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# Optimal Arrangement of Components via Pairwise Rearrangements

by

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# ABSTRACT

We introduce the notion of comparison of the criticality of two nodes in a coherent system, and develop a monotonicity property of the reliability function under component pairwise rearrangement. We use this property to find the optimal component arrangement. Worked examples illustrate the methods proposed.

## 1. Introduction

Optimal assignment of components (or optimal assembly) of a coherent system has become increasingly important in reliability theory. Generally speaking, given a coherent system with n nodes and N components with respective reliabilities $p_1,...,p_N$, we wish to find the optimal assignment of components to nodes to maximize system reliability. For a series, parallel, and k-out-of-n system, the optimal assignment was obtained by Derman, Lieberman, and Ross (1974). Recently El-Neweihi, Proschan, and Sethuraman (1986) developed results for parallel-series and series-parallel systems. The problem received greater attention after the publication of the Derman, Lieberman, and Ross (1982) paper developing the optimal arrangement for a "2-consectuve failures-out- of-n fails system" and stating their famous conjecture on the "k-consecutive failures-out-of-n fails system."

If the numbers of nodes and components are the same (i.e., n = N) and if each component is to be assigned to one node, then an assignment policy is just an arrangement policy. In this case, we wish to find the arrangement which maximizes system reliability. Using the notation in Barlow and Proschan (1981), we let $\phi(\underline{x})$ denote the structure function of the system and $\underline{p} = (p_1,...,p_n)$ denote the vector of component reliabilities. Without loss of generality we assume that

$$0 < p_1 \leq p_2 \leq \cdots \leq p_n < 1 \qquad (1)$$

and for obvious reasons, we avoid the trivial case $p_i = 0$ or $p_i = 1$. In real-life applications the true values of the $p_i$'s may or may not be known, and in most cases they are unknown. Thus to obtain the main results in this paper, we assume knowledge of only the ranks of the component reliabilities and not their actual values. This situation occurs, for example, when the ages of the components are known and when their common life distribution has an increasing failure rate. In this case with mission length fixed, the component reliabilities can be inversely ordered according to their ages.

Let $\underline{\pi} = (\pi_1,...,\pi_n)$ denote a permutation of (1,2,...,n) and let $\underline{\pi}(\underline{p})$ denote the vector $(p_{\pi_1}, p_{\pi_2},...,p_{\pi_n})$. Then for a givn permutation $\underline{\pi}$, the corresponding reliability function of the system is $h(\underline{\pi}(\underline{p})) = E_{\underline{\pi}(\underline{p})}\phi(\underline{X})$.

Assume a coherent system $\phi(\underline{x})$ and component reliability vector $\underline{p}$ satisfying (1). Then we state:

**Definition 1.** $\underline{\pi}^{(0)} = (\pi_1^{(0)},...,\pi_n^{(0)})$ is said to be an _optimal_ _permutation_ if

$$h(\underline{\pi}^{(0)}(\underline{p})) = \max_{\underline{\pi}} h(\underline{\pi}(\underline{p})). \qquad (2)$$

In this paper we provide a method for obtaining the optimal permutation (or one of the optimal permutations if it is not unique) by a process of elimination. This process depends on a notion of criticality of the nodes of a coherent system, to be defined in Section 2. Our main theorem says that if the node i is more critical than node j and if under the current permutation $\pi$ a less reliable component is assigned to node i, then an improvement is made by interchanging the components assigned to nodes i and j; we denote this interchange by $\pi_{ij}$ and call the resulting permuation a <u>pairwise rearrangement</u>. Consequently, $\pi$ can be eliminated from further consideration. The optimal permutation $\pi^{(0)}$ is then determined from the permutations not yet eliminated.

To illustrate how an optimal permutation depends on the criticality of the nodes we consider the following example:

**Example 1.** Suppose that oil is to be pumped from location A to location B through either nodes 1 and 2 or nodes 1 and 3.
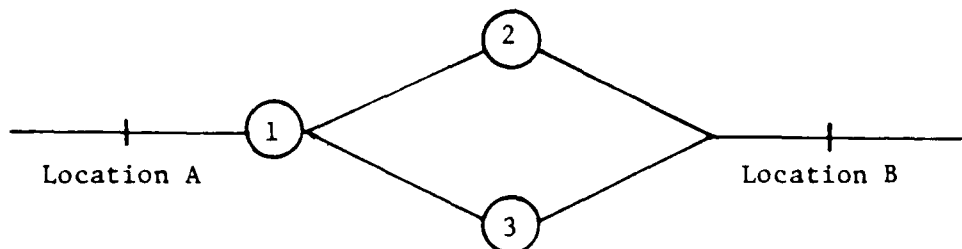


Figure 1.

If pumps with respective reliabilities $p_1 \leq p_2 \leq p_3$ are to be assigned to nodes 1, 2, and 3, the problem is to find the assignment for which the reliability of the coherent system with structure function

$$\phi(x_1, x_2, x_3) = x_1(x_2 \amalg x_3)$$

is maximized. We shall see that according to Definition 2 to be given in the next section, node 1 is more critical than either of nodes 2 and 3. Thus by pairwise rearrangements an optimal permutation is found; under it, the most reliable pump is assigned to node 1. The details will be given in Example 1'. □


The definition of criticality of a node is given in Section 2. The main theorem concerning criticality and pairwise rearrangements is then established. The relationship between criticality and the well-known definition of structural importance of nodes is examined in Section 3. Finally, in Section 4 we state a procedure for obtaining the optimal permutation via criticality and rearrangements, and discuss examples and applications.

## 2. Criticality of Nodes and Pairwise Rearrangements

Let $\underline{x} = (x_1,...,x_n)$ be a vector of binary variables and $\phi(\underline{x})$ the structure function of a coherent system. The components (with reliability vector $\underline{p}$) are assumed to function independently. For notational convenience, let $(1_i,0_j,\underline{y}^{(ij)})$ be the vector $\underline{x}$ for which $x_i = 1$ and $x_j = 0$, and $\underline{x}^{(ij)}$ is the vector obtained by deleting $x_i$ and $x_j$ in $\underline{x}$. The vectors $(0_i,1_j,\underline{x}^{(ij)})$, etc. and $\underline{p}^{(ij)}$ are defined similarly.

**Definition 2.** Node i is more critical than node j for the structure function $\phi$ $(i \overset{c}{>} j)$ if $\phi(1_i,0_j,\underline{x}^{(ij)}) \geq \phi(0_i,1_j,\underline{x}^{(ij)})$ holds for all $\underline{x}^{(ij)}$ and strict inequality holds for some $\underline{x}^{(ij)}$.

Note that this definition depends only on the structure function $\phi(\underline{x})$ of a coherent system, and not on the component reliability vector $\underline{p}$.

Now let $\underline{\pi} = (\pi_1,...,\pi_n)$ and $\underline{\pi}_{ij}$ denote permutations as defined in Section 1. If $\pi_i < \pi_j$, then from (1) we have $p_{\pi_i} \leq p_{\pi_j}$. Thus under the present permutation $\underline{\pi}$ a less reliable component is assigned to node i. If the components assigned to nodes i and j are interchanged and the components assigned to the other nodes remain the same, then the reliability function changes from $h(\underline{\pi}(\underline{p}))$ to $h(\underline{\pi}_{ij}(\underline{p}))$, and under the new permutation $\underline{\pi}_{ij}$ a more reliable component is assigned to node i.

**Theorem 1.** Let $\underline{\pi}$ be any permutation such that $\pi_i < \pi_j$. Then $h(\underline{\pi}_{ij}(\underline{p})) \geq h(\underline{\pi}(\underline{p}))$ holds for all $\underline{p}$ satisfying $0 < p_{\pi_i} \leq p_{\pi_j} < 1$, with strict inequality for some $\underline{p}$, if and only if, $i \overset{c}{>} j$ holds.

**Proof.** (a) (1) It is easy to see that for $q_r = 1 - p_r$ $(r = 1,...,n)$,

$$h(\underline{\pi}(\underline{p})) = p_{\pi_i}p_{\pi_j}E_{\underline{\pi}(\underline{p})^{(ij)}}\phi(1_i,1_j,\underline{X}^{(ij)}) + q_{\pi_i}q_{\pi_j}E_{\underline{\pi}(\underline{p})^{(ij)}}\phi(0_i,0_j,\underline{X}^{(ij)})$$

$$+ p_{\pi_i}q_{\pi_j}E_{\underline{\pi}(\underline{p})^{(ij)}}\phi(1_i,0_j,\underline{X}^{(ij)}) + q_{\pi_i}p_{\pi_j}E_{\underline{\pi}(\underline{p})^{(ij)}}\phi(0_i,1_j,\underline{X}^{(ij)})$$

and that $\underline{\pi}(\underline{p})^{(ij)} = \underline{\pi}_{ij}(\underline{p})^{(ij)}$. Thus if $\phi(1_i,0_j,\underline{x}^{(ij)}) \geq \phi(0_i,1_j,\underline{x}^{(ij)})$ holds for all $\underline{x}^{(ij)}$, then

$$h(\underline{\pi}_{ij}(\underline{p})) - h(\underline{\pi}(\underline{p})) = (p_{\pi_j}q_{\pi_i} - p_{\pi_i}q_{\pi_j})E_{\underline{\pi}(\underline{p})^{(ij)}}[\phi(1_i,0_j,\underline{X}^{(ij)}) - \phi(0_i,1_j,\underline{X}^{(ij)})] \geq 0.$$

(a) (2) Furthermore, if $\phi(1_i,0_j,\underline{x}_0^{(ij)}) = 1 > \phi(0_i,1_j,x_0^{(ij)}) = 0$ for some $\underline{x}_0^{(ij)}$, then

we have

$$1 = h(\underline{\pi}_{i,j}(\underline{p})) > h(\underline{\pi}(\underline{p})) = 0,$$

where $\underline{p}$ is a vector of 0's and 1's such that $p_{\pi_i} = 0$, $p_{\pi_j} = 1$, and $P_{\underline{\pi}(\underline{p})}[\underline{X}^{(i,j)} = \underline{x}_0^{(i,j)}] = 1$. Since $h(\underline{p})$ is a continuous function of $\underline{p}$ for $\underline{p} \in [0,1]^n$, there exists a $\underline{p} \in (0,1)^n$ such that $h(\underline{\pi}_{i,j}(\underline{p})) - h(\underline{\pi}(\underline{p})) > 0$.

(b) If $i \overset{c}{>} j$ does not hold, then either $\phi(1_i,0_j,\underline{x}^{(i,j)}) = \phi(0_i,1_j,\underline{x}^{(i,j)})$ for all $\underline{x}^{(i,j)}$ (in this case $h(\underline{\pi}_{i,j}(\underline{p})) = h(\underline{\pi}(\underline{p}))$) or there exists an $\underline{x}_0^{(i,j)}$ such that $\phi(1_i,0_j,\underline{x}_0^{(i,j)}) = 0$ and $\phi(0_i,1_j,\underline{x}^{(i,j)}) = 1$. In the latter case we conclude by an argument similar to that used in (a) (2), that there exists a $\underline{p} \in (0,1)^n$ such that $h(\underline{\pi}_{i,j}(\underline{p})) < h(\underline{\pi}(\underline{p}))$ holds. This completes the proof. □

For certain structure functions $\phi(\underline{x})$ the minimum path vectors and minimum cut vectors are easy to find. In these cases the criticality of two nodes may be established in terms of such vectors.

**Theorem 2.** Let $P_1,...,P_R$ be the minimum path sets of a coherent system with nodes $1,...,n$ and structure function $\phi(\underline{x})$. Let $A_i = \{P_r: i \in P_r, r = 1,...,R\}$. If $A_j$ is a proper subset of $A_i$ $(A_i \supset A_j)$, then $i \overset{c}{>} j$.

**Proof.** If $A_i \supset A_j$, then without loss of generality we may assume that

$$A_j = \{P_1,...,P_{r_1}\}, \quad A_i = \{P_1,...,P_{r_2}\}$$

where $r_1 < r_2$. Denote $\rho_r(\underline{x}) = \underset{k \in P_r}{\Pi} x_k$. Then for every fixed $\underline{x}^{(i,j)}$, we have

$$\rho_r(1_i,0_j,\underline{x}^{(i,j)}) = \rho_r(0_i,1_j,\underline{x}^{(i,j)}) = 0 \quad \text{for } r \leq r_1 \text{ and}$$
$$\rho_r(1_i,0_j,\underline{x}^{(i,j)}) \geq \rho_r(0_i,1_j,\underline{x}^{(i,j)}) = 0 \quad \text{for } r_1 < r \leq r_2,$$

with strict inequality holding for some $r$ and $\underline{x}^{(i,j)}$. Moreover,

$$\rho_r(1_i,0_j,\underline{x}^{(i,j)}) = \rho_r(0_i,1_j,\underline{x}^{(i,j)}) \quad \text{for } r_2 < r \leq R.$$

It follows that

$$\phi(1_i,0_j,\underline{x}^{(i,j)}) = \overset{R}{\underset{r=1}{\amalg}} \rho_r(1_i,0_j,\underline{x}^{(i,j)}) \geq \overset{R}{\underset{r=1}{\amalg}} \rho_r(0_i,1_j,\underline{x}^{(i,j)}) = \phi(0_i,1_j,\underline{x}^{(i,j)})$$

holds for all $\underline{x}^{(i,j)}$, with strict inequality holding for some $\underline{x}^{(i,j)}$. □

By a similar proof we immediately have

**Theorem 3.** If in Theorem 2 the words "minimum path sets" are replaced by "minimum cut sets," then the theorem remains true.

To see that the converse of Theorem 2 (and of Theorem 3) is false, see the example below:

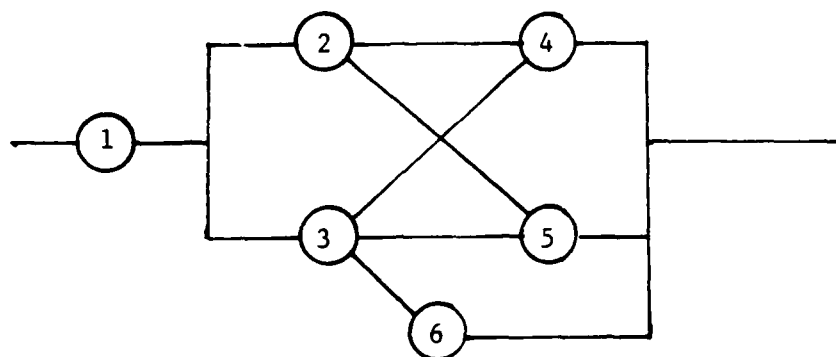**Example 2.** Consider a system of 6 nodes connected as shown in the following diagram:



Figure 2.

It is easy to verify that $3 \overset{c}{>} 2$ and that $P_1 - \{1,2,4\}$ is a minimum path set that does not contain 3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 3. Criticality and Structural Importance

A well-known concept for comparing the relative importance of nodes in a coherent system with structure function $\phi(\underline{x})$ is based on the index of structural importance. The structural importance index of node i is given by (Barlow and Proschan (1981), p. 13)):

$$I_\phi(i) - \frac{1}{2^{n-1}} \sum_{\{\underline{x}:x_i=1\}} [\phi(1_i,\underline{x}) - \phi(0_i,\underline{x})],$$

where $(1_i,\underline{x})$ $((0_i,\underline{x}))$ denote the vector of $\underline{x}$ such that $x_i = 1$ $(x_i = 0)$. For two nodes i and j, we say that i is structurally more important than j if $I_\phi(i) > I_\phi(j)$. In the following theorem we show that the notion of criticality is in some sense stronger than that of structural importance.

**Theorem 4.** Let i and j be two nodes of a coherent system with structure function $\phi(\underline{x})$. If $i \overset{c}{>} j$, then $I_\phi(i) > I_\phi(j)$.

**Proof.** The result is immediate from

$$I_\phi(i) - I_\phi(j) - \frac{1}{2^{n-2}} \sum_{\underset{\sim}{\mathbf{x}}^{(i,j)}} [\ \phi(1_i,0_j,\underset{\sim}{\mathbf{x}}^{(i,j)}) - \phi(0_i,1_j,\underset{\sim}{\mathbf{x}}^{(i,j)})] > 0. \qquad \square$$

To see that $I_\phi(i) > I_\phi(j)$ does <u>not</u> imply $i \overset{c}{>} j$, simply note the following example.

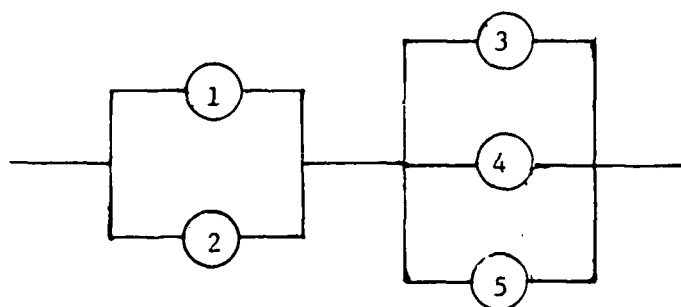**Example 3.** Consider a system of 5 nodes connected as shown in the following diagram:



Figure 3.

It is easy to verify that $I_\phi(1) > I_\phi(3)$. But

$$0 = \phi(1,1,0,0,0) < \phi(0,1,1,0,0) = 1,$$
$$1 = \phi(1,0,0,1,0) > \phi(0,0,1,1,0) = 0.$$

So neither $1 \overset{c}{>} 3$ nor $3 \overset{c}{>} 1$ holds. $\qquad \square$

### 4. <u>Optimal Arrangement Via Pairwise Rearrangements</u>

In this section we give a procedure for obtaining the optimal permutation (or one of them if it is not unique) by a process of elimination of inadmissible permutations via pairwise rearrangements. Toward this end we first state a definition of and develop a result for the permutation equivalence of nodes.

**Definition 3.** Let $h(\underset{\sim}{\pi}(\underset{\sim}{p})) = E_{\underset{\sim}{\pi}(\underset{\sim}{p})} \phi(\underset{\sim}{X})$ be the reliability function of a coherent system under permutation $\underset{\sim}{\pi}$. Two nodes $i$ and $j$ are said to be <u>permutation equivalent</u> (p.e.) under $\underset{\sim}{\pi}$ if $h(\underset{\sim}{\pi}(\underset{\sim}{p})) = h(\underset{\sim}{\pi}_{i,j}(\underset{\sim}{p}))$ holds for all $\underset{\sim}{p}$.

The next theorem provides a characterization of permutation equivalence in

terms of the structure function $\phi(\underline{x})$.

**Theorem 5.** Nodes i and j are p.e. under any permutation $\underline{\pi}$ if and only if $\phi(\underline{x})$ is permutation symmetric in $x_i$ and $x_j$; that is, $\phi(x_i, x_j, \underline{x}^{(i,j)}) = \phi(x_j, x_i, \underline{x}^{(i,j)})$ holds for all $\underline{x}^{(i,j)}$.

**Proof.** (a) If $\phi(\underline{x})$ is permutation symmetric in $x_i$ and $x_j$, then

$$h(\underline{\pi}(\underline{p})) = E_{\underline{\pi}(\underline{p})}\phi(X_i, X_j, \underline{X}^{(i,j)}) = E_{\underline{\pi}(\underline{p})}\phi(X_j, X_i, \underline{X}_{(i,j)})$$

$$= E_{\underline{\pi}_{i,j}(\underline{p})}\phi(X_i, X_j, \underline{X}^{(i,j)}) = h(\underline{\pi}_{i,j}(\underline{p})).$$

(b) If $\phi(1_i, 0_j, \underline{x}_0^{(i,j)}) = 1$, $\phi(0_i, 1_j, \underline{x}_0^{(i,j)}) = 0$ for some $x_0^{(i,j)}$, then for the vector $\underline{p}$ with elements 0's and 1's such that $p_{\pi_i} = 0$, $p_{\pi_j} = 1$, $P_{\underline{\pi}(\underline{p})}[\underline{X}^{(i,j)} = \underline{x}_0^{(i,j)}] = 1$, we have $h(\underline{\pi}(\underline{p})) = 0$, $h(\underline{\pi}_{i,j}(\underline{p})) = 1$. By continuity there exists a $\underline{p} \in (0,1)^n$ such that $h(\underline{\pi}(\underline{p})) < h(\underline{\pi}_{i,j}(\underline{p}))$. $\square$

Let $\Pi$ denote the set of all n! permutations.

**Definition 4.** A permutation $\underline{\pi} \in \Pi$ is said to be <u>inadmissible</u> if there exists a $\underline{\pi}' \in \Pi$ such that $h(\underline{\pi}(\underline{p})) \le h(\underline{\pi}'(\underline{p}))$ holds for all $\underline{p}$ satisfying (1) and strict inequality holds for some $\underline{p}$.

From Theorem 2 we immediately obtain:

**Corollary 1.** Let $\underline{\pi}$ be any permutation such that $\pi_i < \pi_j$. If $i \overset{c}{>} j$, then $\underline{\pi}$ is inadmissible. As a consequence, $\underline{\pi}$ should be eliminated from further consideration because $\underline{\pi}_{i,j}$ is a uniformly better permutation.

This suggests the following procedure for obtaining an optimal permutation from the subset of permutations not yet eliminated:

**Procedure.** (a) Eliminate all inadmissible permutations via the pairwise rearrangement principle.

(b) Delete all but one of the permutations which are permutation equivalent.

(c) Let $\Pi_0 \subset \Pi$ denote the subset of permutations which are not yet eliminated or deleted. Find a permutation $\underline{\pi}^{(0)}$ in $\Pi_0$ satisfying

$$h(\underset{\sim}{\pi}^{(0)}(\underset{\sim}{p})) = \max_{\underset{\sim}{\pi} \in \Pi_0} h(\underset{\sim}{\pi}(\underset{\sim}{p}))$$

either analytically or (when necessary) from numerical calculations where the $p_i$ values are known. Then $\underset{\sim}{\pi}^{(0)}$ is an optimal permutation.

We now illustrate this approach in the examples given below.

**Example 1'.** Consider the system given in Example 1. Since the structure function is $\phi(\underline{x}) = x_1(x_2 \; \underline{u} \; x_3)$, it is easy to see that nodes 2 and 3 are permutation equivalent and that $1 \overset{c}{>} 2$, $1 \overset{c}{>} 3$. Thus among the 6 permutations in $\Pi$, 123, 132, 213, 231 are inadmissible, and both 312 and 321 are optimal.

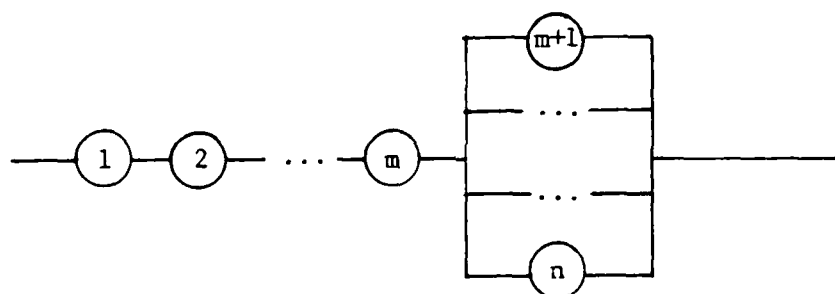**Example 4.** Consider the more general system given below:



Figure 4.

From the structure function of the system it is easy to verify that (1) nodes i and j are permutation equivalent for $1 \leq i < j \leq m$ and $m+1 \leq i < j \leq n$, and (2) $i \overset{c}{>} j$ for all $i \leq m$ and $j > m$. Thus under an optimal permutation the m most reliable components are assigned to nodes 1,2,3,...,m. □

**Example 5.** Consider a system consisting of 7 components connected in the following fashion:
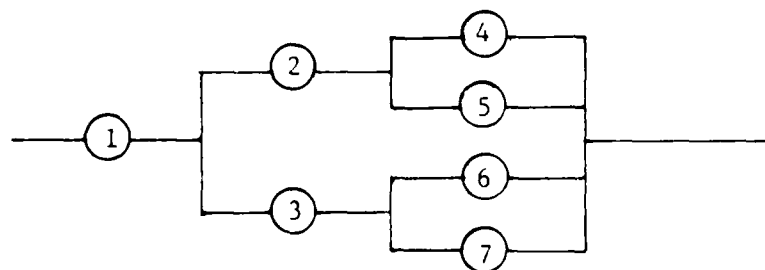


Figure 5.

Here $1 \overset{c}{>} j$ for all $j > 1$; $2 \overset{c}{>} 4$; $2 \overset{c}{>} 5$; $3 \overset{c}{>} 6$; and $3 \overset{c}{>} 7$. Furthermore, the modules $\{2,4,5\}$ and $\{3,6,7\}$ are permutation equivalent. Thus any permutation $\underline{\pi} \in \Pi_0$ must satisfy: $\pi_1 = 7$, $\pi_2 > \max(\pi_4, \pi_5)$, and $\pi_3 > \max(\pi_6, \pi_7)$. It follows that there are only $\binom{6}{3}/2 = 10$ permutations left in $\Pi_0$ for consideration. [For example, (7361245), (7461235) are in $\Pi_0$ but (7341265) is inadmissible and is not in $\Pi_0$.] Without this elimination process we would need to consider all $7!/(2!)^3 = 630$ permutations. If the $p_i$ values are known and the values of $h(\underline{\pi}(\underline{p}))$ are to be computed on a computer for selecting the best permutation, then the use of the procedure will reduce the computer time by a factor $> 50$. Of course, when combining with other results (see e.g., El-Neweihi, Proschan, and Sethuraman (1986)), we can reduce the number of permutations to be considered even further. But that is a separate problem and will not be treated here. □

**Example 6.** Consider a consecutive-k-out-of-n system or a "k-consecutive failures-out-of-n fails system" with nodes connected linearly. Tong (1985) shows that under the optimal permutation policy: $\pi_1 < \pi_2 < \cdots < \pi_{[n+1]/2}$ and $\pi_{[n+]/1} > \pi_{[n+3]/2} > \cdots > \pi_n$ when $k \geq n/2$. The proof uses the explicit expression of the system reliability function. We now show that using the more general principle of pairwise rearrangements, the Tong result follows without knowledge of the reliability function.

To see this, note that for the consecutive k-out-of-n system (the k-consecutive-failures-out-of-n-fails system) the minimum path sets (the minimum cut sets) are: $(1,2,\ldots,k)$, $(2,3,\ldots,k+1)$, $(3,4,\ldots,k+2),\ldots,(n-k+1,\ldots,n)$, where $k \geq n/2$. Thus, in the symbols of Theorem 2 (Theorem 3), $A_i \supseteq A_j$ holds for all $1 \leq j < i \leq \frac{n+1}{2}$ and $\frac{n+1}{2} \geq i > j \geq n$. Consequently $i \overset{c}{>} j$ for all $i,j$ satisfying these inequalities (unless $i$ and $j$ are p.e.). The rearrangement inequality in Tong (1985) now follows. □

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