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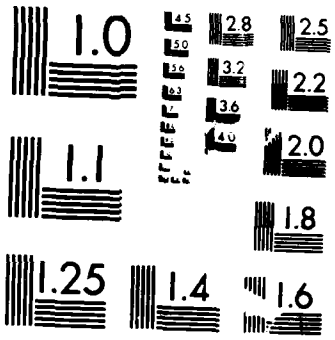

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ASYMPTOTIC EXPANSIONS FOR LARGE DEVIATION PROBABILITIES IN THE STRONG LAW OF LARGE NUMBERS

by

James Allen Fill
Stanford University

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TECHNICAL REPORT NO. 4
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1. Introduction and formulation of results.

Throughout this paper we use the notation $E(Y; A)$ as shorthand for $E(YI_A)$.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with common distribution function F and zero mean. Denote the random walk of partial sums by $S_n = \sum_{k=1}^{n} X_k$, with $S_0 \equiv 0$.

Let $\epsilon > 0$ and $\alpha \in \mathbb{R}$ be given, and denote by g the straight line boundary

$$g(t) = \alpha + \epsilon t.$$

From the weak and strong laws of large numbers (WLLN and SLLN), respectively, it follows that the sequences of boundary crossing probabilities

$$P\{S_m > g(m)\} \tag{1.1}$$

and

$$p_m = P\{S_n > g(n) \text{ for some } n \geq m\} \tag{1.2}$$

converge to zero as $m \to \infty$. Under certain assumptions on the moment generating function of F, Fill and Wichura (1987, theorems 0 and 1), extending work of Bahadur and Ranga Rao (1960) and Siegmund (1975), identified for each of (1.1) and (1.2) the rate of convergence to zero by providing an explicit expression equal to the probability in question up to a multiplicative factor $1 + o(1)$ as $m \to \infty$. The results differ according as F is non-lattice or lattice; the results in the lattice case simplify appreciably when ϵ is in the lattice for F and α is an integer multiple of the span.

The objective of this paper is to produce complete asymptotic expansions for (1.1) and (1.2) when appropriate conditions on F are met. When $\alpha = 0$ our results for (1.1) essentially reproduce those of Bahadur and Ranga Rao (1960); however, we indicate how to simplify their presentation in the lattice case. The asymptotic expansions for (1.2) are (beyond the dominant term) entirely new. We remark that analogous results can be obtained for the probabilities $P\{S_m \geq g(m)\}$ and $p_m^+ := P\{S_n \geq g(n) \text{ for some } n \geq m\}$, but we omit the details.

We suppose at the outset that the reader is familiar with the more basic paper by Fill and Wichura (1987), which treats primarily the lattice case but also summarizes previous work in the non-lattice case. We shall use freely the notation of that paper and shall refer to the list of assumptions in section 1 there, reproduced here for convenience and augmented here by the special case 2a′ of assumption 2a. (Here $K(\xi) = \log E(e^{\xi X_1})$ is the cumulant generating function (cgf) of X_1, and $I = \{\xi : K(\xi) < \infty\}$ with interior I^0.)

Assumption 0: $EX_1 = 0, \quad \text{Var } X_1 = 1.$

Assumption 1: There exists $\xi_0 \neq 0$ in I^0 for which $K'(\xi_0) = \epsilon$.

Assumption 1′: There exists $\xi_1 \neq 0$ in I^0 for which $K(\xi_1) = \epsilon\xi_1$.

Assumption 2a: F is non-lattice.

Assumption 2a′: F is strongly non-arithmetic, i.e.,

$$\limsup_{|t|\to\infty} |Ee^{itX_1}| < 1.$$

(This is also called Cramér's condition (C) for F.)

Assumption 2b: F is concentrated on a lattice $\{b + jh : j \in \mathbb{Z}\}$ having span $h > 0$ and (for definiteness) $b \in [0, h)$.

Assumption 2b′: Assumption 2b is met, ϵ is in the lattice for F, and α is an integer multiple of h.

Assumption 2a′ is met, for example, when F is absolutely continuous. We shall give asymptotic expansions for (1.1) and (1.2) when assumptions 0, 1, and either 2a′ or 2b are met. The results in the lattice case 2b simplify appreciably in the special case 2b′.

Assumption 1′ will not be used in this paper. Although the proofs of the lead-term results for p_m in Siegmund (1975) and Fill and Wichura (1987) require assumption 1′ to hold, the results themselves make no reference to any quantity derived from ξ_1. Indeed, a new argument for analyzing the difference $p_m - P\{S_m > g(m)\}$ due in part to D. Siegmund and incorporated into this paper renders assumption 1′ unnecessary for the results of both the earlier papers and the present work.

We shall refer to sections, displays, and results from three other papers as indicated by the following examples. Section F2.1 refers to section 2.1 in Fill and Wichura (1987). (B1) refers to display (1) in Bahadur and Ranga Rao (1960). Theorem S1 refers to theorem 1 in Siegmund (1975).

Theorem 1. *If assumptions* 0, 1, *and either* 2a′ *or* 2b′ *are met, then there exist constants* $c_1, c_2, \ldots$ *and* $d_1, d_2, \ldots$ *(all depending on* F, α, *and* ϵ*) such that, for any given positive integer* k,

$$P\{S_m > g(m)\} = a_m \left(1 + \sum_{j=1}^{k} c_j m^{-j} + O(m^{-(k+1)})\right) \tag{1.3}$$

and

$$p_m = (1+\gamma) a_m \left(1 + \sum_{j=1}^{k} d_j m^{-j} + O(m^{-(k+1)})\right) \tag{1.4}$$

as $m \to \infty$. *Here* γ *is defined by* (F1.12) *and* a_m *by* (F1.11) *and, according as assumption* 2a′ *or* 2b′ *is met,* (F1.13a) *or* (F1.13b). ∎

The coefficients c_j are given by (3.16) in case 2a′ and by (3.12) in case 2b′. The coefficients d_j are given by (4.54), with $(b_j^{\sim})$ defined by (4.52) and (2.1), in case 2a′ and by (4.26), with (ρ_j) defined by (4.40) and $(b_j^{\sim}\,|_{w=1})$ defined by (4.36) with $w = 1$ and (2.1), in case 2b′. The common ingredients $\mu_{r,s}$, ℓ_j, and $\nu_{k,\ell}$ in these formulas are defined by (3.9), (3.7), and (4.35), respectively; the polynomials $\mathcal{P}_s$ appearing in (3.9) are described in Section 2, and n is the standard normal density function.

The results are somewhat more complicated when assumption 2b, but not 2b′, is met.

Theorem 2. *If assumptions* 0, 1, *and* 2b *are met, then for each* $j = 1, 2, \ldots$ *there exist bounded sequences* $(c_{j,m})_{m\geq 1}$ *and* $(d_{j,m})_{m\geq 1}$ *(depending on* F, α, *and* ϵ*) such that, for any given positive integer* k,

$$P\{S_m > g(m)\} = a_m(1+\epsilon_m) \left(1 + \sum_{j=1}^{k} c_{j,m} m^{-j} + O(m^{-(k+1)})\right) \tag{1.5}$$

and

$$p_m = (1+\gamma)a_m(1+\lambda_m)\left(1+\sum_{j=1}^{k} d_{j,m}m^{-j} + O(m^{-(k+1)})\right) \tag{1.6}$$

as $m \to \infty$. *Here* γ *is defined by* (F1.12), a_m *by* (F1.11) *and* (F1.13b), e_m *by* (F1.15), *and* λ_m *by* (F1.16). ∎

The coefficients $c_{j,m}$ are given by (3.11); η_m therein is defined in (3.10) and (3.4). The coefficients $d_{j,m}$ are given by (4.25), with $\rho_{j,m}$ given by (4.34), wherein the sequences $(u_n)_{n\geq 0}$ and $(V_n^{(j)}(0)/j!)_{n\geq 0}$ have the respective generating functions (F1.17) and (4.38). In (4.38) the sequence $(b_j^{\sim})$ depends on w and is defined by (4.36) and (2.1). The other basic ingredients to these formulas are indexed following Theorem 1 above.

If $h^{-1}(\epsilon - b)$ is a rational number expressible as p/q ($p \in \mathbb{Z}$, $q \in \mathbb{Z}$) in lowest terms, then the sequences (e_m) and (λ_m) and, for each fixed j, $(c_{j,m})$ and $(d_{j,m})$ are periodic with period q. In particular, when assumption 2b′ is met (1.5) and (1.6) reduce to (1.3) and (1.4), respectively.

The main task of Section 2 is to record the refinements of the central limit theorem needed in analyzing (1.1) and (1.2). The analysis of (1.1) is carried out in Section 3; Sections 3.1 and 3.2 treat the cases 2b (including 2b′) and 2a′, respectively. The analysis of (1.2) is carried out in Section 4 and is likewise separated into the lattice and strongly non-arithmetic cases.

2. Preliminaries.

Throughout the sequel we suppose that assumptions 0 and 1 are in force. We use freely, and often without comment, the notation in sections F1 and F2. Much of this notation, including all of section F2.0, applies to all distributions F, both latice and general non-lattice. We write n for the standard normal density function.

We shall on several occasions make use of the following observation. Let $b_1, b_2, \ldots$ be a given sequence of complex numbers, and suppose that the power series $\sum_{k=1}^{\infty} b_k z^k$ is convergent for all complex z satisfying $|z| < R$ for some given $R > 0$. Then there exists a unique sequence $b_0^{\sim}, b_1^{\sim}, b_2^{\sim}, \ldots$ of complex numbers such that

$$\exp\left(\sum_{k=1}^{\infty} b_k z^k\right) = \sum_{j=0}^{\infty} b_j^{\sim} z^j \tag{2.1}$$

for $|z| < R$. The coefficients b_k and $b_j^{\sim}$ can be explicitly related. For example, the first few $b_j^{\sim}$'s are given in terms of the b_k's by

$$\begin{aligned} b_0^{\sim} &= 1, \\ b_1^{\sim} &= b_1, \\ b_2^{\sim} &= b_2 + \frac{1}{2} b_1^2, \\ b_3^{\sim} &= b_3 + b_1 b_2 + \frac{1}{6} b_1^3, \\ b_4^{\sim} &= b_4 + \frac{1}{2} b_2^2 + b_1 b_3 + \frac{1}{2} b_1^2 b_2 + \frac{1}{24} b_1^4. \end{aligned} \tag{2.2}$$

Expressions for $b_j^{\sim}$ up through $j = 10$ in terms of the b_k's can be obtained from display (3.37) in Kendall and Stuart (1977) by substituting $j! b_j^{\sim}$ and $k! b_k$, respectively, for μ_j' and κ_k there.

We shall have particular interest in the case that for $k \geq 0$, $b_k = \Lambda_{k+2} w^{k+2}$, where $(k+2)! \Lambda_{k+2}$ is the $(k+2)$nd cumulant of $\mathcal{L}_0(X/\sigma)$: in the notation of (F2.2) and (F1.6)

$$\phi_0(\eta/\sigma) = \frac{1}{2}\eta^2 + \sum_{k=3}^{\infty} \Lambda_k \eta^k \tag{2.3}$$

for η near 0. In this case $b_j^{\sim}$ is a polynomial in w of degree $3j$, even or odd according as j is

even or odd, which we shall denote by $\mathcal{P}_j$. The first few polynomials are

$$
\begin{aligned}
&\mathcal{P}_0(w) = w^0 \equiv 1, \\
&\mathcal{P}_1(w) = \Lambda_3 w^3, \\
&\mathcal{P}_2(w) = \Lambda_4 w^4 + \frac{1}{2}\Lambda_3^2 w^6, \\
&\mathcal{P}_3(w) = \Lambda_5 w^5 + \Lambda_3\Lambda_4 w^7 + \frac{1}{6}\Lambda_3^3 w^9 \\
&\mathcal{P}_4(w) = \Lambda_6 w^6 + \left(\frac{1}{2}\Lambda_4^2 + \Lambda_3\Lambda_5\right) w^8 + \frac{1}{2}\Lambda_3^2\Lambda_4 w^{10} + \frac{1}{24}\Lambda_3^4 w^{12}.
\end{aligned}
\tag{2.4}
$$

Except for the following modifications, the display (2.4) is identical with (B17). We have changed λ_j and P_j, respectively, in (B17) to Λ_j and $\mathcal{P}_j$ here to avoid notational conflict with (F1.16) and (F2.1). In (2.4) we have corrected a typographical error in the expression for $P_4(w)$ in (B17) by supplying the missing factor 1/2 in the coefficient of w^{10}.

At the heart of the proofs of Theorems 1 and 2 lies the use of certain refinements of the central limit theorem (CLT). Consider first the lattice case 2b. Under the distribution P_0 of (F1.3) (indeed, under any P_θ of (F2.1)), $(\sigma^2 m)^{-1/2} S_m$ is confined to the lattice $L_m := \{(\sigma^2 m)^{-1/2}(\alpha - h\delta_m + hr) : r \in \mathbb{Z}\}$, with δ_m given by (F1.14). As shown, e.g., by theorem 1 of section 51 in Gnedenko and Kolmogorov (1954), the local CLT in the lattice case has for each $k = 0, 1, \ldots$ the refinement

$$
P_0\{S_m = (\sigma^2 m)^{1/2} x\} = (\sigma^2 m)^{-1/2} h \left[\sum_{j=0}^{k} m^{-j/2} \mathrm{n}(x)\mathrm{p}(x) + O(m^{-(k+1)/2})\right] \tag{2.5}
$$

for $x \in L_m$. Here $\mathrm{p}_0(x) \equiv 1$ and for $j \geq 1$, p_j is the polynomial Feller (1971), in his theorem XVI.2.2, calls P_{j+2}; his F is our $\mathcal{L}_0(X)$. As Feller shows (but incorrectly states at the end of his section XVI.2),

$$
\mathrm{n}(x)\mathrm{p}_j(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-itx} \mathcal{P}_j(it)\mathrm{n}(t)dt \quad (x \in \mathbb{R}), \tag{2.6}
$$

where $\mathcal{P}_j$ is the polynomial described following (2.3) above. Although Gnedenko and Kolmogorov (1954) do not explicitly so state, the result (2.5) holds – as their proof shows –

uniformly in $x \in L_m$. Combining (2.5)-(2.6),

$$
\begin{aligned}
&P_0\{S_m = (\sigma^2 m)^{1/2} x\} \\
&\qquad = (2\pi\sigma^2 m)^{-1/2} h \left[\int_{-\infty}^{\infty} e^{-itx} f(t; m, k) dt + O(m^{-(k+1)/2}) \right]
\end{aligned} \tag{2.7}
$$

uniformly in $x \in L_m$, where for abbreviation we write

$$
f(t; m, k) := \left[\sum_{j=0}^{k} m^{-j/2} \mathcal{P}_j(it) \right] \mathrm{n}(t). \tag{2.8}
$$

There are local expansions analogous to (2.5) for non-lattice distributions, but the standard results (e.g., Feller, 1971, theorem XVI.2.2) require more stringent conditions for their validity than our assumption 2a′. The method used by Bahadur and Ranga Rao (1960) in our case 2a′ – and also, unnecessarily, in our case 2b – is to reduce the analysis of $P\{S_m > g(m)\}$, via an appropriate integration by parts, to the use of a certain refinement of the global CLT. This refinement, which we too shall employ in our case 2a′, is due to Cramér and may be found in Feller (1971, theorem XVI.4.3). In the notation we have set up, the conclusion is that

$$
P_0\{S_m \le (\sigma^2 m)^{1/2} x\} = \sum_{j=0}^{k} m^{-j/2} \int_{-\infty}^{x} \mathrm{n}(\xi) \mathrm{p}_j(\xi) d\xi + O(m^{-(k+1)/2}) \tag{2.9}
$$

uniformly in $x \in \mathbb{R}$. We observe, according to Feller (1971, theorem XVI.4.1), that the instance of (2.9) obtained by setting $k = 1$ applies to the more general case 2a, again uniformly in $x \in \mathbb{R}$ provided that $O(m^{-1})$ is changed to $o(m^{-1/2})$.

3. Weak-law large deviation probabilities.

The strategy for proving (1.3) and (1.5) is to tilt from P_{θ_0} in (F2.7) to P_0 and then apply (2.7) or (after an integration by parts) (2.9) and (2.6) according as assumption 2b or 2a′ obtains. The tilting is accomplished by (F2.11):

$$P\{S_m > g(m)\} = P_{\theta_0}\{S_m > \alpha\} = \exp(-m\phi(\theta_0))E_0(\exp(-|\theta_0|S_m); S_m > \alpha). \tag{3.1}$$

In Sections 3.1 and 3.2 we treat separately cases 2b and 2a′, respectively.

3.1. The lattice case. Suppose for Section 3.1 that assumption 2b is met. In this case

$$\begin{aligned} E_0(\exp(-|\theta_0|S_m); S_m > \alpha) &= \sum_{r=1}^{\infty} \exp[-|\theta_0|(\alpha - h\delta_m + hr)]P_0\{S_m = \alpha - h\delta_m + hr\} \\ &= \exp(-|\theta_0|\alpha)(1+e_m)\sum_{r=1}^{\infty}\exp(-|\theta_0|hr)P_0\{S_m = \alpha - h\delta_m + hr\}. \end{aligned} \tag{3.2}$$

Here δ_m and e_m are given, respectively, by (F1.14) and (F1.15). Now we apply (2.7) and Fubini's theorem to obtain

$$\begin{aligned} &(2\pi\sigma^2 m)^{1/2}h^{-1}\sum_{r=1}^{\infty}\exp(-|\theta_0|hr)P_0\{S_m = \alpha - h\delta_m + hr\} \\ &= \sum_{r=1}^{\infty}\int_{-\infty}^{\infty}\exp(-|\theta_0|hr)\exp(-ith(\sigma^2 m)^{-1/2}r)\exp(-it(\sigma^2 m)^{-1/2}(\alpha - h\delta_m))f(t;m,k)dt \\ &\quad + O(m^{-(k+1)/2}) \\ &= \int_{-\infty}^{\infty}[1-\exp(-|\theta_0|h)\exp(-ith(\sigma^2 m)^{-1/2})]^{-1}\exp(-|\theta_0|h)\exp(-ith(\sigma^2 m)^{-1/2}) \\ &\quad \times \exp(-it(\sigma^2 m)^{-1/2}(\alpha - h\delta_m))f(t;m,k)dt + O(m^{-(k+1)/2}) \\ &= [\exp(|\theta_0|h) - 1]^{-1} \\ &\quad \times \left\{\int_{-\infty}^{\infty}\frac{(1-y)\exp(-it(\sigma^2 m)^{-1/2}(\alpha + h\delta'_m))}{1 - y\exp(-it(\sigma^2 m)^{-1/2}h)}f(t;m,k)dt + O(m^{-(k+1)/2})\right\}. \end{aligned} \tag{3.3}$$

In (3.3) we have put

$$\delta'_m = 1 - \delta_m, \tag{3.4}$$

$$y = \exp(-|\theta_0|h) \tag{3.5}$$

for abbreviation. Combining (3.1)–(3.3),

$$(3.6)\quad P\{S_m > g(m)\} = a_m(1+\epsilon_m) \times \left\{ \int_{-\infty}^{\infty} \frac{(1-y)\exp(-it(\sigma^2 m)^{-1/2}(\alpha + h\delta'_m))}{1 - y\exp(-it(\sigma^2 m)^{-1/2}h)} f(t;m,k)dt + O(m^{-(k+1)/2}) \right\};$$

the remainder term $O(m^{-(k+1)/2})$ in (3.6) is $O(m^{-(k+1)/2})$ uniformly in $\alpha \in \mathbb{R}$.

To complete the proof of (1.5) we follow (B47)–(B50) in section B4. For any η and any $j = 0, 1, 2, \ldots$ let

$$(3.7)\qquad \ell_j(\eta) = \frac{1}{j!}\left\{ \frac{d^j}{dw^j}\left(\frac{(1-y)e^{-\eta w}}{1 - ye^{-w}} \right) \right\}_{w=0}.$$

(See (B52) with $u = y/(1-y)$ for examples.) It then follows easily from (B31) and (3.6) that

$$(3.8)\qquad \begin{aligned} &P\{S_m > g(m)\} \\ &= a_m(1+\epsilon_m)\left[\sum_{0\le j<(k+1)/2} m^{-j} \left\{ \sum_{r+s=2j} (h/\sigma)^r \ell_r(\eta_m)\mu_{r,s} \right\} + O(m^{-(k+1)/2}) \right], \end{aligned}$$

where (cf. (B30) and see (B35) and (B36) for examples)

$$(3.9)\qquad \mu_{r,s} := \int_{-\infty}^{\infty} (it)^r \mathcal{P}_s(it)\mathrm{n}(t)dt \quad (r, s = 0, 1, 2, \ldots)$$

and where

$$(3.10)\qquad \eta_m := h^{-1}\alpha + \delta'_m.$$

By replacing k with $2k+1$ in (3.8) we obtain (1.5), where

$$(3.11)\qquad c_{j,m} := \sum_{r+s=2j} (h/\sigma)^r \ell_r(\eta_m)\mu_{r,s}.$$

In passing from (3.6) to (1.5) we lose the uniformity in $\alpha \in \mathbb{R}$.

If assumption 2b′ is met, then for every m we have $\delta_m = \epsilon_m = 0$, $\delta'_m = 1$, and $\eta_m = h^{-1}\alpha + 1$, whence (1.5) reduces to (1.3) with

$$(3.12)\qquad c_j := \sum_{r+s=2j} (h/\sigma)^r \ell_r(h^{-1}\alpha + 1)\mu_{r,s}.$$

3.2. The strongly non-arithmetic case. Suppose for Section 3.2 that assumption 2a′ is met. In this case we find

$$
\begin{aligned}
E_0(\exp(-|\theta_0|S_m); S_m > \alpha) &= \int_{(0,\infty)} \exp(-|\theta_0|(\alpha + s))P_0\{S_m - \alpha \in ds\} \\
&= (\sigma^2 m)^{1/2}|\theta_0| \exp(-|\theta_0|\alpha) \\
&\quad \times \int_0^\infty \exp(-(\sigma^2 m)^{1/2}|\theta_0|x)[P_0\{S_m \le (\sigma^2 m)^{1/2} \times [(\sigma^2 m)^{-1/2}\alpha + x]\} \\
&\quad - P_0\{S_m \le (\sigma^2 m)^{1/2} \times (\sigma^2 m)^{-1/2}\alpha\}]dx
\end{aligned}
$$

by an integration by parts. Now we apply (2.9) to obtain

$$
\begin{aligned}
&\int_0^\infty \exp(-(\sigma^2 m)^{1/2}|\theta_0|x)[P_0\{S_m \le (\sigma^2 m)^{1/2}[(\sigma^2 m)^{-1/2}\alpha + x]\} \\
&\qquad - P_0\{S_m \le (\sigma^2 m)^{1/2} \times (\sigma^2 m)^{-1/2}\alpha\}]dx \\
&\quad = \int_0^\infty \exp(-(\sigma^2 m)^{1/2}|\theta_0|x) \left[\sum_{j=0}^{k} m^{-j/2} \int_{(\sigma^2 m)^{-1/2}\alpha}^{(\sigma^2 m)^{-1/2}\alpha + x} \mathrm{n}(\xi)\mathrm{p}_j(\xi)d\xi\right] dx \\
&\qquad + O(m^{-(k/2+1)}).
\end{aligned}
$$

By reversing the foregoing calculations we deduce

$$
\begin{aligned}
&E_0(\exp(-|\theta_0|S_m); S_m > \alpha) \\
&\quad = \exp(-|\theta_0|\alpha) \times \left\{\int_0^\infty \exp(-|\theta_0|s)\left[(\sigma^2 m)^{-1/2} \sum_{j=0}^{k} m^{-j/2}\mathrm{n}((\sigma^2 m)^{-1/2}(\alpha + s))\right.\right. \\
&\qquad \left.\left. \times \mathrm{p}_j((\sigma^2 m)^{-1/2}(\alpha + s))\right] ds + O(m^{-(k+1)/2})\right\}.
\end{aligned}
$$

Next we substitute (2.6) and use Fubini's theorem:

$$
\begin{aligned}
&(2\pi\sigma^2 m)^{1/2} \exp(|\theta_0|\alpha)E_0(\exp(-|\theta_0|S_m); S_m > \alpha) \\
&\quad = \int_0^\infty \int_{-\infty}^\infty \exp(-|\theta_0|s) \exp(-it(\sigma^2 m)^{-1/2}s) \\
&\qquad \times \exp(-it(\sigma^2 m)^{-1/2}\alpha)f(t; m, k)dt\ ds + O(m^{-k/2}) \\
&\quad = \int_{-\infty}^\infty [|\theta_0| + it(\sigma^2 m)^{-1/2}]^{-1} \exp(-it(\sigma^2 m)^{-1/2}\alpha)f(t; m, k)dt + O(m^{-k/2}) \\
&\quad = |\theta_0|^{-1}\left\{\int_{-\infty}^\infty [1 + it|\theta_0|^{-1}(\sigma^2 m)^{-1/2}]^{-1}\right. \\
&\qquad \left. \times \exp(-it(\sigma^2 m)^{-1/2}\alpha)f(t; m, k)dt + O(m^{-k/2})\right\}.
\end{aligned} \tag{3.13}
$$

Combining (3.1) and (3.13),

$$(3.14)\; P\{S_m > g(m)\} = a_m \times \left\{ \int_{-\infty}^{\infty} [1 + it|\theta_0|^{-1}(\sigma^2 m)^{-1/2}]^{-1} \exp(-it(\sigma^2 m)^{-1/2}\alpha) f(t; m, k) dt + O(m^{-k/2}) \right\};$$

the remainder term $O(m^{-k/2})$ in (3.14) is $O(m^{-k/2})$ uniformly in $\alpha \in \mathbb{R}$.

To complete the proof of (1.3) we follow (B29)–(B34) in section B3. It follows easily from (3.14) and (B31) that

$$P\{S_m > g(m)\} = a_m \left[\sum_{0 \le j < k/2} m^{-j} \left\{ \sum_{r+s=2j} \left(-\frac{1}{\sigma|\theta_0|} \right)^r \mu_{r,s} \sum_{r'=0}^{r} \frac{(|\theta_0|\alpha)^{r'}}{(r')!} \right\} + O(m^{-k/2}) \right]. \tag{3.15}$$

By replacing k with $2(k+1)$ in (3.15) we obtain (1.3), where

$$c_j := \sum_{r+s=2j} \left(-\frac{1}{\sigma|\theta_0|} \right)^r \mu_{r,s} \sum_{r'=0}^{r} \frac{(|\theta_0|\alpha)^{r'}}{(r')!}. \tag{3.16}$$

Unlike (3.14), (1.3) does not hold uniformly in $\alpha \in \mathbb{R}$.

4. Strong-law large deviation probabilities.

Throughout Section 4 we suppose that assumptions 0 and 1 are met. As in (F2.16), let

$$T_m = \inf\{n : n \geq m, S_n > \alpha\}, \tag{4.1}$$

the inf of the empty set being $+\infty$. Then (F2.17) holds:

$$p_m = P\{S_m > g(m)\} + P_{\theta_0}\{m < T_m < \infty\}. \tag{4.2}$$

The first term was analyzed in Section 3; we shall use the results (3.6) (in the lattice case 2b) and (3.14) (in the strongly non-arithmetic case 2a′). For the second term we observe

$$\begin{aligned} P_{\theta_0}\{m < T_m < \infty\} &= \int_{[0,\infty)} P_{\theta_0}\{m < T_m < \infty \mid \alpha - S_m = x\} P_{\theta_0}\{\alpha - S_m \in dx\} \\ &= \int_{[0,\infty)} P_{\theta_0}\{M > x\} P_{\theta_0}\{\alpha - S_m \in dx\}, \end{aligned} \tag{4.3}$$

where

$$M := \sup\{S_n : n \geq 0\} \tag{4.4}$$

is the (P_{θ_0}-a.s. finite and achieved) maximum value of the random walk S. In analyzing (4.3) we shall make use of a joint generating function under P_{θ_0} for M and the first epoch

$$L := \inf\{n \geq 0 : M = S_n\} \tag{4.5}$$

at which S achieves its maximum value, namely,

$$E_{\theta_0}(z^L \exp(\zeta M)) = \sum_{n=0}^{\infty} z^n E_{\theta_0}(\exp(\zeta M); L = n). \tag{4.6}$$

We now show that for complex z and ζ satisfying $|z| \leq 1$ and $\operatorname{Re} \zeta \leq |\theta_0|$ the random variable $z^L \exp(\zeta M)$ has under P_{θ_0} the finite expectation

$$\begin{aligned} \sum_{n=0}^{\infty} z^n E_{\theta_0}(\exp(\zeta M); L = n) &= (1+\gamma)[\tilde{u}(1)]^{-1} \\ &\times \exp\left[\sum_{n=1}^{\infty} z^n n^{-1} \exp(-n\phi(\theta_0)) E_0(\exp(-(|\theta_0| - \zeta)S_n); S_n > 0)\right]. \end{aligned} \tag{4.7}$$

Both the claim of finiteness and all interchanges in the following calculations proving (4.7) are justified by performing the same calculations when $z = 1$ and $\zeta = |\theta_0|$ and invoking Fubini's theorem. Letting $\tau^{(k)}$ denote the kth (strict ascending) ladder epoch for S (with $\tau^{(0)} = 0$) and $\tau = \tau^{(1)}$ we have

$$\begin{aligned}
\sum_{n=0}^{\infty} z^n E_{\theta_0}(\exp(\zeta M); L = n) &= E_{\theta_0}(z^L \exp(\zeta M)) \\
&= \sum_{k=0}^{\infty} E_{\theta_0}(z^{\tau^{(k)}} \exp(\zeta S_{\tau^{(k)}}); \tau^{(k)} < \infty, \tau^{(k+1)} = \infty) \\
&= \sum_{k=0}^{\infty} [E_{\theta_0}(z^{\tau} \exp(\zeta S_{\tau}); \tau < \infty)]^k P_{\theta_0}\{\tau = \infty\} \\
&= P_{\theta_0}\{\tau = \infty\}/[1 - E_{\theta_0}(z^{\tau} \exp(\zeta S_{\tau}); \tau < \infty)]. \qquad (4.8)
\end{aligned}$$

By a standard result in the theory of random walk (compare Chung (1974, theorem 8.4.2)),

$$\begin{aligned}
1 - E_{\theta_0}(z^{\tau} \exp(\zeta S_{\tau}); \tau < \infty) &= \exp\left[-\sum_{n=1}^{\infty} z^n n^{-1} E_{\theta_0}(\exp(\zeta S_n); S_n > 0)\right] \\
&= \exp\left[-\sum_{n=1}^{\infty} z^n n^{-1} \exp(-n\phi(\theta_0)) E_0(\exp(-(|\theta_0| - \zeta) S_n); S_n > 0)\right], \qquad (4.9)
\end{aligned}$$

which by (F2.31) reduces to

$$\begin{aligned}
P_{\theta_0}\{\tau = \infty\} &= \exp\left[-\sum_{n=1}^{\infty} n^{-1} \exp(-n\phi(\theta_0)) P\{S_n > \epsilon n\}\right] \\
&= (1 + \gamma)/\tilde{u}(1) \qquad (4.10)
\end{aligned}$$

upon setting $z = 1$ and $\zeta = 0$. Combination of (4.8)–(4.10) yields (4.7). We note from (F1.16) that in the lattice case 2b

$$1/\tilde{u}(1) = (1 + \lambda_m) \Big/ \left[\sum_{n=0}^{\infty} (1 + e_{m+n}) u_n\right]. \qquad (4.11)$$

In Sections 4.1–4.2 and 4.3–4.4 we analyze (4.3) separately for the cases 2b and 2a′, respectively. Our calculations will also demonstrate that (F1.10) and (F1.21) remain true in their respective setings when assumption 1′ is dropped.

4.1. The lattice case. Suppose for Sections 4.1–4.2 that assumption 2b is met. In this

case

$$P_{\theta_0}\{m < T_m < \infty\} = \sum_{0 \le x \in B_m} P_{\theta_0}\{M > x\} P_{\theta_0}\{\alpha - S_m = x\}, \tag{4.12}$$

where (see (F2.24)) B_m is the lattice for $\mathcal{L}_{\theta_0}(\alpha - S_m)$ (in fact, for $\mathcal{L}_\theta(\alpha - S_m)$ for any θ). To approximate $P_{\theta_0}\{\alpha - S_m = x\}$ we tilt from P_{θ_0} to P_0 and apply (2.7), yielding

$$\begin{aligned} P_{\theta_0}\{\alpha - S_m = x\} &= \exp(-m\phi(\theta_0)) \exp(-|\theta_0|(\alpha - x)) P_0\{\alpha - S_m = x\} \\ &= h(2\pi\sigma^2 m)^{-1/2} \exp(-m\phi(\theta_0) - |\theta_0|\alpha) \exp(|\theta_0| x) \\ &\quad \times \left\{ \int_{-\infty}^{\infty} \exp(-it(\sigma^2 m)^{-1/2}(\alpha - x)) f(t; m, k) dt + O(m^{-(k+1)/2}) \right\}, \end{aligned} \tag{4.13}$$

uniformly in $x \in B_m$ and $\alpha \in \mathbb{R}$. Combining (4.12)–(4.13),

$$\begin{aligned} P_{\theta_0}\{m < T_m < \infty\} &= a_m[\exp(|\theta_0| h) - 1] \\ &\quad \times \left\{ \int_{-\infty}^{\infty} Y_m(|\theta_0| + it(\sigma^2 m)^{-1/2}) \exp(-it(\sigma^2 m)^{-1/2}\alpha) f(t; m, k) dt + O(m^{-(k+1)/2}) \right\} \end{aligned} \tag{4.14}$$

holds uniformly in $\alpha \in \mathbb{R}$, where for each n we define

$$Y_n(\zeta) := \sum_{0 \le x \in B_n} P_{\theta_0}\{M > x\} \exp(\zeta x) \tag{4.15}$$

when $\operatorname{Re} \zeta \le |\theta_0|$.

In Section 4.2 below we show that the series defining $Y_n(\zeta)$ converges absolutely when $\operatorname{Re} \zeta \le |\theta_0|$ and that if also $\exp(\zeta h) \ne 1$ then

$$Y_m(\zeta) = [\exp(\zeta h) - 1]^{-1} \left[\sum_{n=0}^{\infty} \exp(\zeta h \delta_{m+n}) E_{\theta_0}(\exp(\zeta M); L = n) - \exp(\zeta h \delta_m) \right], \tag{4.16}$$

where we recall from (4.7) and (4.11) that the sequence $(E_{\theta_0}(\exp(\zeta M); L = n))_{n \ge 0}$ has generating function

$$\begin{aligned} &(1 + \gamma)(1 + \lambda_m) \left[\sum_{n=0}^{\infty} (1 + e_{m+n}) u_n \right]^{-1} \\ &\qquad \times \exp\left[\sum_{n=1}^{\infty} z^n n^{-1} \exp(-n\phi(\theta_0)) E_0(\exp(-(|\theta_0| - \zeta) S_n); S_n > 0) \right], \end{aligned} \tag{4.17}$$

valid for all complex z with $|z| \leq 1$. In particular, with y defined by (3.5) we find after some rearrangement

$$(4.18)\quad Y_m(|\theta_0| + it(\sigma^2 m)^{-1/2}) = \exp(-|\theta_0|h)$$
$$\times \exp(-it(\sigma^2 m)^{-1/2}h)[1 - y\exp(-it(\sigma^2 m)^{-1/2}h)]^{-1}$$
$$\times [(1+\gamma)(1+\lambda_m)\rho_m(t(\sigma^2 m)^{-1/2}) - (1+e_m)\exp(it(\sigma^2 m)^{-1/2}h\delta_m)].$$

Here for $x \in \mathbb{R}$

$$(4.19)\qquad \rho_m(x) := \left[\sum_{n=0}^{\infty}(1+e_{m+n})u_n\right]^{-1}\sum_{n=0}^{\infty}(1+e_{m+n})\exp(ixh\delta_{m+n})U_n(x),$$

the sequence $(U_n(x))_{n\geq 0}$ having generating function

$$(4.20)\qquad \tilde{U}(z;x) = \exp\left[\sum_{k=1}^{\infty} z^k k^{-1}\exp(-k\phi(\theta_0))E_0(\exp(ixS_k); S_k > 0)\right],$$

valid for all complex z with $|z| \leq 1$.

Now substitute (4.18) into (4.14) to obtain

$$(4.21)\, P_{\theta_0}\{m < T_m < \infty\} = a_m\left\{\int_{-\infty}^{\infty}\frac{(1-y)\exp(-it(\sigma^2 m)^{-1/2}(\alpha+h))}{1-y\exp(-it(\sigma^2 m)^{-1/2}h)}\right.$$
$$\times [(1+\gamma)(1+\lambda_m)\rho_m(t(\sigma^2 m)^{-1/2}) - (1+e_m)\exp(it(\sigma^2 m)^{-1/2}h\delta_m)]f(t;m,k)dt$$
$$\left. + O(m^{-(k+1)/2})\right\}.$$

Combine (4.2), (3.6), and (4.21) to yield

$$(4.22)\qquad p_m = (1+\gamma)a_m(1+\lambda_m)\left\{\int_{-\infty}^{\infty}\frac{(1-y)\exp(-it(\sigma^2 m)^{-1/2}(\alpha+h))}{1-y\exp(-it(\sigma^2 m)^{-1/2}h)}\right.$$
$$\left.\times\, \rho_m(t(\sigma^2 m)^{-1/2})f(t;m,k)dt + O(m^{-(k+1)/2})\right\},$$

the remainder estimate holding uniformly in $\alpha \in \mathbb{R}$. From here we shall travel momentarily a now familiar route. We shall expand the factor multiplying $f(t;m,k)$ in the integrand appearing in (4.22) into a power series in it and integrate term-by-term using (3.9) (recall (2.8)) to derive

$$(4.23)\qquad p_m = (1+\gamma)a_m(1+\lambda_m) \times \left[\sum_{0\le j<(k+1)/2} m^{-j} \left\{ \sum_{r+s=2j} \sigma^{-r}\mu_{r,s} \sum_{r'=0}^{r} h^{r'}\ell_{r'}(h^{-1}\alpha+1)\rho_{r-r',m} \right\} + O(m^{-(k+1)/2}) \right];$$

we have employed the notations (3.9), (3.7), and

$$(4.24)\qquad \rho_{j,m} := \rho_m^{(j)}(0)/(i^j j!),$$

the superscript (j) denoting, as usual, jth derivative. By replacing k with $2k+1$ in (4.23) we then obtain (1.6), where

$$(4.25)\qquad d_{j,m} := \sum_{r+s=2j} \sigma^{-r}\mu_{r,s} \sum_{r'=0}^{r} h^{r'}\ell_{r'}(h^{-1}\alpha+1)\rho_{r-r',m}\ ;$$

in passing from (4.22) to (1.6) we lose the uniformity in $\alpha \in \mathbb{R}$.

If assumption 2b′ is met, then none of $\delta_m \equiv 0$, $e_m \equiv 0$, $\lambda_m \equiv 0$, $\rho_m(x)$, and $\rho_{j,m}$ depends on m, whence (1.6) reduces to (1.4), where (with $\rho_{j,m} \equiv \rho_j$)

$$(4.26)\qquad d_j := \sum_{r+s=2j} \sigma^{-r}\mu_{r,s} \sum_{r'=0}^{r} h^{r'}\ell_{r'}(h^{-1}\alpha+1)\rho_{r-r'}.$$

We now show how to derive (4.23) from (4.22); the difficulty in comparison with the derivation of (3.8) from (3.3) is the dependence of the function ρ_m on m. We also indicate how in practice to compute the coefficients $\rho_{j,m}$ of (4.24).

We begin with some observations related to ρ_m. Recall that the functions $U_n : \mathbb{R} \to \mathbb{C}$ appearing in (4.19) are given implicitly by

$$(4.27)\qquad \sum_{n=0}^{\infty} U_n(x)z^n = \exp\left[\sum_{k=1}^{\infty} z^k k^{-1} \exp(-k\phi(\theta_0)) E_0(\exp(ixS_k); S_k > 0) \right],\ |z| \le 1.$$

Define functions $V_n : \mathbb{R} \to [0,\infty]$ implicitly by

$$(4.28)\qquad \sum_{n=0}^{\infty} V_n(x)z^n = \exp\left[\sum_{k=1}^{\infty} z^k k^{-1} \exp(-k\phi(\theta_0)) E_0(\exp(xS_k); S_k > 0) \right],\ |z| \le 1.$$

Now $0 \leq E_0(\exp(xS_k); S_k > 0) \leq E_0 \exp(xS_k) = \exp(k\phi(x))$ and $\phi(0) = 0$, so it follows easily that $\sum_{n=0}^{\infty} V_n(x)$ is a finite nondecreasing function of $x \in (-\infty, x_0)$ for some $x_0 > 0$. It is not much harder to see that each $\sum_{n=0}^{\infty} V_n^{(j)}(x)$, $j = 0, 1, \ldots$, is a finite nondecreasing function of $x \in (-\infty, x_0)$ for the same x_0. Each $V_n(x)$ is a linear combination with nonnegative coefficients of products of the expressions $k^{-1}\exp(-k\phi(\theta_0))E_0(\exp(xS_k); S_k > 0)$, $k \geq 0$; the corresponding $U_n(x)$ is the same linear combination of the same products of the expressions $k^{-1}\exp(-k\phi(\theta_0))E_0(\exp(ixS_k); S_k > 0)$, $k \geq 0$. It follows easily that

$$|U_n^{(j)}(x)| \leq V_n^{(j)}(0) \tag{4.29}$$

for every n and j and $x \in \mathbb{R}$, and that

$$U_n^{(j)}(0) = i^j V_n^{(j)}(0). \tag{4.30}$$

Likewise

$$\left|\frac{d^j}{dx^j}[\exp(ixh\delta_{m+n})U_n(x)]\right| \leq \left|\left\{\frac{d^j}{dx^j}[\exp(xh\delta_{m+n})V_n(x)]\right\}_{x=0}\right| \tag{4.31}$$

for every m and n and j and $x \in \mathbb{R}$, and

$$\left\{\frac{d^j}{dx^j}[\exp(ixh\delta_{m+n})U_n(x)]\right\}_{x=0} = i^j\left\{\frac{d^j}{dx^j}[\exp(xh\delta_{m+n})V_n(x)]\right\}_{x=0}. \tag{4.32}$$

We therefore draw the following conclusions:

(i) ρ_m is infinitely differentiable;

(ii) the derivatives of $[\sum_{n=0}^{\infty}(1+\epsilon_{m+n})u_n]\rho_m(x) = \sum_{n=0}^{\infty}(1+\epsilon_{m+n})\exp(ixh\delta_{m+n})U_n(x)$ can be computed term-by-term;

(iii) for every m and j and $x \in \mathbb{R}$

$$|\rho_m^{(j)}(x)| \leq j!\rho_{j,m}; \tag{4.33}$$

(iv) for every m and j

$$\rho_{j,m} = \left[\sum_{n=0}^{\infty}(1+\epsilon_{m+n})u_n\right]^{-1}\sum_{n=0}^{\infty}(1+\epsilon_{m+n})\left\{(j!)^{-1}\frac{d^j}{dx^j}[\exp(h\delta_{m+n}x)V_n(x)]\right\}_{x=0}$$

$$= \left[\sum_{n=0}^{\infty}(1+\epsilon_{m+n})u_n\right]^{-1}\sum_{n=0}^{\infty}(1+\epsilon_{m+n})\left\{\sum_{k=0}^{j}\frac{(h\delta_{m+n})^k}{k!}\frac{V_n^{(j-k)}(0)}{(j-k)!}\right\}. \tag{4.34}$$

Moreover, with

$$\nu_{k,0} := P_0\{S_k > 0\}, \quad \nu_{k,\ell} := E_0(S_k^+)^\ell \quad (\ell \geq 1) \tag{4.35}$$

and

$$b_\ell := \frac{1}{\ell!}\sum_{k=1}^{\infty} w^k k^{-1}\exp(-k\phi(\theta_0))\nu_{k,\ell} \quad (\ell \geq 1) \tag{4.36}$$

(with dependence of b_ℓ on w suppressed), we have (recalling (F1.17))

$$\begin{aligned}\sum_{n=0}^{\infty}V_n(x)w^n &= \exp\left[\sum_{k=1}^{\infty}w^k k^{-1}\exp(-k\phi(\theta_0))\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\nu_{k,\ell}\,x^\ell\right]\\ &= \tilde{u}(w)\exp\left(\sum_{\ell=1}^{\infty}b_\ell\,x^\ell\right)\\ &= \tilde{u}(w)\sum_{j=0}^{\infty}b_j^{\sim}x^j \end{aligned} \tag{4.37}$$

for $x < x_0$ in the notation of (2.1), dependence of $b_j^{\sim}$ on w being suppressed. It follows that $(V_n^{(j)}(0)/j!)_{n\geq 0}$ has generating function

$$\sum_{n=0}^{\infty}\frac{V_n^{(j)}(0)}{j!}w^n = \tilde{u}(w)b_j^{\sim}. \tag{4.38}$$

From (4.34) we thus obtain the bounds

$$0 < \exp(-|\theta_0|h)b_j^{\sim}\Big|_{w=1} < \rho_{j,m} < \exp(|\theta_0|h)\sum_{k=0}^{j}\frac{h^k}{k!}b_{j-k}^{\sim}\Big|_{w=1} < \infty. \tag{4.39}$$

At last we are in position to complete the derivation of (4.23). If one substitutes

$$A(t;m,k) := \left[\sum_{j=0}^{k}\ell_j(h^{-1}\alpha+1)(it(\sigma^2 m)^{-1/2}h)^j\right] \times \left[\sum_{j=0}^{k}\rho_{j,m}(it(\sigma^2 m)^{-1/2})^j\right] \times f(t;m,k)$$

for the integrand appearing in (4.22), the error incurred in calculating the integral is by the above discussion (especially (4.33) and (4.39))

$$O\left(\int_{-\infty}^{\infty} |t(\sigma^2 m)^{-1/2}|^{k+1} \mathrm{n}(t)dt\right) = O(m^{-(k+1)/2}).$$

But $\int_{-\infty}^{\infty} A(t; m, k)dt = \sum_{0\le j<(k+1)/2} d_{j,m} m^{-j} + O(m^{-(k+1)/2})$, so (4.23) follows.

Remark. (a) Note $\rho_{0,m} = 1$ for every m. To compute $d_{1,m}$ (respectively, $d_{2,m}$) of (4.25), the values of $\rho_{j,m}$ for $j = 0, 1, 2$ (respectively, $j = 0, 1, 2, 3, 4$) are required, and so one needs to compute $b_j^{\sim}$ in (4.38) for $j = 0, 1, 2$ (respectively, $j = 0, 1, 2, 3, 4$) from the b_ℓ's, $\ell = 1, 2$ (respectively, $\ell = 1, 2, 3, 4$), in (4.36) à la (2.2).

(b) If assumption 2b′ is met, then (4.34) reduces to

$$\rho_j = b_j^{\sim}\Big|_{w=1}. \tag{4.40}$$

More generally, if (recalling the notation of assumption 2b) $h^{-1}(\epsilon - b)$ is rational, then for each j the numbers $V_n^{(j)}(0)/j!$ appearing in (4.34) can be expressed explicitly, rather than implicitly as in (4.38). This is done using the Fourier method of section F4; we omit the details.

(c) By taking $k = 0$ in (1.6) it follows that assumption 1′ is extraneous for (F1.21) and in case 2b′ for (F1.10)

4.2. Proof of (4.16). Both the claim of absolute convergence following (4.15) and all interchanges in the following calculations proving (4.16) are justified by performing the same calculations when $\zeta = |\theta_0|$ and invoking Fubini's theorem. We begin by recalling (4.5) and observing from (4.15)

$$\begin{aligned} Y_m(\zeta) &= \sum_{0\le x\in B_m} \exp(\zeta x) \sum_{n=0}^{\infty} P_{\theta_0}\{L = n, M > x\} \\ &= \sum_{0\le x\in B_m} \exp(\zeta x) \sum_{n=0}^{\infty} \sum_{y: y>x,\ o-y\in B_n} P_{\theta_0}\{L = n, M = y\} \\ &= \sum_{n=0}^{\infty} \sum_{y: y>0,\ o-y\in B_n} P_{\theta_0}\{L = n, M = y\} \sum_{x: 0\le x<y,\ x\in B_m} \exp(\zeta x). \end{aligned} \tag{4.41}$$

Generalizing the last calculation preceding (F2.41) we find for $\alpha - y \in B_n$

$$\sum_{x:0\le x<y,\ x\in B_m} \exp(\zeta x) = [\exp(\zeta h) - 1]^{-1}[\exp(\zeta h\delta_{m+n})\exp(\zeta y) - \exp(\zeta h\delta_m)],$$

provided, of course, that $\exp(\zeta h) \neq 1$. Thus

$$\begin{aligned}
(4.42a)\quad Y_m(\zeta) &= [\exp(\zeta h) - 1]^{-1} \sum_{n=0}^{\infty} \sum_{y:y>0,\ \alpha-y\in B_n} P_{\theta_0}\{L = n, M = y\} \\
&\quad \times [\exp(\zeta h\delta_{m+n})\exp(\zeta y) - \exp(\zeta h\delta_m)] \\
(4.42b)\qquad &= [\exp(\zeta h) - 1]^{-1} \left[\sum_{n=0}^{\infty} \exp(\zeta h\delta_{m+n}) E_{\theta_0}(\exp(\zeta M); L = n) - \exp(\zeta h\delta_m) \right].
\end{aligned}$$

In passing from (4.42a) to (4.42b), i.e., to the desired (4.16), we have relaxed the inequality $y > 0$ in (4.42a) to $y \ge 0$. This does not change the value of the sum in (4.42a), because if $\alpha \in B_n$, then $\delta_{m+n} = \delta_m$.

4.3. The strongly non-arithmetic case. Suppose for Sections 4.3–4.4 that assumption 2a′ is met. In this case we integrate (4.3) by parts:

$$(4.43)\quad P_{\theta_0}\{m < T_m < \infty\} = \int_{(0,\infty)} [P\{S_m > \alpha - y + \epsilon m\} - P\{S_m > \alpha + \epsilon m\}] P_{\theta_0}\{M \in dy\}.$$

Now use (3.14) together with the fact that the remainder estimate therein holds uniformly in α to deduce via Fubini's theorem that

$$\begin{aligned}
&P\{S_m > \alpha - y + \epsilon m\} - P\{S_m > \alpha + \epsilon m\} \\
&\quad = a_m \Big\{ \int_{-\infty}^{\infty} [1 + it|\theta_0|^{-1}(\sigma^2 m)^{-1/2}]^{-1} \exp(-it(\sigma^2 m)^{-1/2}\alpha) \\
&\qquad \times [\exp((|\theta_0| + it(\sigma^2 m)^{-1/2})y) - 1] f(t; m, k)\,dt + O(m^{-k/2}) \Big\} \\
(4.44)\qquad &\quad = a_m \Big\{ \int_0^y \int_{-\infty}^{\infty} |\theta_0| \exp((|\theta_0| + it(\sigma^2 m)^{-1/2})x) \exp(-it(\sigma^2 m)^{-1/2}\alpha) \\
&\qquad \times f(t; m, k)\,dt\,dx + O(m^{-k/2}) \Big\}
\end{aligned}$$

holds uniformly in $\alpha \in \mathbb{R}$ and $y \in \mathbb{R}$. By reversing the foregoing calculations and again using

Fubini's theorem we find that

$$P_{\theta_0}\{m < T_m < \infty\} = a_m \Big\{ \int_0^\infty P_{\theta_0}\{M > x\} \int_{-\infty}^\infty |\theta_0| \exp((|\theta_0| + it(\sigma^2 m)^{-1/2})x)$$
$$\times \exp(-it(\sigma^2 m)^{-1/2}\alpha) f(t; m, k) dt\, dx + O(m^{-k/2}) \Big\}$$

$$(4.45) \quad = a_m \Big\{ \int_{-\infty}^\infty |\theta_0| Y(|\theta_0| + it(\sigma^2 m)^{-1/2}) \exp(-it(\sigma^2 m)^{-1/2}\alpha) f(t; m, k) dt + O(m^{-k/2}) \Big\}$$

holds uniformly in $\alpha \in \mathbb{R}$ where we define

$$(4.46) \qquad Y(\zeta) := \int_0^\infty P_{\theta_0}\{M > x\} \exp(\zeta x) dx$$

when $\operatorname{Re} \zeta \le |\theta_0|$.

In Section 4.4 below we show that the integral defining (4.46) converges absolutely when $\operatorname{Re} \zeta \le |\theta_0|$ and that if also $\zeta \ne 0$ then

$$(4.47) \quad Y(\zeta) = \zeta^{-1} \Big\{ (1+\gamma)[\tilde{u}(1)]^{-1}$$
$$\times \exp\Big[\sum_{n=1}^\infty n^{-1} \exp(-n\phi(\theta_0)) E_0(\exp(-(|\theta_0| - \zeta) S_n); S_n > 0) \Big] - 1 \Big\}.$$

In particular,

$$|\theta_0| Y(|\theta_0| + it(\sigma^2 m)^{-1/2})$$
$$(4.48) \qquad = [1 + it|\theta_0|^{-1}(\sigma^2 m)^{-1/2}]^{-1} \{(1+\gamma)[\tilde{u}(1)]^{-1} \tilde{U}(1; t(\sigma^2 m)^{-1/2}) - 1\};$$

we have employed the notation (4.20).

Now substitute (4.48) into (4.45) to obtain

$$(4.49) \qquad P_{\theta_0}\{m < T_m < \infty\} = a_m \Big\{ \int_{-\infty}^\infty [1 + it|\theta_0|^{-1}(\sigma^2 m)^{-1/2}]^{-1}$$
$$\times [(1+\gamma)(\tilde{u}(1))^{-1} \tilde{U}(1; t(\sigma^2 m)^{-1/2}) - 1]$$
$$\times \exp(-it(\sigma^2 m)^{-1/2}\alpha) f(t; m, k) dt + O(m^{-k/2}) \Big\}.$$

Combine (4.2), (3.14), and (4.49) to yield

$$(4.50)\qquad p_m = (1+\gamma)a_m\left\{\int_{-\infty}^{\infty}[1+it|\theta_0|^{-1}(\sigma^2 m)^{-1/2}]^{-1}(\tilde{u}(1))^{-1}\tilde{U}(1;t(\sigma^2 m)^{-1/2})\right.$$
$$\left.\times\exp(-it(\sigma^2 m)^{-1/2}\alpha)f(t;m,k)dt + O(m^{-k/2})\right\},$$

the remainder estimate holding uniformly in $\alpha \in \mathbb{R}$.

Finally, from the discussion in Section 4.1 (see especially (4.29) and (4.38)) above it follows that

$$(4.51)\qquad (\tilde{u}(1))^{-1}\tilde{U}(1;x) = \sum_{j=0}^{k}(ix)^j b_j^{\sim} + O(x^{k+1}),$$

where in the present situation $b_j^{\sim}$ $(j \geq 0)$ is defined in terms of

$$(4.52)\qquad b_\ell := \frac{1}{\ell!}\sum_{k=1}^{\infty}k^{-1}\exp(-k\phi(\theta_0))\nu_{k,\ell}\quad(\ell \geq 1),$$

with $\nu_{k,\ell}$ given by (4.35), via (2.1). It follows routinely from this that

$$(4.53)\quad p_m = (1+\gamma)a_m$$
$$\times\left[\sum_{0\leq j<k/2}m^{-j}\left\{\sum_{r+s=2j}\left(\frac{-1}{\sigma|\theta_0|}\right)^r\mu_{r,s}\sum_{r'=0}^{r}|\theta_0|^{r'}\sum_{r''=0}^{r'}\frac{\alpha^{r''}}{(r'')!}(-1)^{r'-r''}b_{r'-r''}^{\sim}\right\}\right.$$
$$\left.+O(m^{-k/2})\right].$$

By replacing k with $2(k+1)$ in (4.53) we obtain (1.4), where

$$(4.54)\qquad d_j := \sum_{r+s=2j}\left(\frac{-1}{\sigma|\theta_0|}\right)^r\mu_{r,s}\sum_{r'=0}^{r}|\theta_0|^{r'}\sum_{r''=0}^{r'}\frac{\alpha^{r''}}{(r'')!}(-1)^{r'-r''}b_{r'-r''}^{\sim};$$

in passing from (4.50) to (1.4) we lose the uniformity in $\alpha \in \mathbb{R}$.

Remark. If in the above calculations assumption 2a′ is relaxed to assumption 2a, then (3.14) continues to hold uniformly in $\alpha \in \mathbb{R}$ for $k = 1$ when $O(m^{-1/2})$ is changed to $o(1)$. Thus (4.44) remains true when $O(m^{-1/2})$ is changed to $o(1)$, as therefore does (4.53). We conclude that assumption 1′ is extraneous for (F1.10) in case 2a and hence (by remark (c) at the end of Section 4.1 above) in general.

4.4. Proof of (4.47). Both the claim of absolute convergence following (4.46) and the interchange in the following calculations proving (4.47) are justified by performing the same calculations when $\zeta = |\theta_0|$ and invoking Fubini's theorem. Observe from (4.46)

$$\begin{aligned} Y(\zeta) &= \int_0^\infty \int_{(x,\infty)} P_{\theta_0}\{M \in dy\} \exp(\zeta x) dx \\ &= \int_{(0,\infty)} \int_0^y \exp(\zeta x) dx \; P_{\theta_0}\{M \in dy\} \\ &= \zeta^{-1} \int_{(0,\infty)} [\exp(\zeta y) - 1] P_{\theta_0}\{M \in dy\}, \end{aligned} \tag{4.55}$$

provided, of course, that $\zeta \neq 0$. Thus, since the range of integration in the last integral can be expanded to $[0, \infty)$ without effect, we have

$$Y(\zeta) = \zeta^{-1}[E_{\theta_0} \exp(\zeta M) - 1]. \tag{4.56}$$

Setting $z = 1$ in (4.7) we find

$$\begin{aligned} &E_{\theta_0} \exp(\zeta M) \\ &= (1+\gamma)[\tilde{u}(1)]^{-1} \exp\left[\sum_{n=1}^\infty n^{-1} \exp(-n\phi(\theta_0)) E_0(\exp(-(|\theta_0| - \zeta)S_n); S_n > 0)\right]. \end{aligned} \tag{4.57}$$

Combine (4.56) and (4.57) to obtain the desired (4.47).

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20. Abstract.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with common distribution function F having zero mean, and let (S_n) be the random walk of partial sums. The weak and strong laws of large numbers, respectively, imply that for any $\alpha \in \mathbb{R}$ and $\epsilon > 0$ the probabilities $P\{S_m > \alpha + \epsilon m\}$ and

$$p_m := P\{S_n > \alpha + \epsilon n \text{ for some } n \geq m\}$$

tend to 0 as m tends to ∞. Building upon work of Bahadur and Ranga Rao [Ann. Math. Statist. **31**(1960): 1015–1027], Siegmund [Z. Wahrscheinlichkeitstheorie verw. Gebiete **31**(1975): 107–113], and Fill and Wichura [University of Chicago Technical Report No. 211 (1987)], we produce complete asymptotic expansions for the probabilities $P\{S_m > \alpha + \epsilon m\}$ and p_m.

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