (14b)$$

$$\frac{\partial V}{\partial \eta} = 0, \quad \eta = \epsilon \quad (14c)$$

$$\frac{\partial^2 V}{\partial \eta^2} = 0, \quad \eta = \epsilon. \quad (14d)$$

$V$  can be represented as a cubic polynomial of  $\eta$ :

$$V = a + b\eta + c\eta^2 + d\eta^3. \quad (15)$$

With the application of eq 14a-d, the coefficients  $a$ ,  $b$ ,  $c$  and  $d$  were found as

$$a = 0 \quad (16a)$$

$$b = F(0)\theta, \quad (16b)$$

$$c = -\frac{1}{\epsilon} F(0)\theta, \quad (16c)$$

$$d = \frac{1}{3\epsilon^2} F(0)\theta, \quad (16d)$$

Substituting eq 16a-d into eq 15, it follows that

$$V = F(0)\theta \left[ \eta - \frac{\eta}{\epsilon} + \frac{\eta^3}{3\epsilon^2} \right]. \quad (17)$$

With eq 17 and 14d, eq 12 can be integrated into

$$\frac{F(0)}{F'(0)} \epsilon - 2 \left[ \frac{F(0)}{F'(0)} \right]^2 \ln \left[ \frac{2 + \frac{F'(0)}{F(0)} \epsilon}{2} \right] = \frac{3}{F(0)} \tau. \quad (18)$$

If we take the reference properties of  $k$ ,  $c_p$  and  $\rho$  to be those at the surface, then  $F(0) = 1$ , and eq 18 can be further simplified to

$$\epsilon - \frac{2}{F'(0)} \ln \left[ \frac{2 + F'(0)\epsilon}{2} \right] = 3F'(0)\tau. \quad (19)$$

The computed values of  $\epsilon$  as a function of  $\tau$ , with  $F'(0)$  ranging from 0.01 to 2.0, are shown in Table 1 and in Figure 1 for selected values of  $F'(0)$ . From eq 10 we have

$$\frac{\partial V}{\partial \eta} = F(\eta) \theta(\eta). \quad (20)$$

Subsequently from eq 17 and 20, the temperature profile is

$$\theta = \frac{F(0)\theta_s}{F(\eta)} \left( 1 - \frac{2\eta}{\epsilon} + \frac{\eta^2}{\epsilon^2} \right). \quad (21)$$

#### Constant surface heat flux

The same analysis as in the case of constant surface temperature can be applied to the case of constant surface heat flux. The corresponding boundary conditions on  $V$  become

$$\frac{\partial V}{\partial \eta} = F'(0) \frac{\partial V}{\partial \eta} = H, \quad \eta = 0 \quad (22)$$

where  $H$  is the dimensionless heat flux defined as

$$H = \frac{q''}{k_s T_s} \quad (23)$$

in which  $q''$  is the heat flux. The temperature distribution is given as

$$\theta = \frac{H}{F(\eta)[2 + F'(0)\epsilon]} \left[ -\epsilon + 2\eta - \frac{\eta^2}{\epsilon} \right] \quad (24)$$

**Table 1. Computed values of dimensionless boundary layer thickness  $\epsilon$  as functions of dimensionless time  $\tau$  and  $F'(0)$  for a constant surface temperature and  $F(0) = 1.0$ .**

$\tau$	$F'(0)$						
	0.01	0.05	0.10	0.20	0.5	1.0	2.0
0	0	0	0	0	0	0	0
0.01	0.348	0.348	0.348	0.351	0.357	0.367	0.388
0.05	0.775	0.779	0.785	0.795	0.825	0.878	0.987
0.10	1.102	1.105	1.115	1.136	1.198	1.304	1.528
0.50	2.460	2.500	2.551	2.655	2.973	3.537	4.750
1.00	3.479	3.565	3.667	3.875	4.529	5.695	8.222
2.00	4.940	5.095	5.310	5.731	7.075	9.500	14.758
4.00	6.760	7.330	7.752	8.613	11.390	16.445	27.348
6.00	8.605	9.095	9.726	11.038	15.295	23.057	39.710
8.00	9.960	10.620	11.460	13.228	18.995	29.515	51.975
10.00	11.150	11.980	13.040	15.273	22.585	35.885	64.188
14.00	13.250	14.420	15.901	19.073	29.510	46.375	88.500
20.00	15.900	17.550	19.730	24.350	29.560	67.090	124.850

in which  $\epsilon$  is given by the transcendental expression

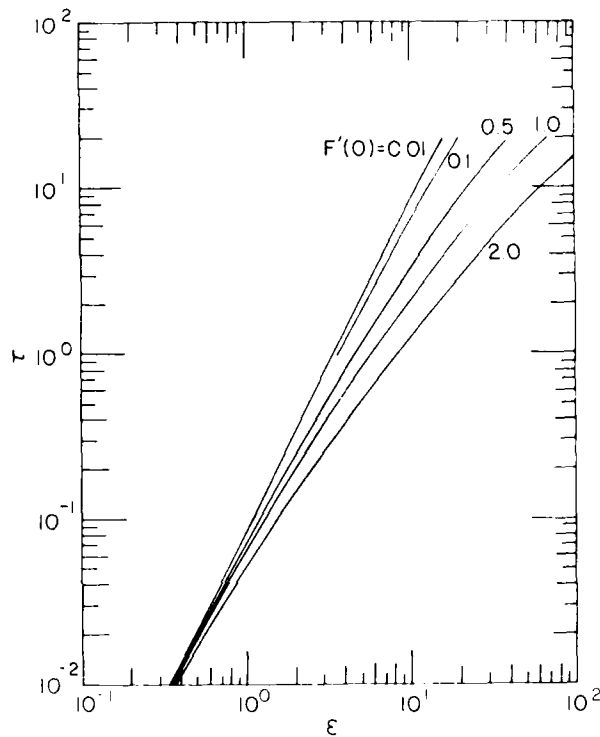
$$\epsilon - \frac{4}{F'(0)} \left[ \frac{1}{2} - \frac{1}{2 + F'(0)\epsilon} \right] = 3F'(0)\tau. \quad (25)$$

The computed values of  $\epsilon$  as function of  $\tau$  are shown in Table 2. Similarly a plot of  $\tau$  vs  $\epsilon$  is shown in Figure 2 for some selected values of  $F'(0)$ . Appendix A shows the derivation of eq 25.

#### Comparison with exact solution

For the case of constant surface temperature ( $T_s \neq 0$ ) that was initially at zero temperature, the exact solution for the temperature distribution in a semi-infinite solid, with  $k(x) = k_0(1 + \beta x)$  but with  $\rho$  and  $c_p$  being constant, has been reported by Carslaw and Jaeger (1959). The numerical solution reported by Jaeger (1956) is compared with those obtained by this approximation method for a linear dependency of  $k(x)$  on  $x$ . The independent variable  $y$  becomes

$$y = \int_0^x \frac{dx}{(1 + \beta x)} = \frac{1}{\beta} \ln(1 + \beta x). \quad (26)$$



*Figure 1.  $\tau$  vs  $\epsilon$  for constant surface temperature,  $F(0) = 1$ .*

**Table 2. Computed values of dimensionless boundary layer thickness  $\epsilon$  as functions of dimensionless time  $\tau$  and  $F'(0)$  for a constant surface heat flux and  $F(0) = 1.0$ .**

$\tau$	$F'(0)$						
	0.01	0.05	0.10	0.20	0.05	1.0	2.0
0	0	0	0	0	0	0	0
0.01	0.246	0.246	0.246	0.248	0.253	0.261	0.277
0.05	0.549	0.552	0.555	0.563	0.587	0.628	0.718
0.10	0.775	0.782	0.790	0.806	0.853	0.939	1.131
0.50	1.740	1.770	1.809	1.890	2.148	2.638	3.792
1.00	2.465	2.526	2.607	2.768	3.313	4.376	6.875
2.00	3.495	3.618	3.778	4.116	5.275	7.582	12.930
4.00	4.959	5.209	5.554	6.245	8.745	13.745	24.965
6.00	6.092	6.468	6.968	8.066	12.005	19.815	36.980
8.00	7.198	7.555	8.233	9.732	15.170	25.860	48.980
10.00	7.898	8.534	9.390	11.318	18.285	31.890	60.990
14.00	9.379	10.276	11.505	14.283	24.430	41.915	84.980
20.00	11.259	12.560	14.358	18.485	33.581	61.960	120.990

If the characteristic length is  $\beta^{-1}$ , then

$$\eta = y\beta = \ln(1 + \gamma) \quad (27)$$

or

$$1 + \gamma = e^\eta \quad (28)$$

where  $\gamma = x\beta$  and the function  $F(\eta)$  becomes

$$F(\eta) = (1 + \beta x) = (1 + \gamma) = e^\eta. \quad (29)$$

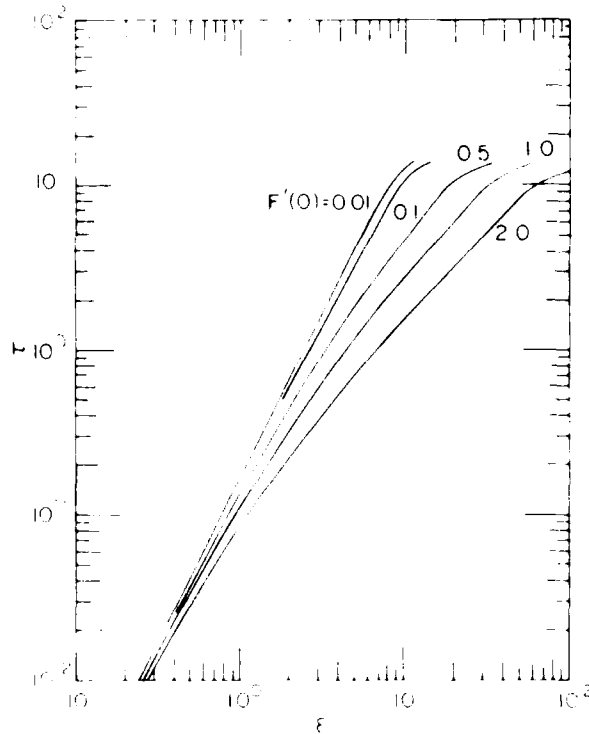
For the case of  $k(x) = k_0(1 + \beta x)$ , it then follows that  $F(0) = 1$ ,  $F'(0) = 1$  and  $F'(\eta) = e^\eta$ , and eq 21 reduces to

$$\theta = \frac{\theta_s}{e^\eta} \left( 1 - \frac{2\eta}{\epsilon} + \frac{\eta^2}{\epsilon^2} \right) \quad (30)$$

for the case of constant surface temperature. Similarly for constant surface heat flux, eq 24 becomes

$$\theta = \frac{H}{e^\eta(2 + \epsilon)} \left( -\epsilon + 2\eta \frac{\eta^2}{\epsilon} \right). \quad (31)$$

The exact solution, in this case, is obtained by extending the solution given by Jaeger (1956) with the following substitutions. The temperature at  $x$  and  $t$  is given by



**Figure 2.  $\tau$  vs  $\epsilon$  for constant surface heat flux,  $F(0) = 1$ .**

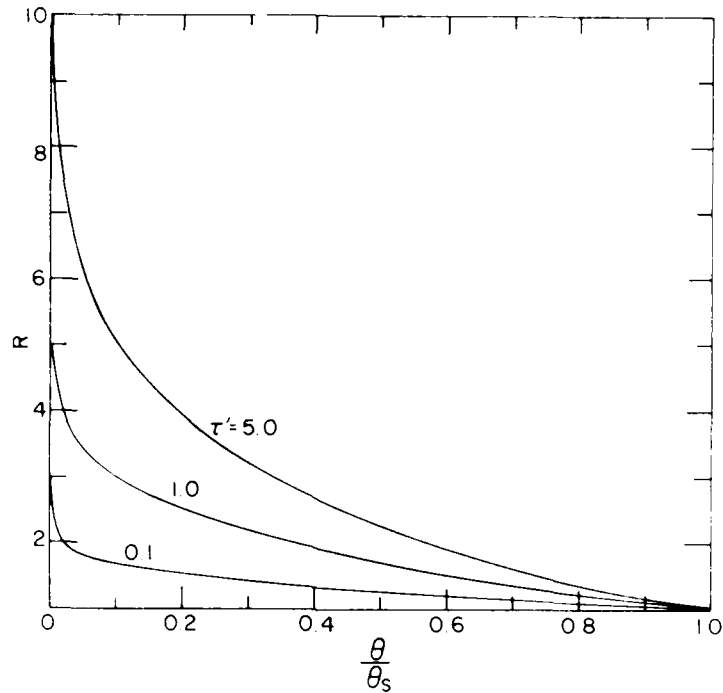


Figure 3.  $R$  vs  $\theta/\theta_s$  for  $\tau = 0.4, 4.0$  and  $20.0$  ( $\tau' = 0.1, 1.0$  and  $5.0$ ).

$$R = (1 + \beta x)^{1/2} \quad \text{and} \quad \tau' = \frac{\beta^2 t k_0}{4 \rho c_p} \quad (32)$$

in which  $R = r/a$  where  $a$  is the radius of the cylinder.  $R$  can be rewritten as

$$R = (1 + \gamma)^{1/2}. \quad (33)$$

But from eq 27

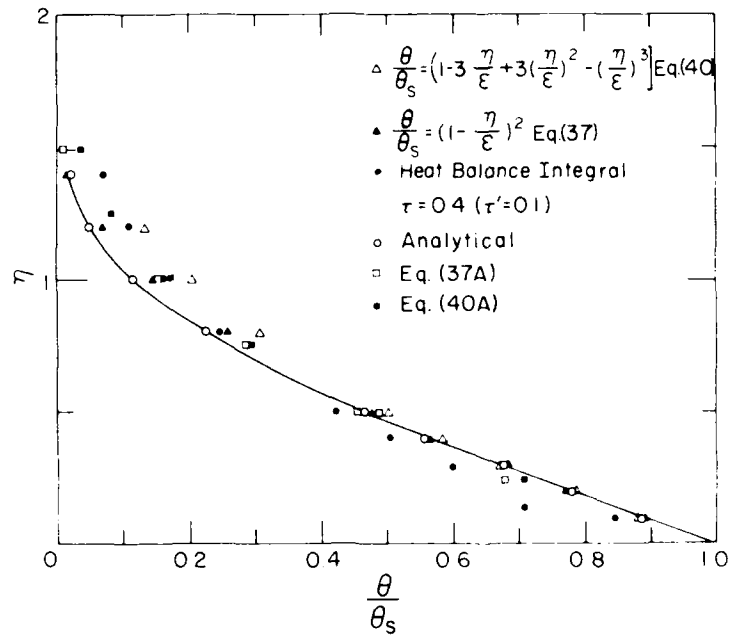
$$\eta = \ln(1 + \gamma). \quad (34)$$

Therefore  $R = (e^\eta)^{1/2}$ .

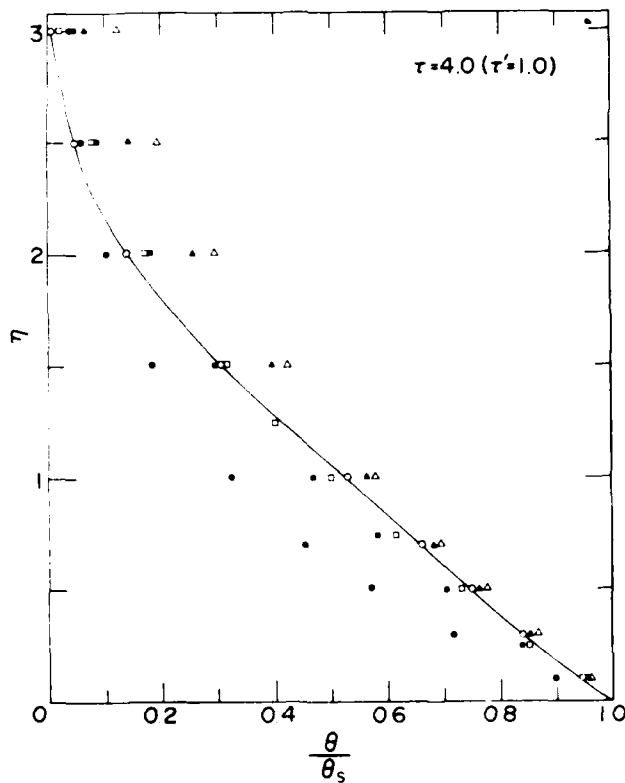
Since  $\rho$  and  $c_p$  are constant and can be considered as the values taken at the surface, the  $\tau'$  defined in eq 32 can be expressed in terms of  $\tau$  as

$$\tau' = \frac{\epsilon \alpha_0 t}{4(1/\beta)^2} = \frac{\tau}{4}. \quad (35)$$

The tabulated values of Jaeger (1956) for  $\tau' = 0.1, 1.0$  and  $5.0$  (corresponding to the  $\tau$  values in Jaeger's table; corresponding values of  $\tau$  in our approximation method are  $\tau = 0.4, 4.0$  and  $20.0$  respectively) are shown in Figure 3 to facilitate the computation of  $\theta/\theta_s$  for varied values of  $\eta$  with eq 34 to get the analytical results. Figure 4 shows comparisons of dimensionless temperature distribution of  $\theta/\theta_s$  vs  $\eta$ . Only the case of constant surface temperature  $b$  is shown for  $\tau' = 0.1, 1.0$  and  $5.0$  ( $\tau = 0.4, 4.0$  and  $20$  respectively). For each specific  $\tau$  value, a corresponding  $\epsilon$  for the specific value of  $F'(0) = 1$  is obtained from Table 1 and subsequently substituted into eq 30 to evaluate the ratio of  $\theta/\theta_s$  as a function of  $\eta$ . In Figure

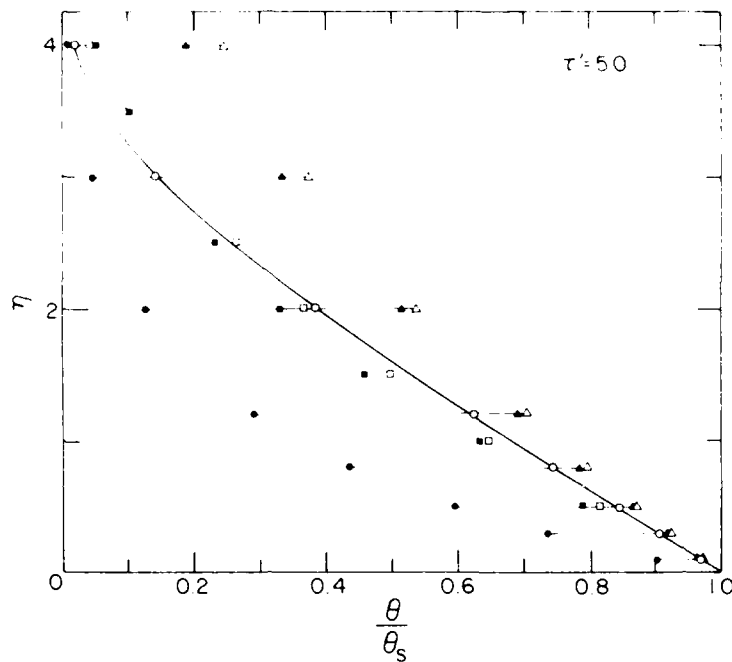


a.  $\tau = 0.4 (\tau' = 0.1)$ .



b.  $\tau = 4.0 (\tau' = 1.0)$ .

Figure 4. Comparison of  $\eta$  vs  $\theta/\theta_s$ .



c.  $\tau = 20.0$  ( $\tau' = 5.0$ ).

Figure 4 (cont'd.).

4, it can be seen that for small values of  $\tau$  (i.e.  $\tau' = 0.1$ , or  $\tau = 0.4$ ), the results from the analytical method are nearly equal to those from the heat balance integral method (HBIM), though it seems that the HBIM gives somewhat lower values of  $\theta/\theta_s$  for smaller values of  $\eta$  and larger values when  $\eta$  is greater than 0.5. A similar trend exists for  $\tau = 4.0$  ( $\tau' = 1.0$ ) except that the discrepancy between analytical and HBIM grows larger and the inverse occurs at much larger values of  $\eta$ . A similar statement can be made for  $\tau = 20$  ( $\tau' = 5.0$ ). The HBIM gives much lower values of  $\theta/\theta_s$ , and the inverse occurs at much higher values of  $\eta$ .

#### ALTERNATIVE METHOD

As shown in Figure 4 for  $\tau = 0.4$ , 4 and 20, the deviations of HBIM from the analytical method grow greater and greater as  $\tau$  increases. With this type of approximation method, one can only expect a fair degree of accuracy for the variable itself, which in this case is

$$V = \int_0^{\eta} F(y) \theta(y) dy.$$

Furthermore, it is well known that the process of differentiation only increases the error. Therefore a fair approximation of  $V$  does not necessarily guarantee a similar fair approximation of its derivatives; in this case, it means the error for  $\theta$  will be greater since

$$\theta = \frac{1}{F(\eta)} \cdot \frac{\partial V}{\partial \eta}.$$

Consequently one would expect a better result if the approximation of  $\theta$  can be used instead.

Assuming a temperature profile of

$$\frac{\theta}{\theta_s} = \left(1 - \frac{\eta}{\epsilon}\right)^2 \quad (36)$$

and integrating eq 6 for 0 to  $\epsilon$ ,

$$\int_0^{\epsilon} \frac{1}{F(\eta)} \frac{\partial^2 \theta}{\partial \eta^2} d\eta = \int_0^{\epsilon} \frac{\partial \theta}{\partial \tau} d\eta.$$

The final result is

$$\frac{2}{\epsilon^2} (1 - e^{-\gamma}) = \frac{1}{3} \frac{d\epsilon}{d\tau}$$

or

$$\tau = \int_0^{\epsilon} \frac{\epsilon^2 d\epsilon}{6(1 - e^{-\gamma})}. \quad (37)$$

However, for  $F(\eta) = e^\eta$ , an alternative solution with a closed form can be obtained as

$$\tau = \ln \epsilon - \frac{2}{\epsilon} + \frac{2e^\epsilon}{\epsilon} - Ei(\epsilon) + \gamma - 2 \quad (37a)$$

which is briefly derived in Appendix C. Similarly for a cubic representation of temperature, i.e.

$$\theta = a + b \frac{\eta}{\epsilon} + c \left(\frac{\eta}{\epsilon}\right)^2 + d \left(\frac{\eta}{\epsilon}\right)^3 \quad (38)$$

the dimensionless temperature is found to be

$$\frac{\theta}{\theta_s} = 1 - 3 \frac{\eta}{\epsilon} + 3 \left(\frac{\eta}{\epsilon}\right)^2 - \left(\frac{\eta}{\epsilon}\right)^3. \quad (39)$$

The final result is

$$\tau = \int_0^{\epsilon} \frac{\epsilon^3}{24(e^{-\eta} + \epsilon - 1)} d\eta. \quad (40)$$

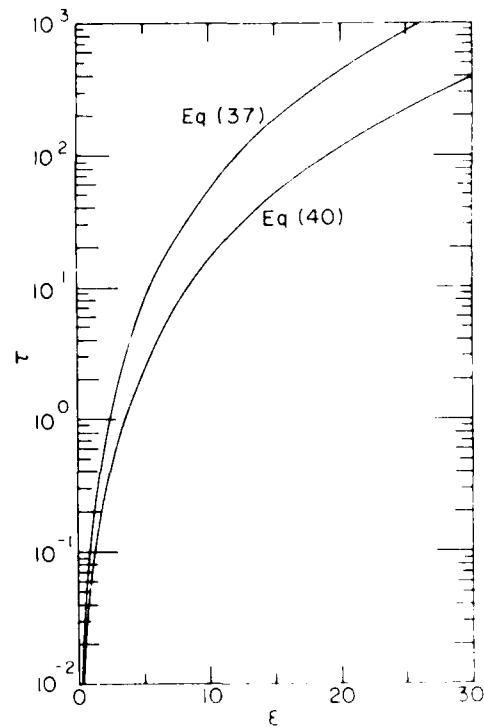
For this case and under the same condition, i.e.  $F(\eta) = e^\eta$ , the alternative closed form solution is

$$\tau = \frac{3}{\epsilon^2} (e^\epsilon - 1) + \frac{1}{\epsilon} (e^\epsilon - 4) + \ln \epsilon - Ei(\epsilon) + \gamma - 2.5. \quad (40a)$$

Equations 37, 37a, 40 and 40a are derived under the condition that  $F(\eta) = e^\eta$ ,  $F(0) = F'(0) = 1$ . Somewhat detailed derivations of these two equations are given in Appendices B and C. Numerical values of  $\tau$  and  $\epsilon$  evaluated from eq 37 and 40 are listed in Table 3 and graphically shown in Figure 5 to facilitate the determination of the specific values of  $\tau$ . For  $\tau = 0.4$ , 4

**Table 3. Values of  $\epsilon$  at specific values of  $\tau$  from eq 37 and 40.**

$\tau$	$\epsilon$	
	Equation 37	Equation 40
0.01	0.328	0.466
0.04	—	—
0.05	0.693	—
0.08	—	1.220
0.10	0.943	1.347
0.20	—	1.822
0.40	—	2.440
0.50	1.850	—
0.60	—	2.880
0.80	—	3.234
1.0	2.426	3.533
2.0	3.147	4.623
4.0	4.045	6.000
6.0	4.668	6.965
8.0	5.161	7.732
10.0	5.575	8.380
14.0	6.258	—
20.0	7.067	10.724
30.0	—	12.366
40.0	—	13.672
60.0	—	15.738
80.0	—	17.381
100.0	12.148	18.768
200.0	15.316	23.796
400.0	19.303	30.126
600.0	22.099	34.565
800.0	24.324	38.099
1000.0	26.203	41.082



*Figure 5.  $\tau$  vs  $\epsilon$  from numerical integration of eq 37 and 40.*

and 20, values of  $\epsilon$  are evaluated from Figure 5 and from eq 37a and 40a, and substituted into eq 36 and 39. The dimensionless temperature ratio  $\theta/\theta_s$  is plotted in Figure 4. It is evident that, for small values of  $\tau$ , the results from HBIM, alternative HBIM (AHBIM) and the analytical are close to each other. At smaller values of  $\eta$ , AHBIM gives almost the same values of  $\theta/\theta_s$  as the analytical method. It is also noted that the simpler temperature representation (i.e. eq 36) gives a better approximation to the analytical results. In general, AHBIM always gives higher ratios of  $\theta/\theta_s$  for all  $\eta$ . As in the HBIM, the discrepancy between the AHBIM and the analytical results increases as  $\tau$  increases. However, it can be observed from Figure 4b and c that the AHBIM gives a much superior approximation than HBIM for smaller  $\eta$  values in all the three cases considered. But it is evident that for some special functions of  $F(\eta)$ , where a closed form solution is feasible, the results are nearly identical to those from the analytical solution especially for large  $\tau$  values.

## CONCLUSION AND COMMENTS

The HBIM has been extended to the solution of heat transfer problems in media with variable properties, such as snow. The essence of the mathematical treatment involves the introduction of a new variable  $V$  and subsequently obtaining the temperature profile by differentiation. In AHBIM, integration was carried out directly and the results thus obtained have

proven to be closer to the analytical results than those from the HBIM. The results obtained from second-degree polynomial temperature representation were found to be nearly as good as the results from the cubic temperature representation, though the latter involves much work in deriving eq 4 and its numerical evaluation of  $\epsilon$ . However, the AHBIM is more restricted because the integration cannot be carried out unless the functional dependence of  $F(\eta)$  is explicitly given. For a special case of  $F(\eta) = e^\eta$ , a closed form solution was obtained, which provides results nearly identical to those from analytical methods.

Only the case of a semi-infinite solid with constant boundary conditions has been considered, but it can be easily extended to the case of a finite slab. In such a case the finite slab would behave exactly the same as a semi-infinite solid until  $\delta = L$  ( $L$  is the thickness of the slab). The conditions at  $x = L$  would have to be replaced, one of which would be the additional boundary condition at  $x = L$  and the other determined from the heat balance integral.

#### LITERATURE CITED

**Carslaw, H.S. and J.C. Jaeger** (1959) *Conduction of Heat in Solids*. Oxford: Clarendon Press, Second Edition.

**Goodman, T.R.** (1958) The heat balance integral and its application to problems involving a change of phase. *Transactions of the American Society of Mechanical Engineers*, **80**: 355-342.

**Goodman, T.R.** (1964) Application of integral methods to transient nonlinear heat transfer. In *Advances in Heat Transfer* (T.F. Irvine and J.P. Hartnett, Ed.). New York: Academic Press, Vol. 1, p. 55-122.

**Jaeger, J.C.** (1956) Numerical values for the temperature in radial heat flow. *Journal of Mathematical Physics*, **34**: 316-321.

**Pohlhausen, K.** (1921) Zur naherungsweise Integration der Differentialgleichung der laminaren Reibungsschicht. *Zeitschrift Angewandte Mathematik Mechanik*, **1**: 252-268.

**Von Kármán, T.** (1921) Über laminare und turbulente Reibung. *Zeitschrift Angewandte Mathematik Mechanik*, **1**: 233-252. Also National Advisory Committee for Aeronautics, Technical Memorandum 1092 (1946).

## APPENDIX A: DERIVATION OF EQUATION 25

For constant surface heat flux, the boundary condition is

$$-k \frac{\partial T}{\partial x} = q. \quad (\text{A1})$$

The corresponding condition on  $V$  can be derived from the fact that

$$\frac{\partial \theta}{\partial \eta} = \frac{1}{F(\eta)} \left[ \frac{\partial^2 V}{\partial \eta^2} - \frac{F'(\eta)}{F(\eta)} \frac{\partial V}{\partial \eta} \right], \quad \text{at } \eta = 0. \quad (\text{A2})$$

Since

$$\frac{\partial \theta}{\partial \gamma} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial \gamma} = \frac{\partial \theta}{\partial \eta} \frac{k_0}{k(x)} \quad (\text{A3})$$

but  $k = k_0$  at  $x = 0$ , it follows that

$$\frac{\partial \theta}{\partial \eta} = \frac{\epsilon}{T_r} \frac{\partial T}{\partial x} \Big|_0 = - \frac{\epsilon q}{k_0 T_r} = H$$

where  $H$  is the dimensionless heat flux. Substituting  $\partial \theta / \partial \eta = H$  into eq A2 and noting that  $F(0) = 1$ , it becomes

$$\frac{\partial^2 V}{\partial \eta^2} - F'(0) \frac{\partial V}{\partial \eta} = H, \quad \eta = 0. \quad (\text{A4})$$

Assuming a cubic temperature representation of  $V$ ,

$$V = a + b\eta + c\eta^2 + d\eta^3 \quad (\text{A5})$$

with the compatibility condition on  $V$  as

$$\begin{aligned} V &= 0, & \eta &= 0 \\ \frac{\partial^2 V}{\partial \eta^2} \Big|_0 - F'(0) \frac{\partial V}{\partial \eta} \Big|_0 &= H, & \eta &= 0 \\ \frac{\partial V}{\partial \eta} &= 0, \quad \frac{\partial^2 V}{\partial \eta^2} = 0, & \eta &= \epsilon. \end{aligned} \quad (\text{A6})$$

With conditions of eq A6, the constants  $a$ ,  $b$ ,  $c$  and  $d$  in eq A5 can be found and the temperature  $V$  can be represented by

$$V = \frac{H}{2 + F'(0)\epsilon} \left( -\epsilon\eta + \eta^2 - \frac{\eta^3}{3\epsilon} \right). \quad (\text{A7})$$

Substituting appropriate derivatives of  $V$  from eq A7 and with  $F(0) = 1$ , eq 12 becomes

$$\frac{2H}{2+F'(0)\epsilon} + F'(0) \frac{H\epsilon}{2+F'(0)\epsilon} = \frac{d}{d\tau} \left[ \frac{H\epsilon^2}{3[2+F'(0)\epsilon]} \right] \quad (\text{A8})$$

which can be simplified to

$$3d\tau = \frac{d\epsilon}{f'(0)} - \frac{4}{F'(0)} \frac{d\epsilon}{[2+F'(0)\epsilon]^2}. \quad (\text{A9})$$

Integrating eq A9 from  $\tau = 0$  and  $\epsilon = 0$  to  $\epsilon$ , we have

$$\epsilon - \frac{4}{F'(0)} \left[ \frac{1}{2} - \frac{1}{2+F'(0)\epsilon} \right] = 3F'(0)\tau. \quad (\text{25})$$

## APPENDIX B: DERIVATIONS OF EQUATIONS 37 AND 40

Integrating eq 6 from 0 to  $\epsilon$ , we have

$$\int_0^{\epsilon} \frac{1}{F(\eta)} \frac{\partial^2 \theta}{\partial \eta^2} d\eta = \int_0^{\epsilon} \frac{\partial \theta}{\partial \eta} d\eta \quad (6)$$

if the temperature profile is represented by eq 36, i.e.

$$\theta = \theta_s \left(1 - \frac{\eta}{\epsilon}\right)^2 \quad (36)$$

With the use of Leibniz's rule for a general function,

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t) dx = f(b,t) \frac{db}{dt} - f(a,t) \frac{da}{dt} + \int_a^b \frac{\partial f}{\partial t}(x,t) dx.$$

The right-hand side of eq 6 then becomes

$$\frac{d}{d\tau} \int_0^{\epsilon} \theta d\eta = \int_0^{\epsilon} \frac{\partial \theta}{\partial \tau} d\eta + \theta(\epsilon) \frac{d\eta}{d\tau} \quad (B1)$$

but  $\theta(\epsilon) = 0$ ; therefore

$$\int_0^{\epsilon} \frac{\partial \theta}{\partial \tau} d\eta = \frac{d}{d\tau} \int_0^{\epsilon} \theta d\eta. \quad (B2)$$

By substitution of eq 36 into B2 and integration, eq B2 becomes

$$\int_0^{\epsilon} \frac{\partial \theta}{\partial \tau} d\eta = \theta_s \frac{d}{d\tau} \left(\frac{\epsilon}{3}\right). \quad (B3)$$

The left-hand side of eq 6 is integrated by parts with the function  $F(\eta) = e^{-\eta}$  (see eq 29) as

$$\int_0^{\epsilon} \frac{1}{F(\eta)} \frac{\partial^2 \theta}{\partial \eta^2} d\eta = \frac{2\theta_s}{\epsilon^2} (1 - e^{-\epsilon}). \quad (B4)$$

Combining eq B3 and B4, i.e.

$$\frac{2\theta_s}{\epsilon^2} (1 - e^{-\epsilon}) = \theta_s \frac{d}{d\tau} \left(\frac{\epsilon}{3}\right), \quad (B5)$$

and integrating respectively for  $\tau$  from 0 to  $\tau$  and  $\epsilon$  from 0 to  $\epsilon$ , we have

$$\tau = \int_0^{\epsilon} \frac{\epsilon^2 d\epsilon}{6(1 - e^{-\epsilon})}. \quad (37)$$

For the derivation of eq 40, the only difference is the use of a cubic temperature profile instead of eq 36, i.e.

$$\frac{\theta}{\theta_s} = \left[ 1 - 3 \frac{\eta}{\epsilon} + 3 \left( \frac{\eta}{\epsilon} \right)^2 - \left( \frac{\eta}{\epsilon} \right)^3 \right] \quad (39)$$

The right-hand side of eq 6 becomes

$$\int_0^{\epsilon} \frac{\partial \theta}{\partial \eta} d\eta = \theta_s \frac{d}{d\tau} \left( \frac{\epsilon}{4} \right) \quad (B6)$$

and the left-hand side is

$$\int_0^{\epsilon} \frac{1}{F(\eta)} \frac{\partial^2 \theta}{\partial \eta^2} d\eta = \frac{6\theta_s}{\epsilon^2} \left( \frac{\epsilon - 1 + e^{-\epsilon}}{\epsilon} \right). \quad (B7)$$

Combining eq B6 and B7, it follows that

$$\frac{6\theta_s}{\epsilon^2} \left( \frac{\epsilon - 1 + e^{-\epsilon}}{\epsilon} \right) = \theta_s \frac{d}{d\tau} \left( \frac{\epsilon}{4} \right). \quad (B8)$$

Integrating eq B8 from 0 to  $\tau$  and 0 to  $\epsilon$  respectively, we have

$$\tau = \frac{1}{24} \int_0^{\epsilon} \frac{\epsilon^3}{(\epsilon - 1 + e^{-\epsilon})} d\epsilon. \quad (40)$$

### APPENDIX C: DERIVATION OF EQUATIONS 37a and 40a

Integrating eq 6 from 0 to  $\epsilon$ , i.e.

$$\int_0^{\epsilon} \frac{1}{F(\eta)} \frac{\partial^2 \theta}{\partial \eta^2} d\eta = \int_0^{\epsilon} \frac{\partial \theta}{\partial \eta} d\eta. \quad (6)$$

For a temperature profile of

$$\theta = \theta_0 \left(1 - \frac{\eta}{\epsilon}\right)^2 \quad (36)$$

and the application of Leibniz's rule [and from the fact  $\theta(\epsilon) = 0$ ] the right-hand side of eq 6 becomes

$$\int_0^{\epsilon} \frac{\partial \theta}{\partial \tau} d\eta = \frac{d}{d\tau} \int_0^{\epsilon} \theta d\eta. \quad (C1)$$

Equation 6 can then be rewritten as

$$\int_0^{\epsilon} \frac{\partial^2 \theta}{\partial \eta^2} d\eta = \frac{d}{d\tau} \int_0^{\epsilon} F(\eta) \theta d\eta. \quad (C2)$$

For  $\theta$  given by eq 36 and  $F(\eta) = e^\eta$ , eq C2 can be transformed to

$$\frac{2}{\epsilon} = \frac{d}{d\epsilon} \left( -1 - \frac{2}{\epsilon} - \frac{2}{\epsilon^2} + \frac{2e^\epsilon}{\epsilon^2} \right) \frac{d\epsilon}{d\tau}. \quad (C3)$$

Upon differentiation and subsequent integration eq C3 becomes

$$\tau = \ln \epsilon - \frac{2}{\epsilon} + \frac{2e^\epsilon}{\epsilon} - Ei(\epsilon) + \gamma - \lim_{\delta \rightarrow 0} \left[ \ln \delta - \frac{2}{\delta} + \frac{2e^\delta}{\delta} - Ei(\delta) + \gamma \right]. \quad (C4)$$

The limit of the last term can be written as

$$\lim_{\delta \rightarrow 0} [-Ei(\delta) + \ln \delta + \gamma] + \lim_{\delta \rightarrow 0} \left[ \frac{2}{\delta} (e^\delta - 1) \right]. \quad (C5)$$

The first part of eq C5 is zero and the second part approaches a value of 2 when  $\delta \rightarrow 0$ . Therefore, the final expression for  $\tau$  is

$$\tau = \ln \epsilon - \frac{2}{\epsilon} + \frac{2e^\epsilon}{\epsilon} - Ei(\epsilon) + \gamma - 2 \quad (37a)$$

where  $Ei(\epsilon)$  is an exponential integral and  $\gamma$  is the Euler's constant ( $\gamma = 0.57722$ ).

For the cubic temperature profile

$$\frac{\theta}{\theta_s} = \left[ 1 - 3 \frac{\eta}{\epsilon} + 3 \left( \frac{\eta}{\epsilon} \right)^2 - \left( \frac{\eta}{\epsilon} \right)^3 \right] \quad (39)$$

Following exactly the same procedures involved in deriving eq 37a, eq 6 becomes

$$\frac{3}{\epsilon} = \frac{d}{d\epsilon} \left[ -1 - \frac{3}{\epsilon} - \frac{6}{\epsilon^2} - \frac{6}{\epsilon^3} + \frac{6e'}{\epsilon^3} \right] \frac{d\epsilon}{d\tau} \quad (C6)$$

which can be differentiated and then integrated to

$$\begin{aligned} \tau &= \frac{3e'}{\epsilon^2} + \frac{e'}{\epsilon} + \ln \epsilon - \frac{4}{\epsilon} - \frac{3}{\epsilon^2} - Ei(\epsilon) + \gamma \\ &- \lim_{\delta \rightarrow 0} \left[ \frac{3e^\delta}{\delta^2} + \frac{e^\delta}{\delta} + \ln \delta - \frac{4}{\delta} - \frac{3}{\delta^2} - Ei(\delta) + \gamma \right]. \end{aligned} \quad (C7)$$

The last term can be grouped as

$$\lim_{\delta \rightarrow 0} [\ln \delta - Ei(\delta) + \gamma] + \lim_{\delta \rightarrow 0} \left[ \frac{3}{\delta^2} (e^\delta - 1) + \frac{1}{\delta} (e^\delta - 4) \right]. \quad (C8)$$

The first term in eq C8 is zero. The limiting value of the second term can be obtained by Taylor expansion of

$$e^\delta = \left( 1 + \delta + \frac{\delta^2}{2} + \frac{\delta^3}{6} + \frac{\delta^4}{24} + \dots \right), \text{ i.e.}$$

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \left[ \frac{3}{\delta^2} (e^\delta - 1) + \frac{1}{\delta} (e^\delta - 4) \right] \\ &= \lim_{\delta \rightarrow 0} \left[ \frac{3}{\delta^2} \left( 1 + \delta + \frac{\delta^2}{2} + \frac{\delta^3}{6} + \frac{\delta^4}{24} + \dots - 1 \right) \right. \\ &\quad \left. + \frac{1}{\delta} \left( 1 - 4 + \delta + \frac{\delta^2}{2} + \frac{\delta^3}{6} + \frac{\delta^4}{24} + \dots \right) \right] \\ &= \lim_{\delta \rightarrow 0} \left[ \left( \frac{3}{\delta} + \frac{3}{2} + \frac{\delta}{2} + \frac{\delta^2}{8} + \dots \right) + \left( -\frac{3}{\delta} + 1 + \frac{\delta}{2} + \frac{\delta^2}{6} + \dots \right) \right] \\ &= 2.5. \end{aligned}$$

Therefore, for the cubic temperature profile, the final expression of  $\tau$  is

$$\tau = \frac{3}{\epsilon^2} (e' - 1) + \frac{1}{\epsilon} (e' - 4) + \ln \epsilon - Ei(\epsilon) + \gamma - 2.5. \quad (40a)$$

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iv, 26 p., illus.; 28 cm. (CRREL Report 87-19.)

Bibliography: p. 12.

1. Conduction. 2. Heat transfer. 3. Mathematical analysis. 4. Snow. I. United States. Army. Corps of Engineers. II. Cold Regions Research and Engineering Laboratory, Hanover, N.H. III. Series: CRREL Report 87-19.

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