SECURITY CLASSIFICATION OF THIS PAGE

DTIC FILE COPY

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

AD-A187 430

1b. RESTRICTIVE MARKINGS

3. DISTRIBUTION/AVAILABILITY OF REPORT
Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)

5. MONITORING ORGANIZATION REPORT NUMBER(S)
AFOSR-TR- 87-1440

6a. NAME OF PERFORMING ORGANIZATION

6b. OFFICE SYMBOL (If applicable)

7a. NAME OF MONITORING ORGANIZATION

6c. ADDRESS (City, State, and ZIP Code)

7b. ADDRESS (City, State, and ZIP Code)

8a. NAME OF FUNDING/SPONSORING ORGANIZATION

8b. OFFICE SYMBOL (If applicable)

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR-87-0073

8c. ADDRESS (City, State, and ZIP Code)

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO	WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)

12. PERSONAL AUTHOR(S)

13a. TYPE OF REPORT

13b. TIME COVERED FROM ______ TO ______

14. DATE OF REPORT (Year, Month, Day)
July 30, 1987

15. PAGE COUNT

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

DTIC
ELECTE
NOV 1 6 1987
S D
H

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT
☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION

22a. NAME OF RESPONSIBLE INDIVIDUAL

22b. TELEPHONE (Include Area Code)

22c. OFFICE SYMBOL

DD Form 1473, JUN 86 Previous editions are obsolete. SECURITY CLASSIFICATION OF THIS PAGE to appear in Proceedings of First International Conference in Advances in Communications and Control Systems

AFOSR-TR- 87-1440

ON OBSERVER PROBLEMS FOR SYSTEMS GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS *

J.S. Baras† and A. Bensoussan††

†University of Maryland

††University Paris Dauphine
and
INRIA

1. Introduction

The problem of observers has been introduced in the control literature by D. Luenberger [15]. Let us consider a dynamic system which is deterministic, but whose inital state is unknown. An observer is a model which mimics the behavior of the physical system, and in particular its state becomes closer and closer as time evolves to the state of the physical system. There is a great deal of freedom in such a design and it is important to investigate various kinds of observers.

Since, after all, the observer prblem presents analogies with the filtering problem (estimating the state of a stochastic dynamic system), although there are no stochastic disturbances, it is natural to exploit the analogy. This idea has been used by J.S. Baras and P.S. Krishnaprasad [4] and leads to an observer which is different from Luenberger's observer. It presents several advantages. In particular, it is obtained in a constructive way and it has robustness properties.

From the very definition it applies identically when there are disturbances, whereas the Luenberger observer is strictly limited to the deterministic case and is not obtained in a constructive way. Also it may apply to more general cases, in the sense that when the Luenberger observer exists, the observer arising from the Kalman filter theory exists as well.

In this article we consider dynamic systems whose evolution is governed by a parabolic partial differential equation, or more generally a differential operational equation in the sense of J.L. Lions [12]

The Luenberger theory has been extended to infinite dimensional systems (see in particular M.J. Chapman and A.J. Pritchard [6], A. Ichikawa and A.J. Pritchard [10]). We explore here the observer based upon Kalman filter theory.

Let us discuss here an example to present the main reults of the paper. Let Ω be a smooth domain of R^n. Consider the P.D.E.

*Partial Sponsorship from: AFOSR-870073, NSF ECS-8219123, NSF INT-84-93793.

$$\frac{\partial y}{\partial t} - \Delta y = 0 \quad \text{in } \Omega ,$$

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Gamma} = 0 ,$$

$$y(x, o) = y_o(x) \in H = L^2(\Omega) , \tag{1.1}$$

where $\Gamma = \partial\Omega$ denotes the boundary of Ω. The state y_o is unknown. If instead of (1.1) we were to have

$$\frac{\partial y}{\partial t} - \Delta y + \lambda y = 0 , \quad \lambda > 0 ,$$

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Gamma} = 0 ,$$

$$y(x, o) = y_o , \tag{1.2}$$

then the system is stable and $y(x, T) \to 0$ as $T \to \infty$. In other words, whatever y_o is, the state comes closer and closer to 0 as T evolves and thus becomes "more and more known." An observer could be simply the model (1.2) itself with an arbitrary value of y_o independent of the available observation.

Such a stability property is not present in the model (1.1). Let us assume that we observe

$$z = y|_{\Gamma} \tag{1.1'}$$

i.e., the value of the state on the boundary.

An observer in the spirit of Luenberger would be the following model

$$\frac{\partial m}{\partial t} - \Delta m = 0 \quad \text{in } \Omega ,$$

$$\frac{\partial m}{\partial \nu} = (z - m)_{\Gamma} ,$$

$$m(x, o) = m_o(x) , \tag{1.3}$$

where m_o is arbitrary. The error $\eta = y - m$ appears as the solution of

$$\frac{\partial \eta}{\partial t} - \Delta \eta = 0 \quad \text{in } \Omega ,$$

$$\left.\frac{\partial \eta}{\partial \nu} + \eta\right|_{\Gamma} = 0 ,$$

$$\eta(x, o) = \eta_o(x) = y_o - m_o . \tag{1.4}$$

Multiplying (4) by η integrating over Ω yields

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} \eta^2 \, dx + \int_{\Omega} |D\eta|^2 \, dx + \int_{\Gamma} \eta^2 \, d\Gamma = 0 .$$

But

$$\int_{\Omega} |D\eta|^2 \, dx + \int_{\Gamma} \eta^2 \, d\Gamma \geq \beta \int_{\Omega} \eta^2 \, dx , \quad \beta > 0 ,$$

and thus

$$\int_{\Omega} \eta^2(x, t)\, dx \;\le\; Ce^{-2\beta t} ,$$

which proves the exponential decay of the error.

The theory developed in this paper leads to the following observer:

$$\frac{\partial m}{\partial t} - \Delta m \;=\; \int_{\Gamma} P(x, \xi, t)(z(\xi, t) - m(\xi, t))\, d\xi ,$$

$$\left.\frac{\partial m}{\partial \nu}\right|_{\Gamma} = 0 , \tag{1.5}$$

$$m(x, o) \;=\; m_o(x) ,$$

u.c.P where $p(x, \xi, t)$ appears as the solution of a Riccati equation, connected to a filtering problem, or by duality to a control problem. We study the type of control problems which may be introduced in order to derive exponential decay for the error.

2. Setting of the Problem

2.1 Notation and Assumptions

Let V, H be two separable real Hilbert spaces, such that identifying H and its dual H', one has

$$V \subset H = H' \subset V' \tag{2.1}$$

each space being dense in the next one with continuous injection.

We denote by $((\ ,\ ))$ $\|\ \|$ and $(\ ,\ )$ $|\ |$ the scalar product and norm in V, H, respectively. For another Hilbert space X we shall use the notation $(\ ,\ )_X$ and $|\ |_X$. We denote by $<\ ,\ >$ the duality V, V'.†

Let $A(t)$ be a family of operators such that

$$A(\cdot) \;\in\; L^{\infty}(o, \infty; V; V'))$$

$$<A(t)\phi, \phi> + \lambda|\phi|^2 \;\ge\; \alpha\|\phi\|^2 , \qquad \forall\, \phi \in V , \tag{2.2}$$

$$\alpha > 0,\ \lambda \ge 0,\ \forall\, t .$$

We consider a dynamic system whose evolution is governed by the differential operational equations (see J.L. Lions [12])

$$\frac{dy}{dt} + A(t)y = f ,$$

$$y(o) = y_o , \tag{2.3}$$

† More generally, $<\ ,\ >_X$ will represent the duality between a Hilbert space X ands its dual X' and Λ_X the canonical isomorphism between X and X'.

where

$$f \in L_2(o,\infty; V') \quad \text{given}\,, \tag{2.4}$$

$$y_o \in H\,, \quad y_o \text{ unknown}\,. \tag{2.5}$$

It is well known that for any y_o, (2.3) defines the state $y(\cdot)$ in the sense

$$y \in L^2_{loc}(o,\infty; V)\,, \quad \frac{dy}{dt} \in L^2_{loc}(o,\infty; V')\,. \tag{2.6}$$

We perform an observation on the state $y(\cdot)$ as follows:

$$z(t) = C(t)y(t)\,, \tag{2.7}$$

where

$$C(\cdot) \in L^\infty(o,\infty; \mathcal{L}(V;F))\,,$$

F being a given Hilbert space.

2.2 Definition of an Observer

An observer is a function $m(t)$ measurable with values in H, whose value can be computed at each time t in terms of the known data (in particular the observation $z(\cdot)$) and such that the error

$$e(t) = y(t) - m(t)$$

satisfies

$$|e(t)| \to 0 \quad \text{as } t \to \infty\,.$$

Therefore $m(t)$ will reasonably estimate the state of the system at time t (at $t = 0$ this estimate may be very bad, but it improves more and more as $t \to \infty$; it is of course nice to get an exponential decay for $|e(t)|$).

This problem has been extensively studied in the finite dimensional case, starting with the seminal work of D. Luenberger [15]. For infinite dimensional systems, the Luenberger observer has been exended in a natural fashion, however the research has mostly concentrated on the design of compensation (i.e., use the possibility of controlling the system, so that the global system made of the system itself and of the observer is stable).† While the control is generally present in such contexts, it is worthwhile to separate the observer problem from the compensation problem, and consider cases similar to (2.3), (2.7) when there is no control. In particular, nothing can be done to stabilize the system itself.

We shall thus prefer to introduce observers constructed in a different way, following ideas introduced by J.S. Baras and P.S. Krishnaprasad [4], which consist in artificially randomizing the problem and using Kalman filter theory for infinite dimensional systems.

† For details in this direction see M.J. Chapman and A.J. Pritchard [6], A. Ichikawa and A.J. Pritchard [10].

2.3 A Randomizing System

The theory of linear stochastic infinite dimensional systems has attracted a considerable interest in the literature. One of the main objectives has been to develop a rigorous theory of the Kalman filter applicable to distributed parameter systems. Among the main contributions one can refer to A.V. Balakrishnan [1], [2], A. Bensoussan [5], R.F. Curtain [8], R.F. Curtain and A.J. Pritchard [9].

Let us describe briefly what we shall need for our purpose. Let $(\Omega, \mathfrak{F}, P)$ be a probability space equipped with a filtration $\mathfrak{F}^t$, satisfying the usual conditions.

Let E be a Hilbert space, a *generalized E Wiener space* is a stochastic process indexed by an element $e_\bullet(\cdot) \in L^2_{loc}(o, \infty; E')$, denoted by $\mu_t^{e_\bullet}(\omega)$ satisfying

$$\mu_t^{e_\bullet} \quad \text{is a } \mathfrak{F}^t \text{ Wiener, } \forall\, e_\bullet \;, \tag{2.8}$$

$$E\,\mu_t^{e_\bullet}\,\mu_s^{e_\bullet} = \int_o^{t\wedge s} \langle e_\bullet(\lambda), \Lambda_E^{-1} e_\bullet(\lambda)\rangle \, d\lambda \;, \tag{2.9}$$

$$\text{the map } e_\bullet \to \mu_t^{e_\bullet} \text{ is linear .} \tag{2.10}$$

We assume that such a generalized E Wiener process exists. Similarly we shall assume the existence of a generalized F Wiener process $\nu_t^{h_\bullet}$ and we suppose that

$$\mu_t^{e_\bullet} \text{ and } \nu_t^{f_\bullet} \text{ are independent .} \tag{2.11}$$

Let $G(\cdot) \in L^\infty(o, \infty; \mathcal{L}(E; V'))$ and $M(\cdot) \in L^\infty(o, \infty; \mathcal{L}(F; F))$, we shall consider the stochastic proceses $\mu_t^{G^* v}$ and $\nu_t^{M^* f_\bullet}$ indexed respectively by $v(\cdot) \in L^2_{loc}(o, \infty; V)$ and $f_\bullet(\cdot) \in L^2_{loc}(o, \infty; F')$. They are clearly independent $\mathfrak{F}^t$ processes and one has

$$E\,\mu_t^{G^* v_1}\mu_s^{G^* v_2} = \int_o^{t\wedge s} \langle G\Lambda_E^{-1} G^* v_1(\lambda), v_2(\lambda)\rangle \, d\lambda \;. \tag{2.12}$$

Finally, let there be a family of random variables indexed by $h \in H$ denoted by ξ^h such that

$$\begin{aligned} &\forall\, h\,, \quad \xi^h \text{ is a Gaussian with mean } O, \text{ and} \\ &E\,\xi^h\xi^{\tilde h} = (P_o h, \tilde h)\,, \quad P_o \in \mathcal{L}(H; H)\,, \end{aligned} \tag{2.13}$$

symmetric, semipositive definite.

$$\xi^h \text{ is independent from } \mu_t^{e_\bullet} \text{ and } \nu_t^{f_\bullet}. \tag{2.14}$$

Let $\Gamma(t, s)$, $t \geq s$ be the Green's operator corresponding to $A(t)$, i.e.,

$$\Gamma(t, s) \in \mathcal{L}(H; H) \quad \text{and} \quad \alpha(t) = \Gamma(t, s)h \tag{2.15}$$

is the solution of

$$\frac{d\alpha}{dt} + A\alpha = 0\,, \quad t > s,\ \alpha(s) = h\,.$$

For simplicity we write $\Gamma(t) = \Gamma(t, o)$.

A *stochastic linear system* is a process y_t^h indexed by $h \in H$, defined by the formula

$$y_t^h = (\bar{y}(t), h) + \xi^{\Gamma^*(t)h} + \mu_t^{G^*\Gamma^*(t,\cdot)h}, \tag{2.16}$$

where $\bar{y}(t)$ is the solution of

$$\frac{d\bar{y}}{dt} + A(t)\bar{y} = f(t), \quad \bar{y}(o) = y_o, \qquad y_o \text{ given in } H. \tag{2.17}$$

We are to be careful in the interpretation of $\mu_t^{G^*\Gamma^*(t,o)h}$. In fact, consider $s \to G^*(s)\Gamma^*(t, s)h$ for $s < t$; it coincides with $G^*(s)\beta(s)$, where β is the solution of

$$\frac{\partial\beta}{\partial s} + A^*(s)\beta = 0, \quad \beta(t) = h,$$

and $\beta(s) \in L^2(o,t,V)$, hence $s \to G^*(s)\Gamma^*(t,s)h = e_\cdot^t(s)$ belongs to $L^2(o,t;E')$. Since $\mu_t^{e_\cdot}$ depends only on the restriction of $e_\cdot$ on (o, t), the quantity $\mu_t^{e_\cdot^t}$ is well defined for any t.

We define the observation at time T as follows. Let $f_\cdot \in L^2_{loc}(o, T; F')$ and η be the solution of

$$-\frac{d\eta}{dt} + A^*\eta = C^*f_\cdot, \qquad \eta(t) = 0. \tag{2.18}$$

Then one sets

$$z_T^{f_\cdot} = \int_o^T \langle f_\cdot(t), C(t)\bar{y}(t)\rangle\, dt + \xi^{\eta(o)} + \mu_T^{G^*\eta} + \nu_T^{M^*f_\cdot}. \tag{2.19}$$

2.4 Kalman Filter

This problem can be stated as follows. Let

$$Z^T = \sigma(z^{f_\cdot}, f_\cdot \in L_2(o, T; F')).$$

Find

$$\hat{y}_T^h = E[y_T^h \mid Z^T]. \tag{2.20}$$

Without redoing the theory, for which we refer to A. Bensoussan [5], we shall only recall that it can be obtained by solving a deterministic control problem related to maximum likelihood. We assume that M is invertible as well as P_o and set

$$R = M^{*-1}\Lambda_F M^{-1} \in \mathcal{L}(F, F'). \tag{2.21}$$

We introduce the control problem

$$\frac{dy}{dt} + Ay = f + Ge\,, \qquad e(\cdot) \in L^2(o, T; E)\,,$$
$$y(o) = y_o + \xi\,, \tag{2.22}$$

in which ξ and $e(\cdot)$ are the decision variables.

We are interested in minimizing the following cost:

$$J(\xi, e(\cdot)) = \int_o^T [<\Lambda_E e, e> + <R(\zeta - Cy), \zeta - Cy>]\, dt$$
$$+ (P_o^{-1}\xi, \xi)\,, \tag{2.23}$$

where on the right side of (2.23) the function $\zeta(\cdot)$ is given in $L_2(o, T; F)$.

Define

$$D_1 = G\Lambda_E^{-1}G^*\,,$$
$$D_2 = C^*RC\,, \tag{2.24}$$

and consider the pair $y(\cdot)$, $p(\cdot)$ defined by the system of coupled equations

$$\frac{dy}{dt} + A(t)y + D_1(t)p = f\,,$$
$$-\frac{dp}{dt} + A^*(t)p - D_2(t)y = -C^*R\zeta\,,$$
$$y(o) = y_o - P_o p(o)\,,$$
$$p(T) = 0\,. \tag{2.25}$$

Then the optimal control is given by

$$\hat{e}(t) = -\Lambda_E^{-1}G^*(t)p(t)\,,$$
$$\xi = -P_o p(o)\,. \tag{2.26}$$

The decoupling theory leads to the Riccati equation (written formally)

$$\frac{dP}{dt} + AP + PA^* + PD_2P = D_1\,,$$
$$P(o) = P_o\,, \tag{2.27}$$

and the linear equation

$$\frac{dr}{dt} + Ar = f + PC^*R(\zeta - Cr)\,,$$
$$r(o) = y_o\,. \tag{2.28}$$

It can be proven that if $\phi^h_\cdot \in L_2(o, T; F')$ is defined by

$$\int_o^T <\phi^h_\cdot, \zeta>\, dt = (r(T) - \bar{y}(T), h)\,, \qquad \forall\, \zeta\,,$$

then the quantity (2.20) is given by

$$\hat{y}_T^h = z^{\phi^h_*} + (\bar{y}(T), h) .$$

3. Study of the Operator $P(T)$

3.1 Definition of $P(T)$

Consider the coupled system

$$\frac{d\hat{\alpha}}{dt} + A\hat{\alpha} + D_1\hat{\beta} = 0 ,$$

$$-\frac{d\hat{\beta}}{dt} + A^*\hat{\beta} - D_2\hat{\alpha} = 0 ,$$

$$\hat{\alpha}(o) = -P_o\hat{\beta}(o) ,$$

$$\hat{\beta}(T) = h . \tag{3.1}$$

Then we set

$$P(T)h = -\hat{\alpha}(T) . \tag{3.2}$$

The system (3.1) is related to the following control problem:

$$\frac{d\alpha}{dt} + A(t)\alpha = Ge ,$$

$$\alpha(o) = \xi , \tag{3.3}$$

$$J_T^h(\xi, e(\cdot)) = \frac{1}{2}\left\{(P_o^{-1}\xi, \xi) + \int_o^T [|e|^2 + \langle D_2\alpha, \alpha\rangle]\, dt\right\} + (h, \alpha(T)) .$$

We deduce easily that

$$\frac{1}{2}(P(T)h, h) = -\inf_{\xi, e(\cdot)} J_T^h(\xi, e(\cdot)) . \tag{3.4}$$

We can also characterize this quantity in a different way. Consider the control problem

$$-\frac{d\beta}{dt} + A^*\beta = C^*\phi_* , \quad \phi_* \in L^2(o, T; F') ,$$

$$\beta(T) = h , \tag{3.5}$$

and the cost

$$\tilde{J}_T^h(\phi_*(\cdot)) = (P_o\beta(o), \beta(o)) + \int_o^T \langle R^{-1}\phi_*, \phi_*\rangle\, dt + \int_o^T \langle D_1\beta, \beta\rangle\, dt .$$

Then one has also

$$(P(T)h, h) = \inf \tilde{J}_T^h(\phi_*(\cdot)) . \tag{3.6}$$

3.2 *Detectability and Stabilizability*

DEFINITION 3.1. We shall say that the pair $A(\cdot)$, $C(\cdot)$ is detectable if $\forall\, h$, $\exists\, \phi_\bullet^{T,h}$ such that

$$\int_o^T |\phi_\bullet^{T,h}|_{F'}^2\, dt + \int_o^T |\beta^{T,h}|_H^2\, dt < K_h \tag{3.7}$$

independent of T, where $\beta^{T,h}$ represents the solution of (3.5) corresponding to $\phi_\bullet^{T,h}$. □

In the stationary case, i.e., A, C independent of time, it is sufficient to assume the following

DEFINITION 3.2. We say A^*, C^* is stabilizable if $\forall\, h$, $\exists\, \phi_\bullet^h \in L^2(o, \infty; F')$ such that the solution γ^h of

$$\frac{d\gamma^h}{dt} + A^*\gamma^h = C^*\phi_\bullet^h ,$$

$$\gamma^h(o) = h ,$$

satisfies $\gamma^h \in L^2(o, \infty; H)$. □

PROPOSITION 3.1. In the stationary case, if A^*, C^* is stabilizable, then A, C is detectable.

Proof. Define

$$\phi_\bullet^{T,h}(t) = \phi_\bullet^h(T-t) , \qquad \beta^{T,h}(t) = \gamma^h(T, t) .$$

Then clearly $\beta^{T,h}(t)$ is the solution of (2.5) corresponding to $\phi_\bullet^{T,h}(t)$, and

$$\int_o^T |\phi_\bullet^{T,h}|_{F'}^2\, dt = \int_o^T |\phi_\bullet^h|^2\, dt < \int_o^\infty |\phi_\bullet^h|^2\, dt ,$$

$$\int_o^T |\beta^{T,h}|^2\, dt = \int_o^T |\gamma^h|^2\, dt < \int_o^\infty |\gamma^h|^2\, dt ,$$

and the desired property follows. □

THEOREM 3.1. Assume that $A(\cdot)$, $C(\cdot)$ is detectable, then

$$|P(T)|_{\mathcal{L}(H;H)} \le p . \tag{3.8}$$

Proof. It follows from (3.5) that

$$\tfrac{1}{2}|\beta(o)|^2 + \int_o^T \langle A^*\beta, \beta\rangle\, dt = \tfrac{1}{2}|h|^2 + \int_o^T \langle \beta, C^*\phi_\bullet\rangle\, dt .$$

Hence

$$\tfrac{1}{2}|\beta(o)|^2 + \alpha\int_o^T \|\beta\|^2\, dt \le \tfrac{1}{2}|h|^2 + \int_o^T \langle \beta, C^*\phi_\bullet\rangle\, dt + \lambda\int_o^T |\beta|^2\, dt$$

and the detectability condition implies immediately

$$|\beta^{T,h}(o)| \leq K_h' , \quad \int_o^T \|\beta^{T,h}\|^2 \, dt \leq K_h' .$$

Therefore from the definition (3.6) one has

$$(P(T)h, h) \leq \tilde{J}_T^h(\phi_*^{T,h}) \leq K_h'' .$$

This and the fact that $P(T)$ is symmetric positive semidefinite implies the desired result. □

THEOREM 3.2. If there exists a family $\Gamma(\cdot) \in L^\infty(o, \infty; \mathcal{L}(F; V'))$ such that

$$<(A(t) + \Gamma(t)C(t)\xi, \ \xi> \ \geq \ \alpha_o \|\xi\|^2 \tag{3.9}$$

then the pair $A(\cdot)$, $C(\cdot)$ is detectable.

Proof. Indeed pick the feedback

$$\phi_* = -\Gamma^*\beta .$$

The corresponding trajectory is given by

$$-\frac{d\beta}{dt} + (A^* + C^*\Gamma^*)\beta = 0 , \quad \beta(T) = h ,$$

and thus $\int_o^T \|\beta\|^2 dt$ is bounded by a constant independent of T. The desired result follows. □

3.3 *Invertibility*

We turn now to the question of invertibility of the operator $P(T)$. We shall need another property. Consider the dynamic system

$$\begin{aligned} &\frac{d\alpha}{dt} + A\alpha = Ge , \\ &\alpha(o) = \xi . \end{aligned} \tag{3.10}$$

DEFINITION 3.3. We shall say that the pair $A(\cdot)$, $G(\cdot)$ is controllable if $\forall \ h, \exists \ e^{T,h}$ and $\xi^{T,h}$ such that

$$|\xi^{T,h}|^2 + \int_o^T |e^{T,h}|^2 \, dt + \int_o^T |\alpha^{T,h}|^2 \, dt \leq L_h \tag{3.11}$$

and

$$\alpha^{T,h}(T) = h , \tag{3.12}$$

where $\alpha^{T,h}$ designates the solution of (3.10) corresponding $\xi^{T,h}$ and $e^{T,h}$. The constant L_h is independent of T. □

We can give an example of controllability (probably the only one really implementable).

PROPOSITION 3.2. Assume that

$$D_1 \text{ is invertible .} \tag{3.13}$$

Then the pair $A(\cdot)$, $G(\cdot)$ is controllable.

Proof. Let

$$\phi_h \in L_2(o,\infty, V)\,, \quad \frac{d\phi_h}{dt} \in L_2(o,\infty; V')\,, \quad \phi_h(o) = h\,.$$

We set

$$\alpha^{T,h}(t) = \phi_h(T-t)\,, \quad \xi^{T,h} = \alpha^{T,h}(o) = \phi_h(T)$$

and

$$e^{T,h}(t) = \Lambda_E^{-1} G^*(t) D_1(t)^{-1} \left(\frac{d\alpha^{T,h}}{dt} + A(t)\alpha^{T,h}(t)\right)\,. \tag{3.14}$$

Then

$$|\xi^{T,h}|^2 = |\phi_h(T)|^2 = 2\int_o^T \left\langle \phi_h, \frac{d\phi_h}{dt}\right\rangle dt + |h|^2$$

$$< |h|^2 + 2\int_o^\infty \left\langle \phi_h, \frac{d\phi_h}{dt}\right\rangle dt\,,$$

$$\int_o^T |\alpha^{T,h}|^2\,dt = \int_o^T |\phi_h(T-t)|^2\,dt = \int_o^T |\phi_h(t)|^2\,dt < \int_o^\infty |\phi_h|^2\,dt\,.$$

and

$$\int_o^T |e^{T,h}(t)|^2\,dt \le C\left[\int_o^T \left\|\frac{d\alpha^{T,h}}{dt}\right\|_{V'}^2 dt + \int_o^T \|\alpha^{T,h}(t)\|^2\,dt\right]$$

$$\le C\left[\int_o^\infty \left\|\frac{d\phi_h}{dt}\right\|_{V'}^2 dt + \int_o^\infty \|\phi_h\|^2\,dt\right]\,.$$

Finally from (3.4) we obtain

$$G(t)e^{T,h}(t) = \frac{d\alpha^{T,h}}{dt} + A(t)\alpha^{T,h}(t)$$

and

$$\alpha^{T,h}(T) = h\,.$$

The desired result has been established. □

REMARK 1. As is well known (cf. R. Lattes and J.L. Lions [11]), we cannot solve *a priori* the backward problem

$$\frac{d\alpha}{dt} + A\alpha = Ge\,, \quad \alpha(T) = h\,. \tag{3.15}$$

The situation is different from ordinary differential equations. The problem (3.15) (for given e) is *a priori* ill posed. We refer to J.L. Lions [14] for a detailed study of these ill posed problems in the context of control theory. □

We shall now consider control problems similar to those described in Section 3.1. The response is described by

$$\begin{aligned} \frac{d\alpha}{dt} + A\alpha &= Ge , \\ \alpha(o) &= \xi . \end{aligned} \tag{3.16}$$

We impose the constraint

$$\alpha(T) = h \tag{3.17}$$

and minimize the cost (recall that P_o is invertible (cf. Section 2.4))

$$K_T^h(\xi, e(\cdot)) = (P_o^{-1}\xi, \xi) + \int_o^T [|e|_E^2 + \langle D_2\alpha, \alpha\rangle]\, dt . \tag{3.18}$$

Note that (3.17) must be considered as a constraint and not an initial condition.

THEOREM 3.3. Assume that the pair $A(\cdot)$, $G(\cdot)$ is controllable. Then the control problem (3.16), (3.17), (3.18) has a unique solution. There exists a unique pair $\overline{\alpha}, \overline{\beta}$ such that

$$\begin{aligned} \frac{d\overline{\alpha}}{dt} + A\overline{\alpha} + D_1\overline{\beta} &= 0 , \\ -\frac{d\overline{\beta}}{dt} + A^*\overline{\beta} - D_2\overline{\alpha} &= 0 , \\ \overline{\alpha}(o) &= -P_o\overline{\beta}(o) , \\ \overline{\alpha}(T) &= h , \end{aligned} \tag{3.19}$$

and the optimal control is

$$\begin{aligned} \overline{e}(t) &= -\Lambda_E^{-1}G^*(t)\overline{\beta}(t) , \\ \overline{\xi} &= -P_o\overline{\beta}(o) . \end{aligned} \tag{3.20}$$

Proof We follow the technique introduced by J.L. Lions [13] to deal with this type of ill posed problem. We shall penalize the constraint (3.17).

It relie
on the u
of the penal
technique

a. Existence and Uniqueness of the Optimal Control

Let $\Gamma_h = \{e(\cdot), \xi \,|\, \alpha(T) = h\}$. By the controllability assumption Γ_h is not empty. It is a convex closed subset of the Hilbert space $L_2(o,T; E) \times H$ and the functional (3.18) is a coercive quadratic functional. Hence the existence and uniqueness of the optimal control $\overline{e}(\cdot), \overline{\xi}$.

b. Necessary and Sufficient Conditions of Optimality

Consider Γ_o (i.e., Γ_h with $h = 0$). It is obviously a closed subvector space of $L^2(o,T;\, E) \times H$. Noting that $\bar{e}(\cdot) + \lambda e(\cdot)$, $\bar{\xi} + \lambda\bar{\xi}$ $\forall$ $e(\cdot)$, $\bar{\xi}$ belonging to Γ_o, we deduce easily from the relation

$$K_T^h(\bar{\xi} + \lambda\xi,\ \bar{e}(\cdot) + \lambda e(\cdot)) \geq K_T^h(\bar{\xi}, \bar{e}(\cdot))$$

that the following condition must hold

$$(P_o^{-1}\bar{\xi}, \xi) + \int_o^T <\Lambda_E \bar{e}, e> dt + \int_o^T <D_2\bar{\alpha}, \alpha> dt = 0\,, \qquad (3.21)$$

$$\forall\ \xi,\ e(\cdot) \text{ in } E_o\,,$$

α being the corresponding solution of (3.16),

and

$\bar{\alpha}$ designates the optimal trajectory.

This condition is sufficient, since $\forall$ $\tilde{\xi}$, $\tilde{e}(\cdot) \in E_h$, $\tilde{\xi} - \bar{\xi}$, $\tilde{e}(\cdot) - \bar{e}(\cdot) \in E_o$. Hence

$$K_T^h(\bar{\xi}, \bar{e}(\cdot)) = (P_o^{-1}\bar{\xi}, \tilde{\xi}) + \int_o^T <\Lambda_E \bar{e}, \tilde{e}> dt + \int_o^T <D_2\bar{\alpha}, \tilde{\alpha}> dt$$

and

$$K_T^h(\tilde{\xi}, \tilde{e}(\cdot)) > K_T^h(\bar{\xi}, \bar{e}(\cdot))\,.$$

c. Adjoint System. Uniqueness

Assume that we have a solution of (3.19). Then define $\bar{e}(\cdot)$ and $\bar{\xi}$ by (3.20). Let us show that it is optimal. Clearly $\bar{\alpha}$ is the trajectory corresponding to the control $\bar{e}(\cdot)$, $\bar{\xi}$ and it is admissible (belongs to Γ_h). It is easy to check that it satisfies (3.21). It is thus the optimal control and therefore $\bar{e}(\cdot)$, $\bar{\xi}$ and $\bar{\alpha}(\cdot)$ are uniquely defined. It is also the case of $\bar{\beta}(o)$ and $-(d\bar{\beta}/dt) + A^*\bar{\beta}$. This implies that $\bar{\beta}$ is unique as well.

d. Adjoint System. Existence

It remains to prove the existence of the pair $\bar{\alpha}$, $\bar{\beta}$ solution of (3.19). The penalty technique is now used. Consider the functional

$$K_T^{h,\varepsilon}(\varepsilon, e(\cdot)) = K_T^h(\xi, e(\cdot)) + \frac{1}{\varepsilon}|\alpha(T) - h|^2\,. \qquad (3.22)$$

Then the control problem (3.16), (3.22) becomes classical, and there exists a pair α^ε, β^ε solution of

$$\frac{d\alpha_\varepsilon}{dt} + A\alpha_\varepsilon + D_1\beta_\varepsilon = 0\,,$$
$$-\frac{d\beta_\varepsilon}{dt} + A^*\beta_\varepsilon - D_2\alpha_\varepsilon = 0\,,$$
$$\alpha_\varepsilon(o) = -P_o\beta_\varepsilon(o)\,,$$
$$\beta_\varepsilon(T) = \frac{1}{\varepsilon}(\alpha_\varepsilon(T) - h)\,, \qquad (3.23)$$

and the optimal control is

$$e_\varepsilon = -\Lambda_E^{-1}G^*\beta_\varepsilon\,, \quad \xi_\varepsilon = -P_o\beta_\varepsilon(o)\,.$$

Now from

$$K_T^{h,\varepsilon}(\varepsilon, e(\cdot)) \le K_T^h(\xi^{T,h}, e^{T,h}) \le C_h$$

we deduce

$$|\xi_\varepsilon|_H\,, \quad \int_o^T |e_\varepsilon|_E^2\,dt \int_o^T \langle D_2\alpha_\varepsilon, \alpha_\varepsilon\rangle\,dt < C_h$$
$$\frac{1}{\varepsilon}|\alpha_\varepsilon(T) - h|^2 < C_h\,, \qquad (3.24)$$

and since α_ε is the trajectory corresponding to e_ε and ξ_ε we have

$$\int_o^T \left\|\frac{d\alpha_\varepsilon}{dt}\right\|^2 dt + \int_o^T \|\alpha_\varepsilon\|^2\,dt < C_{h,T} \qquad (3.25)$$

(this last constant may depend on T). Now for any ξ, $e(\cdot)$ we have (necessary condition of optimality for the problem (3.16), (3.22)),

$$(P_o^{-1}\xi_\varepsilon, \xi) + \int_o^T \langle \Lambda_E e_\varepsilon, e\rangle\,dt + \int_o^T \langle D_2\alpha_\varepsilon, \alpha\rangle\,dt$$
$$+ \frac{1}{\varepsilon}(\alpha_\varepsilon(T) - h, \alpha(T)) = 0\,. \qquad (3.26)$$

Let us pick $\xi = \xi^{T,k}$, $e = e^{T,k}$ where k is arbitrary in H. We deduce from (3.26) and the estimates (3.24), (3.25), that

$$\left|\left(\frac{\alpha_\varepsilon(T) - h}{\varepsilon}, k\right)\right| \le C_{h,T,k} = C_k\,,$$

since in this context h, T are fixed. From the Banach-Steinhaus theorem it follows that $|(\alpha_\varepsilon(T) - h)/\varepsilon| \le C_{h,T}$. Therefore, also using (3.23), one has

$$\int_o^T \left\|\frac{d\beta_\varepsilon}{dt}\right\|^2 dt + \int_o^T \|\beta_\varepsilon\|^2\,dt < C_{h,T}\,. \qquad (3.27)$$

With the estimates (3.24), (3.25), (3.27) we can extract a subsequence α_ε, β_ε, converging weakly to $(d\bar{\alpha}/dt)$, $(d\bar{\beta}/dt)$, in $L_2(o, T; V')$. We obtain easily that $\bar{\alpha}, \bar{\beta}$ is a solution of (3.19), the proof has been completed. □

THEOREM 3.4. Under the assumptions of Theorem 3.3 one has

$$\bar{\beta}(T) = -Q(T)h\,, \tag{3.28}$$

where $Q(T) \in \mathcal{L}(H;H)$ symmetric positive semidefinite. Morever,

$$|Q(T)|_{\mathcal{L}(H;H)} \le C\,. \tag{3.29}$$

Proof. Consider (3.23). the quantities $\alpha_\varepsilon, \beta_\varepsilon$ are linear continuous functions of h. Hence we can write

$$\beta_\varepsilon(T) = -Q_{\varepsilon}(T)h\,, \tag{3.30}$$

subindex missing

where it is easily seen that $Q_\varepsilon(T) \in \mathcal{L}(H;H)$ symmetric, positive semidefinite. Moreover, one easily computes

$$\inf_{\xi, e(\cdot)} K_T^{h,\varepsilon}(\xi, e(\cdot)) = (Q_\varepsilon(T)h, h)\,. \tag{3.31}$$

Since the left hand side increases as ε decreases, it follows that $Q_\varepsilon(T)$ is an increasing family of symmetric positive semidefinite operators in $\mathcal{L}(H;H)$.

But

$$K_T^{h,\varepsilon}(\xi, e(\cdot)) = K_T^h(\xi, e(\cdot)) \qquad \forall\, \xi, e(\cdot) \text{ in } \Gamma_h\,.$$

Therefore

$$\begin{aligned}(Q_\varepsilon(T)h, h) &\le \inf_{\xi, e(\cdot)\in\Gamma_h} K_T^h(\xi, e(\cdot)) \\ &\le K_T^h(\xi^{T,h}, e^{T,h}) \le C_h\end{aligned} \tag{3.32}$$

independent of T and ε. Necessarily $Q_\varepsilon(T) \uparrow Q(T) \in \mathcal{L}(H;H)$ positive semidefinite and

$$|Q(T)|_{\mathcal{L}(H;H)} \le C \tag{3.33}$$

independent of T. But

$$\inf_{\xi, e(\cdot)} K_T^{h,\varepsilon}(\xi, e(\cdot)) \to \inf_{\xi, e(\cdot)\in\Gamma_h} K_T^h(\xi, e(\cdot))\,. \tag{3.34}$$

Indeed,

$$(P_o^{-1}\xi_\varepsilon, \xi_{\varepsilon}) + \int_o^T |e_\varepsilon|^2\, dt + \int_o^T \langle D_2\alpha_\varepsilon, \alpha_\varepsilon\rangle\, dt + \tfrac{1}{\varepsilon}|\alpha_\varepsilon(T) - h|^2$$

c-overbars missing

ε missing

$$\le (P_o^{-1}\bar{\xi}, \bar{\xi}) + \int_o^T |\bar{e}|^2\, dt + \int_o^T \langle D_2\bar{\alpha}, \bar{\alpha}\rangle\, dt\,,$$

and from the weak convergence

$$\le \underline{\lim}\left\{(P_o^{-1}\xi_\varepsilon, \xi_{\varepsilon}) + \int_o^T |e_\varepsilon|^2\, dt + \int_o^T \langle D_2\alpha_\varepsilon, \alpha_\varepsilon\rangle\, dt\right\}.$$

missing

This implies the strong convergence of ξ_ε, e_ε, α_ε, to $\bar{\xi}$, $\bar{e}$, $\bar{\alpha}$, and also that $(1/\varepsilon)|\alpha_\varepsilon(T) - h|^2 \to 0$.† Therefore (3.34) is demonstrated and thus

$$\inf_{\xi, e(\cdot) \in \Gamma_h} K_T^h(\xi, e(\cdot)) = (Q(T)h, h)\,. \tag{3.35}$$

Moreover, since $\beta_\varepsilon(T) \to \bar{\beta}(T)$ in H weakly, we deduce the property (3.28), too. □

REMARK 3.2. Before it was used in treating ill posed problems, the penalty technique (which goes back to Courant [7]), has been widely used as an approximation technique for the control of systems governed by partial differential equations, see A.V. Balakrishnan [3], J.L. Lions [13]. □

We can now compare $Q(T)$ and $P(T)$.

THEOREM 3.5. Assume that $A(\cdot)$, $C(\cdot)$ is detectable and $A(\cdot)$, $G(\cdot)$ is controllable. Then one has

$$Q(T)P(T) = P(T)Q(T) = I\,, \tag{3.36}$$

$$|Q(T)|_{\mathcal{L}(H;H)} \le q\,, \quad |P(T)|_{\mathcal{L}(H;H)} \le p\,, \tag{3.37}$$

where p, q are constants independent of T.

Proof. The property (3.37) has already been proven. The property (3.36) follows by comparing (3.1), (3.2) to (3.19), (3.28). Indeed in (3.19) set $h = -P(T)h$. Then we have $\bar{\alpha} = \hat{\alpha}$ and $\bar{\beta} = \hat{\beta}$ (by uniqueness) and

$$\bar{\beta}(T) = h = -Q(T)(-P(T)h) = Q(T)P(T)h\,.$$

A similar proof is made to prove that $PQ = I$. □

4. Observer Based Upon Kalman Filter

4.1 The Model

Motivated by the form of the Kalman filter see (3.28)), we shall define the *observer* by the solution of the equation

$$\frac{dm}{dt} + Am = f + PC^*R(z - Cm)\,, \tag{4.1}$$

$$m(o) = m_o$$

where m_o is arbitrary and z is the observation corresponding to the state (2.3).

The writing (4.1) is somewhat formal, since P is not defined on V'. In fact, we shall give a meaning to

$$y - m = \eta\,, \tag{4.2}$$

† This follows also directly from the fact that $|(\alpha_\varepsilon(T)-h)/\varepsilon|$ is bounded in ε.

where η appears as the solution of

$$\frac{d\eta}{dt} + (A + PD_2)\eta = 0\,, \qquad \eta(o) = y_o - m_o\,. \tag{4.3}$$

The solution $\eta(t)$ of (4.3) is defined by duality. Indeed, considering (3.1) we see that the equation

$$-\frac{d\beta^T}{dt} + (A^* + D_2P)\beta^T = 0\,, \qquad \beta^T(T) = h\,, \tag{4.4}$$

has a solution in the functional space

$$W_P(o,T) = \left\{\phi \in L^2(o,T;V),\ \frac{d\phi}{dt} \in L^2(o,T;V'),\ P\phi \in L^2(o,T;V)\right\}. \tag{4.5}$$

Then the value $\eta(T)$ is defined by

$$(\eta(T), h) = (\beta^T(o), y_o - m_o)\,, \qquad \forall\, h\,, \tag{4.6}$$

which defines $\eta(T)$ uniquely in H.

Our problem amounts to studying the behavior of $\eta(T)$ as $T \to 0$.

4.2 *Estimates*

We shall prove the following (main) result.

THEOREM 4.1. Assume $A(\cdot)$, $C(\cdot)$ detectable, and

$$\langle D_1(t)v, v\rangle \geq k|v|_H^2 \qquad \forall\, v \in V\,. \dagger \tag{4.7}$$

Then one has

$$|y(t) - m(t)| \leq C|y_o - m_o|e^{-\gamma t}\,, \qquad \gamma > 0\,. \tag{4.8}$$

Proof Considering the system (3.1) and the relation (4.6), we know that

$$(P(T)h, h) = (P_o\beta^T(o), \beta^T(o)) + \int_o^T \langle D_1\beta^T(s), \beta^T(s)\rangle\, ds + \int_o^T \langle D_2P\beta^T(s), P\beta^T(s)\rangle\, ds\,,$$

where we have written $\beta^T(t)$ instead of β to emphasize the dependence on T. Recall that $P\beta^T = -\bar{\alpha}$.

† Of course if $\langle D_1(t)v, v\rangle \geq k\|v\|_V^2$, (4.7) holds as well as the controllability property (see Proposition 3.2). But (4.7) is weaker.

In a similar manner we can write the relation

$$(P(T)h, h) = (P(t)\beta^T(t), \beta^T(t)) + \int_t^T \langle D_1\beta^T(s), \beta^T(s)\rangle\, ds$$
$$+ \int_t^T \langle D_2P\beta^T(s), P\beta^T(s)\rangle\, ds\,, \qquad (4.9)$$

which holds for any $t \in (o, T)$. Therefore it follows that

$$\frac{d}{dt}(P(t)\beta^T(t), \beta^T(t)) = \langle D_1(t)\beta^T(t), \beta^T(t)\rangle$$
$$+ \langle D_2(t)P(t)\beta^T(t), P(t)\beta^T(t)\rangle$$

and from (4.7)

$$\frac{d}{dt}(P(t)\beta^T(t), \beta^T(t)) \geq k|\beta^T(t)|^2\,. \qquad (4.10)$$

Note that, from Theorem 3.1 one has

$$(P(t)\beta^T(t), \beta^T(t)) \leq p|\beta^T(t)|^2\,.$$

Therefore (4.10) implies

$$\frac{d}{dt}|P^{1/2}(t)\beta^T(t)|^2 \geq \frac{k}{p}|P^{1/2}(t)\beta^T(t)|^2\,.$$

Hence

$$(P_o\beta^T(o), \beta^T(o)) \leq (P(T)h, h)e^{-(k/p)T} \leq p\, e^{-(k/p)T}\, |h|^2\,. \qquad (4.11)$$

Therefore, if

$$(P_o\xi, \xi) \geq \nu_o|\xi|^2\,,$$

we have

$$|\beta^T(o)|^2 \leq \frac{p}{\nu_o} e^{-(k/p)T}\, |h|^2\,.$$

From (4.6) it follows that

$$|\eta(T)| \leq |y_o - m_o| \sqrt{\frac{p}{\nu_o}}\, e^{-(k/2p)T}\,. \ \square$$

4.3 Example

Let us turn to the example considered in the introduction. We shall take

$$H = L^2(\Omega)\,, \qquad V = H^1(\Omega)\,,$$

$$E = H\,, \quad G = I\,, \quad \text{hence } D_1 = I\,,$$

$$F = L^2(\Gamma)\,, \quad C = \gamma = \text{trace operator}\,.$$

The system of optimality (3.1) looks as follows:

$$\frac{\partial \hat{\alpha}}{\partial t} - \Delta\hat{\alpha} + \hat{\beta} = 0, \quad \frac{\partial \hat{\alpha}}{\partial \nu}\Big|_{\Gamma} = 0,$$

$$-\frac{\partial \hat{\beta}}{\partial t} - \Delta\hat{\beta} = 0, \quad \frac{\partial \hat{\beta}}{\partial \nu}\Big|_{\Gamma} = \hat{\alpha},$$

$$\hat{\alpha}(o) = -P_o\hat{\beta}(o),$$

$$\hat{\beta}(T) = h. \tag{4.12}$$

The assumptions are satisfied. Indeed, the detectability condition is satisfied, since using the stationarity we can apply Proposition 3.1, and we have seen that the system

$$-\frac{\partial \gamma}{\partial t} - \Delta\gamma = 0,$$

$$\frac{\partial \gamma}{\partial \nu}\Big|_{\Gamma} + \gamma|_{\Gamma} = 0,$$

$$\gamma(o) = h,$$

has a solution in $L^2(o, \infty; V)$.

We can check, at least formally, that the operator $P(t)$ will satisfy the equation

$$\left(\frac{\partial}{\partial t}Pz, \zeta\right) + \left(\frac{\partial}{\partial x_i}Pz, \frac{\partial \zeta}{\partial x_i}\right) + \left(\frac{\partial z}{\partial x_i}, \frac{\partial P\zeta}{\partial x_i}\right) + \int_{\Gamma} PzP\zeta \, d\Gamma$$

$$= (z, \zeta) \qquad \forall\, z, \zeta \text{ in } H^1. \tag{4.13}$$

Representing $P(t)$ by a kernel

$$P(t)h(x) = \int P(x, \xi, t)\, h(\xi)\, d\xi$$

yields the equation for the kernel

$$\frac{\partial P}{\partial t} - \Delta_x P - \Delta_\xi P + \int_{\Gamma} P(x, \eta, t)\, P(\eta, \xi, t)\, d\Gamma = \delta(x - \xi), \tag{4.14}$$

with the boundary conditions

$$\frac{\partial P}{\partial \nu_x}(x, \xi, t) = 0, \quad \forall\, x \in \Gamma, \xi \text{ in } \Omega,$$

$$\frac{\partial P}{\partial \nu_\xi}(x, \xi, t) = 0, \quad \forall\, x \in \Omega, \xi \text{ in } \Gamma,$$

$$P(x, \xi, t) = P(\xi, x, t),$$

$$P(x, \xi, t) = P_o(x, \xi) \quad \text{(kernel of } P_o\text{)}. \tag{4.15}$$

The observer has been defined in (1.5) (the writing is formal).

REMARK 4.1. It is important to notice that the assumption (4.7) does not imply controllability (the situation is not similar to that of ordinary differential equations).

REMARK 4.2. It is fair to mention that (1.5) leads to more complex calculation than (1.3), but it is less affected by disturbances. Moreover, it respects the boundary condition of the initial system.

References

[1] Balakrishnan, A.V. *Applied Functional Analysis*. Springer Verlag, 1976.

[2] Balakrishnan, A.V. 1972. Stochastic control, a function space approach. *SIAM J. Control*, 10:285-297.

[3] Balakrishnan, A.V. 1968. On a new computing technique in optimal control. *SIAM J. Control*, 6:149-173.

[4] Baras, J.S., and Krishnaprasad, P.S. Dynamic observers as asymptotic limits of recursive filters. *IEEE Proc. 21st CDC, Orlando, Florida*. Dec 1982.

[5] Bensoussan, A. *Filtrage optimal des systèmes linéaires*. Paris: Dunod, 1971.

[6] Chapman, M.J., and Pritchard, A.J. Finite dimensional compensators for nonlinear infinite dimensional systems. Control Theory Centre, Report 110, Warwick University, 1982.

[7] Courant, R. 1943. Variational methods for the solution of problems of equilibrium and vibrations. *Bull. Amer. Math. Societ.* 49:1-23.

[8] Curtain, R.F. 1977. Stochastic evolution equations with general white noise disturbances. *J. Math. Analysis Applicat.*, 60:570-595.

[9] Curtain, R.F., and Pritchard, A.J. *Infinite Dimensional Linear Systems Theory*. Berlin: Springer Verlag, 1978.

[10] Ichikawa, A., and Pritchard, A.J. Existence, uniqueness and stability of nonlinear evolution equations. Control Theory Centre, Report 65, Warwick University, June 1977.

[11] Lattes, R., and Lions, J.L. *Méthode de quasi réversibilité et applications*. Paris: Dunod, 1967.

[12] Lions, J.L. *Equations différentielles opérationnelles et problèmes aux limites*. Springer Verlag, 1961.

[13] Lions, J.L. *Contrôle Optimal de systèmes gouvernés par des équations aux dérivées partielles*. Paris: Dunod, 1968.

[14] Lions, J.L. *Contrôle des systèmes distribués singuliers*. Paris: Gauthier Villars, 1983.

[15] Luenberger, D. 1966. Observer for multivariable systems. *IEEE Trans. Automatic Control*, vol. AC-11: 190-199.

[16] Sorine, M. 1981. Un résultat d'existence et d'unicité pour l'équation de Riccati stationnaire. *Rapp. INRIA* 55.