

NUSC Technical Report 8027
11 September 1987

AD-A187 718

**Accurate Efficient Evaluation of
Bessel Transform; Programs
and Error Analysis**

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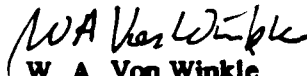
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Preface

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator, Dr. Albert H. Nuttall (Code 3314). This technical report was prepared with funds provided by the NUSC In-House Independent Research and Independent Exploratory Development Program, sponsored by the Chief of Naval Research.

The Technical Reviewer for this report was Dr. E. S. Eby (Code 3314).

Reviewed and Approved: 11 September 1987



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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS A187718									
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.									
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE												
4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR 8027			5. MONITORING ORGANIZATION REPORT NUMBER(S)									
6a. NAME OF PERFORMING ORGANIZATION Naval Underwater Systems Center		6b. OFFICE SYMBOL (if applicable)	7a. NAME OF MONITORING ORGANIZATION									
6c. ADDRESS (City, State, and ZIP Code) New London Laboratory New London, CT 06320			7b. ADDRESS (City, State, and ZIP Code)									
8a. NAME OF FUNDING / SPONSORING ORGANIZATION		8b. OFFICE SYMBOL (if applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER									
8c. ADDRESS (City, State, and ZIP Code)			10. SOURCE OF FUNDING NUMBERS									
			<table border="1"> <tr> <td>PROGRAM ELEMENT NO.</td> <td>PROJECT NO.</td> <td>TASK NO.</td> <td>WORK UNIT ACCESSION NO.</td> </tr> <tr> <td></td> <td></td> <td></td> <td></td> </tr> </table>		PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.				
PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.									
11. TITLE (Include Security Classification) ACCURATE EFFICIENT EVALUATION OF BESSEL TRANSFORM; PROGRAMS AND ERROR ANALYSIS												
12. PERSONAL AUTHOR(S) Albert H. Nuttall												
13a. TYPE OF REPORT		13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Year, Month, Day) 1987 September 11	15. PAGE COUNT								
16. SUPPLEMENTARY NOTATION												
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)									
FIELD	GROUP	SUB-GROUP	Bessel Transform Parabolic Approximation									
			Filon Integration Aliasing									
			Linear Approximation Error Dependence									
19. ABSTRACT (Continue on reverse if necessary and identify by block number)												
The method of Filon numerical integration for Fourier transforms is extended to Bessel transforms of the form $G(\omega) = \int_0^{\infty} dx J_0(\omega x) g(x) ,$ for general $g(x)$. Specifically, for the two cases where $g(x)$ is approximated by												
20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ OTIC USERS			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED									
22a. NAME OF RESPONSIBLE INDIVIDUAL Albert H. Nuttall			22b. TELEPHONE (Include Area Code) (203) 440-4618	22c. OFFICE SYMBOL Code 3302								

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18. SUBJECT TERMS (Cont'd.)

Panel Width
Error Maintenance

19. ABSTRACT (Cont'd.)

~~(a)~~ straight lines, or

~~(b)~~ parabolas.

over abutting panels, the corresponding integrals in the Bessel transform $G(\phi)$ are evaluated exactly (within computer round-off error). Although these integrals cannot be expressed in closed form (as for Filon's case), a recursive procedure and an asymptotic expansion yield rapid accurate evaluation of the required quantities.

Programs are furnished for both cases ~~(a)~~ and (b) in BASIC. Furthermore, two versions of each are furnished: a faster one requiring considerable storage, and a slower one requiring very little storage. The presence and location of aliasing is predicted and its magnitude is investigated numerically. The error dependence on the panel width used in both cases (a) and (b) is established by means of numerical examples, one with a very fast decay with ω , the other with a very slow decay with ϕ . Comparisons with standard Trapezoidal and Simpson's rules reveal that the new procedures are error maintenance procedures, tending to keep the absolute error for larger ω comparable to that near $\omega = 0$, whereas the standard rules are subject to aliasing errors that become very significant for larger ω .

Extensions to more general Bessel transforms are possible and procedures for obtaining them are outlined.

TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	iii
LIST OF SYMBOLS	iv
INTRODUCTION	1
LINEAR APPROXIMATION	5
Special Function Definitions	6
Abutting Point	8
Approximation to Integral	9
Sampling Increment for ω	10
Computation Time Considerations	10
Behavior for Small θ	12
PARABOLIC APPROXIMATION	13
Approximation to Integral	14
Sampling Increment for ω	16
Behavior for Small θ	17

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TABLE OF CONTENTS (Cont'd)

EXAMPLES	18
Aliasing	19
Graphical Results	20
Comparison of Procedures	22
Error Dependence on Sampling Increment	24
 SUMMARY	 29
 APPENDIX A. NUMERICAL EVALUATION PROCEDURE FOR BESSEL INTEGRALS	 A-1
 APPENDIX B. DERIVATION OF INTEGRATION RULE FOR STRAIGHT LINE FITS TO $g(x)$	 B-1
 APPENDIX C. DERIVATION OF INTEGRATION RULE FOR PARABOLIC FITS TO $g(x)$	 C-1
 REFERENCES	 R-1

LIST OF ILLUSTRATIONS

Figure		Page
1	Linear Approximations to $g(x)$	5
2	Parabolic Approximations to $g(x)$	13
3	Errors for Rayleigh Function $g(x)$	21
4	Errors for Gaussian Function $g(x)$	21
5	Errors for Four Procedures, $\omega < 120$	23
6	Errors for Four Procedures, $\omega > 120$	23
7	Linear Procedure; Rayleigh $g(x)$	25
8	Parabolic Procedure; Rayleigh $g(x)$	25
9	Linear Procedure; Gaussian $g(x)$	27
10	Parabolic Procedure; Gaussian $g(x)$	27
C-1	Categorization of Sample Points	C-1

LIST OF SYMBOLS

x	variable of integration, (2)
$g(x)$	function to be transformed, (2)
$J_0(u)$	zero-th order Bessel function, (2)
ω	transform variable, (2)
$G(\omega)$	Bessel transform of $g(x)$, (2)
x_l	lower limit of integral, (11)
x_r	upper limit of integral, (11)
h	panel width, sampling increment, (12)
x_n	general sample point in x , figure 1
g_n	sample value $g(x_n)$, figure 1
$A(u)$	indefinite integral of J_0 , (13)
$J_1(u)$	first order Bessel function, (14)
$B_0(u)$	auxiliary function, (16)
$B_1(u)$	auxiliary function, (17)
I_n	integral contribution, (18)
y	normalized variable, (19)
l, r	left and right integer limits, (20)
θ	product ωh , (22)
Δ	sampling increment in ω , (25)
k	integer location of sample in ω , (25)
K_1, K_2	limits on k , (25)
S_l, S_r	auxiliary quantities, (31)
Q_l, Q_r	auxiliary quantities, (31)
D_n, F_n, R_n	auxiliary sequences, (32)

LIST OF SYMBOLS (Cont'd)

$a(b)c$	sequence $a, a+b, a+2b, \dots, c-b, c$, (33)
$\sum$	Summation over every other term, (30)
$I_0(u)$	zero-th order modified Bessel function, (40)

ACCURATE EFFICIENT EVALUATION OF BESSEL
TRANSFORM; PROGRAMS AND ERROR ANALYSIS

INTRODUCTION

The method of Filon integration for Fourier transforms [1], [2; pages 408-409], [3; pages 67-75], [4; page 400], [5; page 890], [6; pages 62-66], [7]

$$\int_{-\infty}^{+\infty} dx \exp(i\omega x) g(x) \quad (1)$$

is well established and very useful for accurate numerical work. Instead of the standard Simpson's rule, which would approximate the complete integrand $\exp(i\omega x) g(x)$ by parabolas over abutting pairs of panels, Filon's method approximates only the function $g(x)$ by parabolas, and carries out the corresponding integrals in (1) analytically. These closed form integrals are then evaluated with computer aid. Since the exponential in (1) is being handled exactly for all ω , the hope is that the error of approximating (1) by means of Filon's method will be substantially the same for larger ω as for small ω (where all the error arises from approximating $g(x)$). That is, Filon's method is expected to be an error maintenance procedure, whereby the absolute error does not increase significantly with ω . Certainly that is not the case for the Trapezoidal and Simpson rules, where significant aliasing severely limits the accuracy of the results for larger ω .

An alternative simpler procedure to Filon's method for Fourier transforms is to approximate $g(x)$ by straight lines over abutting panels, and again to evaluate the resultant integrals in (1) analytically in closed form. This (less-accurate) procedure is documented in [8; pages 418-419], for example.

Here, we will extend these two procedures to a Bessel transform of the form

$$G(\omega) = \int_0^{\infty} dx J_0(\omega x) g(x) , \quad (2)$$

where $g(x)$ is an arbitrary given function, and J_0 is the zeroth-order Bessel function. One of the major differences we encounter, relative to Filon's method, is that the resultant integrals cannot all be evaluated in closed form. In order to circumvent this problem, we use a combination of a downward recursion and an asymptotic expansion, which are limited in accuracy only by the inherent round-off error of the computer utilized, thereby obtaining an efficient useful procedure for numerical evaluation of the pertinent integrals and functions.

To give a physical application where the Bessel transform arises, consider that we are interested in two-dimensional Fourier transform

$$\iint_{-\infty}^{+\infty} dx dy \exp(iux + ivy) f_2(x,y) , \quad (3)$$

where function f_2 has isotropic behavior. That is, suppose the dependence of f_2 is solely on the distance from the origin of coordinates:

$$f_2(x,y) = f_1(\sqrt{x^2 + y^2}) . \quad (4)$$

Then (3) becomes

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx \, dy \exp(iux + ivy) f_1(\sqrt{x^2 + y^2}) = \\ = 2\pi \int_0^{\infty} dr J_0(\omega r) r f_1(r) , \end{aligned} \quad (5)$$

where we changed to cylindrical coordinates and have defined

$$\omega = (u^2 + v^2)^{1/2} . \quad (6)$$

Thus, (5) is of the form of (2), upon identification of $g(x)$ as $x f_1(x)$.

Suppose in (3) that the f_2 dependence on x, y is more general than (4), namely of the form

$$f_2(x, y) = f_1 \left(\left[\left(\frac{x - x_0}{a} \right)^2 + \left(\frac{y - y_0}{b} \right)^2 - 2\rho \left(\frac{x - x_0}{a} \right) \left(\frac{y - y_0}{b} \right) \right]^{1/2} \right) , \quad (7)$$

which allows for a general center point of symmetry x_0, y_0 , as well as a tilted elliptical shape. Then substitution in (3) yields, after a cylindrical coordinate change, the result

$$\frac{2\pi ab}{(1 - \rho^2)^{1/2}} \exp(iux_0 + ivy_0) \int_0^{\infty} dr J_0(\omega r) r f_1(r) , \quad (8)$$

where now

$$\omega = \left[\frac{a^2 u^2 + b^2 v^2 + 2\rho abuv}{1 - \rho^2} \right]^{1/2} . \quad (9)$$

Again, the fundamental Bessel transform of the form of (2) results, where $g(x)$ is $x f_1(x)$.

On the other hand, if $G(\omega)$ is specified in (2) for $\omega > 0$, the corresponding solution to this integral equation is

$$g(x) = x \int_0^{\infty} d\omega J_0(x\omega) \omega G(\omega) , \quad (10)$$

which is again a Bessel transform of the form of (2).

Thus, we have presented several instances where the transform given by (2) is of interest and must be accomplished accurately for large as well as small arguments of the transform variable ω .

LINEAR APPROXIMATION

The integral of interest here is

$$G(\omega) = \int_{x_l}^{x_r} dx J_0(\omega x) g(x) , \quad (11)$$

where left-end point x_l could be zero, and right-end point x_r could be taken so large that $g(x)$ is essentially zero for $x > x_r$. (If x_l is negative, the values of g could be folded over to the positive x -axis, using $g(x) + g(-x)$ as the new integrand, since $J_0(\omega x)$ is even in x .) We break interval x_l, x_r into a number of abutting panels, each of the same width h , and fit $g(x)$ by straight lines over each of those panels. The fits for the left-end point and an abutting (internal) point x_n are depicted in figure 1, where it is temporarily presumed that the adjacent sample values of

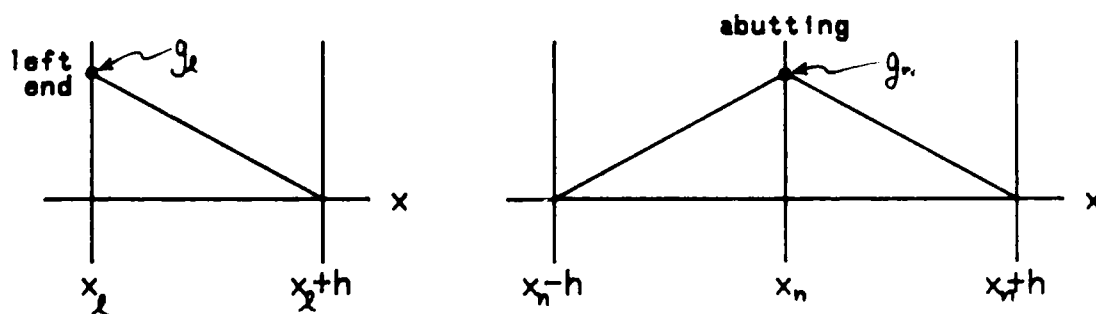


Figure 1. Linear Approximations to $g(x)$

function $g(x)$ are zero; this allows us to isolate the contribution of each sample of $g(x)$ to the total desired in (11). The straight lines pass through the function value $g_n = g(x_n)$ at sample value x_n , and are zero at the adjacent sample points. h is the sampling increment in x applied to $g(x)$. The situation at the right end is the mirror image of that at the left end, depicted in figure 1.

If ω is zero in (11), the approximation afforded to the integral by means of figure 1 is obviously

$$\begin{aligned} G(0) &\approx h \left[\frac{1}{2} g_l + g_{l+1} + \dots + g_{r-1} + \frac{1}{2} g_r \right] = \\ &= h \left[\frac{1}{2} g(x_l) + g(x_{l+1}) + \dots + g(x_{r-1}) + \frac{1}{2} g(x_r) \right] \quad \text{for } \omega = 0. \end{aligned} \quad (2)$$

which is just the Trapezoidal rule. For $\omega > 0$, considerably more effort is required; there is no need to consider $\omega < 0$, since $J_0(\omega x)$ is even in ω . Before we get into that derivation, we must introduce some auxiliary functions.

SPECIAL FUNCTION DEFINITIONS

Define the integral

$$A(u) = \int_0^u dt J_0(t). \quad (13)$$

This function cannot be evaluated in closed form; a table of $A(u)$ is available in [5; pages 492-493]. On the other hand, the integral

$$\int_0^u dt \, t \, J_0(t) = u \, J_1(u) \quad (14)$$

is immediately available by use of [5; 9.1.30]. And two integrations by parts, coupled with (13), yields the result

$$\int_0^u dt \, t^2 \, J_0(t) = u^2 \, J_1(u) + u \, J_0(u) - A(u) . \quad (15)$$

We will also find use for the auxiliary functions

$$B_0(u) \equiv A(u) - u \, J_0(u) = \int_0^u dt \, t \, (u - t) \, J_0(t) , \quad (16)$$

and

$$B_1(u) \equiv A(u) - J_1(u) = \int_0^u dt \, \left(1 - \frac{t}{u}\right) J_0(t) . \quad (17)$$

All of these functions, A , B_0 , B_1 , are zero at the origin and are odd.

Numerical evaluation of these functions is considered in appendix A.

ABUTTING POINT

For an abutting (internal) point x_n in the interval (x_l, x_r) , as depicted on the right-hand side of figure 1, the contribution to integral (11), due to this single sample point $g_n = g(x_n)$, is

$$I_n = \int_{x_n-h}^{x_n} dx J_0(\omega x) g_n (1+y) + \int_{x_n}^{x_n+h} dx J_0(\omega x) g_n (1-y), \quad (18)$$

where we have defined

$$y = \frac{x - x_n}{h}. \quad (19)$$

We now assume that the n -th sample point x_n is taken such that

$$x_n = n h \quad \text{for } l \leq n \leq r. \quad (20)$$

This makes

$$x_l = l h, \quad x_r = r h, \quad \text{i.e.} \quad \frac{x_r}{x_l} = \frac{r}{l} = \text{rational}. \quad (21)$$

This constitutes a restriction on ratio x_r/x_l in (11); it has been adopted here in order to minimize the number of calculations of the Bessel function J_0 later, when we consider the multiple values of ω desired for (11). (The procedure presented here can be extended to the general case where x_l is arbitrary and $x_n = x_l + nh$, if desired.) If x_l is zero, then the choice in (20) is no restriction at all.

APPROXIMATION TO INTEGRAL

An important parameter in this numerical integration procedure is the quantity

$$\Theta = \omega h \quad (22)$$

which is the product of "radian frequency" ω and the sampling increment h . As we shall see, values of Θ near π and 2π will constitute points of considerable aliasing; see [4; page 400] for a discussion of the Fourier transform case.

When the procedure in (18)-(19) is extended to include the left-end and right-end points of integral (11), and the various integrals evaluated with the help of (13)-(17), the total approximation is given by appendix B in several alternative forms, one of which is (B-7):

$$\begin{aligned} \omega G(\omega) \cong & \left[\ell g_{\ell+1} - (\ell + 1) g_{\ell} \right] B_1(\ell\Theta) - g_{\ell} J_1(\ell\Theta) + \\ & + \left[r g_{r-1} - (r - 1) g_r \right] B_1(r\Theta) + g_r J_1(r\Theta) + \\ & + \sum_{n=\ell+1}^{r-1} n \left[g_{n+1} - 2g_n + g_{n-1} \right] B_1(n\Theta) \end{aligned} \quad (23)$$

where

$$g_n = g(x_n) = g(nh) . \quad (24)$$

Reasons for this grouping of terms, including speed of execution and storage requirements, are discussed below.

SAMPLING INCREMENT FOR ω

When output variable ω in integral (11) is restricted to multiples of a sampling increment Δ , according to

$$\omega = k\Delta \quad \text{for } 1 \leq K_1 \leq k \leq K_2, \quad (25)$$

then $n\Theta = nk\Delta h$, meaning that the arguments of the $B_1(u)$ function in (23) are limited to integer multiples of $h\Delta$, the product of the sampling increment in input variable x and the sampling increment in output (transform) variable ω . The explicit relationship for $G(\omega) = G(k\Delta)$ is given by specializing (23) to the values (24), thereby obtaining

$$\begin{aligned} k\Delta G(k\Delta) \cong & \left[\ell g_{\ell+1} - (\ell+1)g_{\ell} \right] B_1(\ell k\Delta h) - g_{\ell} J_1(\ell k\Delta h) + \\ & + \left[r g_{r-1} - (r-1)g_r \right] B_1(rk\Delta h) + g_r J_1(rk\Delta h) + \\ & + \sum_{n=\ell+1}^{r-1} n \left[g_{n+1} - 2g_n + g_{n-1} \right] B_1(nk\Delta h) \quad \text{for } k \geq 1. \quad (26) \end{aligned}$$

COMPUTATION TIME CONSIDERATIONS

Thus, we need evaluate $B_1(u)$ only at $u = m\Delta h$, where m is an integer. Furthermore, not all values of integer m will be encountered as n and k sweep out their respective values given by (20) and (25). And since $B_1(u)$, defined in (17) and (13), is the most time-consuming aspect of the computation of (26), it behooves us not to compute $B_1(m\Delta h)$ at values of m that will not be encountered, and not to recompute $B_1(m\Delta h)$ at values of m

that are encountered more than once. This latter situation arises when m is highly composite; for example, $m = 12 = 4 \cdot 3 = 6 \cdot 2 = 12 \cdot 1$ could be encountered several times as n and k vary in (26).

In order to incorporate this time-saving feature into the Bessel integral evaluations required by (26), the values of $B_1(nk\Delta h)$ are computed only once and stored in a one-dimensional array at linear location $m = nk$. Unfortunately, this speed-up feature is achieved at the expense of considerable storage, for if n and k range up to N and K , respectively, the one-dimensional storage array must have NK cells, of which most are empty when N and K are large.

When N and K are so large that storage is not feasible, such as when x_r in (11) is large, and large ω is desired in (25), then the alternative procedure of direct brute-force evaluation of (26) for $B_1(nk\Delta h)$, repeated as often as necessary, but without storage, is employed. Recomputation of $B_1(m\Delta h)$ for some m values occurs, but evaluation at unused m values never does.

Thus we have two alternatives and two corresponding programs for (26): one faster routine which may require considerable storage, and a slower procedure utilizing very little storage. The former is recommended when feasible, while the latter furnishes a back-up position. Programs for both procedures are listed in appendix B.

BEHAVIOR FOR SMALL Θ

When Θ is small, differences of functions with similar values are required in (23), as may be observed by the linear ω dependence on the left-side. The appropriate series development for this linear approximation approach to (11) is given in (B-11)-(B-12), through order Θ^2 . Additional terms to order Θ^4 , Θ^6 can be derived by extending the approach given there; however, an easier technique will be developed in the next section.

PARABOLIC APPROXIMATION

The integral of interest is again

$$G(\omega) = \int_{x_l}^{x_r} dx J_0(\omega x) g(x) . \quad (27)$$

However, now we approximate $g(x)$ by parabolas over abutting pairs of panels, each of width h . The fits for a mid-point, an abutting point, the left-end point, and the right-end point are illustrated in figure 2. Again, the

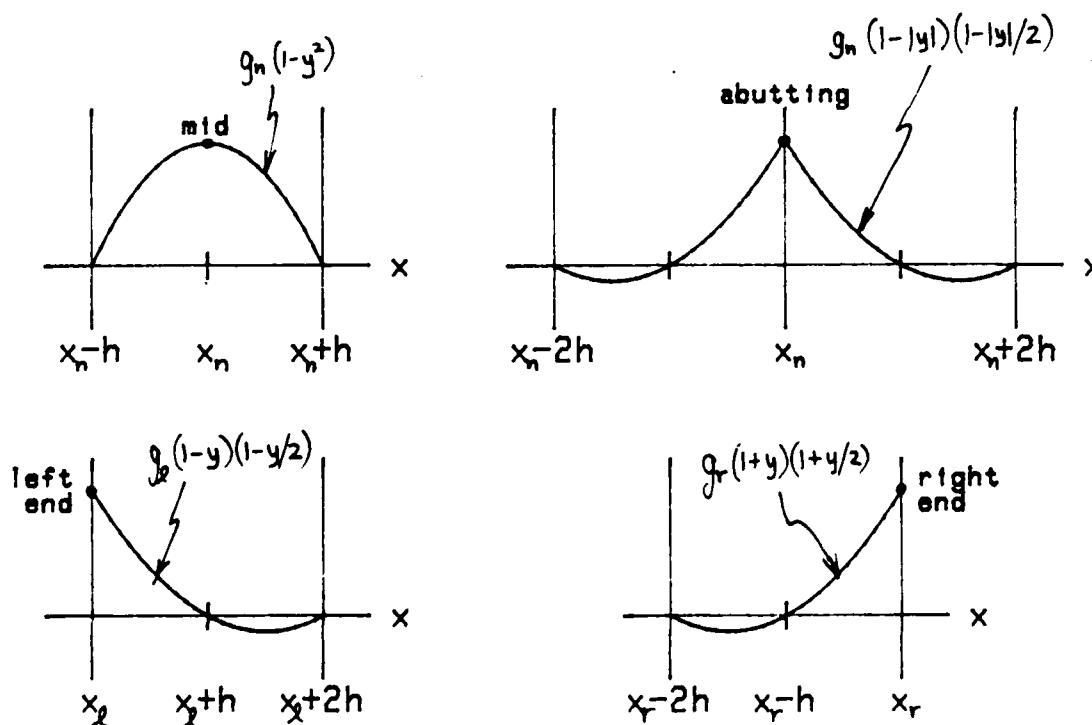


Figure 2. Parabolic Approximations to $g(x)$

contribution of each sample value $g_n = g(x_n)$ is isolated, by temporarily presuming that the neighboring sample values of $g(x)$ are zero. The variable y in figure 2 is again the normalized quantity

$$y = \frac{x - x_n}{h}, \quad (28)$$

where h is the sampling increment in x applied to $g(x)$.

If ω is zero in (27), the approximation afforded to the integral by means of figure 2 is

$$\begin{aligned} G(0) &\cong \frac{h}{3} [g_{\ell} + 4g_{\ell+1} + 2g_{\ell+2} + \dots + 2g_{r-2} + 4g_{r-1} + g_r] = \\ &= \frac{h}{3} [g(x_{\ell}) + 4g(x_{\ell+1}) + 2g(x_{\ell+2}) + \dots + 2g(x_{r-2}) + 4g(x_{r-1}) + g(x_r)] , \end{aligned} \quad (29)$$

which is Simpson's rule.

APPROXIMATION TO INTEGRAL

Since J_0 in (27) is even in ω , we only need to consider $\omega > 0$ in the following. The derivation of the approximation to integral (27), by means of the parabolic fits in figure 2, is carried out in appendix C, culminating in (C-10)-(C-12):

$$\begin{aligned}
2\omega G(\omega) \cong & \frac{1}{\theta^2} S_{\ell} B_0(\ell\theta) - Q_{\ell} B_1(\ell\theta) - 2g_{\ell} J_1(\ell\theta) - \\
& - \frac{1}{\theta^2} S_r B_0(r\theta) + Q_r B_1(r\theta) + 2g_r J_1(r\theta) + \\
& + \frac{1}{\theta^2} \sum_{n=\ell+2}^{r-2} D_n B_0(n\theta) - \sum_{n=\ell+2}^{r-2} R_n B_1(n\theta) .
\end{aligned} \tag{30}$$

The auxiliary sequences utilized in (30) are defined below:

$$\begin{aligned}
S_{\ell} &= g_{\ell+2} - 2g_{\ell+1} + g_{\ell} \\
S_r &= g_r - 2g_{r-1} + g_{r-2} \\
Q_{\ell} &= \ell(\ell+1)g_{\ell+2} - 2\ell(\ell+2)g_{\ell+1} + (\ell+2)(\ell+1)g_{\ell} \\
Q_r &= (r-2)(r-1)g_r - 2r(r-2)g_{r-1} + r(r-1)g_{r-2}
\end{aligned} \tag{31}$$

and

$$\left. \begin{aligned}
D_n &= g_{n+2} - 2g_{n+1} + 2g_{n-1} - g_{n-2} \\
F_n &= g_{n+2} - 4g_{n+1} + 6g_n - 4g_{n-1} + g_{n-2} \\
R_n &= n^2 D_n + n F_n
\end{aligned} \right\} \begin{array}{l} \text{for } n = \\ (\ell+2)(2)(r-2) . \end{array} \tag{32}$$

The functions $B_0(u)$ and $B_1(u)$ are those defined in (13)-(17), and the slash on the summation symbol in (30) denotes skipping every other term, after starting at $n = \ell + 2$. A shorthand notation that will be used here is

$$n = \ell + 2, \ell + 4, \dots, r - 4, r - 2 = (\ell + 2)(2)(r - 2) . \tag{33}$$

Several important observations should be made about the result in (30)-(32). The four quantities in (31) are evaluated only once at the end points $n = \ell$ and r . The sequences in (32) must be evaluated at all the points listed in (33), that is, at every other interior point. All of these computations should be done once and stored, when given the function $g(x)$, the limits x_ℓ, x_r , and sampling increment h , prior to ever considering which ω values will be of interest in (30). Input function $g(x)$ must be evaluated at all $x = x_n = nh$ for $n = \ell(1)r$.

The time-consuming calculations of $B_0(u)$ and $B_1(u)$ in (30) are only necessary at the values $u = n\theta$ for $n = \ell(2)r$, and need not be evaluated at any of the in-between points $n = (\ell + 1)(2)(r - 1)$. The Bessel function $J_1(u)$ need only be evaluated at end points $u = \ell\theta$ and $r\theta$; however, this quantity shows up as a free by-product of evaluating $B_0(u)$ and $B_1(u)$, by the method indicated in appendix A.

SAMPLING INCREMENT FOR ω

When output variable ω in desired integral (27) is restricted to multiples of a sampling increment Δ , according to

$$\omega = k\Delta \quad \text{for } 1 \leq K_1 \leq k \leq K_2, \quad (34)$$

then

$$\theta = \omega h = k\Delta h, \quad (35)$$

and (30) takes on the form

$$\begin{aligned}
& 2k\Delta G(k\Delta) \cong \\
& = \frac{1}{(k\Delta h)^2} S_\ell B_0(\ell k\Delta h) - Q_\ell B_1(\ell k\Delta h) - 2g_\ell J_1(\ell k\Delta h) - \\
& - \frac{1}{(k\Delta h)^2} S_r B_0(rk\Delta h) + Q_r B_1(rk\Delta h) + 2g_r J_1(rk\Delta h) + \\
& + \frac{1}{(k\Delta h)^2} \sum_{n=\ell+2}^{r-2} D_n B_0(nk\Delta h) - \sum_{n=\ell+2}^{r-2} R_n B_1(nk\Delta h) . \tag{36}
\end{aligned}$$

At this point, the discussion in the sequel to (26) is directly relevant and should be reviewed. The only change in the presentation is to replace $B_1(u)$, there, by both $B_0(u)$ and $B_1(u)$ here. We again end up with two alternatives and two corresponding programs for evaluation of (36): one faster routine which may require considerable storage, and a slower procedure utilizing very little storage. Programs for both procedures are listed in appendix C.

BEHAVIOR FOR SMALL Θ

When Θ is small, differences of functions with similar values are required in (30), as may be observed by the linear ω dependence on the left side and the $1/\Theta^2$ dependence on the right side. This behavior is also typical for Filon's method, and indicates the need for a series expansion in powers of Θ for the right-hand side of (30) when Θ is small; see [5; (25.4.53)], for example. The appropriate series development for this parabolic approximation approach to (27) is given in (C-15)-(C-17), through order Θ^2 . Additional terms to order Θ^4 , Θ^6 can be derived by an obvious extension of the approach given there.

EXAMPLES

Two examples will be considered in this section; the first is a Rayleigh function,

$$g(x) = x \exp(-x^2/2) \quad \text{for } x > 0, \quad (37)$$

for which Bessel transform (11) is [9; 6.631 4]

$$G(\omega) = \exp(-\omega^2/2). \quad (38)$$

The second is a Gaussian function,

$$g(x) = \exp(-x^2) \quad \text{for } x > 0, \quad (39)$$

leading to [9; 6.618 1]

$$G(\omega) = 1/2\sqrt{\pi} \exp(-\omega^2/8) I_0(\omega^2/8). \quad (40)$$

These two examples are very different, in that transform (38) decays very quickly for large ω , whereas (40) decays very slowly for large ω . In fact, for the latter case [5; 9.7.1],

$$G(\omega) \sim 1/\omega \quad \text{as } \omega \rightarrow +\infty. \quad (41)$$

This difference will enable us to investigate both absolute and relative errors of the approximate numerical integration procedures developed earlier, over a wide range of values of ω .

ALIASING

The Bessel function J_0 is rather similar to a sinusoid; in fact, for large z [9; 9.2.1],

$$J_0(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi}{4}\right) \text{ as } z \rightarrow +\infty. \quad (42)$$

Then when argument x in transform (11) is sampled at increment h , we encounter the behavior

$$J_0(\omega x_n) = J_0(\omega h n) = J_0(\theta n) \sim \left(\frac{2}{\pi \theta n}\right)^{1/2} \cos\left(\theta n - \frac{\pi}{4}\right) \quad (43)$$

for large θn . Now when $\theta = 2\pi$, the cosine yields the same values as for $\theta = 0$; this leads us to expect larger errors for the numerical integration procedure when θ is near 2π .

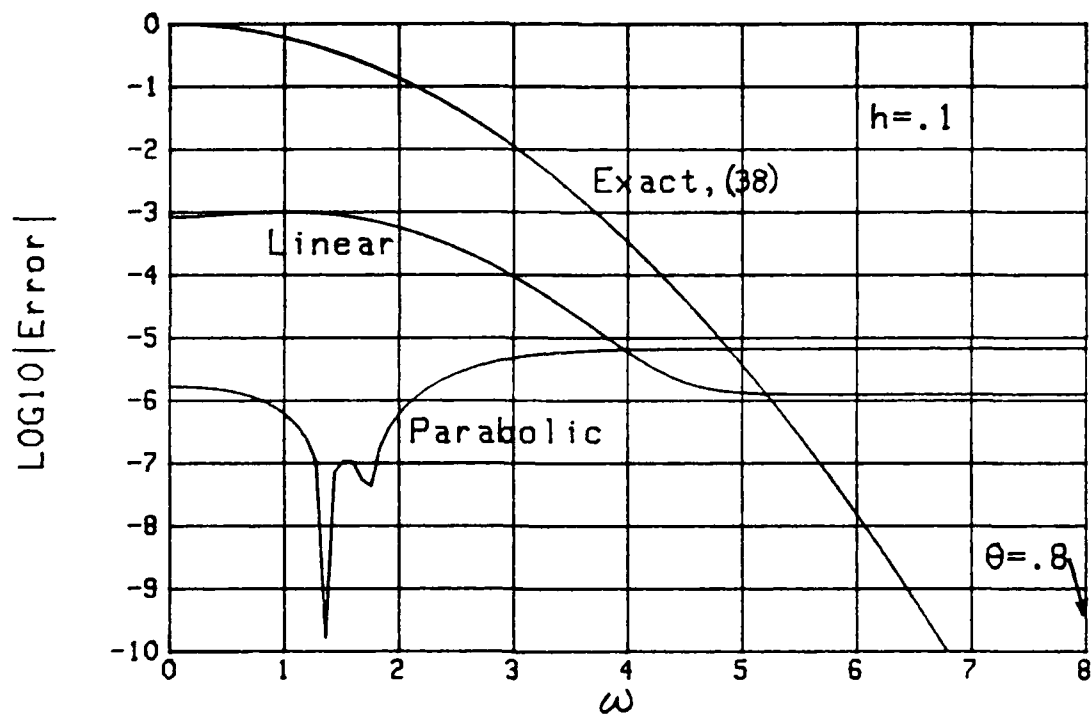
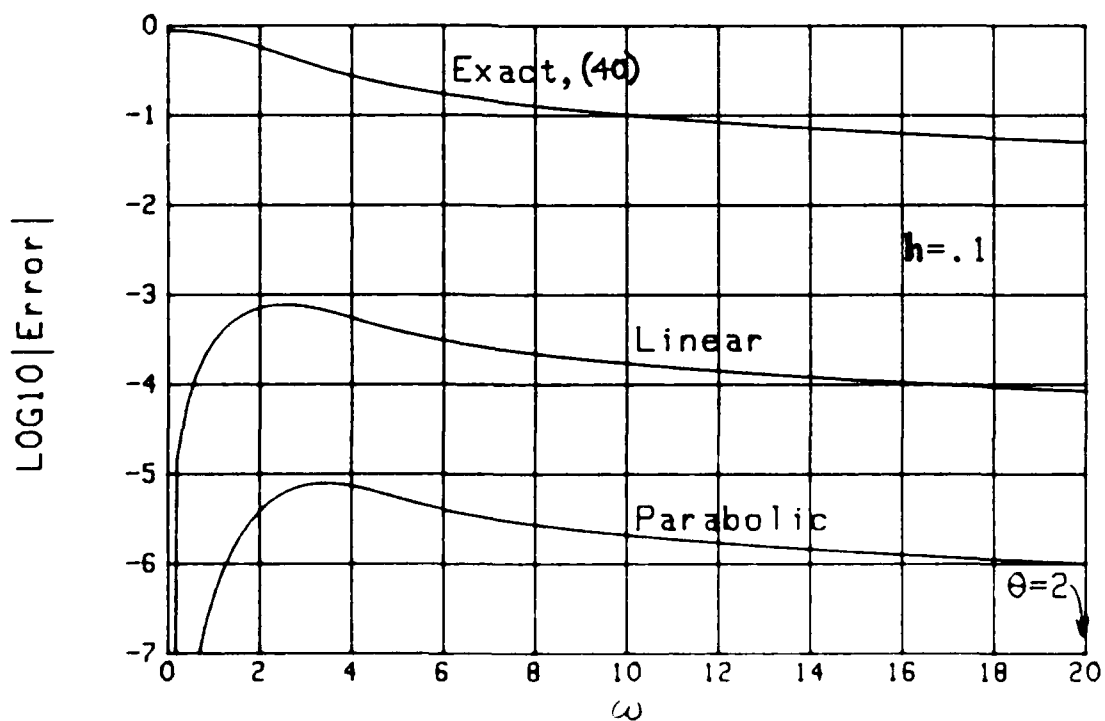
For a Fourier transform, this aliasing effect was studied quantitatively in [10; appendix A] for both the Trapezoidal rule and Simpson's rule. The former rule was shown to have a large aliasing lobe at $\theta = \omega h = 2\pi$, while the latter rule had an additional large lobe at $\theta = \pi$, due to the alternating character of the Simpson weights; see [10; (A-6) and (A-8)]. This leads us to anticipate that the linear approximation procedure developed here for Bessel transform (11) will be subject to aliasing near

$\Theta = 2\pi$, while the parabolic approximation will be degraded earlier, namely near $\Theta = \pi$. This will be borne out by the numerical examples to follow.

GRAPHICAL RESULTS

The Bessel transform numerical integration rule for the linear approximation to $g(x)$ is given by (23) or (26), while the rule for the parabolic approximation to $g(x)$ is given by (30) or (36). The exact transforms (38) and (40), and the absolute errors associated with these two rules, are depicted in figures 3 and 4 for the Rayleigh and Gaussian functions $g(x)$ of (37) and (39), respectively, with sampling increment $h = .1$. The ordinates in all figures are the logarithm to the base 10 of the corresponding results, while the abscissas are linear in ω or Θ . The upper limit, x_p , of integration in (2) or (11) is taken large enough to guarantee a negligible contribution (less than $1E-20$) to the truncation error.

In figure 3, the error for the parabolic fits is initially lower (for small ω) than for the linear fits; however, the linear error decays rapidly with ω , and stays below the parabolic error for larger ω . Both absolute errors flatten out and are not increasing with ω , at least for this range of ω values. The maximum value of Θ is .8, as indicated in the figure.

Figure 3. Errors for Rayleigh Function $g(x)$ Figure 4. Errors for Gaussian Function $g(x)$

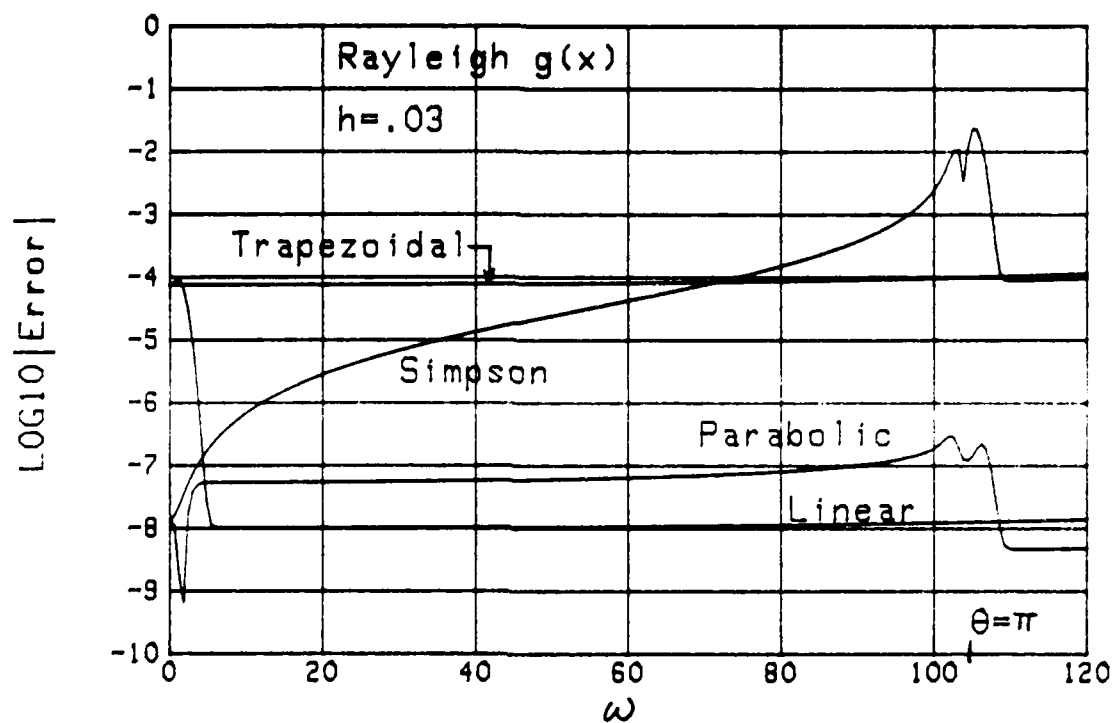
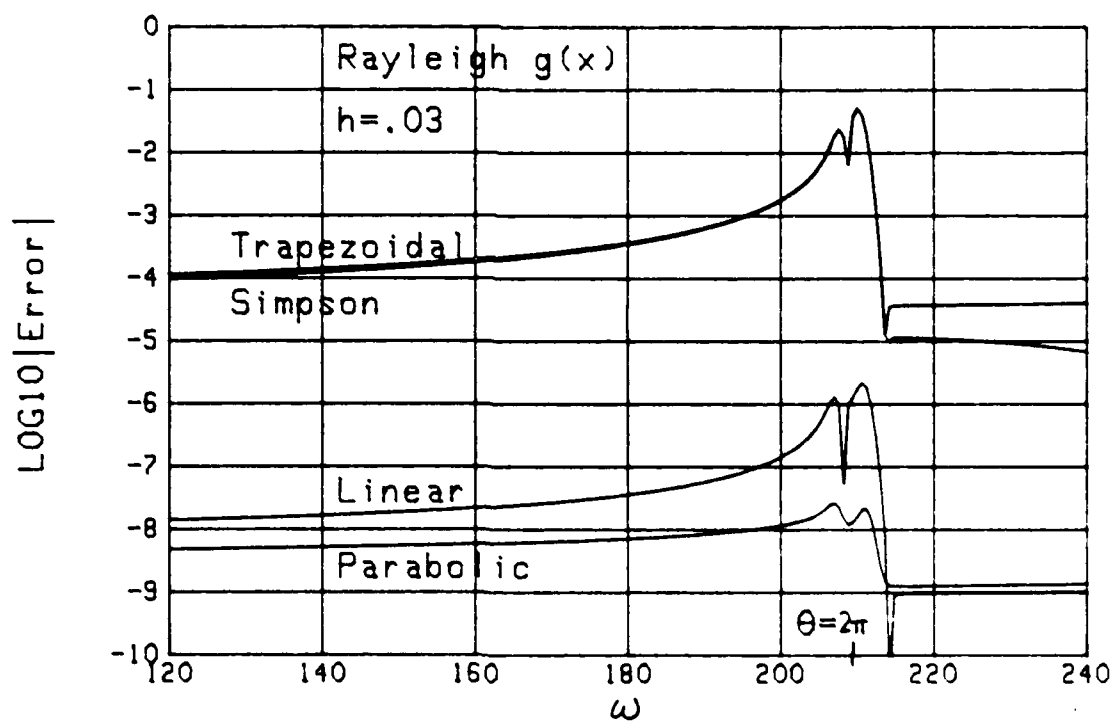
For the Gaussian function $g(x)$, the parabolic error in figure 4 is everywhere less than the linear error. Both errors near and at $\omega = 0$ are extremely small; this fortuitous result for the linear fits is fully explained in [6; pages 92-93], especially in the paragraph under (3.4.5). It has to do with the fact that the integrand in (11) for this Gaussian case, namely $J_0(\omega x) \exp(-x^2)$, has zero odd derivatives at the limits of integration. This is not the case for the Rayleigh function; hence the much larger errors at $\omega = 0$ in figure 3 result.

COMPARISON OF PROCEDURES

To demonstrate the benefits to be accrued from the fitting procedures derived in this study, a comparison of the absolute errors for four different procedures is presented in figure 5 for the Rayleigh function (37). The sampling increment in x is $h = .03$. The variable ω now covers the range $(0, 120)$; the point where $\theta = \pi$ is indicated by a tic mark on the abscissa.

The Trapezoidal result is obtained by applying it to the complete integrand $J_0(\omega x) g(x)$ of (11). The error is essentially constant for all ω , including the region near $\theta = \pi$; thus, as expected, aliasing is not significant at $\theta = \pi$ for the Trapezoidal rule.

Application of the standard Simpson's rule to the complete integrand of (11) yields a very small error near $\omega = 0$, but a rapidly increasing error with ω , and a very large aliasing lobe centered around $\theta = \pi$. This confirms the expectations presented earlier in this section.

Figure 5. Errors for Four Procedures; $\omega < 120$ Figure 6. Errors for Four Procedures; $\omega > 120$

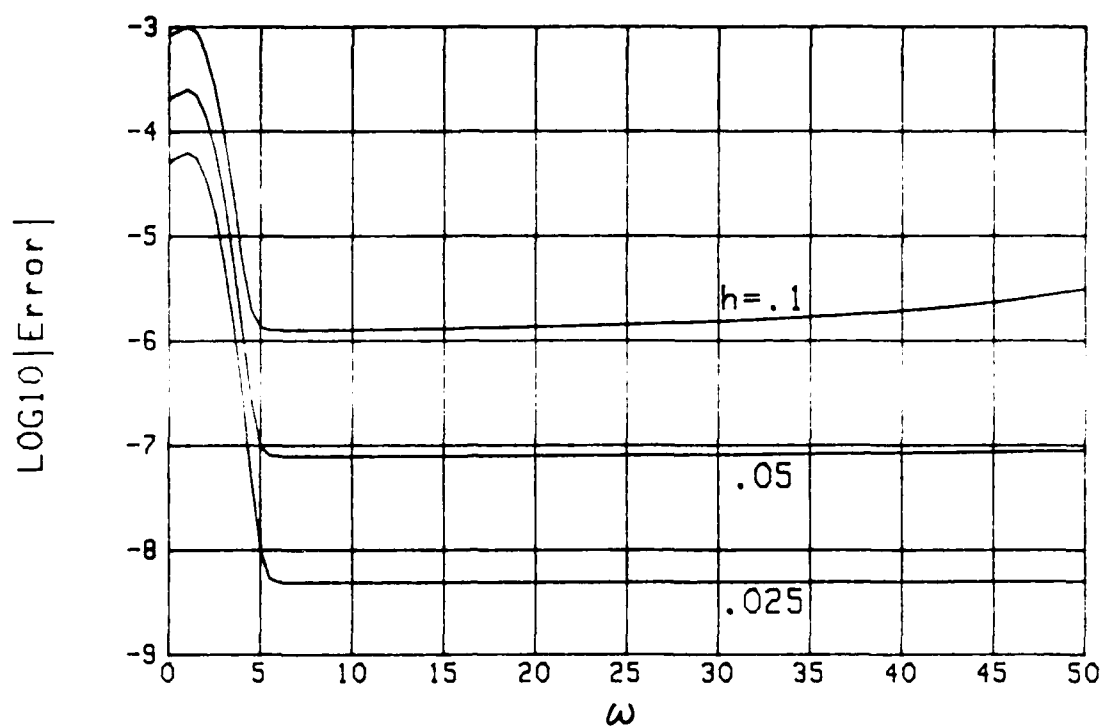
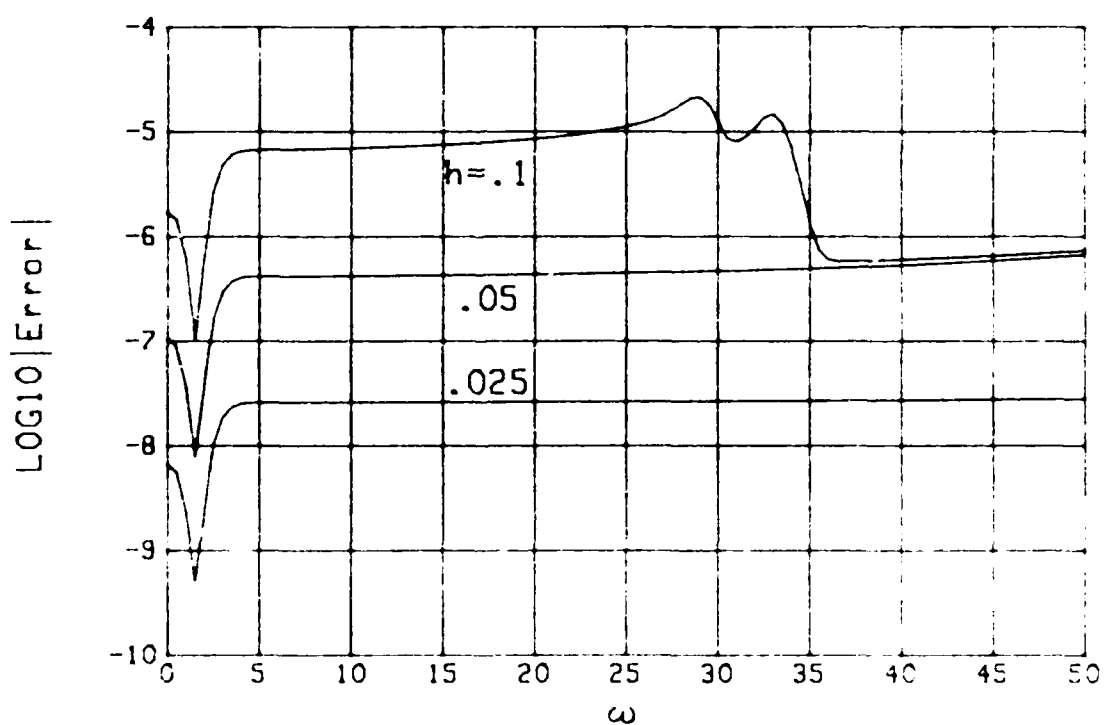
For the case of linear fits to $g(x)$, rather than $J_0(\omega x) g(x)$, the error drops dramatically, by four orders of magnitude as ω increases, similar to figure 3. Furthermore, there is no aliasing at $\theta = \pi$.

The situation for the parabolic fits is that the absolute error starts out small and remains so, for all $\omega < 120$, there being a slight aliasing effect near $\theta = \pi$. However, it is 5 orders of magnitude smaller than the Simpson error in this region of ω .

The results in figure 6 extend the abscissa to cover the range of (120,240) in ω ; that is, these curves are an extension of those in figure 5. Now all rules suffer aliasing in the neighborhood of $\theta = 2\pi$. The absolute error for the linear procedure increases by 2 orders of magnitude near $\theta = 2\pi$, while the parabolic error is just slightly larger; however, the latter is 6 orders of magnitude better than the standard Trapezoidal and Simpson rules for numerical integration. All of these results confirm the predicted presence and location of aliasing discussed earlier.

ERROR DEPENDENCE ON SAMPLING INCREMENT

In figure 7, we investigate the dependence of the error on increment h employed to sample x in (11). Here we apply the linear fit procedure to the Rayleigh function (37). The absolute error for small ω (< 2) decreases by a factor of 4 as h is halved; that is, the large error bump near $\omega = 0$ behaves as h^2 for small increments h . On the other hand, for larger ω (> 5),

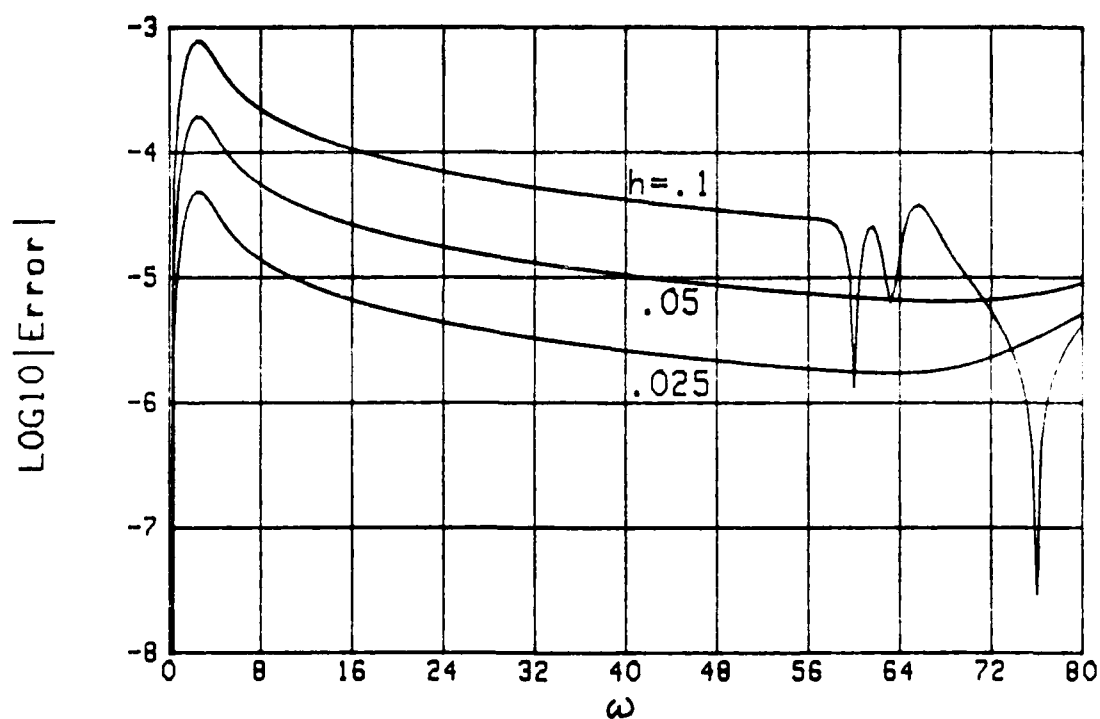
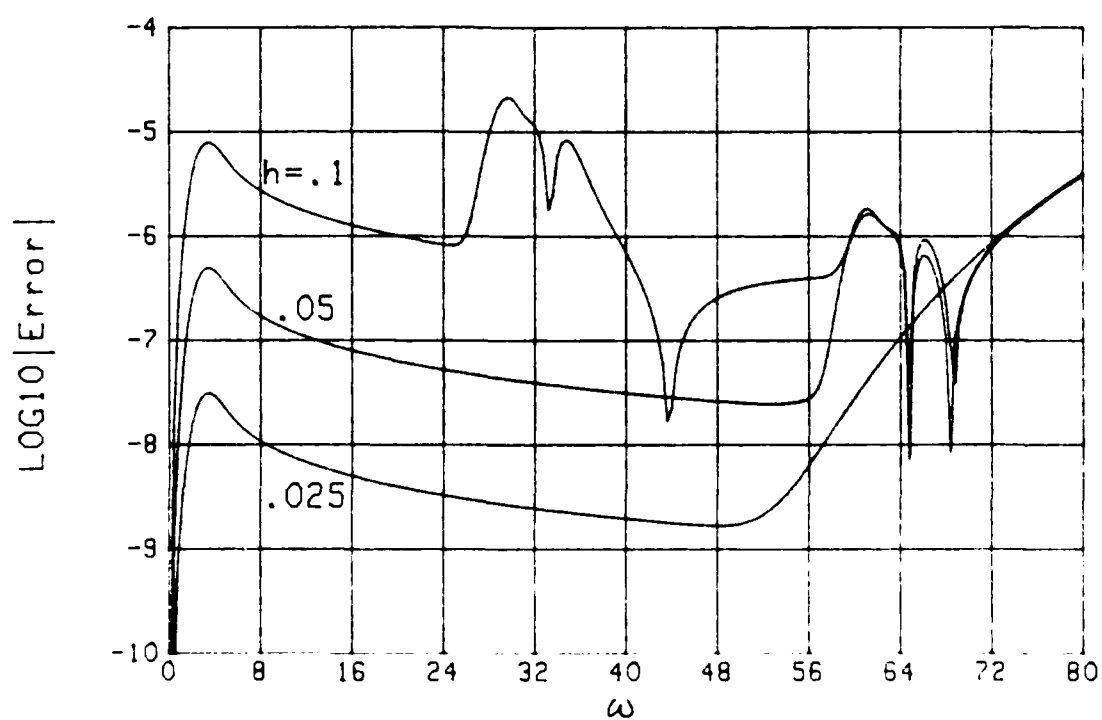
Figure 7. Linear Procedure, Rayleigh $g(x)$ Figure 8. Parabolic Procedure, Rayleigh $g(x)$

the error decreases by a factor of 16 when h is halved; that is, the "saturation" level of error behaves as h^4 for small h . The slight flare in the error curve near $\omega = 50$, for $h = .1$, is an indication of the beginning of aliasing; that is, $\Theta = 5$ here, which is near the $\Theta = 2\pi$ location.

Still considering the Rayleigh function (37), but now switching to the parabolic procedure, the results in figure 8 demonstrate that the error drops by a factor of 16 as h is halved; thus, the error dependence is h^4 for all ω . The wiggles in the $h = .1$ curve near $\omega = 30$ are due to aliasing, since $\Theta = \pi$ for $\omega = 10\pi = 31.4$.

When the function $g(x)$ is changed to the Gaussian example of (39), and the linear fitting procedure is employed, the errors are depicted in figure 9. Here, the error dependence is according to h^2 for all ω , until aliasing sets in. Aliasing is present in the $h = .1$ curve near $\omega = 64$, since $\Theta = 2\pi$ at $\omega = 62.8$ for that curve. Comparison of these errors with the exact answer in figure 4 reveals that the relative error is constant in the range $4 < \omega < 56$.

When the parabolic procedure is used instead on the Gaussian example, the error dependence is again according to h^4 , until aliasing becomes dominant. The aliasing lobes in the $h = .1$ curve in figure 10 are centered at $\Theta = \pi$ and 2π , as before. The large increase in the error for the $h = .025$ curve, when ω exceeds 50, is a feature not seen previously. It may be due to the sum of distant aliasing of sidelobes which decay very slowly

Figure 9. Linear Procedure, Gaussian $g(x)$ Figure 10. Parabolic Procedure, Gaussian $g(x)$

with ω ; in fact, from (41), the exact answer only decays as $1/\omega$. The rapid decay of the Rayleigh transform, (38), apparently precluded this type of error from appearing in any of the numerical cases considered here for the Rayleigh $g(x)$.

SUMMARY

There is a marked difference between the form of these results and the Filon equations; namely, the term multiplying sample value $g_n = g(x_n)$ (in (B-3), for example) varies with n in such a fashion that no simplification or factoring is possible. In order to better explain this complication, let us investigate the evaluation of (18) when $J_0(\omega x)$ is replaced by $\exp(i\omega x)$; that is, consider evaluation of a Fourier transform, rather than a Bessel transform, for the moment. When the linear fits to $g(x)$ in (18) are then integrated, there follows

$$I_n = g_n h \exp(in\theta) \left[\frac{\sin(\theta/2)}{\theta/2} \right]^2. \quad (44)$$

But the bracketed term here is a common factor (independent of n) that can be removed from the summation on n . This fortuitous simplification does not hold for the corresponding result (B-3) here, because whereas $\exp(iu)$ is periodic, $J_0(u)$ and $A(u)$ are not.

In an effort to recover some of this loss in execution time, we therefore grouped the terms in (B-6) in an alternative form, pivoted around $B_1(n\theta)$ rather than g_n ; see (B-7). Perhaps another rearrangement of terms would be more advantageous for some purposes.

It is possible to extend the results here to other Bessel transforms. For example, suppose we are interested in the evaluation of first-order transform

$$\int dx J_1(\omega x) g(x) , \quad (45)$$

and we approximate $g(x)$ either by straight lines or parabolas. The integrals in (13)-(15) are then replaced by

$$\int_0^u dt J_1(t) = 1 - J_0(u) ,$$

$$\int_0^u dt t J_1(t) = B_0(u) ,$$

$$\int_0^u dt t^2 J_1(t) = u^2 J_2(u) = 2u J_1(u) - u^2 J_0(u) , \quad (46)$$

where we used [5; (11.1.6) and (9.1.30)] and (16). Since all of these terms have already been encountered here, extension to transform (45) would not be difficult.

For the evaluation of the alternative transform

$$\int dx \frac{J_1(\omega x)}{x} g(x) , \quad (47)$$

we need the additional result [5; (11.1.1)]

$$\int_0^u dt \frac{J_1(t)}{t} = \frac{4}{u} \sum_{k=1}^{\infty} k J_{2k}(u) =$$

$$= \frac{4}{u} [J_2(u) + 2 J_4(u) + 3 J_6(u) + \dots] . \quad (48)$$

But this type of term is easily evaluated by means of the downward recurrence technique given in appendix A. In fact, immediately following the single line $Se = Se + E$, we have merely to add the line $Sx = Sx + Se$; when the downward recurrence is completed, the bracketed term in (48) results in storage location Sx (after the scaling correction).

APPENDIX A NUMERICAL EVALUATION PROCEDURE FOR BESSEL INTEGRALS

The three fundamental Bessel integrals that must be evaluated are given by (13)-(17) as

$$A(u) = \int_0^u dt J_0(t) , \quad (A-1)$$

$$B_0(u) = A(u) - u J_0(u) = \int_0^u dt t (u - t) J_0(t) , \quad (A-2)$$

$$B_1(u) = A(u) - J_1(u) = \int_0^u dt \left(1 - \frac{t}{u}\right) J_0(t) . \quad (A-3)$$

By expanding J_0 in a power series [5; (9.1.10)], and integrating term by term, there follows from (A-1),

$$A(u) = \frac{u}{2} \sum_{k=0}^{\infty} \frac{(-u^2/4)^k}{k! k! (k + \frac{1}{2})} = u - \frac{u^3}{12} + \frac{u^5}{320} - \dots . \quad (A-4)$$

When this result is coupled with the series expansions of J_0 and J_1 in (A-2) and (A-3) respectively, there follows

$$B_0(u) = \frac{u^3}{4} \sum_{k=0}^{\infty} \frac{(-u^2/4)^k}{k! (k+1)! (k + \frac{3}{2})} = \frac{u^3}{6} - \frac{u^5}{80} + \frac{u^7}{2688} - \dots \quad (A-5)$$

and

$$B_1(u) = \frac{u}{4} \sum_{k=0}^{\infty} \frac{(-u^2/4)^k}{k! (k+1)! (k + \frac{1}{2})} = \frac{u}{2} - \frac{u^3}{48} + \frac{u^5}{1920} - \dots \quad (A-6)$$

Although these power series could be used for small and moderate values of u , they are not useful for large u , due to the loss of significant digits caused by the alternating character of series (A-4)-(A-6). In fact, we will find that a downward recurrence will yield all the values of A , B_0 , B_1 , J_0 , and J_1 very efficiently for small u , while an asymptotic expansion is equally attractive for large u .

DOWNWARD RECURRENCE

We start with [5; (11.1.2)] and (A-1):

$$A(u) = 2[J_1(u) + J_3(u) + J_5(u) + \dots] \quad (A-7)$$

Thus if we can evaluate all the odd-order Bessel functions, we can get $A(u)$ from their sum. Also, $B_0(u)$ and $B_1(u)$ follow immediately from (A-2) and (A-3), if we can additionally get $J_0(u)$.

But the Bessel functions satisfy the downward recurrence [5; (9.1.2)],

$$J_{k-1}(u) = \frac{2k}{u} J_k(u) - J_{k+1}(u)$$

$$J_m(u) = \frac{2}{u} (m+1) J_{m+1}(u) - J_{m+2}(u) \quad (A-8)$$

for $m \geq 0$. This recurrence can be started by guessing at $J_M(u) = 0$, $J_{M-1}(u) = 1E-250$ for example, and evaluating downward via (A-8) to $m = 0$. Since the error increases much slower than the size of the terms in (A-8) [5; table 9.4], the relative error of the terms is very small for the smaller values of m , if M is chosen large enough to start with. In order to accurately establish the absolute level of the sequence of $\{J_m\}$ values, we then use the check sum formula [5; (9.1.46)]

$$J_0(u) + 2[J_2(u) + J_4(u) + \dots] = 1 \quad (A-9)$$

In order to realize 15 decimal accuracy in A , B_0 , B_1 , J_0 , J_1 , it has been found sufficient to choose even integer M as

$$M = M(u) = 2 \text{ INT} \left(20 + .56u - \frac{175}{12 + u} \right) + 12 \quad \text{for } 0 \leq u < 45 \quad (A-10)$$

While conducting the downward recurrence on m in (A-8), an even sum of $J_M + J_{M-2} + \dots$, and an odd sum of $J_{M-1} + J_{M-3} + \dots$, are maintained. After completion to $m = 0$, the even sum is subject to constraint (A-9), in order to establish the scale factor that must be applied to all the desired outputs; this is to correct for the initial arbitrary (incorrect) guess of $J_{M-1}(u) = 1E-250$. With this scale factor in hand, the odd sum in (A-7) can then be modified by means of one multiplication for the correct absolute level for $A(u)$. Since the last two quantities yielded by recurrence (A-8)

are $J_1(u)$ and $J_0(u)$ (after scaling), we then have all the necessary ingredients to determine $B_0(u)$ and $B_1(u)$.

No array declarations or array storage is necessary in this procedure, since there is never any need to "go back up" the recurrence and correctly scale the $\{J_m(u)\}$ terms. This has been guaranteed (through numerical investigation) by the choice of M in (A-10). A further economy in the program for this two-term recurrence (A-8) has been achieved by splitting it into even and odd versions, thereby avoiding the usual temporary storage of the left-hand side of (A-8) until the right-hand side is updated. This compact program is listed below as subroutine SUB Besj. For given u , it outputs values for $J_0(u)$, $J_1(u)$, $A(u)$, $B_0(u)$, $B_1(u)$, provided that $0 \leq u < 45$.

ASYMPTOTIC EXPANSION

For large u , the starting integer M in (A-10) gets too large to make downward recurrence a viable procedure. Instead, we resort to the asymptotic expansion [5; (11.1.11)]

$$A(u) = \int_0^u dt J_0(t) \sim 1 - \left(\frac{2}{\pi u}\right)^{1/2} \left[\cos\left(u - \frac{\pi}{4}\right) \sum_{k=0}^{\infty} \frac{(-1)^k a_{2k+1}}{u^{2k+1}} - \sin\left(u - \frac{\pi}{4}\right) \sum_{k=0}^{\infty} \frac{(-1)^k a_{2k}}{u^{2k}} \right] \quad (\text{A-11})$$

as $u \rightarrow +\infty$; here, we also used the definite integral result that $A(\infty) = 1$ [5; (11.4.17)]. The values of the coefficients are [5; (11.1.2)]

$$a_k = \left(\frac{1}{2}\right)_k \sum_{s=0}^k \left(\frac{1}{2}\right)_s \frac{1}{2^s s!} \quad (\text{A-12})$$

and are conveniently obtained by recursion

$$T_s \equiv \left(\frac{1}{2}\right)_s \frac{1}{2^s s!} = \frac{s - \frac{1}{2}}{2s} T_{s-1} \quad \text{for } s \geq 1. \quad (\text{A-13})$$

The number of terms required in the summations in (A-11) depends on the value of u and the desired accuracy. For $u > 45$ and 15 decimal accuracy, it has been sufficient to terminate (A-11) at $k = \text{INT}(u/2)$.

Since (A-11) yields only $A(u)$, it is necessary to calculate $J_0(u)$ and $J_1(u)$ additionally; this has been accomplished by use of [11; section 6.8]. All of these quantities are evaluated by means of subroutine SUB Bessel listed below. For input $u \geq 0$, this subroutine yields values of $J_0(u)$, $J_1(u)$, $A(u)$, $B_0(u)$, $B_1(u)$.

```

10  SUB Bessel(X, J0, J1, A, B0, B1)  !  A = INTEGRAL(0, X) dt Jo(t)
20  DOUBLE K, I                        !  INTEGERS
30  IF X>45. THEN 60
40  CALL Besj(X, J0, J1, A, B0, B1)  !  DOWNWARD RECURRENCE 9.1.27, 1
50  SUBEXIT
60  I=INT(X)/2                        !  ASYMPTOTIC SERIES 11.1.11 & 12
70  Rx=1./X
80  F=.5*Rx
90  T=.25
100 A=1.25
110 Re=.625*Rx
120 Im=P=1.
130 FOR K=1 TO I
140  P=-P
150  Sn=K+K
160  F5=Sn-.5
170  F=F+F5*Rx
180  T=T+F5/(Sn+Sn)
190  A=A+T
200  Be=F*A
210  Im=Im+P*Be
220  Sn=Sn+1.
230  F5=Sn-.5
240  F=F+F5*Rx
250  T=T+F5/(Sn+Sn)
260  A=A+T
270  Bo=F*A
280  Re=Re+P*Bo
290  IF Be*Be+Bo*Bo<1.E-26 THEN 310
300  NEXT K
310  F=X-.78539816339744828
320  T=.79788456080286541
330  A=1.-T*SQR(Rx)*(Re*COS(F)-Im*SIN(F))
340  J0=FNJo(X)                       !  J0 = Jo(X)
350  J1=FNJ1(X)                       !  J1 = J1(X)
360  B0=A-X*J0                         !  B0 = A(X) - X Jo(X)
370  B1=A-J1                          !  B1 = A(X) - J1(X)
380  SUBEND
390  !
400  SUB Besj(U, J0, J1, A, B0, B1)  !  J0 = Jo(U), J1 = J1(U)
410  IF U>0. THEN 450                !  A = A(U) = INTEGRAL(0, U) dt Jo(t)
420  J0=1.                            !  B0 = A(U) - U Jo(U)
430  J1=A=B0=B1=0.                   !  B1 = A(U) - J1(U)
440  SUBEXIT
450  DOUBLE Mc, Ms                    !  INTEGERS
460  Mc=2*INT(20.+.56*U-175./(12.+U))+12
470  T=2./U
480  Se=E=0.
490  So=0=1.E-250
500  FOR Ms=Mc TO 2 STEP -2
510  E=T*(Ms+1)*O-E                   !  9.1.27, 1
520  Se=Se+E
530  O=T*Ms+E-O                      !  9.1.27, 1
540  So=So+O
550  NEXT Ms
560  E=T*O-E
570  F=1. (Se+Se+E)                  !  9.1.46
580  J0=E+F
590  J1=O+F
600  A=(So+So)*F                     !  11.1.2
610  B0=A-U*J0
620  B1=A-J1
630  SUBEND
640  !

```

```

650 DEF FNJo(X)      ! Jo(X) via Hart #5845, 6546, and 6946
660 Y=ABS(X)
670 IF Y>8. THEN 770
680 T=Y*Y
690 P=2271490439.5536033-T*(.5513584.5647707522-T+5292.6171303845574+
700 P=2334489171877869.7-T*(.47765559442673.588-T+.462172225031.71803-T+P)
710 P=185962317621897804.-T*(.44145829391815982.-T+P)
720 Q=204251483.52134357+T*(.494030.79491813972+T*(.884.72036756175504+T))
730 Q=2344750013658996.8+T*(.15015462449769.752+T*(.64398674535.133256+T*Q))
740 Q=185962317621897733.+T*Q
750 Jo=P/Q
760 RETURN Jo
770 Z=8./Y
780 T=Z*Z
790 Pn=2204.5010439651804+T*(128.67758574871419+T+.90047934748028803)
800 Pn=8554.8225415066617+T*(8894.4375329606194+T+Pn)
810 Pd=2214.0488519147104+T*(130.88490049992388+T)
820 Pd=8554.8225415066628+T*(8903.8361417095954+T+Pd)
830 Qn=13.990976865960680+T*(1.0497327982345548+T+.00935259532940319)
840 Qn=37.510534954957112+T*(.46.093826814625175+T+Qn)
850 Qd=921.56697552653090+T*(74.428389741411179+T)
860 Qd=2400.6742371172675+T*(2971.9837452084920+T+Qd)
870 T=Y-.78539816339744828
880 Jo=.28209479177387820*SQR(Z)*(COS(T)*Pn/Pd+SIN(T)+2*Qn/Qd)
890 RETURN Jo
900 FNEND
910 !
920 DEF FNJ1(X)      ! J1(X) via Hart #6045, 6747, and 7147
930 Y=ABS(X)
940 IF Y>8. THEN 1040
950 T=Y*Y
960 P=.11073522244537306E-10-T+.63194310317443161E-14
970 P=.49105992765551294E-5-T+.93821933651407445E-8-T+P
980 P=.398310798395233-T*(.17057692643496171E-2-T+P)
990 P=5878.7877666568200-T*(.61.218769973569439-T+P)
1000 P=69536422.632983850-T*(.8356785.4873489143-T+.320302.74688539470-T+P)
1010 Q=139072845.26596769+T*(.670534.68354822993+T+.1284.5934539663019+T)
1020 J1=X*P/Q
1030 RETURN J1
1040 Z=8./Y
1050 T=Z*Z
1060 Pn=3132.7529563550695+T*(.174.31379748379025+T+.1.2065053764359043)
1070 Pn=12909.184718961881+T*(13090.420511035065+T+Pn)
1080 Pd=3109.2814167700288+T*(.169.04721775008610+T)
1090 Pd=12909.184718961879+T*(13086.783087844020+T+Pd)
1100 Qn=51.736532818365916+T*(.3.7994453796930673+T+.036363466476034711)
1110 Qn=144.65262874995209+T*(.174.42916890924259+T+Qn)
1120 Qd=1119.1098527047487+T*(.85.223920643413404+T)
1130 Qd=3085.9270133323172+T*(.3734.3401060163013+T+Qd)
1140 T=-2.3561944801823446
1150 J1=.28209479177387820*SQR(Z)*(COS(T)*Pn/Pd+SIN(T)+2*Qn/Qd)
1160 IF J1=0. THEN J1=-J1
1170 RETURN J1
1180 FNEND

```

APPENDIX B

DERIVATION OF INTEGRATION RULE FOR STRAIGHT LINE FITS TO $g(x)$

The situation of interest here is represented in figure 1, where straight lines are fit to $g(x)$ between adjacent samples of $g(x)$, taken at sample points $\{x_n\}$. In particular, the contribution to integral (11) of an (internal) abutting point x_n was set up in (18)-(19). By letting $t = \omega x$ in (18), and using (19), (20), and (22), namely

$$y = \frac{x - x_n}{h}, \quad x_n = nh, \quad \theta = \omega h, \quad (B-1)$$

there follows, for the n -th contribution to the integral,

$$\begin{aligned} I_n = & \frac{g_n}{\omega} \int_{(n-1)\theta}^{n\theta} dt J_0(t) \left(1 - n + \frac{t}{\theta}\right) + \\ & + \frac{g_n}{\omega} \int_{n\theta}^{(n+1)\theta} dt J_0(t) \left(1 + n - \frac{t}{\theta}\right), \end{aligned} \quad (B-2)$$

where $g_n = g(x_n) = g(nh)$. By reference to the auxiliary functions defined in (13)-(17), the sum of these two integrals can be expressed in the compact form

$$I_n = \frac{g_n}{\omega} \left\{ (n+1) B_1[(n+1)\theta] - 2n B_1[n\theta] + (n-1) B_1[(n-1)\theta] \right\} \quad \text{for } j < n < r. \quad (B-3)$$

The procedures in appendix A are now directly applicable to the evaluation of (B-3) for any n .

For the left-end point x_l depicted on the left side of figure 1, the corresponding contribution to desired integral result (11) is, using (B-1) again,

$$\begin{aligned}
 I_l &= \int_{x_l}^{x_l+h} dx J_0(\omega x) g_l (1 - y) = \\
 &= \frac{g_l}{\omega} \int_{l\theta}^{(l+1)\theta} dt J_0(t) \left(l + 1 - \frac{t}{\theta} \right) = \\
 &= \frac{g_l}{\omega} \left\{ (l + 1) B_1[(l + 1)\theta] - (l + 1) B_1[l\theta] - J_1[l\theta] \right\} . \quad (B-4)
 \end{aligned}$$

The corresponding contribution to integral (11) for the right-end point x_r is given by

$$\begin{aligned}
 I_r &= \int_{x_r-h}^{x_r} dx J_0(\omega x) g_r (1 + y) = \\
 &= \frac{g_r}{\omega} \int_{(r-1)\theta}^{r\theta} dt J_0(t) \left(\frac{t}{\theta} - r + 1 \right) = \\
 &= \frac{g_r}{\omega} \left\{ (r - 1) B_1[(r - 1)\theta] - (r - 1) B_1[r\theta] + J_1[r\theta] \right\} . \quad (B-5)
 \end{aligned}$$

(As a check, combination of (B-4) and (B-5), upon replacement of ℓ and r by n , yields (B-3), as it should. The "end correction terms" in J_1 cancel out for all internal points, n .)

The resultant approximation to desired integral (11) is given by the sum of (B-3)-(B-5):

$$G(\omega) = \int_{x_\ell}^{x_r} dx J_0(\omega x) g(x) \cong I_\ell + I_r + \sum_{n=\ell+1}^{r-1} I_n. \quad (B-6)$$

This particular grouping of terms is according to the function sample values $\{g_n\} = \{g(nh)\}$. An alternative grouping, according to the samples of function $B_1(u)$ instead, is given by

$$\begin{aligned} \omega G(\omega) \cong & \left[\ell g_{\ell+1} - (\ell + 1)g_\ell \right] B_1(\ell\theta) - g_\ell J_1(\ell\theta) + \\ & + \left[r g_{r-1} - (r - 1)g_r \right] B_1(r\theta) + g_r J_1(r\theta) + \\ & + \sum_{n=\ell+1}^{r-1} n [g_{n+1} - 2g_n + g_{n-1}] B_1(n\theta). \end{aligned} \quad (B-7)$$

Whereas $B_1(n\theta)$ must be evaluated for all $\ell \leq n \leq r$, the J_1 function need only be evaluated at the end points $\ell\theta$ and $r\theta$.

When ω is restricted to be multiples of a sampling increment Δ , that is

$$\omega = k\Delta \quad \text{for } k = 1, 2, \dots \quad (\text{B-8})$$

then (B-7) yields, for $k \geq 1$, the approximation

$$\begin{aligned} k\Delta G(k\Delta) \cong & \left[\ell g_{\ell+1} - (\ell + 1)g_{\ell} \right] B_1(\ell k\Delta h) - g_{\ell} J_1(\ell k\Delta h) + \\ & + \left[r g_{r-1} - (r - 1)g_r \right] B_1(rk\Delta h) + g_r J_1(rk\Delta h) + \\ & + \sum_{n=\ell+1}^{r-1} n \left[g_{n+1} - 2g_n + g_{n-1} \right] B_1(nk\Delta h) , \end{aligned} \quad (\text{B-9})$$

where

$$g_n = g(nh) . \quad (\text{B-10})$$

Since n and k are integers (see (20) and (B-8)), the evaluation of $B_1(u)$ in (B-9) is confined to integer multiples of Δh , i.e. $u = m\Delta h$. Further discussion on how to take advantage of this feature of (B-9) is given in the sequel to (26). The end result is that we have two alternative procedures for evaluation of (B-9) and two corresponding programs: one faster routine which may require considerable storage, and a slower procedure utilizing very little storage. Programs for both procedures are listed below.

BEHAVIOR FOR SMALL Θ

When Θ is small, the differences of like quantities in (B-3)-(B-5) can be circumvented by expanding B_1 and J_1 in power series in Θ . Using the facts that

$$\begin{aligned} B_1(u) &\sim \frac{u}{2} - \frac{u^3}{48} \quad \text{as } u \rightarrow 0, \\ J_1(u) &\sim \frac{u}{2} - \frac{u^3}{16} \quad \text{as } u \rightarrow 0, \end{aligned} \quad (B-11)$$

the above results reduce to

$$\begin{aligned} I_n &\sim g_n h \left[1 - \frac{1}{4} \Theta^2 \left(n^2 + \frac{1}{6} \right) \right], \\ I_\lambda &\sim \frac{1}{2} g_\lambda h \left[1 - \frac{1}{4} \Theta^2 \left(\lambda^2 + \frac{2}{3} \lambda + \frac{1}{6} \right) \right], \\ I_r &\sim \frac{1}{2} g_r h \left[1 - \frac{1}{4} \Theta^2 \left(r^2 - \frac{2}{3} r + \frac{1}{6} \right) \right], \end{aligned} \quad (B-12)$$

as $\Theta \rightarrow 0$. By use of the power series expansion developed for $B_1(u)$ in appendix A, these results could be extended to order Θ^4 , Θ^6 if desired.

The total contribution to (11) is given by the sum in (B-6). As $\Theta \rightarrow 0$, this reduces to the Trapezoidal rule, (12).

```

10  ! ZERO-TH ORDER BESSEL TRANSFORM USING LINEAR INTERPOLATION.
20  ! INTEGRAL(X1,Xr) dX Jo(WX) g(X) FOR W1<=W<=W2 IS STORED IN
30  ! Gw(Ks), where W = Ks*Delw.      Faster high-storage.
40  Delx=.025      ! INCREMENT (h) IN X
50  L=0            ! X1=L*Delx, L>=0
60  R=400          ! Xr=R*Delx, R>L
70  Delw=.2        ! INCREMENT (Δ) IN W
80  K1=0           ! W1=K1*Delw, K1>=0
90  K2=40          ! W2=K2*Delw, K2>=K1
100 DOUBLE L,P,K1,K2,K0,L1,R1,Ns,Ks,I      ! INTEGERS
110 DIM G(500),Dg(500),B1(50000),J11(100),J1r(100),Gw(100)
120 K0=K1
130 K1=MAX(K1,1)
140 L1=L+1
150 R1=R-1
160 REDIM G(L:R),Dg(L1:R),B1(L*K1:R*K2)
170 REDIM J11(K1:K2),J1r(K1:K2),Gw(K0:K2)
180 FOR Ks=K0 TO K2
190 Gw(Ks)=0.
200 NEXT Ks
210 FOR Ns=L TO R
220 Gx(Ns)=FNG(Ns*Delx)      ! SEE DEF FNG(X) = g(X)
230 NEXT Ns
240 G1=Gx(L)
250 Gr=Gx(R)
260 IF K0>0 THEN 320
270 F=.5*(G1+Gr)
280 FOR Ns=L1 TO R1
290 F=F+Gx(Ns)
300 NEXT Ns
310 Gw(K0)=F*Delx
320 FOR Ns=L1 TO R
330 Dg(Ns)=Gx(Ns)-Gx(Ns-1)
340 NEXT Ns
350 D2=Delw*Delx
360 IF L=0 THEN 410
370 FOR Ks=K1 TO K2
380 I=L*Ks
390 CALL Bessel(I+D2,J0,J11+Fab,A,B0,B1:I)
400 NEXT Ks
410 FOR Ks=K1 TO K2
420 I=R*Ks
430 CALL Bessel(I+D2,J0,J1r+Fab,A,B0,B1:I)
440 NEXT Ks
450 FOR Ns=L1 TO R1
460 FOR Ks=K1 TO K2
470 I=Ns*Ks
480 IF B1(I)=0. THEN 500
490 CALL Bessel(I+D2,J0,J1,A,B0,B1:I)
500 NEXT Ks
510 NEXT Ns

```

```

520   T1=L*Dg(L1)-G1
530   T2=R*Dg(R)-Gr
540   IF L=0 THEN 580
550   FOR Ks=K1 TO K2
560     Gw(Ks)=T1*B1(L+Ks)-G1+J11(Ks)
570   NEXT Ks
580   FOR Ks=K1 TO K2
590     F=T2+B1(R+Ks)-Gr+J1r(Ks)
600     Gw(Ks)=Gw(Ks)-F
610   NEXT Ks
620   FOR Ns=L1 TO R1
630     F=Ns*(Dg(Ns+1)-Dg(Ns))
640     FOR Ks=K1 TO K2
650       Gw(Ks)=Gw(Ks)+F*B1(Ns+Ks)
660     NEXT Ks
670   NEXT Ns
680   FOR Ks=K1 TO K2
690     Gw(Ks)=Gw(Ks)/(Ks*Delw)
700   NEXT Ks
710   PRINT Gw(*)
720   PAUSE
730   END
740   !
750   DEF FNG(X)          ! g(X)
760     Gx=X*EXP(-.5*X*X) ! RAYLEIGH EXAMPLE
770   RETURN Gx
780   FNEED

```

```

10  ! ZERO-TH ORDER BESSEL TRANSFORM USING LINEAR INTERPOLATION.
20  ! INTEGRAL(X1,Xr) dX Jo(WX) g(X) FOR W1<=W<=W2 IS STORED IN
30  ! Gw(Ks), where W = Ks*Delw.      Slower low-storage.
40  Delx=.025                          ! INCREMENT (h) IN X
50  L=0                                ! X1=L*Delx, L>=0
60  R=400                              ! Xr=R*Delx, R>L
70  Delw=.5                            ! INCREMENT (Δ) IN W
80  K1=0                               ! W1=K1*Delw, K1>=0
90  K2=100                            ! W2=K2*Delw, K2>=K1
100 DOUBLE L,R,K1,K2,K0,L1,R1,Ns,Ks   ! INTEGERS
110 DIM Gx(500),Dg(500),Gw(200)
120 K0=K1
130 K1=MAX(K1,1)
140 L1=L+1
150 R1=R-1
160 REDIM Gx(L:R),Dg(L1:R1),Gw(K0:K2)
170 FOR Ks=K0 TO K2
180   Gw(Ks)=0.
190 NEXT Ks
200 FOR Ns=L TO R
210   Gx(Ns)=FNG(Ns*Delx)              ! SEE DEF FNG(X) = g(X)
220 NEXT Ns
230 G1=Gx(L)
240 Gr=Gx(R)
250 IF K0>0 THEN 310
260 F=.5*(G1+Gr)
270 FOR Ns=L1 TO R1
280   F=F+Gx(Ns)
290 NEXT Ns
300 Gw(0)=F*Delx
310 FOR Ns=L1 TO R
320   Dg(Ns)=Gx(Ns)-Gx(Ns-1)
330 NEXT Ns
340 D2=Delw*Delx
350 T1=L*Dg(L1)-G1
360 T2=R*Dg(R1)-Gr
370 IF L=0 THEN 430
380 T=L*D2
390 FOR Ks=K1 TO K2
400   CALL Bessel(T+Ks,J0,J1,A,B0,B1)
410   Gw(Ks)=T1*B1-G1*J1
420 NEXT Ks
430 T=R*D2
440 FOR Ks=K1 TO K2
450   CALL Bessel(T+Ks,J0,J1,A,B0,B1)
460   F=T2*B1-Gr*J1
470   Gw(Ks)=Gw(Ks)+F
480 NEXT Ks
490 FOR Ns=L1 TO R1
500   F=Ns*(Dg(Ns+1)-Dg(Ns))
510   T=Ns*D2
520 FOR Ks=K1 TO K2
530   CALL Bessel(T+Ks,J0,J1,A,B0,B1)
540   Gw(Ks)=Gw(Ks)+F*B1
550 NEXT Ks
560 NEXT Ns
570 FOR Ks=K1 TO K2
580   Gw(Ks)=Gw(Ks)+F*Delw
590 NEXT Ks
600 PRINT Gw(*)
610 END

```

APPENDIX C

DERIVATION OF INTEGRATION RULE FOR PARABOLIC FITS TO $g(x)$

The situation of interest here is depicted in figure 2, where parabolas are fit to the samples of $g(x)$, a pair of adjacent panels at a time. The derivation of the resultant approximation to integral (27) is broken down into the four cases illustrated in that figure.

It is again presumed, as in (20), that sample points of $g(x)$ are taken at increment h , namely

$$x_n = nh \quad \text{for } l \leq n \leq r, \quad (\text{C-1})$$

and that, in addition,

$$r - l \text{ is even.} \quad (\text{C-2})$$

That is, the total number of panels employed in interval x_l, x_r must be even. A breakdown of all the sample points $\{x_n\}$ into the four categories of figure 2 is depicted in figure C-1, where we have used the abbreviations

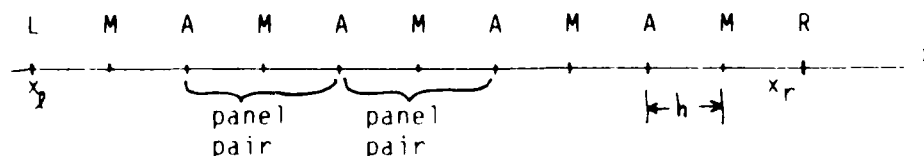


Figure C-1. Categorization of Sample Points

M = mid-point (of a panel pair),

A = abutting point (between two panel pairs),

L = left-end point,

R = right-end point. (C-3)

It is presumed in the following that $\omega > 0$; the case for $\omega = 0$ is given by (29), while $\omega < 0$ is immediately covered by observing that J_0 is even.

Mid-Point

The contribution of a mid-point x_n to integral (27) is (see figure 2)

$$\begin{aligned}
 M_n &= \int_{x_n-h}^{x_n+h} dx J_0(\omega x) g_n (1 - y^2) = \\
 &= \frac{g_n}{\omega} \int_{(n-1)\theta}^{(n+1)\theta} dt J_0(t) \left[1 - \left(\frac{t}{\theta} - n \right)^2 \right], \quad (C-4)
 \end{aligned}$$

where we utilized $t = \omega x$, (C-1), (28), and (22). Upon expansion of the square in (C-4), and use of (13)-(17), (C-4) reduces to the rather compact form

$$M_n = \frac{g_n}{\omega} \left\{ (n^2 - 1) (B_1[(n-1)\theta] - B_1[(n+1)\theta]) - \right. \\ \left. - \frac{1}{\theta^2} (B_0[(n-1)\theta] - B_0[(n+1)\theta]) \right\} . \quad (C-5)$$

This type of term is yielded for $n = \ell + 1, \ell + 3, \dots, r - 3, r - 1$, as reference to figure C-1 will verify. Here, and in the following, for the sake of brevity, we do not document the rather detailed machinations that lead to the compact form (C-5) from the integral definition (C-4). The reader will have to reconstruct those nonprofitable manipulations, if interested.

At this juncture, instead of treating an abutting point with its associated 4 panels (see upper right of figure 2), we split it up into a panel pair with a left-end point and another panel pair with a right-end point. We thus have to consider a general left point and a general right point.

LEFT POINT

This case is obtained by looking at the bottom-left diagram in figure 2 and replacing ℓ by n everywhere. The contribution of this type of panel pair is

$$\begin{aligned}
L_n &= \int_{x_n}^{x_n+2h} dx J_0(\omega x) g_n (1-y) (1-y/2) = \\
&= \frac{g_n}{\omega} \int_{n\theta}^{(n+2)\theta} dt J_0(t) \left[1 - \frac{3}{2} \left(\frac{t}{\theta} - n \right) + \frac{1}{2} \left(\frac{t}{\theta} - n \right)^2 \right] = \\
&= \frac{g_n}{2\omega} \{ (n+1)(n+2) (B_1[(n+2)\theta] - B_1[n\theta]) - \\
&\quad - \frac{1}{\theta^2} (B_0[(n+2)\theta] - B_0[n\theta]) - 2 J_1[n\theta] \} . \tag{C-6}
\end{aligned}$$

This type of term is yielded for $n = \ell, \ell + 2, \dots, r - 4, r - 2$, but not $n = r$; see figure C-1.

RIGHT POINT

This case pertains for the bottom-right diagram in figure 2 when r is replaced by n everywhere. The corresponding contribution to integral (27) is

$$\begin{aligned}
R_n &= \int_{x_n-2h}^{x_n} dx J_0(\omega x) g_n (1+y) (1+y/2) = \\
&= \frac{g_n}{\omega} \int_{(n-2)\theta}^{n\theta} dt J_0(t) \left[1 + \frac{3}{2} \left(\frac{t}{\theta} - n \right) + \frac{1}{2} \left(\frac{t}{\theta} - n \right)^2 \right] = \\
&= \frac{g_n}{2\omega} \{ (n-1)(n-2) (B_1[n\theta] - B_1[(n-2)\theta]) - \\
&\quad - \frac{1}{\theta^2} (B_0[n\theta] - B_0[(n-2)\theta]) + 2 J_1[n\theta] \} . \tag{C-7}
\end{aligned}$$

This type of term is yielded for $n = \ell + 2, \ell + 4, \dots, r - 2, r$, but not $n = \ell$; see figure C-1.

ABUTTING POINT

We can now immediately obtain the integral contribution for an abutting point (top-right diagram of figure 2) by adding (C-6) and (C-7):

$$\begin{aligned} A_n &= L_n + R_n = \\ &= \frac{g_n}{2\omega} \left\{ (n+1)(n+2) B_1[(n+2)\theta] - \frac{1}{\theta^2} B_0[(n+2)\theta] - 6n B_1[n\theta] - \right. \\ &\quad \left. - (n-1)(n-2) B_1[(n-2)\theta] + \frac{1}{\theta^2} B_0[(n-2)\theta] \right\}, \end{aligned} \quad (C-8)$$

which holds only for $n = \ell + 2, \ell + 4, \dots, r - 4, r - 2$; see figure C-1. As a notational shortcut, we say $n = (\ell + 2)(2)(r - 2)$ are the allowed values of n .

At this stage, we have succeeded in evaluating all the types of terms that have been depicted in figures 2 and C-1. The total approximation to integral (27) is therefore

$$G(\omega) \cong \sum_{n=\ell+1}^{r-1} M_n + \sum_{n=\ell+2}^{r-2} A_n + L_\ell + R_r, \quad (C-9)$$

in terms of the contributions in (C-5)-(C-8), where the slash on the summation symbol denotes skipping every other term.

However, this grouping of terms in (C-9) is according to sample values $g_n = g(nh)$ of function $g(x)$. It is advantageous to re-arrange this sum, grouping terms instead according to sample values of functions $B_0(u)$ and $B_1(u)$, defined in (16) and (17). After considerable manipulations, the following alternative to (C-9) is obtained:

$$\begin{aligned}
 2\omega G(\omega) \cong & \frac{1}{\theta^2} S_\ell B_0(\ell\theta) - Q_\ell B_1(\ell\theta) - 2g_\ell J_1(\ell\theta) - \\
 & - \frac{1}{\theta^2} S_r B_0(r\theta) + Q_r B_1(r\theta) + 2g_r J_1(r\theta) + \\
 & + \frac{1}{\theta^2} \sum_{n=\ell+2}^{r-2} D_n B_0(n\theta) - \sum_{n=\ell+2}^{r-2} R_n B_1(n\theta) .
 \end{aligned} \tag{C-10}$$

The auxiliary sequences utilized in (C-10) are defined below:

$$\begin{aligned}
 S_\ell &= g_{\ell+2} - 2g_{\ell+1} + g_\ell \\
 S_r &= g_r - 2g_{r-1} + g_{r-2} \\
 Q_\ell &= \ell(\ell+1)g_{\ell+2} - 2\ell(\ell+2)g_{\ell+1} + (\ell+2)(\ell+1)g_\ell \\
 Q_r &= (r-2)(r-1)g_r - 2r(r-2)g_{r-1} + r(r-1)g_{r-2}
 \end{aligned} \tag{C-11}$$

and

$$\left. \begin{aligned}
 D_n &= g_{n+2} - 2g_{n+1} + 2g_{n-1} - g_{n-2} \\
 F_n &= g_{n+2} - 4g_{n+1} + 6g_n - 4g_{n-1} + g_{n-2} \\
 R_n &= n^2 D_n + n F_n
 \end{aligned} \right\} \begin{array}{l} \text{for } n = \\ (\ell+2)(\ell+1)(r-2) \end{array} \tag{C-12}$$

It is important to observe from (C-10) that the Bessel integrals $B_0(u)$ and $B_1(u)$ need be evaluated only at $u = n\theta$ for $n = \ell(2)r$, and need not be evaluated at the in-between points $n = (\ell + 1)(2)(r - 1)$. Of course, the input function $g(x)$ must be evaluated at all $x = x_n = nh$ for $n = \ell(1)r$. The quantities in (C-11) and (C-12) do not depend on $\theta = \omega h$, and can be computed just once and stored, in preparation for use in (C-10).

If we are interested in evaluating integral $G(\omega)$ in (27) at values of ω equal to integer multiples k of some increment Δ , then we must substitute

$$\omega = k\Delta \quad \text{and} \quad \theta = \omega h = k\Delta h \quad (\text{C-13})$$

into (C-10). Then interest centers on computation of $B_0(u)$ and $B_1(u)$ at $u = m\Delta h$ for certain integers m . This consideration has been discussed in the sequel to (26).

BEHAVIOR FOR SMALL θ

When θ is small, differences of functions with similar values are required in (C-10). This same behavior obtains for Filon's method; see [5; (25.4.53)]. Accordingly, it is useful to have a series expansion for $G(\omega)$ about $\theta = 0$, to be used for small θ .

Since [5; (9.1.12)]

$$J_0(u) \sim 1 - \frac{1}{4} u^2 \quad \text{as } u \rightarrow 0, \quad (\text{C-14})$$

substitution in (C-4), along with the change of variable $y = t/\theta - n$, yields the mid-point contribution

$$\begin{aligned}
 M_n &= g_n h \int_{-1}^1 dy J_0(\theta(n+y)) (1 - y^2) = \\
 &\sim g_n h \int_{-1}^1 dy \left[1 - \frac{1}{4} \theta^2 (n+y)^2 \right] (1 - y^2) = \\
 &= \frac{4}{3} g_n h \left\{ 1 - \frac{1}{4} \theta^2 \left(n^2 + \frac{1}{5} \right) \right\} \text{ as } \theta \rightarrow 0 .
 \end{aligned} \tag{C-15}$$

A similar procedure for left point and right point contributions (C-6) and (C-7) gives

$$L_n = R_n \sim \frac{1}{3} g_n h \left\{ 1 - \frac{1}{4} \theta^2 \left(n^2 - \frac{2}{5} \right) \right\} \text{ as } \theta \rightarrow 0 . \tag{C-16}$$

The total asymptotic contribution to $G(\omega)$ in (27) is therefore given by (a modified version of (C-9))

$$G(\omega) \sim \sum_{n=\ell}^{r-2} L_n + \sum_{n=\ell+2}^r R_n + \sum_{n=\ell+1}^{r-1} M_n \text{ as } \theta \rightarrow 0 , \tag{C-17}$$

using (C-15) and (C-16). For $\theta = 0$, this reduces to Simpson's rule, (29). Additional correction terms involving θ^4 , θ^6 could be derived by using additional terms in expansion (C-14).

When ω is specialized to values $\omega = k\lambda$ in (C-10), the result is as given in (36). Programs for both a faster high-storage procedure and a slower low-storage procedure are listed below.

```

10  ! ZERO-TH ORDER BESSEL TRANSFORM USING PARABOLIC INTERPOLATION.
20  ! INTEGRAL(X1,Xr) dx Jo(WX) g(X) FOR W1<=W<=W2 IS STORED IN
30  ! Gw(Ks), where W = Ks*Delw.      Faster high-storage.
40  Delx=.03                          ! INCREMENT (h) IN X
50  L=0                                ! X1=L*Delx, L>=0
60  R=300                              ! Xr=R*Delx, R-L MUST BE EVEN & >=4
70  Delw=1.                            ! INCREMENT (Δ) IN W
80  K1=0                                ! W1=K1*Delw, K1>=0
90  K2=40                              ! W2=K2*Delw, K2>=K1
100 DOUBLE L,R,K1,K2,K0,L1,L2,R1,R2,Ns,Ks,I ! INTEGERS
110 DIM Gx(800),Gw(500),Sq(500),J11(500),J1r(500)
120 DIM B0(20000),B1(20000)
130 K0=K1
140 K1=MAX(K1,1)
150 L1=L+1
160 L2=L+2
170 R1=R-1
180 R2=R-2
190 REDIM Gx(L:R),Gw(K0:K2),Sq(K1:K2),J11(K1:K2),J1r(K1:K2)
200 REDIM B0(L*K1:R*K2),B1(L*K1:R*K2)
210 FOR Ks=K0 TO K2
220   Gw(Ks)=0.
230 NEXT Ks
240 FOR Ns=L TO R
250   Gx(Ns)=FNG(Ns*Delx)      ! SEE DEF FNG(X) = g(X)
260 NEXT Ns
270 G1=Gx(L)
280 Gr=Gx(R)
290 IF K0>0 THEN 380
300 S1=S2=0.
310 FOR Ns=L1 TO R1 STEP 2
320   S1=S1+Gx(Ns)
330 NEXT Ns
340 FOR Ns=L2 TO R2 STEP 2
350   S2=S2+Gx(Ns)
360 NEXT Ns
370 Gw(0)=(G1+Gr+4.*S1+2.*S2)*Delx/3.
380 G11=Gx(L1)
390 G12=Gx(L2)
400 Gr1=Gx(R1)
410 Gr2=Gx(R2)
420 S1=G12-2.*G11+G1
430 Sr=Gr-2.*Gr1+Gr2
440 O1=L*L1+G12-2.*L*L2+G11+L2*L1+G1
450 Or=R2+R1+Gr-2.*R*R2+Gr1+R*R1+Gr2
460 G12=G1*2.
470 Gr2=Gr*2.
480 D2=Delw*Delx
490 FOR Ks=K1 TO K2
500   F=Ks*D2
510   Sq(Ks)=1./(F*F)
520 NEXT Ks

```

```

530   IF L=0 THEN 580
540   FOR Ks=K1 TO K2
550     I=L*Ks
560     CALL Bessel(I*D2,J0,J1(Ks),A,B0(I),B1(I))
570   NEXT Ks
580   FOR Ks=K1 TO K2
590     I=R*Ks
600     CALL Bessel(I*D2,J0,J1r(Ks),A,B0(I),B1(I))
610   NEXT Ks
620   FOR Ns=L2 TO R2 STEP 2
630     FOR Ks=K1 TO K2
640       I=Ns*Ks
650       IF B0(I)<>0. THEN 670
660       CALL Bessel(I*D2,J0,J1,A,B0(I),B1(I))
670     NEXT Ks
680   NEXT Ns
690   IF L=0 THEN 740
700   FOR Ks=K1 TO K2
710     I=L*Ks
720     Gw(Ks)=Sq(Ks)*S1*B0(I)-Q1*B1(I)-G12*J1(Ks)
730   NEXT Ks
740   FOR Ks=K1 TO K2
750     I=R*Ks
760     F=Sq(Ks)*Sn*B0(I)-Qr*B1(I)-Gr2*J1r(Ks)
770     Gw(Ks)=Gw(Ks)-F
780   NEXT Ks
790   FOR Ns=L2 TO R2 STEP 2
800     G2=Gx(Ns+2)
810     G1=Gx(Ns+1)
820     H1=Gx(Ns-1)
830     H2=Gx(Ns-2)
840     Dn=G2-2.*G1+2.*H1-H2
850     Fn=G2-4.*G1+6.*Gx(Ns)-4.*H1+H2
860     Rn=Ns*(Ns*Dn+Fn)
870     FOR Ks=K1 TO K2
880       I=Ns*Ks
890       Gw(Ks)=Gw(Ks)+Sq(Ks)*(Dn*B0(I)-Rn*B1(I))
900     NEXT Ks
910   NEXT Ns
920   F=Delw*2.
930   FOR Ks=K1 TO K2
940     Gw(Ks)=Gw(Ks)/(Ks*F)
950   NEXT Ks
960   PRINT Gw(*)
970   PAUSE
980   END
990   !
1000  DEF FNG(X)
1010    G=X*EXP(-.5*X+X)
1020  RETURN Gx
1030  FNGEND

```

! g10
! RAYLEIGH EXAMPLE

```

10  ! ZERO-TH ORDER BESSEL TRANSFORM USING PARABOLIC INTERPOLATION.
20  ! INTEGRAL(X1,Xr) dX Jo(WX) g(X) FOR W1<=W<=W2 IS STORED IN
30  ! Gw(Ks), where W = Ks*Delw.      Slower low-storage.
40  Delx=.03      ! INCREMENT (h) IN X
50  L=0           ! X1=L*Delx, L>=0
60  R=300        ! Xr=R*Delx, R-L MUST BE EVEN & >=4
70  Delw=1.      ! INCREMENT (Δ) IN W
80  K1=0         ! W1=K1*Delw, K1>=0
90  K2=120       ! W2=K2*Delw, K2>=K1
100 DOUBLE L,R,K1,K2,K0,L1,L2,R1,R2,Ns,Ks  ! INTEGERS
110 DIM Gx(800),Gw(500),Sq(500)
120 K0=K1
130 K1=MAX(K1,1)
140 L1=L+1
150 L2=L+2
160 R1=R-1
170 R2=R-2
180 REDIM Gx(L:R),Gw(K0:K2),Sq(K1:K2)
190 FOR Ks=K0 TO K2
200   Gw(Ks)=0.
210 NEXT Ks
220 FOR Ns=L TO R
230   Gx(Ns)=FNG(Ns*Delx)      ! SEE DEF FNG(X) = g(X)
240 NEXT Ns
250 G1=Gx(L)
260 Gr=Gx(R)
270 IF K0>0 THEN 360
280 S1=S2=0.
290 FOR Ns=L1 TO R1 STEP 2
300   S1=S1+Gx(Ns)
310 NEXT Ns
320 FOR Ns=L2 TO R2 STEP 2
330   S2=S2+Gx(Ns)
340 NEXT Ns
350 Gw(0)=(G1+Gr+4.*S1+2.*S2)*Delx*.3.
360 G11=Gx(L1)
370 G12=Gx(L2)
380 Gr1=Gx(R1)
390 Gr2=Gx(R2)
400 S1=G12-2.*G11+G1
410 Sr=Gr-2.*Gr1+Gr2
420 O1=L*L1+G12-2.*L*L2+G11+L2*L1+G1
430 Or=R2*R1+Gr-2.*R*R2+Gr1+R*R1+Gr2
440 G12=G1*2.
450 Gr2=Gr*2.
460 D2=Delw*Delx
470 FOR Ks=K1 TO K2
480   F=Ks*D2
490   Sq(Ks)=1. +F*F
500 NEXT Ks

```

```

510 IF L=0 THEN 570
520 T=L*D2
530 FOR Ks=K1 TO K2
540 CALL Bessel(T*Ks,J0,J1,A,B0,B1)
550 Gw(Ks)=Sq(Ks)*S1*B0-O1*B1-G12*J1
560 NEXT Ks
570 T=R*D2
580 FOR Ks=K1 TO K2
590 CALL Bessel(T*Ks,J0,J1,A,B0,B1)
600 F=Sq(Ks)*Sn*B0-Or*B1-Gr2*J1
610 Gw(Ks)=Gw(Ks)-F
620 NEXT Ks
630 FOR Ns=L2 TO R2 STEP 2
640 G2=Gx(Ns+2)
650 G1=Gx(Ns+1)
660 H1=Gx(Ns-1)
670 H2=Gx(Ns-2)
680 Dn=G2-2.*G1+2.*H1-H2
690 Fn=G2-4.*G1+6.*Gx(Ns)-4.*H1+H2
700 Rn=Ns*(Ns*Dn+Fn)
710 T=Ns*D2
720 FOR Ks=K1 TO K2
730 CALL Bessel(T*Ks,J0,J1,A,B0,B1)
740 Gw(Ks)=Gw(Ks)+Sq(Ks)*Dn*B0-Rn*B1
750 NEXT Ks
760 NEXT Ns
770 F=De1w*2.
780 FOR Ks=K1 TO K2
790 Gw(Ks)=Gw(Ks)/Ks*F)
800 NEXT Ks
810 PRINT Gw(*)
820 PAUSE
830 END
840 I
850 DEF FNG(X)
860 G=8*EXP(-.5+X*X)
870 RETURN G)
880 FNGND

```

I g(X)
I PAYLEIGH EXAMPLE

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