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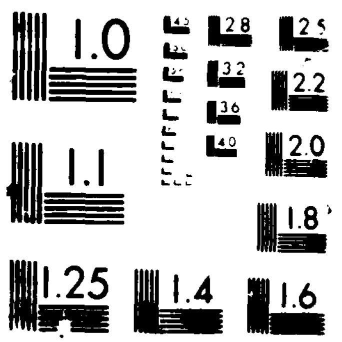
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ON THE MANEUVERING OF VEHICLES*

JAMES C. ALEXANDER† AND J. H. MADDOCKS‡

Abstract. Equations are derived to govern the motion of vehicles which move on rolling wheels. A relation between the centers of curvature of the trajectories of the wheels and the center of rotation of the vehicle is established. From this relation the general kinematic laws of motion are derived. Applications to questions of offtracking (the difference between the trajectories of the front and back wheels of the vehicle) and optimal steering (how to steer around a tight corner) are considered.

Key words. Euler-Savary formulae, offtracking, optimal steering

AMS(MOS) subject classifications. 70B99, 70F25

1. Introduction. The purpose of this paper is to discuss the maneuvering of vehicles such as buses or articulated trucks which ride on wheels. We assume the motion is sufficiently slow that pure rolling occurs; that is, effects such as gliding, skidding and drift of wheels are neglected. It will be seen that there is a condition in order that this assumption be valid—for example it is not exactly true for a truck trailer with two fixed rear axles, but in any event it leads to a good approximation to the true low-speed motion. Our main interest is in vehicles with a fixed axle that are steered by prescribing the motion of some point, such as the hitch of a trailer or the front wheel of an automobile. However we will also discuss the case in which there are no fixed axles.

The motions are assumed to be planar, and each component of the vehicle will be regarded as a planar rigid body. Thus an automobile is regarded as one planar rigid body, and a truck consisting of cab and one trailer is viewed as two connected planar rigid bodies. The first issue is to determine the restricted class of planar rigid-body motions that correspond to trajectories for which all the wheels roll. The problem is purely kinematic with the rolling condition entering as a nonholonomic constraint. For the sake of convenience, the terms "speed" and "velocity" will be used throughout, but time is to be interpreted as an arbitrary parameterization. In other words, we are concerned with the trajectory of the vehicle, but not with the rate at which that trajectory is traversed.

We first work in complete generality, allowing the wheels to pivot like castors, and determine a simple compatibility condition for the rolling condition to be satisfied. Using this, we derive a relation between the radius of rotation of the body and the radii of curvatures of the trajectories of the wheels. The result is completely general. It includes the case of turning axles (i.e., noncircular turns) and emphasizes the fact that the radius of rotation and the radii of curvatures are different. It is a new example of an Euler-Savary formula; such formulae usually arise in the context of the design of mechanisms [6, esp. Chap. VIII]. We show that no generality is lost when only the case of one fixed axle is considered, and we derive a differential equation governing this case. Illustrative applications are presented. In the final section we use the kinematic analysis to solve an optimal steering problem, namely the analogue for rolling bodies of the "mathematician's sofa problem" — how to maneuver an object around a corner.

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A word about the genesis of this paper: Basically the problem treated involves straightforward 19th-century mechanics and the paper might be taken as a case of "reinventing the wheel." However it seems to be quite easy to complicate the analysis or to mis-analyze the situation completely, and in particular to confuse the radius of rotation of the body and the radii of curvature of the wheels. The first-named author, motivated by [8] (see also [7]), submitted a problem to the *SIAM Review* problem section [1] (essentially the corollary to Proposition 2). The second-named author, intrigued by the problem (due in part to his experience backing up boat trailers), submitted a solution. We were surprised when the solutions were published to find it was reported that there were a disproportionate number of incorrect solutions submitted (the second published solution also seems inadequate). Other authors have done extremely complicated or ad hoc analyses to explain some aspect of the problem of maneuvering. Accordingly, the authors have here tried to present the most simple and general discussion of the problem.

There is also a practical aspect to the problem of rolling. The professionals in truck and road design are concerned with "offtracking," the amount the path of a trailer or the rear of a bus diverges from the path of the front. Such people are not interested in exposing the mathematical aspects of the problem, but in quick, reliable rules, usually with a margin of error, for design work. Errors of more than a meter are deemed acceptable in certain cases. A variety of tabular and visual methods have been developed — even a micro-computer program — to determine offtracking, especially for the cases of piecewise circular steering [2 (esp. Chap. II, Fig. II.1–II.11 and Table II.2)], [3], [12], [13], [14], [15], [18].

Our use of centers of curvature, suggested for mathematical reasons, leads to simple general formulae such as (3.1) below, which in turn lead to equations of motion which can be explicitly solved in many cases of interest, as is shown in section 5. Accordingly it may be possible to make more explicit analyses of complicated steering and control problems. There are questions of this kind in the design of roads and vehicles which remain to be studied, especially issues concerning articulated (possibly multiply-articulated) vehicles. See for example [9].

The structure of the paper is as follows. In section 2, we describe our mathematical model of a vehicle on rolling wheels. In section 3 we investigate the general kinematics of such vehicles and in particular develop an Euler-Savary formula relating the center of rotation of the vehicle with the centers of curvature of the trajectories of the axles. In section 4, we consider the most interesting case, that of vehicles with one fixed axle. In section 5, we use the kinematics of sections 3 and 4 to obtain the equations of some special trajectories. In section 6, the offtracking of vehicles going around a corner is discussed. This analysis is more general than that usually found in the literature. In section 7, we determine optimal steering around a corner, and determine how tight a corner a given vehicle can traverse.

2. The model. Each component of a vehicle is modelled as a planar rigid body. Consequently, Chasles' theorem (or Descartes' principle of instantaneous motion) implies that at each instant the motion of a component in the plane is either (i) a pure rotation about some point, or (ii) a pure translation (the limiting case of (i) with center of rotation at infinity). Each component is assumed to be mounted on some number of axles that are parallel to the horisontal plane. Each axle has wheels mounted on it; the wheels are assumed to be disks without thickness which turn independently of each other. Thus the rolling constraint is manifested in the direction in which the axles are pointed. The axles may be rigid in the body, or free to pivot about some point in the body. Since the object of discussion is the motion of the body, not of the wheels, we can simplify the discussion by assuming that each axle has precisely one wheel which is attached directly below the pivot point of the axle. If the axle assembly is more complicated, for example an offset castor, and the motion of the axle assembly is important, it itself can be considered as a separate rigid body. The condition of rolling dictates that the direction of the velocity of the axle pivot point is perpendicular to the the axle. Thus we call the axle pivot points *constraint points* of the component of the body and the directions of the velocities at those points the *constraint directions*.

3. Rolling conditions. The following elementary, but basic, result is an immediate consequence of Chasles' theorem.

PROPOSITION 1. *For rolling to occur, the normals to the constraint velocities through all the constraint points, at any instant, all intersect at one point, namely the instantaneous center of rotation (or infinity in the case of translation).*

For example, the front wheels of a turning automobile cannot be parallel if the wheels are to roll. The line through the rear axle and the perpendiculars to the two front wheels all intersect at one point. The inner wheel must turn at a sharper angle than the outer wheel. The problem of designing a steering linkage to effect the proper angles is an interesting one in its own right (see for example [16], [17], [19] and the references therein).

To consider steering control problems, it is natural to specify the constraint directions with respect to some direction fixed in the body. If this angle is specified as a function of time, we define the (angular) *pivot velocity* as the rate of change of the angle. We next set some notation. Assume that the trajectories of the constraint points are smooth. Let Ω denote the (signed) angular velocity of the body about its center of rotation. For any point x of the body, let $r = r(x)$ denote the distance from that point to the center of rotation. It is convenient to let the r be signed distances such that $\Omega r \geq 0$ (if $\Omega = 0$, then r is infinite). In particular, if x_i is the ith constraint point, let $r_i = r(x_i)$. If $s(x)$ is the arc length on the path traced out by x and $\Omega \neq 0$, then $ds/dt = \Omega r$. Let κ_i denote the curvature of the trajectory traced out by x_i. As we are dealing with planar motion, curvature may (and is) assumed to be a quantity with sign. Let ω_i denote the ith angular pivot velocity. The above quantities are all functions of time, but we suppress the dependence in the notation. The following is the main consequence of Proposition 1.

PROPOSITION 2. *Suppose $\Omega \neq 0$. Then*

$$\kappa_i r_i = \left(1 + \frac{\omega_i}{\Omega}\right). \tag{3.1}$$

Proof. Let Θ_i denote the directed angle from a direction fixed in space to the ith constraint direction. Let Θ_b denote the directed angle from the direction fixed in space to a direction fixed in the body. And let θ_i denote the directed angle from the direction fixed in the body to the ith constraint direction. Differentiation of the relation

$$\Theta_i = \Theta_b + \theta_i \tag{3.2}$$

yields

$$\frac{d\Theta_i}{dt} = \frac{d\Theta_b}{dt} + \frac{d\theta_i}{dt} = \Omega + \omega_i. \tag{3.3}$$

However

$$\frac{ds_i}{dt} = \Omega r_i \tag{3.4}$$

and

$$\kappa_i = \frac{d\Theta_i}{ds_i}. \tag{3.5}$$

Use of the chain rule and equations (3.4)-(3.5) in (3.3) establish the result.

Remarks.(i). If $\Omega = 0$ for some interval of time, the motion is a pure translation and the trajectories of all points are translates of each other (not necessarily straight lines). If the common velocity is v, the common curvature is $\kappa = \kappa_i = \omega/|v|$, where ω is the common value of the pivot velocities ω_i. If Ω is instantaneously zero, then the motion is instantaneously a translation with velocity v. The limit of (3.1), after division by r_i yields the expression

$$\kappa_i = \frac{\omega_i}{|v|} \tag{3.6}$$

for the curvature of the ith path. Here the ω_i may be distinct.

(ii). It is possible that $r_i = 0$ for some i. Proposition 2 then implies that $\omega_i = -\Omega$ in order that the ith constraint path have a smoothly varying tangent.

(iii). It is apparent that there are centers of curvature associated with the paths of the constraint points x_i and also a center of instantaneous rotation associated with the rigid body. These centers need not coincide. The precise connection is given by the proposition. If an axle is fixed in the body, the center of curvature of its path is the instantaneous center of rotation of the body. We formalize this as a corollary.

COROLLARY. *If the ith axle is fixed in the body,*

$$\kappa_i = 1/r_i. \tag{3.7}$$

Contrariwise, whenever an axle is pivoting, its center of curvature and the center of rotation do not coincide. As an example, consider a body with one point that is moving uniformly in a straight line. The body is also rotating around the translating point (if the point is the hitch point of a trailer, the trailer is fishtailing). The radius of curvature of the translating point is constantly infinite. However the center of rotation of the trailer is not at infinity.

We next observe that one can assume, with little loss of generality, that there are precisely two constraint points. If there is only one constraint point, there is no restriction on the trajectory. If there are more than two, the compatibility condition of the Proposition 1 must be satisfied. Two constraints determine the rest provided that the two constraint directions do not share a common normal passing through both constraint points. In particular, suppose there are two constraint points with $i = 0, 1$, a distance L apart in the body. Assume the direction fixed in the body is from x_0 to x_1. Thus θ_0 and θ_1 are defined as signed angles.

PROPOSITION 3. *The distances r_i satisfy*

$$r_i = L\frac{\cos\theta_{1-i}}{\sin(\theta_1 - \theta_0)}, \qquad i = 0, 1. \tag{3.8}$$

Proof. Construct a diagram and use the law of sines. It is also necessary to check that the sign convention is satisfied. There are some particular cases of note. If $\theta_0 = \theta_1 \neq \pm\pi/2$, then the motion of the body is a translation and the r_i are infinite. One of the r_i vanishes if and only if $\theta_{1-i} = \pm\pi/2$, $\theta_i \neq \pm\pi/2$, and then $r_{1-i} = \pm L$. If $\theta_i = \pm\pi/2$ and $\theta_{1-i} = \pm\pi/2$, the formula is completely indeterminate; it can only be said that the center of rotation lies somewhere on the line $\theta_i = 0$. This is precisely the case in which a third constraint point is required to determine the motion.

Propositions 2 and 3 contain the answer to our initial question about rolling motion, namely: What class of planar rigid-body motions can be achieved under the restriction of rolling? The answer is that if there are no further restrictions on the constraint directions θ_0 and θ_1, then any planar rigid body motion can be achieved. This can be seen because the instantaneous center of rotation can be made to lie anywhere in the plane by appropriate choice of θ_0 and θ_1. However if one of the axles is fixed in the body so that one constraint direction is held fixed (which case is of most mathematical and practical importance), not all rigid body motions can be achieved.

4. One fixed axle. Although there are a few types of vehicles with no fixed axles (e.g., certain fire engines, skateboards), most vehicles have one or more axles fixed in the body. In this section we consider the motion of a vehicle with one fixed axle. If a vehicle has two fixed axles, the only motion involving pure rolling is a fixed center of rotation. Thus the wheels on a truck trailer with multiple fixed axles must slip a little in a turn. However, the case of one fixed axle is certainly a valid approximation at moderate speeds.

The corollary to Proposition 2 implies that the center of rotation of the vehicle always lies along the extension of the fixed axle. The center of rotation is also the center of curvature of all the trajectories of the points on the rear axle. One other constraint direction completely determines the instantaneous motion of the vehicle. The center of rotation is the intersection of the extension of the fixed axle and the perpendicular to the other constraint direction. We wish to quantify this statement.

Consider the situation addressed by Proposition 3, namely a body with two constraint points x_1, x_2. We now assume that $\theta_0 \equiv \alpha$, a constant; this fixes one axle in the body. Usually $\alpha = 0$, but not necessarily; however any case $\alpha \neq \pi/2$ can be reduced to the case $\alpha = 0$ with a little trigonometry. We specifically exclude the case $\alpha = \pi/2$, which occurs when the second constraint point lies on the extension of the fixed axle. Let $\theta_1 = \theta$ be a prescribed function of time. This function describes the steering of the vehicle. In practice θ is governed by a steering mechanism in the body, or it is determined by the direction of motion of a second body pulling the first one. We call point x_1 the *steering point.*

We have used L to denote the distance between the two constraint points; let l denote the directed line between them. Let ϕ denote the angle between l and a direction fixed in space. This angle was denoted Θ_b in Proposition 2. There are several quantities that can be used as independent variable. One is ϕ. This is physically reasonable; the driver of the vehicle perceives the orientation of the vehicle in space and adopts his steering strategy $\theta(\phi)$ accordingly. Other possible independent variables are the arc lengths s_i, $i = 0, 1$, of either of the paths of the two constraint points. There are relations among these variables:

$$\frac{ds_i}{d\phi} = r_i, \qquad i = 0, 1, \tag{4.1}$$

$$\frac{ds_1}{ds_0} = \frac{r_1}{r_0} = \frac{\cos\alpha}{\cos\theta}. \tag{4.2}$$

As a consequence, we have the following relations:

$$\frac{d\theta}{d\phi} = \frac{d\theta}{dt}\Big/\frac{d\phi}{dt} = \frac{\omega_1}{\Omega}, \tag{4.3}$$

$$\frac{d\theta}{d\phi} = r_1\frac{d\theta}{ds_1}. \tag{4.4}$$

PROPOSITION 4. *The following formulae hold:*

$$\kappa_0 = \frac{\sin(\theta-\alpha)}{L\cos\theta}, \tag{4.5}$$

$$\sin(\theta-\alpha)\left(1+\frac{d\theta}{d\phi}\right) = L\kappa_1\cos\alpha, \tag{4.6}$$

$$\frac{d\phi}{ds_1} = \frac{\sin(\theta-\alpha)}{L\cos\phi}, \tag{4.7}$$

$$\frac{d\theta}{ds_1} = \kappa_1 - \frac{\sin(\theta-\alpha)}{L\cos\alpha}. \tag{4.8}$$

Proof. Equation (4.5) follows from (3.8) if we note that $\kappa_0 = r_0^{-1}$. Equation (4.6) follows from (3.1) and (3.8) with $i = 1$ together with (4.3). Equation (4.7) follows from (4.1), (4.2) and (4.5). Equation (4.8) follows from (4.6) and (4.4).

In principle, these equations give all the information necessary to determine the motion of the vehicle given the motion of the steering point. From the curvature of the trajectory of the steering point, the angle between the fixed axle and a direction fixed in space can be determined from (4.8) Once this is known, other quantities such as the path of the constraint points can be calculated by straightforward plane geometry and trigonometry. Applications are presented in the remainder of the paper. Conversely (4.6) can be used in the control-theory problem of determining the steering necessary to guide the steering point over a prescribed path. Notice that any path for the steering point can be prescribed.

It is now appropriate to generalize our notion of steering point. In the prior analysis is tacitly assumed that an axle is pivoted at the steering point, but this assumption is immaterial to the mathematical development. Formulae (4.5)–(4.8) can be interpreted as relating the curvature κ_i and tangent angle θ of the trajectory of an *arbitrary* point in the vehicle (away from the extension of the fixed axle) to the curvature κ_0 of a point on the fixed axle.

5. Circular steering. Circular steering means the steering point moves in a circle or straight line. Most analyses, especially for applications, assume that the path of the steering point is piecewise circular. For example, in the literature on vehicles, offtracking is defined only for circular turns (see [3, p. 1].). For circular steering κ_1 ($= \kappa$) is constant, and determines how sharp the turn is. If $\kappa = 0$ the steering point moves in a straight line, and κ is large in absolute value if the turn is sharp. The radius of the turn is the reciprocal of κ. For κ constant the differential equation (4.8) can be explicitly integrated and the solution written in closed form. Since circular steering is well-discussed in the literature, we merely indicate how the results follow from (4.8). For convenience we assume $\alpha = 0$. Let $s = s_1$. The situation can perhaps be most easily visualized as a truck cab moving in a circle pulling a trailer. We are interested in the motion of the trailer.

We let $y = \sin\theta - L\kappa$, so that $\theta = \sin^{-1}(y + L\kappa)$ and

$$d\theta = \frac{dy}{\sqrt{1 - (y + L\kappa)^2}}. \tag{5.1}$$

The differential equation (4.8) thus becomes

$$\frac{dy}{y\sqrt{1 - (y + L\kappa)^2}} = ds, \tag{5.2}$$

which can be integrated explicitly by quadrature. There are three cases.

Case I Shallow turn: $|L\kappa| < 1$.

This is the usual type of turn for actual vehicles. The radius of the circle traced out by the steering point R is greater than L. In particular, if $\kappa = 0$ or equivalently $R = \infty$, the vehicle is being pulled or pushed in a straight line. Let $\theta_\infty^+ = \sin^{-1} L\kappa$, with $|\theta_\infty^+| < \pi/2$, and $\theta_\infty^- = \pi - \theta_\infty^+$. Equation (5.2) integrates to

$$\tan \tfrac{1}{2}(\theta + \theta_\infty^\pm) = \tan\theta_\infty^\pm + ce^{-sL^{-1}\cos\theta_\infty^\pm}, \tag{5.3}$$

where

$$c = c^\pm = \frac{\tan\frac{1}{2}(\theta_0 - \theta_\infty^\pm)}{1 + \tan\theta_\infty^\pm \tan\frac{1}{2}(\theta_0 + \theta_\infty^\pm)} \tag{5.4}$$

is a constant of integration determined by the initial angle $\theta_0 \neq \theta_\infty^\mp$. The angle θ is the angle between the line of the vehicle and the direction of motion of the steering point.

If the vehicle is a trailer pulled by a cab, θ is the angle between the trailer and the cab. Thus for all initial configurations except $\theta = \theta_\infty^-$, the motion of the vehicle relaxes to a circle with $\theta = \theta_\infty$. A little elementary trigonometry shows that the radius of curvature of the relaxed motion of the vehicle is $R^{-1}(R^2 - L^2)$. The constant function $\theta = \theta_\infty^-$ is also a solution. However since $\cos\theta = \theta_\infty^- < 0$, this solution is unstable; nearby solutions run away from it exponentially fast. In fact, the initial conditions with θ_0 not in the range $(-\pi/2, \pi/2)$ correspond to backing up. Anyone who has tried knows there is a way to back up a vehicle in a circle, but that the process is quite unstable. Good feedback to the steering control is essential. Indeed, all the conclusions of Case I are consistent with our physical understanding.

Case II Cab at right angle to trailer: $|L\kappa| = 1$

This is the critical case, where $R = L$. Equation (5.2) integrates to

$$\tan \tfrac{1}{2}\left(\tfrac{\theta-\pi}{2}\right) = \frac{1}{s+c}, \tag{5.5}$$

where

$$c = \cot \tfrac{1}{2}\left(\tfrac{\theta_0-\pi}{2}\right). \tag{5.6}$$

The motion relaxes to a circular motion with the direction of the steering point at right angles to the trailer. The trailer pivots around the constraint point of the fixed axle.

Case III Sharp turn: $|L\kappa| > 1$

In this case the steering point is moving in a tight circle of radius less than L. The model of a truck cab and trailer is not as appropriate for this case. A better model is the following linkage in the plane. Consider a rod of length L with a wheel at one end fixed to roll parallel to the rod. The other end is attached to another rod of length $R = \kappa^{-1}$; the two ends are free to pivot about this joint. The second end of the second rod is fixed at a point and that rod rotates around this point at a uniform angular velocity. Thus the second end of the first rod is driven uniformly in a circle of radius R. We want to describe the motion of the first rod. Let λ be the central angle; note that $\lambda R = s$. The angle θ is the angle between the first rod and the tangent to the circle at the linkage point. Let $\phi = \pi/2 - \theta$ so that ϕ is the angle between the two rods.

Equation (4.8) integrates to

$$\begin{aligned} -\frac{1 - L\kappa\cos\phi}{L\kappa - \cos\phi} &= -\frac{1 - L\kappa\sin\theta}{L\kappa - \sin\theta} \\ &= -\sin\left[\frac{s+c}{L}\sqrt{(L\kappa)^2 - 1}\right] \\ &= -\sin\left[(\lambda + C)\sqrt{1 - \frac{1}{(L\kappa)^2}}\right], \end{aligned} \tag{5.7}$$

where c and C are constants of integration. A change of C amounts to a change of zero angle for λ, so we let $C\sqrt{1 - \frac{1}{(L\kappa)^2}} = -\frac{\pi}{2}$.

Introduce a function

$$\mu(x) = \mu_{L\kappa}(x) = \frac{1 - L\kappa x}{L\kappa - x}, \qquad -1 \le x \le 1 \tag{5.8}$$

for $L\kappa > 1$ Then μ has the following properties:

1. $\mu(\pm 1) = \mp 1$,
2. $\mu^{-1} = \mu$, so the graph of μ is symmetric with respect to the 45° line $y = x$,
3. $\mu' < 0$, so μ is monotonic decreasing,

4. $\mu'' < 0$, so μ is concave down.

The graph is the hyperbola $(x - L\kappa)(y - L\kappa) = (L\kappa)^2 - 1$. For $L\kappa$ near ∞, $\mu(x)$ is near $-x$. For $L\kappa$ near 1, the graph of μ is near the upper and right edges of a square $y = 1$, $-1 \le x \le 1$, and $x = 1$, $-1 \le y \le 1$.

The motion can be understood from the graph of μ with the interpretations $\cos\left[\lambda\sqrt{1 - \frac{1}{(L\kappa)^2}}\right]$ as the y-axis and $\cos\phi$ as the x-axis. Thus ϕ makes one revolution when λ makes $1/\sqrt{1-(L\kappa)^{-2}}$ revolutions. If $L\kappa$ is near ∞, from the graph we see the motion of the first rod is nearly periodic. The rotation of the second rod, which is short compared to the first, causes the first rod to move back and forth like a connecting rod of a piston, with a slow rotation superimposed. On the other hand, if $L\kappa$ is close to one, so that the second rod is only slightly shorter than the first, the motion is completely different. For most of the λ excursion, ϕ is virtually constant at $\phi = 0$; both rods move almost with uniform rotation. The behavior is like that of Case II above with $L\kappa = 1$. However for $\cos\left[\lambda\sqrt{1 - \frac{1}{(L\kappa)^2}}\right]$ near 1, the second rod makes one revolution while the first rod makes one piston-like cycle. The long-term cycle then repeats. As $L\kappa \to 1$, this last behavior is a boundary layer phenomenon. Over the long term, the relative rate of rotation in space of the first rod to the second is $1 - \sqrt{1-(L\kappa)^{-2}}$.

6. Offtracking. In this section we use the results from Case I of the previous section to perform a prototypical analysis of offtracking for noncircular motion. In general, the question is: how much road is needed for the rear wheels of a vehicle in a turn? We make a model analysis for a simplified vehicle in a simplified turn. Suppose the steering point comes in along the positive x-axis to the origin. The constraint point of the fixed axle is also on the x-axis. At the origin, the steering point makes a right-hand turn up the y-axis. Assume the vehicle has semi-width W. We are interested in the path of the point a distance W out from the steering constraint point. The path of this point is the offtracking of the rear wheels.

First assume that $W = 0$. We use (5.3), starting with the steering point at the origin heading up the y-axis. Then $\theta_\infty = 0$ and $\theta_0 = \pi/2$. Thus (5.3) reads

$$\tan\tfrac{1}{2}\theta = e^{-s/L}, \tag{6.1}$$

so that

$$\theta = 2\tan^{-1} e^{-s/L}. \tag{6.2}$$

Note that s is the distance up the y-axis of the steering point, We observe that

$$\begin{aligned}\sin\left(2\tan^{-1} e^{-s/L}\right) &= 2\sin\left(\tan^{-1} e^{-s/L}\right)\cos\left(\tan^{-1} e^{-s/L}\right)\\ &= \frac{2e^{-s/L}}{1+e^{-2s/L}} = \operatorname{sech}\frac{s}{L}.\end{aligned} \tag{6.3}$$

Similarly

$$\cos\left(2\tan^{-1} e^{-s/L}\right) = \tanh\frac{s}{L}. \tag{6.4}$$

Recall that θ is the angle between the y-axis and the line of the vehicle. Let (x_0, y_0) be the coordinates of the steering constraint point. A little trigonometry shows that

$$x_0 = L\sin\theta = L\operatorname{sech}\frac{s}{L}, \tag{6.5}$$

$$y_0 = s - L\cos\theta = s - L\tanh\frac{s}{L}, \tag{6.6}$$

which are parametric equations for the offtracking of a vehicle with no width. The parameter s can be eliminated by solving (6.5) for s and substituting in (6.6). The resulting expression is

$$y_0 = L\left[\operatorname{sech}^{-1}\frac{x_0}{L} - \sqrt{1-\left(\frac{x_0}{L}\right)^2}\right]. \tag{6.7}$$

This is the closed-form expression for the offtracking of the rear constraint point.

We now consider a vehicle with semi-width W. In this case, the above analysis applies to the centerline of the vehicle. Let the coordinates of the point a perpendicular distance W from the constraint point of the axle be (x, y). Elementary trigonometry shows that $x = x_0 + W\cos\theta$ and $y = y_0 + W\sin\theta$, so we obtain the parametric equations for x and y:

$$x = L\operatorname{sech}\frac{s}{L} + W\tanh\frac{s}{L}, \tag{6.8}$$

$$y = s - L\tanh\frac{s}{L} + W\operatorname{sech}\frac{s}{L}. \tag{6.9}$$

If desired, the parameter can be removed as follows: write (6.8) as

$$x = \frac{L + W\sinh\frac{s}{L}}{\cosh\frac{s}{L}},$$

so that

$$\begin{aligned} x^2\cosh^2\frac{s}{L} &= x^2\left(1+\sinh^2\frac{s}{L}\right) \\ &= L^2 + 2LW\sinh\frac{s}{L} + W^2\sinh^2\frac{s}{L}, \end{aligned} \tag{6.10}$$

and

$$\sinh\frac{s}{L} = \frac{LW \pm x\sqrt{W^2+L^2-x^2}}{x^2-W^2}. \tag{6.11}$$

The formula for s is multi-valued because at the beginning of the turn, the inner back wheel reverses direction slightly. Equation (6.11) can be solved for s and substituted in (6.9). We leave the details to the interested reader.

Notice that we have determined in closed form whether a 90° corner in a road can be traversed with a simple turn. Suppose $W = 0$. If the width of the roadway the vehicle is turning from (resp. to) is y_0 (x_0), we see from (6.7) that the turn can be made if and only if

$$y_0 \geq L\left[\operatorname{sech}^{-1}\frac{x_0}{L} - \sqrt{1-\left(\frac{x_0}{L}\right)^2}\right]. \tag{6.12}$$

In the next section we show that (6.12) is actually the necessary and sufficient condition no matter what the steering strategy. If $W \neq 0$, a similar explicit relation holds, but it is more complicated and uses (6.10), (6.11). Of course, most real vehicles cannot make such sharp turns due to a maximum steering lock (one that can is a toy wagon). For piecewise-circular steering, the requisite formulae can again be explicitly derived. Indeed formulae can be derived for multiply-articulated vehicles, such as tandem trailers.

The above analysis determines only whether the rear inside wheel stays on the roadway throughout the turn. If there is rear overhang, and the rear of the vehicle must also remain over the roadway (for example if the roadway is a city alley abutted with buildings, or if the path of oncoming traffic must be considered), a further analysis must be done. In the next section, we consider the general question of optimally steering around a corner.

7. Optimal steering. The problem we examine in this section is whether a vehicle with a single rectangular component and one fixed axle (e.g., a bus) can make a turn between two straight lanes of given width without *any* part of the vehicle leaving the L-shaped region defined by the two lanes. Similar problems were considered in [4], [5], [8]. Typically the two lanes intersect at right angles, but it is instructive to consider the more general case in which the interior angle between the lanes is γ, $0 < \gamma < \pi$. We provide a geometric prescription that determines whether a given vehicle can turn within certain lanes. The calculations of the previous two sections provide closed-form formulae for the resulting trajectories.

We first fix some definitions. Attention is focussed on the outer boundary of the roadway. We assume it is composed of two half-lines with a common vertex meeting at an angle γ, $0 < \gamma < \pi$. It thus divides the plane into two components, the interior and the exterior. We call the two half-lines the *outer rays* of the corner. All points of the vehicle are to remain in the closure of the interior throughout the duration of the turn. Many of the considerations in this section are valid for more general convex outer boundaries, but the entire analysis can be completed only in the present case. We consider trajectories which are asymptotic in negative time to one of the rays and asymptotic in positive time to the other (up to the width of the vehicle). The envelope of the motion is the path of the inner wheel of the fixed axle. The fixed axle can be anywhere in the body; thus rear overhang is an issue in our considerations. A trajectory and resulting envelope are *optimal* if the envelope of any complete trajectory is nowhere closer to the outer boundary than the given one. That is, the vehicle steers around the corner as close as possible to the outer boundary. If the optimal envelope cuts across the inner boundary of the roadway, the vehicle cannot steer around the corner, and conversely. As in the previous section, it is easy to solve the problem for all vehicles once the case of zero-width is solved. Our vehicle has thus been abstracted to the following mathematical object. It is a line segment with a distinguished point (the fixed constraint point) and a distinguished direction at that point (the normal to the constraint direction).

When the restriction of rolling is removed, a larger class of motions is available. The vehicle can "slide" in any direction. The problem reduces to a version of the "mathematician's sofa" problem. In the case of width zero, this becomes the problem of maneuvering a ladder of length L in a corridor. See [10] for further references or [11] for a more general treatment. This last problem is easily solved. The optimal trajectory for the sliding problem is determined by keeping both ends of the ladder on the outer boundary. The optimal envelope consists of two line segments (where it coincides with the outer boundary) separated by a curved region. In the curved region the two ends of the ladder are on the two rays of the outer boundary. All points on the straight parts of the optimal envelope are at a distance greater than L from the other ray of the exterior boundary. When $\pi/2 \leq \gamma < \pi$, the transition points between straight and curved regions of the envelope occur at distance L from the corner, and the optimal optimal envelope has a continuously turning tangent. However, when $0 < \gamma < \pi/2$, the transition point is at a distance greater than L from the corner, and the optimal envelope itself has two corners that arise because the optimal trajectory includes rotations of the ladder about its end points.

If rolling is prescribed, the class of available rigid-body motions is restricted, so in general the optimal sliding trajectory is not attainable. Nevertheless, the previous analysis can be mimicked to obtain the solution for the rolling problem. We shall show that the optimal trajectory is characterized by the following simple geometric property. Consider positions of the vehicle such that the two ends are on the outer rays, one on each, and consider the normals to the rays at these points. By continuity, there is a distinguished position such that these two normals and the normal to the constraint point on the vehicle all intersect at one point. This position is unique. If the vehicle starts at this position, it can, in light of Propositions 1 and 3 and the analysis of Section 4, be maneuvered out to infinity along either ray by steering the end point of the vehicle that is in contact with the ray along the ray. Since the end point of the vehicle moves in a straight line, the analysis of the previous section gives a closed-form expression for the trajectory. Thus one strategy to traverse the corner comprises reversing one of these trajectories until the distinguished position is reached and then continuing along the other trajectory. We call this trajectory the distinguished trajectory.

PROPOSITION 5. *The distinguished trajectory is optimal.*

The proof proceeds in three steps.

LEMMA 1. *Any piecewise smooth optimal trajectory is the union of segments in which either* (a) *at least one end point of the vehicle is in contact with one of the outer rays, or* (b) *the envelope and hence trajectory are piecewise straight.*

Proof. Suppose that a portion of an optimal envelope is not straight and neither end point of the vehicle touches an outer ray. Consider the tangents to two points on the curved portion. They intersect. The trajectory can be modified by moving straight along a tangent, pivoting (i.e., rotating the vehicle with its constraint point fixed), and moving straight along the other tangent to rejoin the original envelope. The modified envelope lies outside the original envelope, and the modification can be made on a sufficiently small scale that the boundary is not touched. The optimality is contradicted, and either (a) or (b) must hold.

Remarks.(1). The analysis of Section 4 shows that the motion in a smooth segment of a trajectory is completely determined if one end point is steered along the boundary.

(2). The analogue of Lemma 1 for sliding motion requires that both end points lie on the exterior boundary at all times. Both Lemma 1 and its sliding analogue are valid when the exterior boundary is more complicated than the union of two rays. However, for the following lemma to be true, the special form of the outer boundary is needed.

LEMMA 2. *The optimal trajectory corresponds to configurations for which one end point is on the boundary at all times.*

Proof. Different segments of the envelope can be joined either with a continuous or discontinuous tangent. In the latter case the envelope contains a corner and the optimal trajectory includes pivoting about the constraint point. Because the exterior boundary is composed of two rays, translational symmetry along an edge can be used to show that a straight segment can exist as part of an optimal trajectory only if one or both ends of such a segment are at a corner of the envelope. Moreover, the corner must be such that the pivoting starts with one end point on the boundary and stops when the other end points hits the boundary. Suppose otherwise. Then the portions of the trajectory in contact with the boundary could be translated toward the apex of the corner, and the straight segments could be shortened to maintain continuity. The modified envelope would lie entirely closer to the outer boundary of the corner, contradicting optimality. The possibility of an optimal envelope with corners can be excluded by direct comparison with the envelope obtained by steering the end-points along the rays.

Finally, observe that the constructions in the proofs of the lemmata show that the optimal trajectory exists and is the distinguished trajectory. Accordingly, Proposition 5 is proved.

If $\pi/2 \leq \gamma < \pi$, the optimal trajectory for nonzero-width vehicles is the same as for the zero-width case. The outer two corners of the vehicle are steered along the boundary in the prescribed fashion, and the optimal envelope is calculated from the inner edge. When $0 < \gamma < \pi/2$, the vehicle is wide and the fixed axle is close to an end, then the prescription given above may fail due to an interior corner of the vehicle touching the exterior boundary of the wedge. We do not pursue the required modifications here.

It is instructive to compare the optimal rolling and sliding envelopes. When the fixed axle is exactly at one end of the body, the rolling and sliding envelopes coincide along one ray. If the fixed axle is not exactly at one end of the body, then the optimal rolling envelope lies everywhere strictly inside the optimal sliding envelope, except at one critical configuration where they coincide. The critical configuration corresponds to the unique point on the optimal rolling envelope at which both end points touch the boundary. The location of this special point is determined by the placement of the fixed axle. Accordingly, if attention is restricted to entrance and exit lanes of equal width, the narrowest lanes can be traversed with the axle at mid-length (such as on school buses). When the axle is not in the center of the vehicle, there is asymmetry between forward and reverse motion If the axle is in the rear half of the vehicle, it is possible to enter a narrower lane by reversing. In particular, trucks with the axles close to the rear can reverse into narrow lanes, albeit that they require a wide exit lane to make the maneuver. In summary, it can be seen that the optimal solution for a zero-width fixed-axle vehicle is very similar to the sliding solution for a ladder. The sliding problem is easily generalized to exterior boundaries that are more general than the union of two rays. However, consideration of an outer boundary with three straight segments shows that the generalization for the rolling problem is not so easily achieved.

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