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The Elastic Analysis of a Thin Toroidal Shell

by

John Dennis Bowen
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John Dennis Bowen
B.S., Mechanical Engineering
United States Naval Academy
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May, 1987

Certified by Dale G Karr
Professor Dale G. Karr
Thesis Supervisor

Accepted by A Douglas Carmichael
Professor A. Douglas Carmichael, Chairman
Ocean Engineering Department Graduate Committee

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by

JOHN DENNIS BOWEN

Submitted to the Department of Ocean Engineering in partial fulfillment of the requirements for the degrees of Naval Engineer and Master of Science in Naval Architecture and Marine Engineering.

ABSTRACT

The specific impetus for this work was a conceptual design of a submarine using the toroid as the pressure hull. The designers could not find a ready body of knowledge to obtain scantlings for thier pressure hull.

This work began with a review of efforts to solve complete toroidal structures. Several works were found which addressed general shells and extended into partial toroids, but the solution of a complete toroid was not found to be a common exercise. Some of the these works are briefly reviewed. An attempt was then made to solve for the displacements in a thin walled circular toroid using the energy method. Several problems were identified associated with the structure geometry which make the solution for the complete toroid difficult. In addition, the functional used for the energy method needs to be more complex than the simple trigonometric or power series functionals used in this work. These two areas, geometry and functionals, are fertile areas for further study

THESIS SUPERVISOR: Dale G. Karr, Ph. D.

TITLE: Assistant Professor
of Ocean Engineering

ACKNOWLEDGEMENTS

Several individuals helped with this work. First and foremost was my wife Agnes who put up with me, and my children who will be happy to see this over with so that daddy's attitude will improve. Professor Karr, the thesis supervisor, was a source of new ways of looking at what I thought I understood. He found out how thickheaded this Irishman really is, something my wife already knew. Janet Jones-Oliveria of the Draper Laboratories provided me with insights into the relationships between structures and mathematics that probably would have taken several more semesters of classwork. Professor Klaus-Jurgen Bathe, who I never talked to about this thesis, showed me in his class that linear algebra is not black magic and how these solution methods really work.

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CHAPTER 1

INTRODUCTION

The purpose of this work was to determine the response of a thin toroidal shell structure. The initial goal seemed achievable until the literature was reviewed. A simple and understandable solution was not available. The disparity in works was apparent in such basic elements as coordinate systems. The solutions were generally directed towards displacements or forces. The solution paths varied from complex algebra to finite difference method. The solutions were as varried as the number of individuals attempting them. It was impossible to expect any simple correlation among the few people who attempted to address this topic.

The result of this work is the identification of two areas where more time and effort is required.

A. The impact of geometric curvature on the solution.

B. The impact of the shell geometry on the assumed functional used in the energy method of the solution.

To be very specific, these two areas have precluded this effort from achieving the initial purpose of this year long effort.

A word needs to be addressed towards assumptions that are made. The assumptions are fairly standard for a shell being analysed in the linear elastic range. The assumptions are:

1. The shell is made of isotropic and homogenious material which obeys Hook's Law.
2. The thickness of the shell is constant.
3. The thickness of the shell is small compared to the radii of curvature.
4. A straight line normal to the middle surface before deformation remains straight and normal to the middle surface after deformation and retains its original length.

In order to establish a consistent presentation, geometry will be discussed here and, unless otherwise noted, all representations of geometry will follow this coordinate system and the definitions. Solutions by other individuals will be translated into this reference system.

DEFINITIONS

a : Ratio of R to r: $a = \frac{R}{r}$

Since $R > r$ to form a complete toroid, a will always be greater than 1.

C : Circumference of the parallel circle: $C = 2 \pi r^*$

r : Radius of curvature in the θ direction of the unloaded shell envelope, or radius of the circle which is rotated about the Z axis (at distance R) to form the toroid.

r^* : Radius to any point on the unloaded toroid measured perpendicularly to the Z axis.

$$r^* = R + r \sin\theta = r (a + \sin\theta).$$

R Radius of rotation about the Z axis of the circle of radius r used to form the toroid.

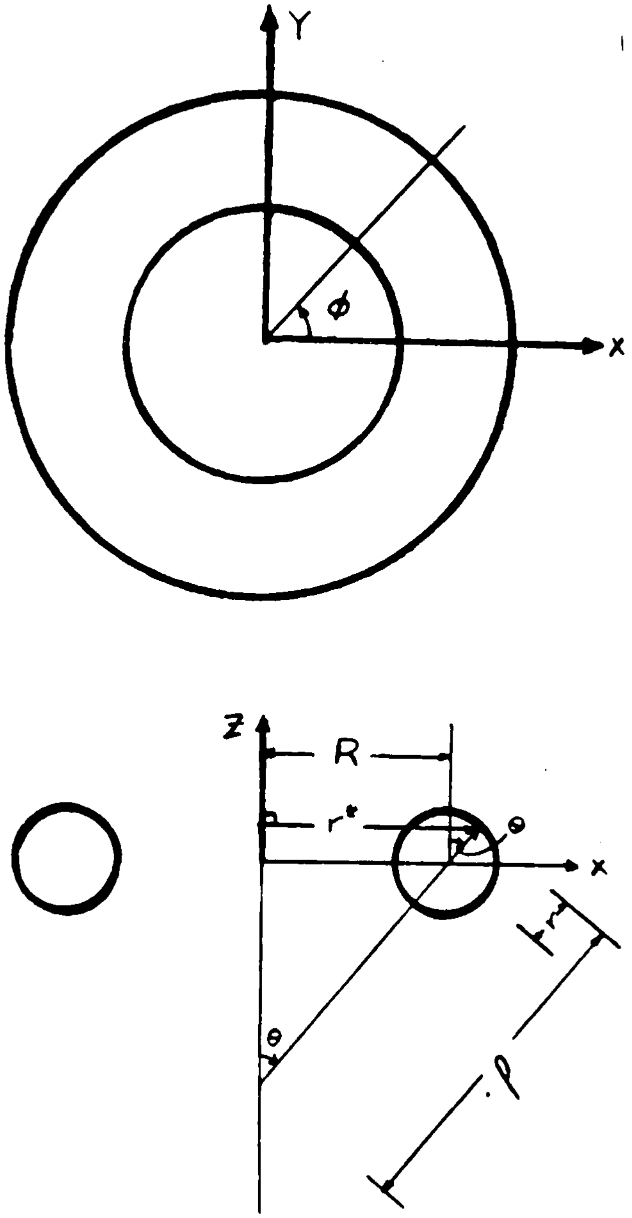
t : Thickness of the plate

φ : Angle of rotation about the Z axis measured counter clockwise from the positive X axis in the X-Y plane.

θ : Angle of rotation measured clockwise from the local perpendicular to the X-Y plane in the positive Z directon at distance R from the origin in any direction φ. See Figure 1. It sounds like a lot, but it is not.

Figure 1

Layout of Variables



ρ : Radius of curvature in the ϕ direction.

$$\rho = \frac{R}{\sin\theta} + r$$

Coordinate System: The standard right-handed XYZ coordinate system is the global reference system. Other coordinates (eg. θ and ϕ) build on this basic reference system.

Geometric Curvature: Curvature of the shell due solely to the geometry or curvature of the unloaded toroid.

Load Curvature: Curvature of the shell caused by displacements in response to the applied load. This could also be looked upon as the change in curvature from the reference (unloaded) curved surface.

Axis of Symmetry: Axis about which the small circle is rotated to form the toroid - the Z axis.

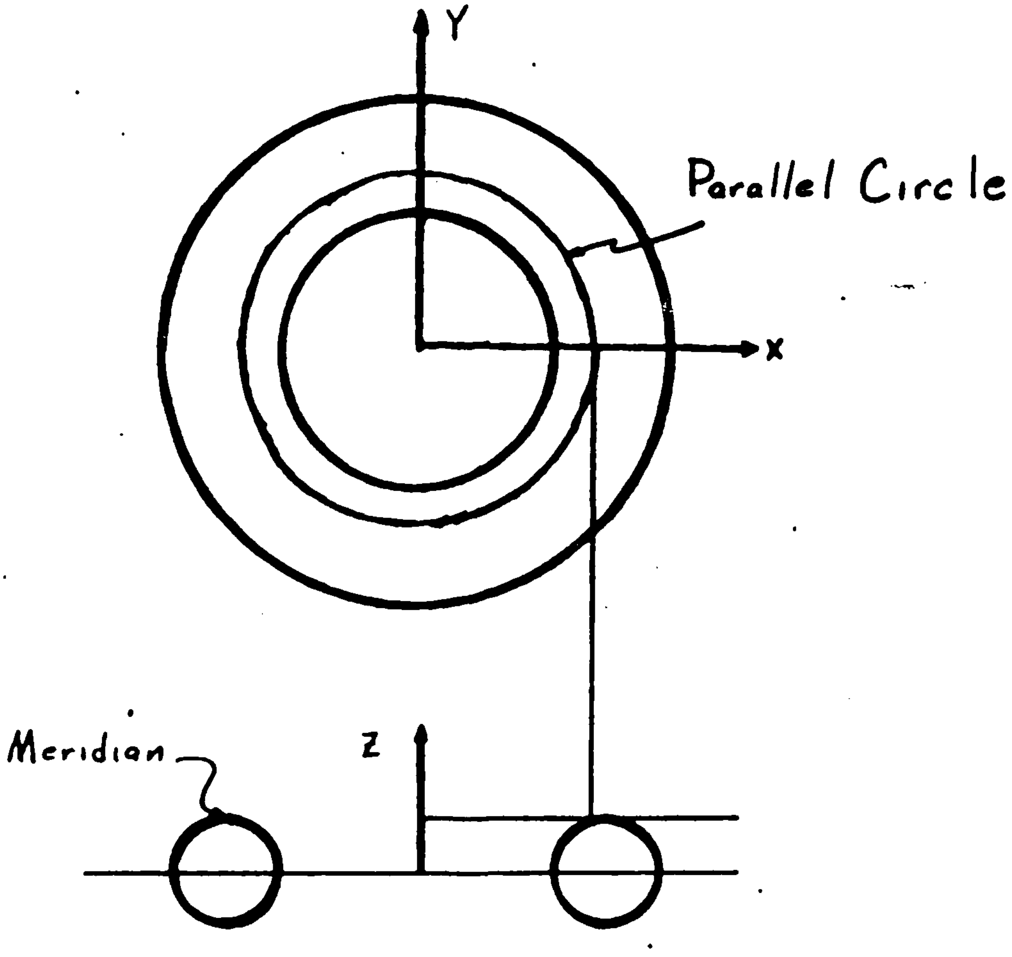
Plane of Symmetry: Plane which bisects the toroid - the X-Y plane

Meridian: Intersection of any plane containing the Z axis with the toroid. On either side of the Z axis, the intersection results in a circle (for the unloaded toroid) with θ as the variable. (See Figure 2)

Parallel Circle: Intersection of any plane parallel to the X-Y plane with the toroid. (See Figure 2)

Figure 2

Parallel Circle and Meridian



CHAPTER 2

HISTORICAL VIEW AND THE PROBLEM

Historical View:

Upon starting to research the analysis of circular toroidal shells, the field was found to be not very large. Books on shell theory and analysis would neglect or only briefly mention the complete toroid. This could imply that the toroid was:

(1) a simple extension of general shell theory or

(2) much more complex than general shell theory.

The latter seems to be the case.

Senjanovic [1] gives a brief history of shell theory in his book. His Russian pride shows in some instances ("...shortcomings were eleminated by representatives of the Russian School..."), but the relative youthfulness of the field is obvious. Initial work in what is now called shell theory was by H. Aron (1874) and A. Love (1888) a mere century ago. Senjanovic does cite Flugge - another work referenced here - as one of the developers of the classical (or western) theory of shells. Yet even Flugge devotes only four pages [2] to the toroid. The following works are illustrative of the background on toroidal shells.

W. Flugge: Flugge starts with a generalized shell and establishes force equlibrium. From this equlibrium he solves differential equations to obtain solutions for forces and displacements. His discussion of the toroid points out some of the problems inherent in the full toroidal solution.

L. Sobel: Sobel [3] was an understudy of Flugge. His doctorial dissertation starts with the same generalized shell theory but is slanted towards the buckling analysis of the toroid. His solution was worked on a computer and is difficult to follow.

I. Senjanovic: Senjanovic starts with a force ballance on a differential element in a localy orthogonal coordinate system, then uses several methods to solve them. Most notably, he employed complex variables and some computer solutions circa 1970.

S. Timoshenko: Timoshenko [4] starts with a plate and develops into a generalized shell. His attention to the toroid is of the same order as Flugge. This writer found Timoshenko's general development of theory easier to follow than others and hence the references to Timoshenko in this area may be more frequent than to other authors listed here.

E. Tsui: Tsui [5] starts with the constitutive relationships and develops them into forces. However for the toridal shape his solutions are on globally orientated forces and displacements. Tsui intended his work more as a handbook than a textbook. His solutions are tables of values from computer analysis of toroidal shells. He does provide however, a consistant development, though shorter and in less detail than some others, for various shells of revolution.

The Problem:

The specific problem to be addressed in this thesis is the response of a toroidal shell to hydrostatic presure. More specifics on the loading will be addressed in chapter three. This is to set the stage for the problem itself and lay out the difficult areas.

The goal of any structural solution is to take into account the loading, geometry of the structure, and the material in the structure and determine the response of the entire structure. The response is then compared to a desired standard or a set of failure criteria and adequacy of the design is determined. This approach is generally slanted towards displacements of the structure since only through displacements are the applied loads distributed and equilibrated.

In any structure, the various elements are in communication with one another. The beam under a simple tensile axial load is the easiest example, but a plate or shell must also satisfy this connectivity. Unfortunately, the complexity of this connectivity or communication among elements increases rapidly. In the axially loaded (tensile) beam, only the axial displacement (say the X direction) is important. Load the same beam laterally and now the X and Z directions are required to describe the beams response. Going to a plate now requires the X, Y, and Z directions. The shell now takes the plate out of the convenient XY plane for a reference.

With appropriate coordinate system selection, some very practical shells can be handled easily. The cylinder and the sphere are classical examples. The transition to a toroid would - by simple implication - be very easy. Start with a cylinder of radius r and length of $2\pi R$ and bend it around upon itself. Discounting a few wrinkles on the inner sections of the cylinder, we now have a toroid. The wrinkles however carry a much deeper implication.

The structure's ability to distribute the applied load changes drastically. The complete toroid has no boundary conditions comparable to the cylinder's boundary conditions at $X=0$ and $X=L$. These are replaced by connectivity (or continuity) requirements. The analysis of toroidal segments is facilited by the ability to insert boundary conditions. But even this facility can be of limited use if too much of the structure is included in the analysis. The simple thin walled analysis on a cylinder for a force balance becomes an algebraic exercise of no small feat for the toroid. (This will be shown in chapter five.)

It should be noted that all of the individuals above stated or implied some difficulty in obtaining a solution for a complete toroid. The most blatant was Tsui [6] when he states that in problems involving the complete toroid, the crowns ($\theta = 0^{o}$and 180^{o} - see FIG 1) can not be solved due to singularities. Flugge [7] gives a solution for the forces but states that they cannot be used "... in the vicinity of the top and bottom circles" (Tsui's crowns) "without additional bending...". He then provides a displacement expression containing terms which blow up at $\theta = 0^{o}$ and 180^{o}.

With all of this going against an easy solution, two courses of action were undertaken:

A. The singularities at the crowns could be avoided by using some expression which avoids the offensive term ($\frac{1}{\sin\theta}$) (see Chapter 6). This was avoided by referencing the differential element to:

$$r \;\; = K + r \sin \theta = r (a + \sin \theta)$$

The biblical addage still holds: If thine eye offends thee, pluck it out.

B. The method of solution selected would be one to avoid further problems as pointed out by Flugge [8]. The energy method would avoid messy mathematical stumbling blocks in the solution, since the form of the solution is assumed going into the energy method. This was almost true as shall be pointed out latter. Additionally, this writer has not seen this attempted by others. Therefore, when the line forms to say that this method will not work, it should be a very short line indeed.

CHAPTER 3

CONSTITUTIVE RELATIONSHIPS

An expression, or series of expressions, relating the material motion to properties such as stress and strain (and further to forces and moments) is required. This follows the definition of constitutive equation of the material as given by Fung [9], "...mathematical expression of the mechanical property of a material."

First we must define our displacements (see Figure 3). Three displacements will be used:

u :Displacement in the positive θ direction

v :Displacement in the positive ϕ direction

w :Displacement in the direction of the positive (ie. outward pointing) normal at the point of interest.

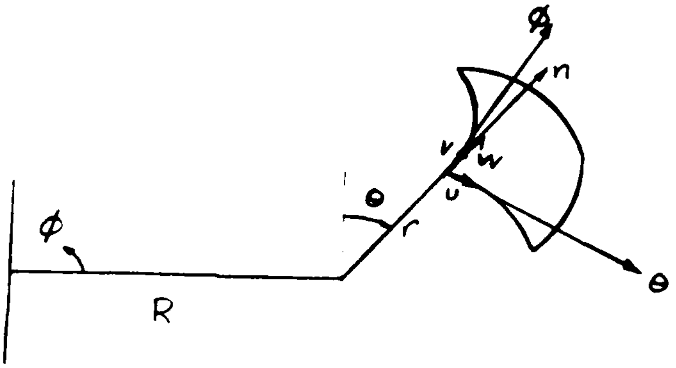
Rotations will be defined as follows (see Figure 3):

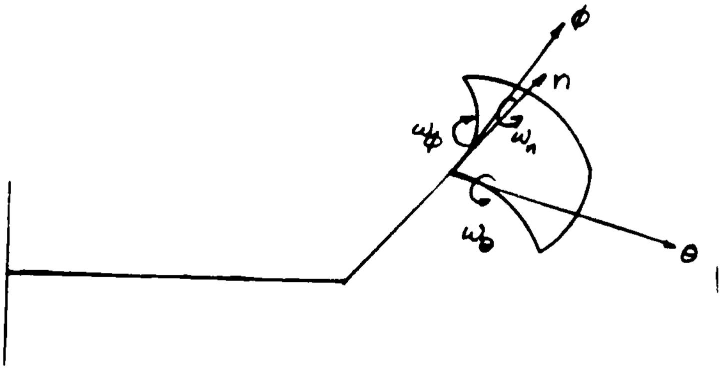
ω_θ : Rotation of a unit normal about the tangent to the meridian in the positive **u** direction: that is, rotation about the positive θ axis.

Figure 3

Dispalcements and Rotations

Displacements





Rotations

ω_φ : Rotation of a unit normal about the tangent to the parallel circle in the positive v direction; that is, about the local positive ϕ axis.

ω_n : Rotation about the positive w direction.

The loading will impact greatly on the final outcome. For this problem, the loading will be assumed to be a constant hydrostatic pressure. This has several implications:

A. Hydrostatic loading always presents a loading normal to the surface.

B. Because of the global nature of the hydrostatic loading, any benifit to be garnered from symmetric conditions would be applicable to this problem.

The second aspect of the loading, axisymmetric loading, results in the number of unknowns being reduced dramatically. Both Timoshenko [10] and Flugge [11] state that the total deformation can be represented by only two displacements; one normal to the surface and one along the meridian in the positive θ direction. Flugge also states [12] that that stresses are independant of ϕ and in fact any derivatives with respect to ϕ are zero. Timoshenko [13] agrees, but the derivative going to zero is implied, not stated as bluntly as in Flugge.

To take this one step further, a lot of things will go to zero because of axisymmetric loading. Sheer stress and strain are cross coordinate terms, ie.

$$[*]_{\theta\phi} = \frac{\delta}{\delta\theta}\left(\frac{\delta}{\delta\phi}[*]\right)$$

But <u>any</u> $\frac{\delta}{\delta\phi} = 0$, so $[*]_{\theta\phi} = [*]_{\phi\theta} = 0$.

Because of the type of loading described, **v** and ω_n are not required for the description of motion. **v** is not required because of the independance of the motion on ϕ, and ω_n is not required because it depends on a partial of ϕ. All of this is the result of axisymmetric loading.

STRAINS

Midplane strains are as follows:

ε_θ : Is the strain in the positive θ direction. ε_θ is made up of two parts (see Figure 4):

1. ε_θ due to **u**

At two points on the meridian: P and Q

$$u_P = u \qquad u_Q = u + \frac{\delta u}{\delta\theta}d\theta$$

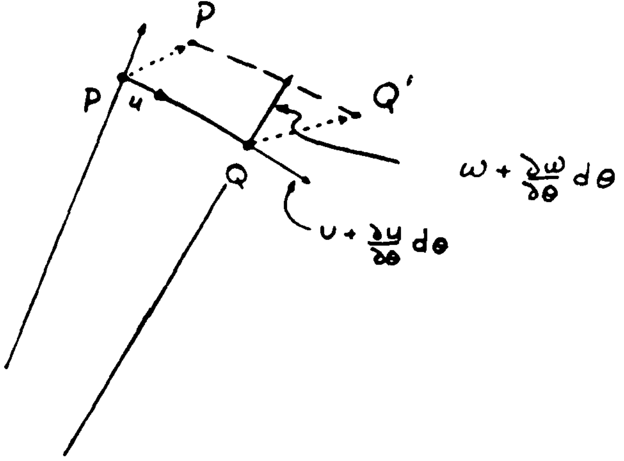
$$\varepsilon = \frac{\delta l}{l} = \frac{1}{r\,d\theta}\left(u + \frac{\delta u}{\delta\theta}d\theta - u\right) = \frac{1}{r}\frac{\delta u}{\delta\theta}$$

2. ε_θ due to **w**

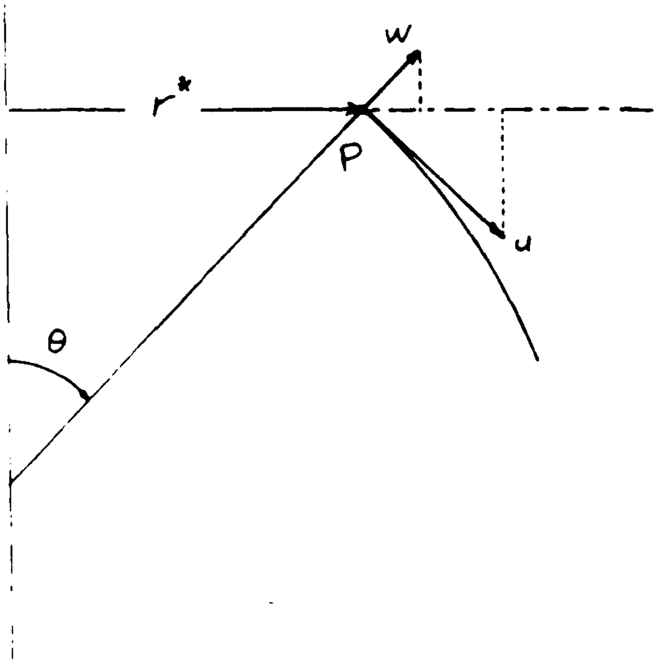
Figure 4

Midplane Strains

ε_θ



ε_ϕ



Length of a segment is equal to the radius times the angular rotation. The original length is $r\ d\theta$. At point P the radius increases by w to $(r + w)$ so the new length would be $(r + w)\ d\theta$. Similarly at point Q the radius increases but is now $(r + w + \frac{\delta w}{\delta\theta} d\theta)$. The new length would be $(r + w + \frac{\delta w}{\delta\theta} d\theta) d\theta$. Averaging the two lengths at P and Q gives an average $\Delta l = w d\theta + \frac{1}{2} \frac{\delta w}{\delta\theta} d\theta$. Dividing by the original $l = r\ d\theta$ gives:

$\varepsilon = \frac{1}{r}\ [w + \frac{1}{2} \frac{\delta w}{\delta\theta} d\theta]$ Neglecting second order terms yields: $\varepsilon = \frac{w}{r}$

Total ε_θ is therefore:

$$\varepsilon_\theta = \frac{1}{r}\ [\frac{\delta u}{\delta\theta} + w]$$

ε_ϕ : Is the strain in the positive ϕ direction (see Figure 4). Since the body is symmetrically loaded, all ϕ's have the same displacements. Therefore ε_ϕ can be calculated by calculating the change in the circumference of the parallel circle at ϕ = constant. Since the circumference is related to the radius, the radius (r^*) will be used to determine strain ε_ϕ.

$$C = 2 \pi r^*$$

At point P, displacements **w** and **u** contribute to the Δr^* as follows:

$$\Delta r^* = u \cos\theta + w \sin\theta$$

$$\Delta C = 2 \pi \Delta r^*$$

$$\varepsilon_\phi = \frac{\Delta C}{C} = \frac{1}{r^*} [u \cos\theta + w \sin\theta]$$

ROTATIONS

ω_ϕ is determined from two points, P and Q, on a meridian seperated by displacement **u** in the positive θ direction (see Figure 5). The unit normal vector at P must rotate through dθ to go from P to Q due to displacement **u**, ie.

$r\,d\theta = u$

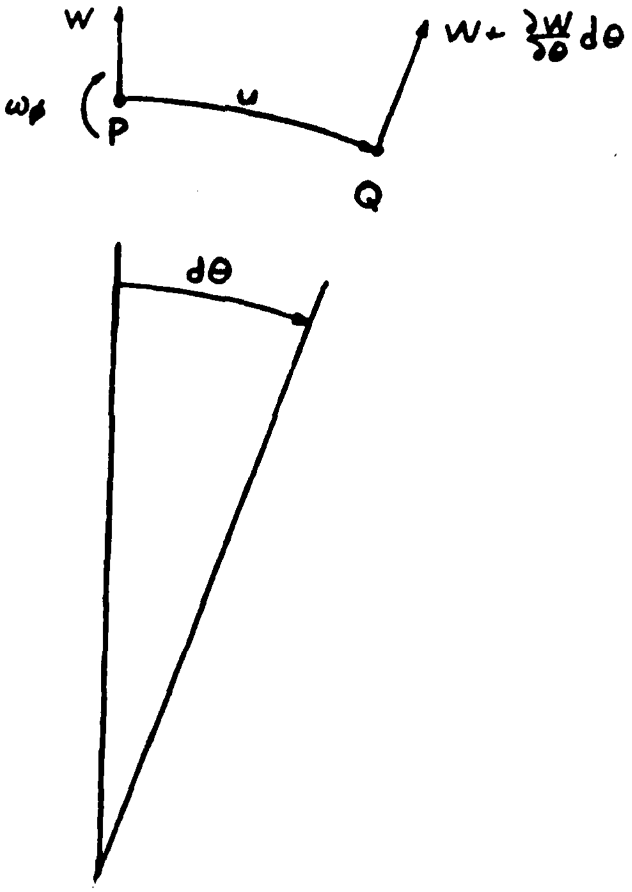
$$\therefore\ d\theta = \frac{u}{r} = \omega_1$$

If **w** was constant from P to Q, no additional rotations would be required since the surface displaces uniformly through dθ. If $\frac{\delta w}{\delta\theta} \neq 0$, then a rotation of the local normal at O will be required to maintain its normal relationship. This rotation will be:

$$\omega_2 = -\frac{\delta w}{\delta\theta} d\theta/(r\, d\theta) = -\frac{1}{r}\frac{\delta w}{\delta\theta}$$

Figure 5

Rotation ω_ϕ



This makes the total ω_ϕ:

$$\omega_\phi = \frac{1}{r} (u - \frac{\delta w}{\delta \theta})$$

Because of the symmetric loading and no subsequent variations in the displacements with ϕ, no v displacements or $\frac{\delta w}{\delta \theta}$ exists. However a ω_θ can exist (see Figure 6).

Due to symmetry in loading, $|w_\phi|$ is constant in ϕ. However, for ω_ϕ to change direction, an additional rotation perpendicular to ω_ϕ must exist. This is the rotation ω_θ.

$$\omega_\theta = - \omega_\phi \cos\theta \, d\phi$$

Figure 6 gives a graphical presentation of ω_θ. Timoshenko [14] gives a similar result but achieves this as a side result of obtaining curvature.

LOAD CURVATURES

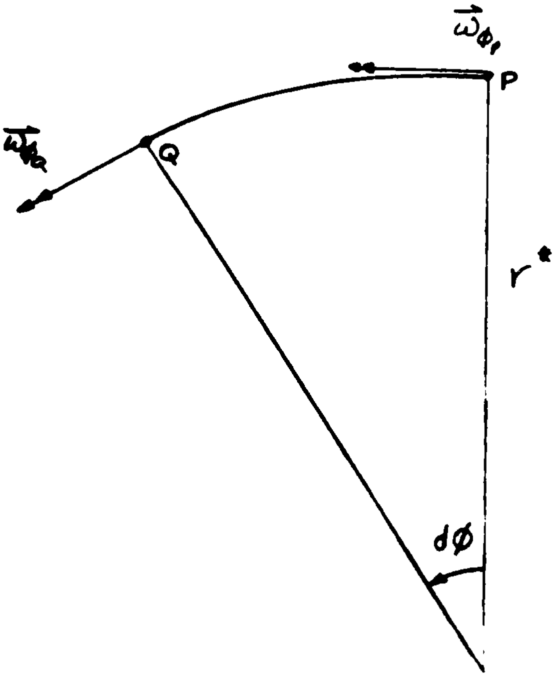
Load curvatures are as follows:

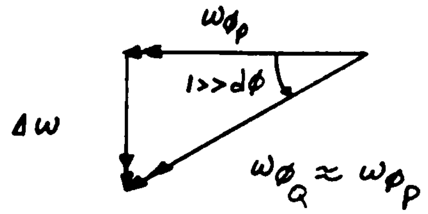
$\varkappa_\theta$: Change in curvature along the positive θ axis due to the applied load.

$\varkappa_\phi$: Change in curvature along the positive ϕ axis due to the applied load.

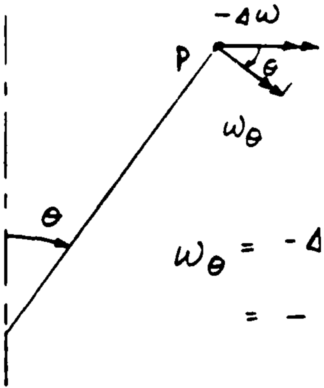
Figure 6

Rotation ω_θ





$$\Delta\omega = \omega_{\phi_P} \sin(d\phi) = \omega_{\phi_P}\, d\phi$$



$$\begin{aligned} \omega_\theta &= -\Delta\omega \cos\theta \\ &= -(\omega_\phi\, d\phi) \cos\theta \\ &= -\omega_\phi \cos\theta\, d\phi \end{aligned}$$

It is significant, to this writer at least, to seperate these curvatures from the geometric curvature defined in Chapter 1. The geometric curvature gives one an indication of how difficult the problem will be [15] and defines the initial or unloaded reference surface for further work. The load curvature is the sole result of the structure's response to the load applied and directly related to the displacements use to equilibrate the load.

Load curvature along the θ direction is the change in rotation about the θ axis $(\frac{\delta\omega_\phi}{\delta\theta})$ divided by the original length $(rd\theta)$.

$$\kappa_\theta = \frac{1}{rd\theta}\,\frac{\delta}{\delta\theta}\,(\omega_\phi)\ d\theta$$

Recalling: $\omega_\theta = \frac{1}{r}\,(u - \frac{\delta w}{\delta\theta})$

$$\kappa_\theta = \frac{1}{r^2}\,(\frac{\delta u}{\delta\theta} - \frac{\delta^2 w}{\delta\theta^2})$$

Timoshenko [16] gives a description of this and κ_ϕ is similarly:

$$\kappa_\phi = \omega_\theta\ d\theta/(r^*\ d\theta)$$

$$= \frac{\cos\theta}{r^2(a + \sin\theta)}\,(u - \frac{\delta w}{\delta\theta})$$

We can now describe all the pertinent engineering parameters in terms of displacements. Adding to this list the general expression for planer stress, we get the following:

$$\sigma_\theta = \frac{E}{1-\nu^2} (\varepsilon_\theta + \nu\varepsilon_\phi)$$

$$= \frac{E}{1-\nu^2} (\frac{1}{r} [\frac{\delta u}{\delta\theta} + w] + \frac{\nu}{r(a + \sin\theta)} [u \cos\theta + w \sin\theta])$$

$$\sigma_\phi = \frac{E}{1-\nu^2} (\frac{1}{r(a + \sin\theta)} [u \cos\theta + w \sin\theta] + \frac{\nu}{r} [\frac{\delta u}{\delta\theta} + w])$$

This almost completes our constitutive relationships. Stress, strain, rotation, and local curvature can be expressed in terms of the displacements u and w. The last step is to proceed to normal forces and moments. This is as follows:

$$N_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i \, dz \qquad ; \; i = \theta, \phi$$

$$M_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i \, z \, dz$$

CHAPTER 4

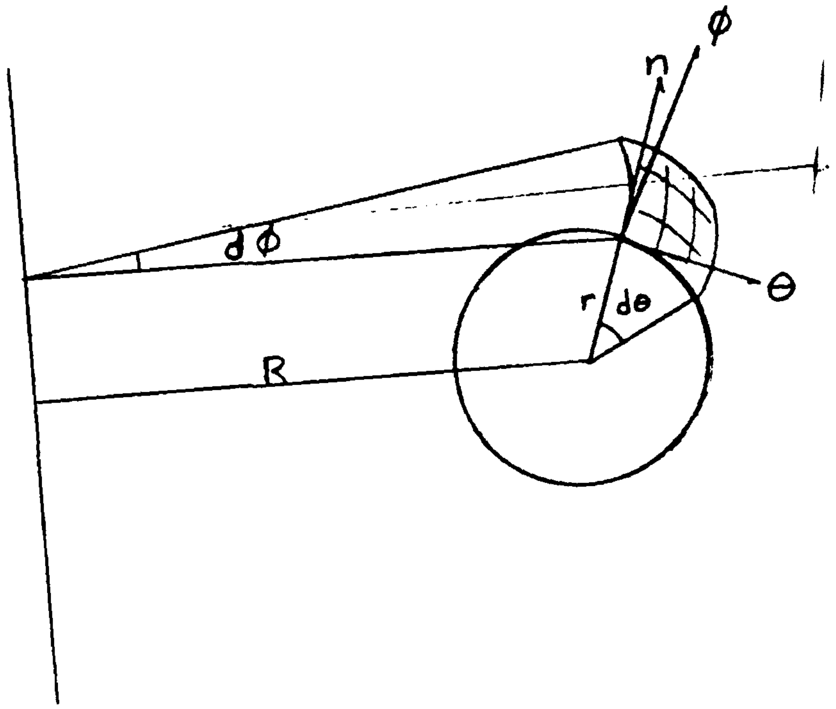
EQULIBRIUM OF FORCES

The next aspect of the solution to be presented will be that of establishing equlibrium conditions for the differential element. The differential element is shown in Figure 7. Geometry rears it's ugly head in that the element needs two reference points to describe it's surface. The first is the axis of rotation of the shell in the θ direction. The second is the origin of the global coordinate system. The first axis can be referenced to the global coordinate system by R and ϕ.

Solutions for partial toroidal shapes all take advantage of the known boundary conditions on the edges. When a complete toroid is analyzed, the boundary conditions double back upon themselves and become internal vice external forces. In these instances, symmetric loading and deformations are invoked. Tsui [17] gives tabulated computer results for both solutions and the differences are

Figure 7

Differential Element



obvious.

In the following discussion, and the associated figures, the following conventions will be used (see Figure 8):

F'_{ι} is a generalized force ie. a force or moment in the ι direction ($\iota = \theta, \phi, n$). The units for F_{ι} are

[Force/unit length].

F_{ι} is the force, ie.

$$F_{\iota} = [\text{Force/unit length}]*[\text{length}] = [\text{Force}]$$

N_{θ}, N_{ϕ} Normal forces in the $\binom{\theta}{\phi}$ direction acting on the side of the element on which $\binom{\phi}{\theta}$ is constant.

Q_{θ} *Sheer force acting on the* ϕ = constant face of the element. Note: Due to symmetry of loading, $Q_{\phi} = 0$.

M_{θ}, M_{ϕ} Moment to induce rotation in the $\binom{\theta}{\phi}$ direction. The moment is defined as positive with respect to the right hand rule as applied about the local positive $\binom{\phi}{\theta}$ direction.

Each force will be addressed graphically with three isometric diagrams and a geometric breakdown to the appropriate F_{ι}. The F_{ι} for each force (N_{θ}, N_{ϕ}, . . .ect.) can then be summed for a force balance.

Force	Figure
N_θ	8
N_ϕ	9
Q_θ	10
P (load)	11
M_θ	12
M_ϕ	13

Figure 8

Forces Defined

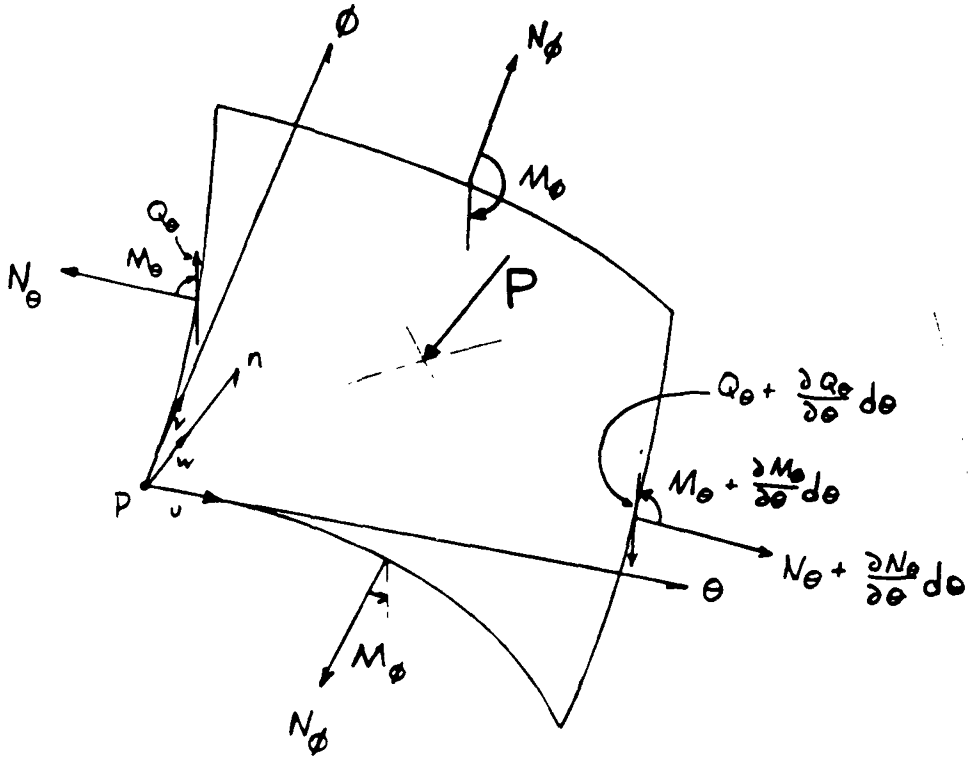
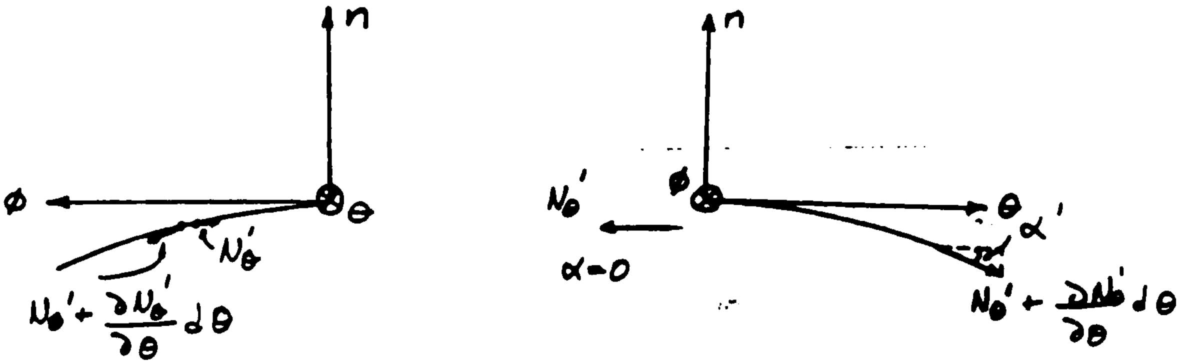


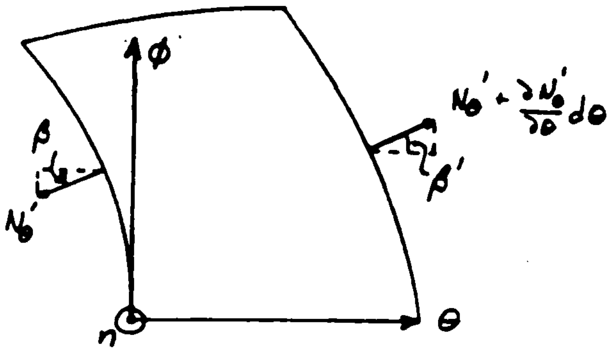
Figure 9

N_θ



$$\alpha = 0; \quad \alpha' = d\theta + \omega_\phi + \frac{\delta\omega_\phi}{\delta\theta} d\theta$$

$$\beta = \beta' = \frac{1}{2}\, d\phi$$



Contribution to F_i doe to N_θ:

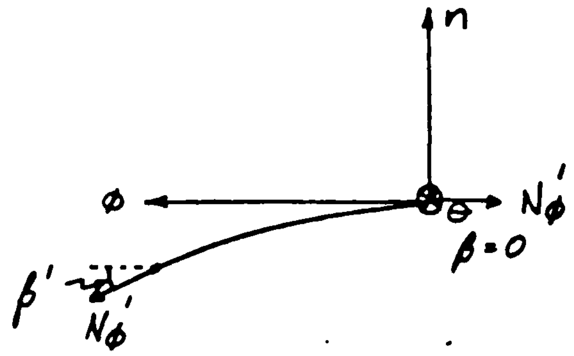
F_ϕ: $-N_\theta' \sin\beta + (N_\theta' + \frac{\delta N_\theta'}{\delta\theta} d\theta) \cos\alpha' \, \sin\beta'$

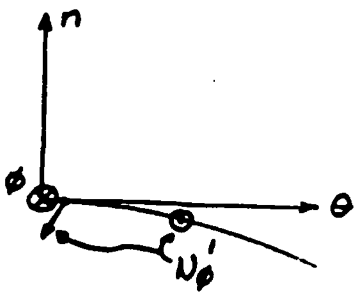
F_θ: $-N_\theta' \cos\beta + (N_\theta' + \frac{\delta N_\theta'}{\delta\theta} d\theta) \cos\alpha' \, \sin\beta'$

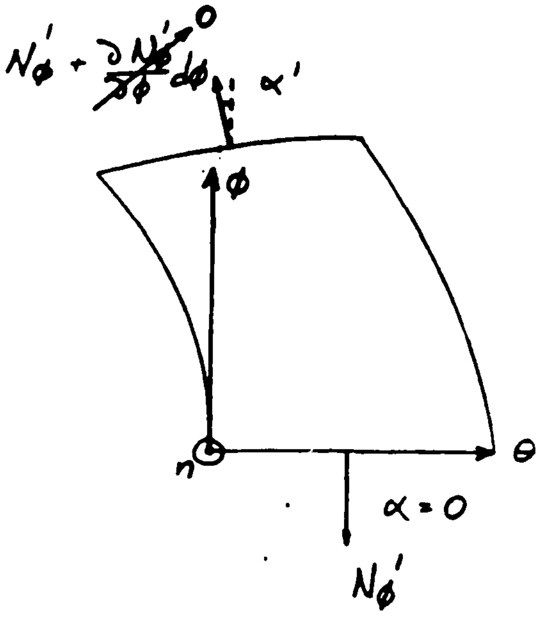
F_n: $N_\theta' \sin\alpha - (N_\theta' + \frac{\delta N_\theta'}{\delta\theta} d\theta) \sin\alpha'$

Figure 10

N_ϕ







$\alpha = 0; \quad \alpha' = d\theta$

$\beta = 0; \quad \beta' = d\phi - \omega_\theta - \frac{\delta\omega_\theta}{\delta\phi} d\phi$

Remember, $\frac{\delta}{\delta\phi} = 0$

Contribution to F_i doe to N_ϕ:

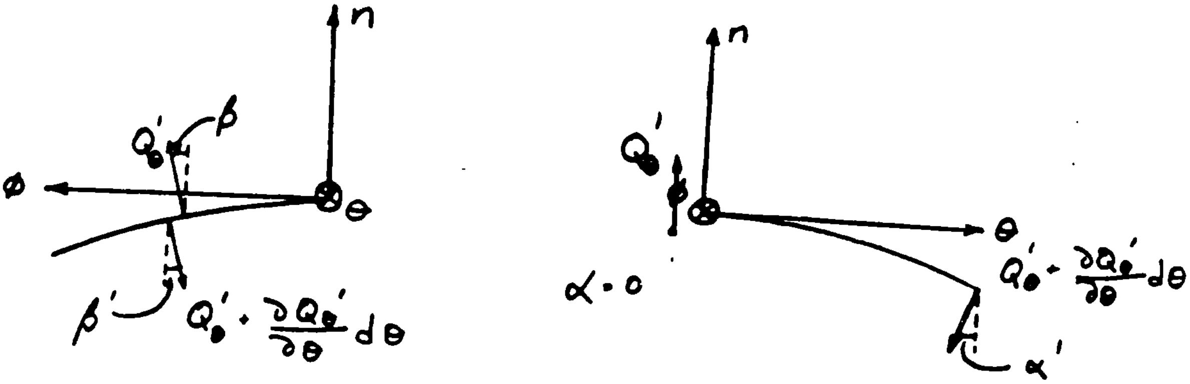
F_ϕ: $-N'_\phi + N'_\phi \cos\alpha' \cos\beta'$

F_θ: $N_\phi \sin\alpha \cos\beta - N'_\phi \cos\beta' \sin\alpha'$

F_n: $N'_\phi \sin\beta - N'_\phi \sin\beta'$

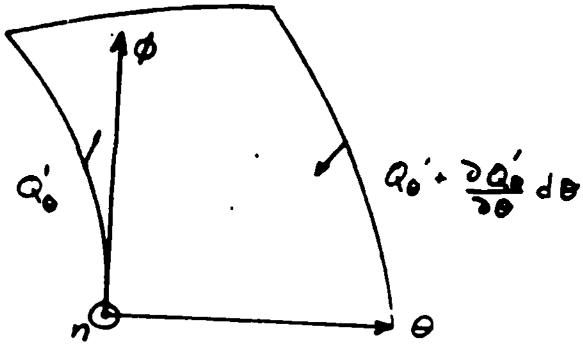
Figure 11

Q_θ



$$\alpha = \frac{1}{2} \left(d\theta + \omega_\phi + \frac{\delta \omega_\phi}{\delta \theta} d\theta \right) \cong \alpha'$$

$$\beta = 0, \quad \beta' = d\phi \, \frac{1}{2} \, \omega_\theta - \frac{\delta \omega_\theta}{\delta \phi} d\phi$$



Contribution to F_i doe to Q_θ:

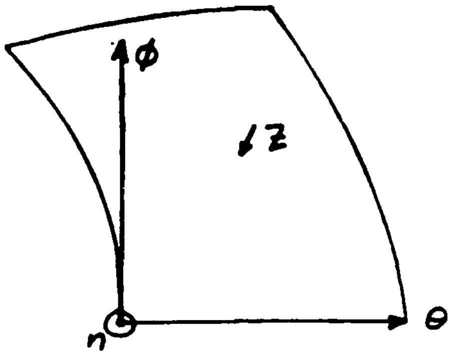
F_ϕ: Q_θ' cosα sinβ + Q_θ' cosα' sinβ'

F_θ: Q_θ' sinα cosβ − Q_θ' cosα' sinβ'

F_n: Q_θ' cosα cosβ + Q_θ' cosα' cosβ'

Figure 12

P



$$\alpha = \frac{1}{2} \left(d\theta + \omega_{\phi} + \frac{\delta\omega_{\phi}}{\delta\theta} d\theta \right)$$

$$\beta = \frac{1}{2} \left(d\phi - \omega_{\theta} - \frac{\delta\omega_{\theta}}{\delta\phi} d\phi \right)$$

NOTE: $Z = P \text{ Area} = P\,[(R + r\sin\theta)d\phi]\,[r\,d\theta]$

Contribution to F_i doe to Z:

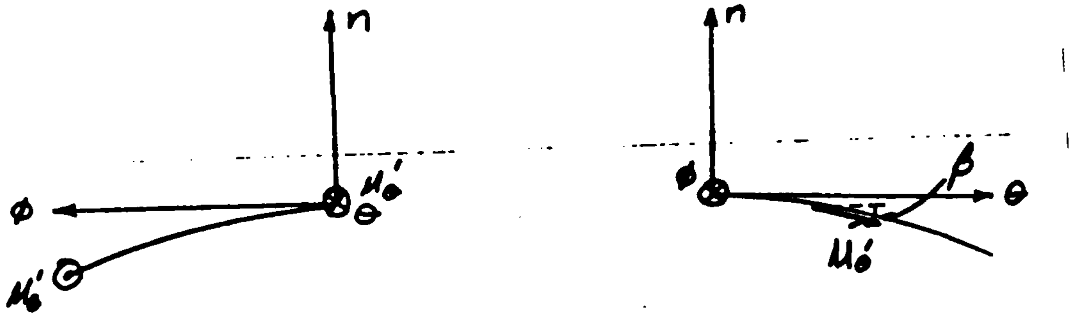
F_{ϕ}: $- Z \cos\alpha \sin\beta$

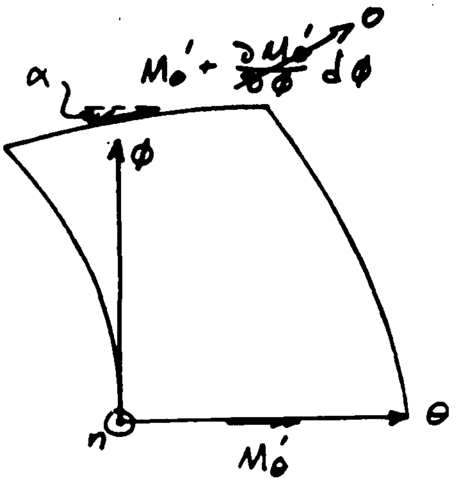
F_{θ}: $- Z \cos\beta \sin\alpha$

F_{n}: $- Z \cos\beta \cos\alpha$

Figure 13

M_θ





$\alpha = 0; \quad \alpha' = d\phi$

$\beta \cong \beta' = \frac{1}{2} (d\theta + \omega_\phi + \frac{\delta\omega_\phi}{\delta\theta} d\theta)$

Contribution to F_ι doe to M_θ:

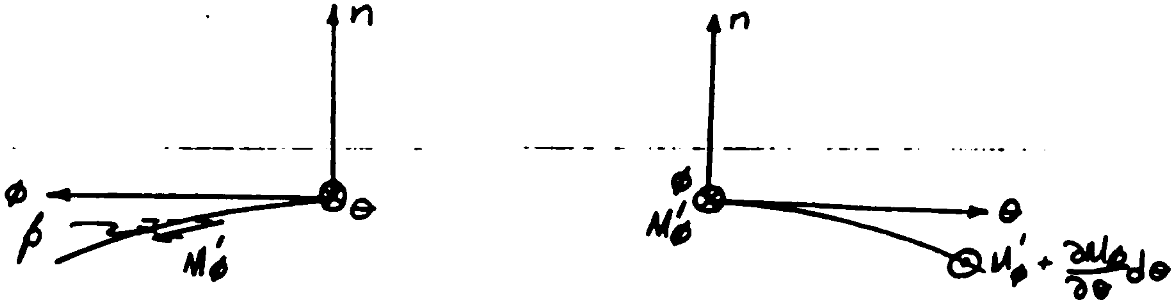
F_ϕ: $M_\theta' \cos\beta - (M_\theta' + \frac{\delta M_\phi'}{\delta\theta} d\theta) \cos\beta' \cos\alpha'$

F_θ: $-(M_\theta' + \frac{\delta M_\theta'}{\delta\theta} d\theta) \cos\beta \sin\alpha$

F_n: $- M_\theta' \sin\beta + (M_\theta' + \frac{\delta M_\theta'}{\delta\theta} d\theta$

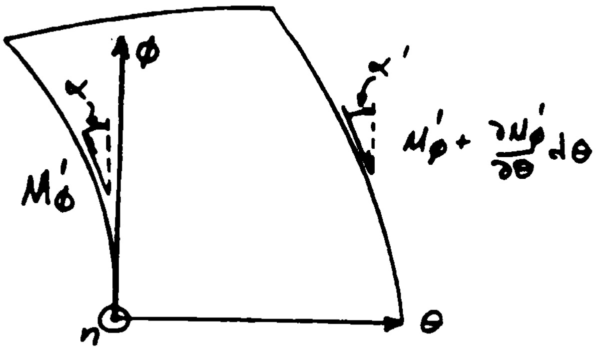
Figure 14

M_{ϕ}



$$\alpha = \frac{1}{2}\, d\phi = \alpha'$$

$$\beta = \frac{1}{2}\left(d\phi - \omega_{\theta} - \frac{\delta \omega_{\theta}}{\delta \phi}\, d\phi\right) \cong \beta'$$



Contribution to F_i doe to M_{ϕ}:

F_{ϕ}: $M_{\phi}' \cos\beta \cos\alpha - \left(M_{\phi}' + \frac{\delta M_{\phi}}{\delta \theta}\, d\theta\right) \cos\beta' \cos\alpha'$

F_{θ}: $- M_{\phi} \cos\beta \sin\alpha + \left(M_{\phi}' + \frac{\delta M_{\phi}}{\delta \theta}\, d\theta\right) \cos\beta \sin\alpha$

F_n: $- M_{\phi}' \cos\alpha \sin\beta + \left(M_{\phi}' + \frac{\delta M_{\phi}}{\delta \theta}\, d\theta\right) \cos\alpha' \sin\beta'$

Relationships between the N'_i and N_i must now be established.

$$N_\theta = N'_\theta \, r^* \, d\theta$$

$$N_\theta + \frac{\delta N_\theta}{\delta\theta} = (N'_\theta + \frac{\delta N'_\theta}{\delta\theta})(r^* + \frac{\delta r^*}{\delta\theta}) \, d\phi$$

$$= N'_\theta r^* d\phi + N'_\theta (\frac{\delta r^*}{\delta\theta}) d\phi + (\frac{\delta N'_\theta}{\delta\theta}) r^* d\phi + (\frac{\delta N'_\theta}{\delta\theta})(\frac{\delta r^*}{\delta\theta}) d\phi$$

$$= (N'_\theta + \frac{\delta N'_\theta}{\delta\theta}) \, r^* \, d\phi,$$ neglecting terms with $\frac{\delta r^*}{\delta\theta}$ as being relatively small

Similarly:

$$N_\phi = N'_\phi \, r \, d\theta$$

$$Q_\theta = Q'_\theta \, r^* \, d\phi$$

$$Q_\theta + (\frac{\delta Q_\theta}{\delta\theta}) = (Q'_\theta + \frac{\delta Q'_\theta}{\delta\theta}) \, r^* \, d\phi$$

$$M_\theta = M'_\theta \, r \, d\phi$$

$$M_\theta + \frac{\delta M_\theta}{\delta\theta} = (M'_\theta + \frac{\delta M'_\theta}{\delta\theta}) \, r^* \, d\phi$$

$$M_\phi = M'_\phi \, r \, d\theta$$

CHAPTER 5

ENERGY METHOD

Any attempted solution must maintain some sort of equlibrium. In the energy method, the equlibrium comes from minimizing the energy from the external load and the internal energy of the shell. For the external energy, a surface integral of the hydrostatic load and the normal displacements gives this value. The internal energy is determined by integrating stress times strain throughout the shell volume. The external energy is subtracted from the internal energy and the resulting functional is generally refered to as Π, the total potential energy of the structure. With V representing the internal energy and W representing the external energy, the functional looks like:

$$\Pi = V - W$$

If the value of Π is at a minimum value, the structure will be in equlibrium. This is the principle of total potential energy [18]. This work will not address buckling or post buckling behavior, only the linear elastic region.

For this situation, equlibrium is achieved when:

$$\frac{\delta \Pi}{\delta q_i} = \frac{\delta V}{\delta q_i} - \frac{\delta W}{\delta q_i} = 0$$

where q_i is any generalized parameter used to determine V and W.

Unfortunately, the parameters used to determine V and W are u and w - which is what we are attempting to solve. To get around this difficulity, a solution will be assumed. Then an energy balance will be conducted.

The assumption of a solution is not without risk. First. the solution must meet known natural boundary conditions. Then the accuracy of the solution is limited by the closeness of the assumed solution to the actual (and unknown) solution.

In the case of the toroid, there are no "boundary conditions" as was pointed out in chapter four. In place of the boundary conditions, we have conditions of compatability to limit the solution. As stated earlier ([19] and [20]), symmetry of deformation can be assumed for the hydrostatic loading. The plane of symmetry is the XY plane.

To determine the internal energy we proceed as follows:

$$V = \int_V \sigma_{ij} \; \varepsilon_{ij} \; dV \quad ; \quad i \;\&\; j = \theta,\ \phi$$

$$= \frac{1}{2} \int_S (N_{ij} \; \varepsilon_{ij} + M_{ij} \; \kappa_{ij}) \; dS$$

Recall that, due to the type of loading,

cross terms $(i \neq j)$ equal zero.

$$V = \frac{1}{2} \int_S (N_\theta \varepsilon_\theta + N_\phi \varepsilon_\phi + M_\theta \kappa_\phi + M_\phi \kappa_\phi) \; dS$$

Eq 1: $N_\theta \varepsilon_\theta = K (\varepsilon_\theta + \nu \varepsilon_\phi) \varepsilon_\theta \quad ; \quad K = \frac{E h}{1-\nu^2}$

$$= K (\varepsilon_\theta^2 + 2 \nu \varepsilon_\theta \varepsilon_\phi + \varepsilon_\phi^2)$$

Similarly:

Eq 2: $N_\phi \varepsilon_\phi = K (\varepsilon_\phi^2 + 2 \nu \varepsilon_\phi \varepsilon_\theta + \varepsilon_\theta^2)$

Eq 3: $M_\theta \kappa_\theta = D (\kappa_\theta + \kappa_\phi) \kappa_\theta \quad ; \quad D = \frac{E h^3}{12(1-\nu^2)} = K (\frac{h^2}{12})$

$$= K (\frac{h^3}{12}) (\kappa_\theta^2 + 2 \nu \kappa_\theta \kappa_\phi + \kappa_\phi^2)$$

Eq 4: $M_\phi \kappa_\phi = K (\frac{h^2}{12}) (\kappa_\phi^2 + 2 \nu \kappa_\phi \kappa_\theta + \kappa_\theta^2)$

Recall from chapter four:

Eq 5,6: $\varepsilon_\theta = \frac{1}{r} (\frac{\delta u}{\delta \theta} + w); \; \varepsilon_\phi = \frac{1}{r(a + \sin\theta)} (u \cos\theta + w \sin\theta)$

Eq 7,8: $\kappa_\theta = \frac{1}{r^2} (\frac{\delta u}{\delta \theta} - \frac{\delta w^2}{\delta \theta^2}); \quad \kappa_\phi = \frac{\cos\theta}{r^2(a + \sin\theta)} (u - \frac{\delta w}{\delta \theta})$

The unknowns u and w still persist. The assumption will be made that u and w are functions of θ. A series solution will be assumed as follows:

$w(\theta) = \Sigma\, w_m \sin(m\theta)$, limits of summation will be addressed later

$$u(\theta) = \Sigma\, u_m \frac{\delta}{\delta\theta} (\sin(m\theta))$$

$$= \Sigma\, u_m\, m \cos(m\theta)$$

This turns out to be not a very good assumption for the the series representation of the displacements. There are some inherent limitations; for instance $\sin(m\theta)$ is always zero at $\theta = 0$. While not in conflict with the conditions of compatability, this is an additional constraint on the solution displacements. If the actual displacements are not zero at $\theta = 0$, then there is a guaranteed error in the assumed solution. The question to be answered, as with all assumed solutions, is how good is the answer obtained compared to the answer that is required. More on this subject in Chapter 6.

For ease of notation, let:

$$\phi_m = \sin(m\theta)$$

$$\phi_m' = m \cos(m\theta)$$

$$\phi_m'' = -m^2 \sin(m\theta)$$

Therefore:

Eq 9: $w(\theta) = \Sigma\ w_m\ \phi_m$

Eq 10: $u(\theta) = \Sigma\ u_m\ \phi_m'$

Boundary conditions for this problem are actually compatability conditions. At $\theta = 90^o$ and 270^o the $u(\theta)$ displacement should be zero and $\frac{\delta}{\delta\theta}(w(\theta))$ should be zero.

$$\frac{\delta w}{\delta\theta} = \frac{\delta}{\delta\theta}(\Sigma\ w_m\ \phi_m) = \Sigma\ w_m\ \frac{\delta}{\delta\theta}(\phi_m) = \Sigma\ w_m\ \phi_m'$$

Since w_m are merely coefficients now in the series solution they are unaffected by the derivative. To look at this in more detail:

$$\frac{\delta w}{\delta\theta} = w_1\ \cos(\theta) + w_2\ 2\ \cos(2\theta) + w_3\ 3\ \cos(3\theta) + \ldots$$

$$\ldots + w_n\ n\ \cos(n\theta)$$

For conveince, let n = 10 for the trial solution. At $\theta = 90^o$ and 270^o, $\cos(m\theta) = 0$ for m = 1, 3, 5, . . and $\cos(m\theta) = \pm 1$ for m = 2, 4, 6, . . . For the series to be equal to zero set $w_m = 0$ for m = 2, 4, 6, . . . Similar reasoning gives $u_m = 0$ for m = 2, 4, 6, . . . (This explains why the functional for u_m was selected as the derivative of the functional for w_m.) As shall be seen later, the method is smart enough to realize this limitation.)

The elemental area (dS) must be investigated for the toroid. $dS = dl_\theta \, dl_\phi$; $dl_\theta = r \, d\theta$; $dl_\phi = (R + r \sin\theta) \, d\phi$

$$= r \, d\theta \, (R + r \sin\theta) \, d\phi$$

As a check: $S = \int_0^{2\pi}\int_0^{2\pi} r \, (R + r \sin\theta) \, d\theta \, d\phi$

$= 4 \, \pi^2 \, R \, r$. This agrees with the CRC Standard Mathematical Tables [21] for the surface area of a toroid. therefore dS is correct.

The ultimate objective is to get $\frac{\delta\Pi}{\delta q_\iota} = 0$. In this case $q_\iota = u_m$ and w_m. So for each specific u_m - that is u_k - one gets an expressison such that:

$$\frac{\delta\Pi}{\delta u_k} = \frac{\delta V}{\delta u_k} - \frac{\delta W}{\delta u_k} = 0$$

Likewise a series of m expressions:

$$\frac{\delta\Pi}{\delta w_k} = \frac{\delta V}{\delta w_k} - \frac{\delta W}{\delta w_k} = 0$$

For a nominal m = 10, this yields 20 equations. Fortunately, u_m and w_m are 20 unknowns.

Now the internal energy can be formated. Equations 9 and 10 are substituted into equations 5 through 8. These equations are then inserted into equations 1 through 4. This is combined with the expression for dS to give V, the internal energy. To facilitate the future derivatives which will be required, the subscripts m and n will be used to indicate the seperate terms in a product, ie.:

$$\varepsilon_\theta^2 = \varepsilon_\theta \, \varepsilon_\theta = \frac{1}{r} (\Sigma \, u_m \, \phi_m'' + \Sigma \, w_m \, \phi_m) \, \frac{1}{r} (\Sigma \, u_n \, \phi_n'' + \Sigma \, w_n \, \phi_n)$$

All of the above yields:

$$V = \pi \, r^2 \, K \int_0^{2\pi} \Big[\frac{1}{r} (\Sigma u_m \phi_m'' + \Sigma w_m \phi_m) \frac{1}{r} (\Sigma u_n \phi_n'' + \Sigma w_n \phi_n)$$

$$+ \frac{2\nu}{r} (\Sigma u_m \phi_m'' \; \Sigma w_m \phi_m) \frac{1}{r(a + \sin\theta)} (\cos\theta \, \Sigma u_n \phi_n' + \sin\theta \, \Sigma w_n \phi_n)$$

$$+ \frac{1}{r(a + \sin\theta)} (\cos\theta \, \Sigma u_m \phi_m' + \sin\theta \, \Sigma \, w_m \phi_m)$$

$$\times \frac{1}{r(a + \sin\theta)} (\cos\theta \, \Sigma u_n \phi_n' + \sin\theta \, \Sigma \, w_n \phi_n)$$

$$+ \left(\frac{h^2}{12}\right) \frac{1}{r^2} (\Sigma u_m \phi_m'' + \Sigma w_m \phi_m'') \frac{1}{r^2} (\Sigma u_n \phi_n'' + \Sigma w_n \phi_n'')$$

$$+ \left(\frac{h^2}{12}\right) \frac{2\nu}{r^2} (\Sigma u_m \phi_m'' + \Sigma w_m \phi_m'') \frac{\cos\theta}{r^2 (a + \sin\theta)} (\Sigma u_n \phi_n' + \Sigma w_n \phi_n')$$

$$+ \left(\frac{h^2}{12}\right) \frac{\cos\theta}{r^2 (a + \sin\theta)} (\Sigma u_m \phi_m' + \Sigma w_m \phi_m')$$

$$\times \; \Big(\frac{\cos\theta}{r^2 (a + \sin\theta)} (\Sigma u_n \phi_n' + \Sigma w_n \phi_n') \Big] \, (a + \sin\theta) \, d\theta$$

Next, carrying out the multiplications yields:

$$
\begin{aligned}
V = \pi\, r^2\, K \int_0^{2\pi} \Big[& \frac{1}{r^2} \Sigma\Sigma(\mathbf{u}_m \mathbf{u}_n\, \phi''_m \phi''_m + \mathbf{u}_m \mathbf{w}_n\, \phi''_m \phi_n + \mathbf{w}_m \mathbf{u}_n\, \phi_m \phi''_n \\
& \quad + \mathbf{w}_m \mathbf{w}_n\, \phi_m \phi_n) \\
& + \frac{2\nu}{r^2(a + \sin\theta)} \Sigma\Sigma(\mathbf{u}_m \mathbf{u}_n\, \phi''_m \phi'_m \cos\theta + \mathbf{u}_m \mathbf{w}_n\, \phi''_m \phi_n \sin\theta \\
& \quad + \mathbf{w}_m \mathbf{u}_n\, \phi_m \phi'_n \cos\theta + \mathbf{w}_m \mathbf{w}_n\, \phi_m \phi_n \sin\theta) \\
& + \frac{1}{r^2(a + \sin\theta)^2} \Sigma\Sigma(\mathbf{u}_m \mathbf{u}_n\, \phi'_m \phi'_m \cos^2\theta \\
& \quad + \mathbf{u}_m \mathbf{w}_n\, \phi'_m \phi_n \cos\theta \sin\theta + \mathbf{w}_m \mathbf{u}_n\, \phi_m \phi'_n \cos\theta \sin\theta \\
& \quad + \mathbf{w}_m \mathbf{w}_n\, \phi_m \phi_n \sin^2\theta) \\
& + \left(\frac{h^2}{12r^4}\right) \Sigma\Sigma(\mathbf{u}_m \mathbf{u}_n\, \phi''_m \phi''_m - \mathbf{u}_m \mathbf{w}_n\, \phi''_m \phi''_n - \mathbf{w}_m \mathbf{u}_n\, \phi''_m \phi''_n \\
& \quad + \mathbf{w}_m \mathbf{w}_n\, \phi''_m \phi''_n) \\
& + \left(\frac{h^2}{12r^4}\right) \frac{2\,\nu \cos\theta}{(a + \sin\theta)} \Sigma\Sigma(\mathbf{u}_m \mathbf{u}_n\, \phi''_m \phi'_m - \mathbf{u}_m \mathbf{w}_n\, \phi''_m \phi''_n \\
& \quad - \mathbf{w}_m \mathbf{u}_n\, \phi''_m \phi'_n + \mathbf{w}_m \mathbf{w}_n\, \phi''_m \phi'_n) \\
& + \left(\frac{h^2}{12r^4}\right) \frac{\cos^2\theta}{(a + \sin\theta)^2} (\mathbf{u}_m \mathbf{u}_n \phi'_m \phi'_m - \mathbf{u}_m \mathbf{w}_n \phi''_m \phi'_n \\
& \quad - \mathbf{w}_m \mathbf{u}_n\, \phi'_m \phi'_n + \mathbf{w}_m \mathbf{w}_n\, \phi'_m \phi'_n) \Big] (a + \sin\theta)\, d\theta
\end{aligned}
$$

This is all well and good but actually $\frac{\delta V}{\delta q_\iota}$ is what we really need. Again, for this problem $q_\iota = \mathbf{u}_k$ and $\mathbf{w}_k$; and $k = 1, 2, \ldots 10$. Applying term by term differentiation is straightforward but tricky and tedious. A couple of examples:

$$
\frac{\delta}{\delta \mathbf{u}_k} (\mathbf{w}_m \mathbf{u}_n \phi_m \phi''_n) = \mathbf{w}_m \phi_m \phi''_n
$$

But observe:

$$\begin{aligned}\frac{\delta}{\delta \mathbf{u}_k}(\mathbf{u}_m\,\mathbf{u}_n\,\phi''_m\,\phi''_n) &= \mathbf{u}_m\,\phi''_m\,\phi''_k + \mathbf{u}_n\,\phi''_k\,\phi''_n \\ &= 2\,\mathbf{u}_m\,\phi''_m\,\phi''_k\end{aligned}$$

Since m and n can both serve as a counter index, the two terms may be combined. The key to this is that both functionals (ϕ_m) have to be of the same order of the derivative. Observe:

$$\begin{aligned}\frac{\delta}{\delta \mathbf{u}_k}(\mathbf{u}_m\,\mathbf{u}_n\,\phi''_m\,\phi'_n) &= \mathbf{u}_m\,\phi''_m\,\phi'_k + \mathbf{u}_n\,\phi''_k\,\phi'_n \\ &= \mathbf{u}_m\,(\phi''_m\,\phi'_k + \phi''_k\,\phi'_m)\end{aligned}$$

Doing all of the above and collection like terms gives the following two expressions:

The partial of V with respect to $\mathbf{u}_k$.

$$\begin{aligned}\frac{\delta V}{\delta \mathbf{u}_k} = \pi\,K\,\Big[\Sigma \mathbf{u}_m \int_0^{2\pi}&(2\,\phi''_m\phi''_k + \frac{2\,\nu\,\cos\theta}{(a+\sin\theta)}(\phi''_k\phi'_m + \phi'_m\phi''_k) \\ &+ \frac{2\,\cos^2\theta}{(a+\sin\theta)^2}\,\phi'_m\,\phi'_k \\ &+ (\frac{h^2}{12})(2\,\phi''_m\phi''_k\;\frac{2\,\nu\,\cos\theta}{(a+\sin\theta)}(\phi'_k\phi_m\,\sin\theta + \phi_m\phi'_k\,\cos\theta) \\ &+ \frac{2\,\cos^2\theta}{(a+\sin\theta)^2}\,\phi'_m\phi'_k))\,(a+\sin\theta)\,d\theta \\ +\Sigma \mathbf{w}_m \int_0^{2\pi}&(2\,\phi''_k\phi_m + \frac{2\,\nu}{(a+\sin\theta)}(\phi''_k\phi_m\,\sin\theta \\ &+ \phi_m\phi'_k\,\cos\theta) + \frac{2\,\cos\theta\,\sin\theta}{(a+\sin\theta)^2}\,\phi'_k\phi_m \\ &+ (\frac{h^2}{12})(-2\,\phi''_k\phi''_m - \frac{2\,\nu\,\cos\theta}{(a+\sin\theta)}(\phi''_k\phi'_m + \phi''_m\phi'_k) \\ &- \frac{2\,\cos^2\theta}{(a+\sin\theta)^2}\,\phi'_k\phi'_m))(a+\sin\theta)\,d\theta\Big]\end{aligned}$$

The partial of V with respect to w_k.

$$\frac{\delta V}{\delta w_k} = \pi\ K\ [\Sigma u_m \int_0^{2\pi} (2\ \phi''_m \phi_k + \frac{2\ \nu}{(a + \sin\theta)} (\phi''_m \phi_k\ \sin\theta + \phi'_m \phi_k\ \cos\theta)$$

$$+ \frac{2\ \cos\theta\ \sin\theta}{(a + \sin\theta)^2}\ \phi'_m \phi_k$$

$$- (\frac{h^2}{12})(2\ \phi''_m \phi''_k\ \frac{2\ \nu\ \cos\theta}{(a + \sin\theta)} (\phi'_k \phi''_m + \phi'_m \phi''_k)$$

$$+ \frac{2\ \cos^2\theta}{(a + \sin\theta)^2}\ \phi'_m \phi'_k))\ (a + \sin\theta)\ d\theta$$

$$+ \Sigma w_m \int_0^{2\pi} (2\ \phi_k \phi_m + \frac{4\ \nu\ \sin\theta}{(a + \sin\theta)}\ \phi_k \phi_m$$

$$+ \frac{2\ \sin^2\theta}{(a + \sin\theta)^2}\ \phi_k \phi_m$$

$$+ (\frac{h^2}{12})(2 \phi''_k \phi''_m + \frac{2\ \nu\ \cos\theta}{(a + \sin\theta)} (\phi''_k \phi'_m + \phi''_m \phi'_k)$$

$$+ \frac{2\ \cos^2\theta}{(a + \sin\theta)^2}\ \phi'_k \phi'_m))(a + \sin\theta)\ d\theta]$$

The external energy (W) should be looked at next. The energy is the product of the load - hydrostatic pressure (P) in this case which is always normal to the surface - times the displacement along the line of action of the force - $w(\theta)$ - in this case.

$$W = \int\ P\ w_m(\theta)\ dS$$

$$= \int_0^{2\pi} \int_0^{2\pi} P\ \Sigma\ w_m\ \phi_m\ r\ (R + r\ \sin\theta)\ d\theta\ d\phi$$

Substitute $R = r\ a$

$$= 2\ \pi\ P\ r^2\ \Sigma\ w_m \int_0^{2\pi} \phi_m\ (a + \sin\theta)\ d\theta$$

And the partials are:

$$\frac{\delta W}{\delta w_k} = 2 \pi P r^2 \int_0^{2\pi} \phi_k (a + \sin\theta) d\theta$$

$$\frac{\delta W}{\delta u_k} = 0$$

The series expressions can now be rearranged:

$$\frac{\delta \Pi}{\delta q_i} = \frac{\delta V}{\delta q_i} - \frac{\delta W}{\delta q_i} = 0$$

$$\frac{\delta V}{\delta q_i} = \frac{\delta W}{\delta q_i} \quad ; \quad q_i = u_k, w_k$$

This can be represented in matrix format as follows:

Eq 11 $$\frac{\delta V}{\delta u_k} = [D] \{u\} + [E] \{w\} = 0 = \frac{\delta W}{\delta u_k}$$

Where [D] represents the coefficients of u_m in $\frac{\delta V}{\delta u_k}$

(see page 9).

[E] represents the coefficients of w_k in $\frac{\delta V}{\delta u_k}$

(see page 9)

$$\{u\}^T = u_m = u_1, u_2, u_3, \ldots u_{10}$$

$$\{w\}^T = w_m = w_1, w_2, w_3, \ldots w_{10}$$

Similarly in matrix format:

Eq 12 $$\frac{\delta V}{\delta w_k} = [F] \{u\} + [G] \{w\} = \{P\} = \frac{\delta W}{\delta w_k}$$

Where [F] represents the coefficients of u_m in $\frac{\delta V}{\delta w_k}$

(see page 10).

[G] represents the coefficients of w_k in $\frac{\delta V}{\delta w_k}$

(see page 10)

$$\{P\}^T = \frac{\delta W}{\delta w_k} = \frac{\delta W}{\delta w_1} + \frac{\delta W}{\delta w_2} + \ . \ . \ + \frac{\delta W}{\delta w_{10}}$$

Solving equation 11 for $\{u\}$ gives:

$$\{u\} = - [D]^{-1} [E] \{w\}$$

Use this expression in equation 12:

$$\{[F] [D]^{-1} [E] + [G]\} \{w\} = \{P\}$$

Solve for $\{w\}$:

$$\{w\} = \{[F] [D]^{-1} [E] + [G]\}^{-1} \{P\}$$

If the solution for w_m and u_m are correct, then these can be used to reconstruct $w(\theta)$ and $u(\theta)$ and from these, using the constitutive relationships in chapter four, stresses in the θ and ϕ directions can be calculated. Roark [22] gives expected values which can be used for comparison.

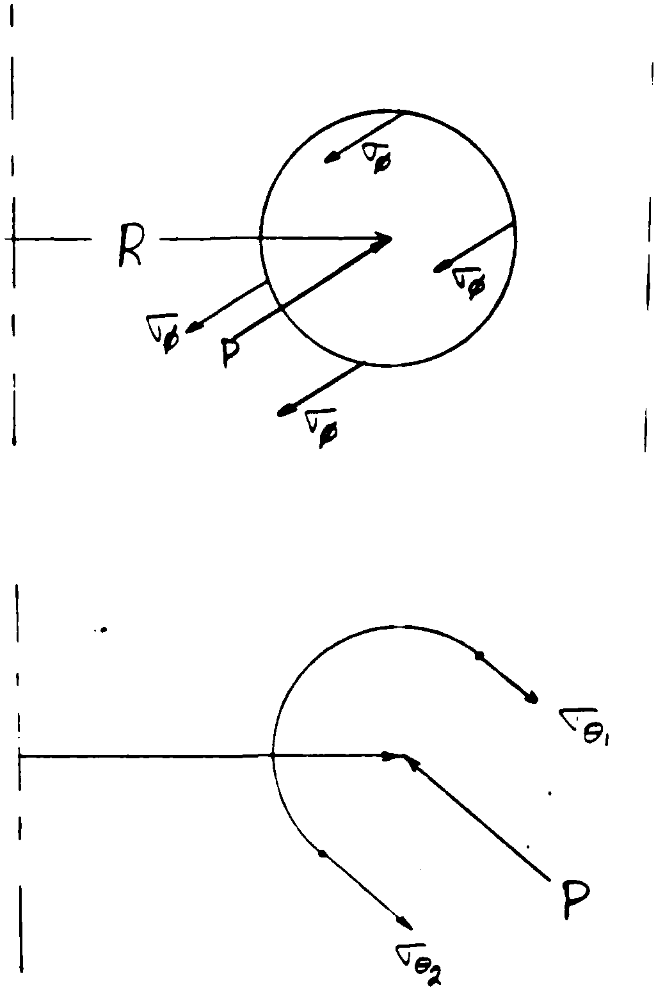
Some justification for Roark's results should be provided since this is how the solution will be checked. He provides no specific reference for these answers in his book, but working backwards one can see that a simple thin walled approximation analysis was performed, similar to that of a cylinder. Roark provides:

$$\sigma_\phi = \frac{Pr}{2t} ; \qquad \sigma_\Theta = \frac{Pr}{t} \left(\frac{a + \frac{1}{2}\sin\theta}{a + \sin\theta} \right)$$

(See Figure 15)

Figure 15

Thin Walled Analysis



σ_ϕ is a simple force balance in which the pressure times the internal cross sectional area equals the stress times the material area.

$$PA = \sigma_\phi \, A_m$$

$$P\,(\pi\, r^2) = \sigma_\phi\,(2\,\pi\, r)\; t$$

$$\sigma_\phi = \frac{Pr}{2t}$$

A similar approach is used on σ_Θ. The pressure acts on the area A, which in this case is (2 r)(2 π R). σ_Θ acts at Θ and $\Theta + 180^\circ$. The thickness is constant at t but the circumference at Θ and $\Theta + 180^\circ$ are different.

$$C(\Theta) = 2\,\pi\, r^* = 2\,\pi\, r(a + \sin\Theta)$$

The metal area at Θ is therefore $A(\Theta) = C(\Theta)\; t$. The force balance is:

$$P\,A = \sigma(\Theta_1)\; C(\Theta_1)\; t + \sigma(\Theta_2)\; C(\Theta_2)\; t$$

with $\Theta_1 = \Theta$ and $\Theta_2 = \Theta + 180^\circ$. Substitute Roark's $\sigma(\Theta)$ into the above equation gives:

$$P\,4\,\pi\, r\, R = \frac{Pr}{t}\left(\frac{a + \frac{1}{2}\sin\Theta_1}{a + \sin\Theta_1}\right) 2\,\pi\, r(a + \sin\Theta_1)\; t$$

$$+ \frac{Pr}{t}\left(\frac{a + \frac{1}{2}\sin\Theta_2}{a + \sin\Theta_2}\right) 2\,\pi\, r(a + \sin\Theta_2)t$$

Cancelling some terms shows the equality:

$$2a = 2a \quad \text{Q.E.D.}$$

This must have been how Roark arrived at these relationships.

This is how the solution will be checked.

THE SOLUTION

The method outlined above requires the integration of several quantities which are not readily found in a table of integrals. The integration was done using Simpsons Rule [23]. It was found, using known integrals, that 50 steps with in the interval 0 to 2π gave good accuracy (5 to 6 significant digits accuracy). A program was written to accomplish the integration and the subsequent matrix manipulations to solve for {u} and {w}.

To check the output from the program, a spreadsheet was set up to handle:

$$w(\theta) = \Sigma\, w_m\, \phi_m \qquad \frac{\delta w(\theta)}{\delta\theta} = \Sigma\, w_m\, \phi_m'$$

$$u(\theta) = \Sigma\, u_m\, \phi_m' \qquad \frac{\delta u(\theta)}{\delta\theta} = \Sigma\, u_m\, \phi_m''$$

The spread sheet also calculated ε_θ, ε_ϕ, σ_θ, and σ_ϕ. As a check, σ_ϕ was averaged and used to calculate a pressure to compare with the input pressure.

THE RESULTS

The results are disheartening. In the sample output provided, only one of the stress values used as a comparison came close to the expected value. σ_ϕ should have been constant and have a value of 5000 psi. It was not constant but it did manage to equlibrate most of the applied pressure (99.9%). This is pretty good for an approximation. The σ_θ is another story.

σ_θ is too low and does not follow the solution in Roark:

$$\sigma_\theta = \frac{P\ r}{t} \left(\frac{a + \frac{1}{2}\sin\theta}{a + \sin\theta} \right)$$

The following are the default values in the program:

GEOMETRY

Radius of rotation : 8 inches

Radius of the circle : 2 inches

Thickness of the shell : .02 inches

MATERIAL

Poisson's ratio : .3 Modulous of elasticity

: 3E+07

SOLUTION

Number of steps in the Simpsons

integral : 50 Number of terms to be

used in

the series approximation : 10

Pressure :-100 psi

The program output for w_m and u_m are:

m	w_m	u_m
1	-1.213117E-03	-1.137496E-03
2	6.043412E-09	1.491326E-09
3	-1.034239E-04	-1.151281E-05
4	3.250964E-09	2.026468E-10
5	-3.933767E-05	-1.579169E-06
6	3.328626E-09	9.227051E-11
7	-1.16143E-05	-2.382749E-07
8	3.831875E-09	5.9864E-11
9	-2.034983E-06	-2.535156E-08
10	2.81034E-09	2.817199E-11

Notice that the even numbered terms, which are to be set to zero as indicated earlier, are several orderes of magnitude below most of the odd values.

The reconstructed output from the program is as follows:

The spreadsheet out put to check the solution is:

$P =$ -100.00 psi   -99.90 < Sln Press (psi)
$\nu =$ 0.30
$\frac{E}{1-\nu^2} =$ 3.3E+07 $E =$ 3E+07
$R =$ 8.00 in $a =$ 4.00
$r =$ 2.00 in
$t =$ 0.02 in

θ (°)	w (in)	$\frac{\delta w}{\delta \theta}$ ($\frac{in}{rad}$)	u (in)	$\frac{\delta u}{\delta \theta}$ ($\frac{in}{rad}$)	σ_θ (psi)	σ_ϕ (psi)
0	0.0E+00	-1.8E-03	-1.2E-03	0.0E+00	-1461	-4870
15	-4.4E-04	-1.4E-03	-1.1E-03	4.2E-04	-1710	-4738
30	-7.2E-04	-8.1E-04	-9.8E-04	6.8E-04	-1946	-4607
45	-9.0E-04	-5.7E-04	-7.8E-04	8.4E-04	-2123	-4405
60	-1.0E-03	-4.2E-04	-5.4E-04	9.6E-04	-2258	-4248
75	-1.1E-03	-2.2E-04	-2.8E-04	1.0E-03	-2345	-4159
90	-1.1E-03	-2.3E-08	-2.8E-08	1.1E-03	-2375	-4131
105	-1.1E-03	2.2E-04	2.8E-04	1.0E-03	-2345	-4159
120	-1.0E-03	4.2E-04	5.4E-04	9.6E-04	-2258	-4248
135	-9.0E-04	5.7E-04	7.8E-04	8.4E-04	-2123	-4405
150	-7.2E-04	8.1E-04	9.8E-04	6.8E-04	-1946	-4607
165	-4.4E-04	1.4E-03	1.1E-03	4.2E-04	-1710	-4738
180	-9.6E-08	1.8E-03	1.2E-03	9.2E-08	-1461	-4870
195	4.4E-04	1.4E-03	1.1E-03	-4.2E-04	-1269	-5190
210	7.2E-04	8.1E-04	9.8E-04	-6.8E-04	-1085	-5498
225	9.0E-04	5.7E-04	7.8E-04	-8.4E-04	-894	-5654
240	1.0E-03	4.2E-04	5.4E-04	-9.6E-04	-747	-5769
255	1.1E-03	2.2E-04	2.8E-04	-1.0E-03	-659	-5854
270	1.1E-03	6.8E-08	8.4E-08	-1.1E-03	-630	-5886
285	1.1E-03	-2.2E-04	-2.8E-04	-1.0E-03	-659	-5854
300	1.0E-03	-4.2E-04	-5.4E-04	-9.6E-04	-747	-5769
315	9.0E-04	-5.7E-04	-7.8E-04	-8.4E-04	-894	-5654
330	7.2E-04	-8.1E-04	-9.8E-04	-6.8E-04	-1085	-5498
345	4.4E-04	-1.4E-03	-1.1E-03	-4.2E-04	-1269	-5190
360	1.9E-07	-1.8E-03	-1.2E-03	-1.8E-07	-1461	-4870

Note: Due to the precision of the machine being used, the forced compatability conditions appear to be non-zero. The forced conditions at $\theta = 90^\circ$ and 270° are significantly less than the rest of the solution and therefore satisfactory.

CHAPTER 6

CONCLUSIONS

The apparent simplicity of the toroidal shell belies the complexity of the mathematics to analyse this shell structure. The analysis of simpler shells (plates, cylinders, and spheres) has the advantage of the geometric curvature remaining either constant or at lease maintaining its sign positive or negative. The geometric curvature of the toroid changes from positive to negative as θ goes from 0 to 2π.

The impact in the change of sign in the radius of curvature about the Z axis has been seen in the results of this work and in applying other's solutions. This impact is emperical and could be the subject for further investigation. The first blatant statement was found in Tsui [24] where he states, "...the relevent differential equations to assume singular solutions, and consequently, problems involving the crowns such as a complete toroid cannot be solved." The culpret was writing the differential equation in terms of (Tsui's nomenclature):

$$r_2 = b\ (1 + \frac{\lambda}{\sin\theta})$$

(This papers nomenclature):

$$\rho = r\ (1 + (\frac{R}{r})/\sin\theta)$$

To avoid the $\frac{1}{\sin\theta}$ term, Tsui shifts variables - both dependant and independant - and provides influence coefficient matrices. (These are outputs from numerical analysis.)

As mentioned earlier, Tsui addresses the significant differences between the partial toroidal shape and the complete toroid. Others may have addressed partial toroidal shapes tangentially but Tsui once again is very blunt in his statements. Again remember Tsui is intending his work as a practical handbook. His intended audiance wants practical answers to real problems. Others, Flugge [25] specifically, attempt to give solutions but he adds, "...this solution cannot be realized...because it again leads to an incompatability of deformations...".

CURVATURE

In attempting to point the finger at why the solution is so difficult, the answer comes back to curvature. This work has seperated curvature into geometric (unloaded) and loaded curvature. This writer feels that the geometric curvature is the culpret. Notice the expression for the radius of curvature in the ϕ direction given in chapter one:

$$\rho = \frac{R}{\sin\theta} + r$$

For the range $0^\circ < \theta < 180^\circ$; $\sin\theta > 0$ and $\rho > 0$. Conversely, for $180^\circ < \theta < 360^\circ$; $\sin\theta < 0$ and $\rho < 0$. At $\theta = 0^\circ$ and 180° $\sin\theta = 0$ and $\rho = \pm \infty$. Two things were observed.

A. In previous solutions obtained in this study- ie. worse than the one presented - the stress outputs for the two regions differed markedly between $0^\circ < \theta < 180^\circ$ and $180^\circ < \theta < 360^\circ$. This could have been due to a <u>lot</u> of reasons, but this was also coincident with ...

B. Flugge's solution for the u displacement [26] was being investigated. This tended to blow up from $\theta = 180^\circ$ to $360^\circ/0^\circ$ due to a term $\ln(\tan\frac{\theta}{2})$. Other problem terms existed ($\cot\theta$) which agrivated the solution at the points $\theta = 0^\circ$ and 180°. In just "playing" with the solution, the points of curvature change kept comming back as critical points requiring further attention.

Something happened at $\theta = 0^{\circ}$ and 180°. In a cylinder these two points are no trouble - but the radii of curvature are ∞ and r - and are constant! In a sphere these two points are no problem - but the radius of curvature is r in all directions - and constant! In the toroid the radii of curvature at both points are $\rho = \pm \infty$ and r. The key difference is that ρ is changing from $\rho > 0$ to $\rho < 0$ and at $\theta = 0^{\circ}$ and 180°, κ_{ϕ} (or $\frac{1}{\rho}$) just happens to be passing through zero.

It was mentioned earlier in Chapter 4 that it required two reference points to define the toroid. Some individuals may take exception to that statement, but here is where it comes into play. Another way to look at ρ, the radius of curvature in the ϕ direction, is that ρ is the length of the arm joining the point in question (say point P) and the axis of symmetry (the Z axis in our case) and coincident with the local unit normal vector at point P. The angle θ can be measured either at the axis of rotation of the small circle forming the shell, or at the intersection of the line ρ and the axis of symmetry for the entire toroid. Now when $\theta = 50^{\circ}$

for instance, the geometry looks like that shown in Figure 16. As θ decreases toward 0, the end of ρ must travel farther and farther down the axis of symmetry. All this time ρ is getting larger and x_ϕ is getting smaller. When θ gets to zero plus (0^+) the situation is one of approximations. ρ is the hypotenuse of the right triangle with the right-angle at the origin. $\sin\theta = \frac{R}{\rho}$ from simple trigonometry, but as ρ goes to ∞, $\sin\theta$ goes to zero and hence θ must go to zero also.

Continuing θ around to 0^-, the closest point on the axis of symmetry in line with the normal vector is now at $+\infty$. This now gives ρ a direction opposite to that of the local normal vector, hence a minus sign, and as θ goes further negative, ρ is now smaller in magnitude. (Similar arguments can be made for the bottom crown where $\theta = 180^\circ$) The sole reason for the $\pm\infty$ trip in ρ, and hence the sign change in x_ϕ, is the offset R. Somehow this must be related to the difficulty in solving the complete toroid.

GEOMETRY

The other subtility of the toroid is the way the mathematical representation of the surface impacted on the attempt to integrate for energy over the toroid surface. This work specifically avoided some problems encountered by other investigators, significantly the expression for the curvature in the ϕ direction. The early development in this work specifically avoided the term $(\frac{1}{\sin\theta})$, selecting instead an expression containing the term $(a + \sin\theta)$. Force balance and constitutive equations were developed using the latter expression. When the constitutive equations were plugged into the energy expression (V), no singularities existed. This was good news.

Unfortunately, the $(a + \sin\theta)$ term also appeared in the expression for the differential surface area (dS). The bad news was that many of the elements in V contained combinations of trigonometric functions not readily found in the CRC Math Tables [27]. This is what forced the numerical integration of the elements in V.

This heavy dependence of the trigonometric functions in describing the problem has an astounding impact on the selection of an assumed functional for the energy method solution. Any combination of $(A\ \theta \sin\theta)$ or $(\theta^{A} \sin\theta)$ goes to zero when integrating from 0 to 2π. Remember that the dS term contains a $\sin\theta$ term so all integrals contain at least one $\sin\theta$ term.

It should be pointed out that the solution to a symmetric problem can be obtained by solving only one of the unique parts of the problem. This was not done in this case because it was hoped that the program being developed would be more general in nature, actually leading to orthotropic analysis of a stiffened toroid.

The only redeeming quality of the integrals also comes from geometry. Geometry terms containing $\cos\theta$, $\cos^2\theta$, $\sin\theta$, $\sin^2\theta$, and $(a + \sin\theta)$ and thier products appear in numerators and denominators through out the integrals. Mister Simpson and the digital computer were allowed to seperate the wheat from the chaff.

An attempt was made to develope a non-trigonometric functional which would satisfy the conditions of connectivity stated earlier and to meet connectivity at $\theta = 0^{\circ}/360^{\circ}$. The conditions to be met were:

$$f(0) = f(360)$$

$$f'(0) = f'(360)$$

$$f(90) = 0$$

$$f'(90) = 0$$

$$f(270) = 0$$

$$f'(270) = 0$$

This resulted in eight conditions. A functional of the following form was generated:

$$f(m\theta) = A + B(m\theta) + C(m\theta)^2 + D(m\theta)^3 + E(m\theta)^4 + F(m\theta)^5 + G(m\theta)^6 + H(m\theta)^7$$

A was arbitrarily set to 1 (to force a non zero solution at $\theta = 0^{\circ}$). Unfortunately, by the time f(mq) was processed, the functional looked the same for all values of m. The coefficients all changed as m changed, but the shape of the functional remained unchanged from one value of m to the next because of the eight conditions imposed upon the functional. This lost the variety of the solution that one expects to obtain using a varying m. See Figure 17.

It is obvious that more complex functionals are needed. This work included an initial evaluation of several functions for this task. A considerable amount of time was put into investigating the hypergeometric functions, particularly the Legendre functions. In addition to serving as a functional for u and w, it was also hoped that this function would satisfy the form of the differential equation for the toroid and hence yield an exact solution. (It was partially the anticipation of using such a complex function that drove the layout of the program (see Appendix) to have a seperate generating section for the functional.) The Legendre function, or any other hypergeometric function, was not used due to compatability conditions being harder to achieve. The next step in this progression would be the assumption of a combined trigonometric function for u and w, (ie. $A_m \sin(m\theta) + B_m \cos(m\theta)$).

Figure 16

Radius of Curvature

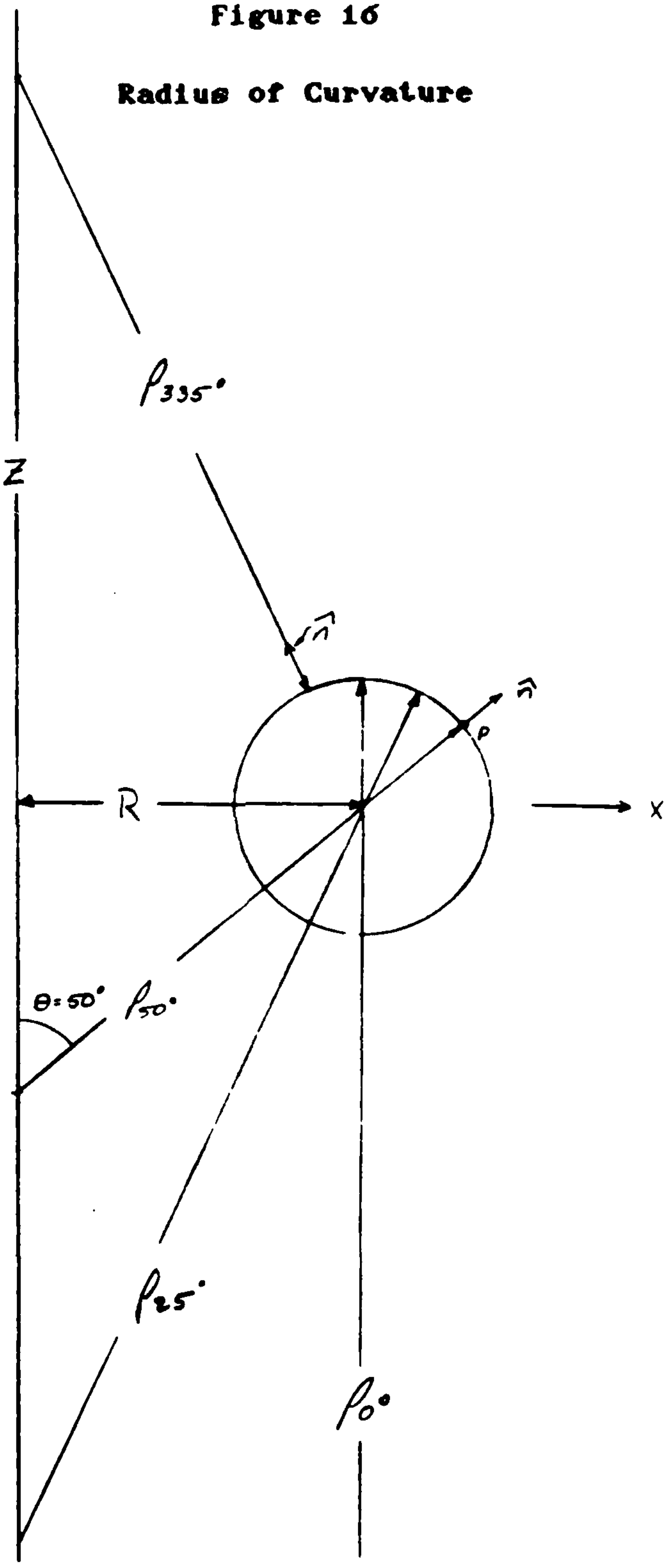
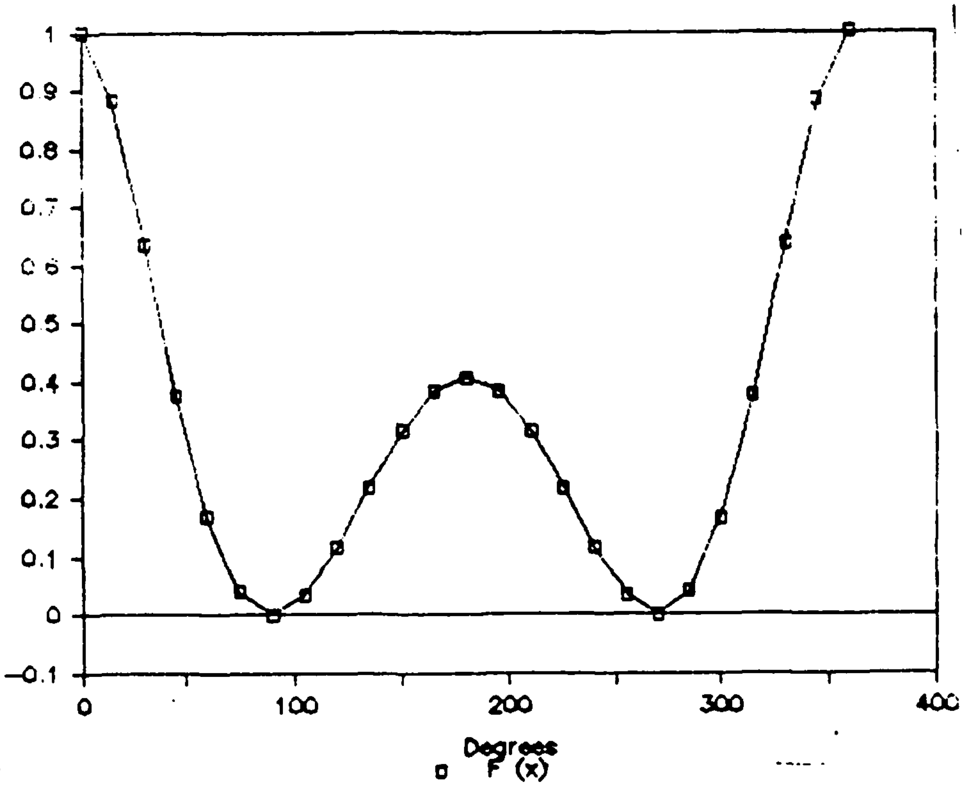


Figure 17

Generated Functional



CLOSING

In Chapter 1 it was stated that the geometric curvature and the impact of shell geometry on the assumed functional were the two problem areas in this thesis. Of the two, the former is the more significant in this writers opinion. The analysis behind Figure 15 is crude, but it is an attempt to represent in graphics and mathematics what the shell is doing. Being unable to do this is what makes the solution so difficult.

In a dynamics class, one student asked the professor if another mathematical technique could be used to analyse a spring. The professor thought briefly, then answered, "You can do all the math you want but you have to think like a spring. The spring knows what it is doing." This writer feels a little short on thinking like a toroid, but also feels in good company.

APPENDIX

THE PROGRAM

This program is to set up the geometry, the energy expression, Simpson's multipliers, and perform the matrix manipulations to solve for the displacements {u} and {w}.

```
50 CLS:PRINT:PRINT:INPUT "WHAT OUTPUT FILE NAME ";A$
60 CLS
1000 NN= 50     *This is the number of steps to be used in the
                  Simpson's integral
1010 MM=10      *This is the number of terms in the series for
                  displacements u ans w.
1020 NU=.3      *Poisson's ratio
1030 R=2        *R from the text.
1040 A=4        *a from the text.
1050 H=.02      *t from the text.
1060 E=3E+07    *Young's modulus
1070 P=-100     *pressure in psi.
1080 CLS:PRINT:PRINT:PRINT
1090 PRINT "The following are the default values in the
             program:"
1100 PRINT:PRINT
1110 PRINT "GEOMETRY"
1120 PRINT "Radius of rotation                  :";A*R;" inches"
1130 PRINT "Radius of the circle                :";R;" inches"
1140 PRINT "Thickness of the shell              :";H;" inches"
1150 PRINT:PRINT
1160 PRINT "MATERIAL"
1170 PRINT "Poisson's ratio                     :";NU
1180 PRINT "Modulous of elasticity              :";E
1190 PRINT:PRINT
1200 PRINT "SOLUTION"
1210 PRINT "Number of steps in the Simpsons"
1220 PRINT "   integral                         :";NN
1230 PRINT "Number of terms to be used in"
1240 PRINT "   the series approximation         :";MM
1250 PRINT "Pressure                            :";P;" psi"
1260 PRINT:PRINT
1270 INPUT "Do you wish to change any of these (Y/N): ";S$
1280 IF S$="N" THEN 1520
1290 CLS
1300 PRINT:PRINT
1310 PRINT "GEOMETRY"
1320 INPUT "Radius of rotation (inches)     :";RR
1330 INPUT "Radius of the circle (inches)   :";R
1340 A=RR/R
1350 IF A > 1 THEN 1380
1360 CLS:PRINT"RADIUS OF ROTATION MUST BE GREATER THAN THE
               RADIUS"
1370 PRINT "OF THE CIRCLE.":GOTO 1300
1380 INPUT "Thickness of the shell (inches) :";H
1390 PRINT:PRINT
1400 PRINT "MATERIAL"
1410 INPUT "Poisson's ratio                     :";NU
1420 INPUT "Modulus of elasticity               :";E
1430 PRINT:PRINT1440 PRINT "SOLUTION"
1450 PRINT "Number of steps in the Simpsons"
```

```
1460 INPUT "   integral                          :";NN
1470 PRINT "Number of terms to be used in"
1480 INPUT "   the series approximation          :";MM
1490 INPUT "Pressure (psi)                       :";P
1500 '
1510 '

1520 'THIS SECTION IS TO SET UP THE PARAMETERS FOR THE
      INTEGRATION SECTION
1530 '
1540 DT=6.283185307#/NN
1550 I=NN+1
1560 DIM  P(I,3,MM)  *Defines the functions φ.
1570 DIM  J(I)       *Simpson's multiplier
```

Arrays MD, ME, MF, and MG correspond to the matrixes in the partial of V with respect to $\mathbf{u}_k$ and $\mathbf{w}_k$. MP is the matrix representing the energy due to the pressure term. MS is a scratch matrix. M*# is a series of matrixes used in the matrix manipulation subroutines. MW and MU are the answers.

```
1580 DIM MD(MM,MM):DIM ME(MM,MM):DIM MF(MM,MM)
1590 DIM MG(MM,MM):DIM MS(MM,MM)
1600 DIM MX#(MM,MM):DIM MY#(MM,MM):DIM MZ#(MM,MM)
               :DIM MI#(MM,MM)
1610 DIM MP(MM):DIM MW(MM):DIM MU(MM)
1620 CLS:PRINT:PRINT:PRINT "GENERATING FUNCTIONS"
1630 FOR I = 1 TO NN+1
1640  T=(I-1)*DT
1650  FOR M = 1 TO MM
1660   P(I,1,M)=SIN(M*T)           'This is the function
1670   P(I,2,M)=M*COS(M*T)         'This is the first
                                        derivative
1680   P(I,3,M)=-M^2*SIN(M*T)      'This is the second
                                        derivative
1690  NEXT M
1700  J(I)=4
1710  IF (I/2-INT(I/2))>0 THEN J(I)=2
1720 NEXT I
1730 J(1)=1:J(NN+1)=1
```

Integrate using Simpson's Rule

```
2000 'INTEGRATE FROM 0 TO 2*PI
2010 IC=3.14159*(E*H/(1-NU^2))*DT/3
2020 BC=H^2/(6*R^3)
2030 PC=-2*3.14159*P*R^2*DT/3
2040 FOR I = 1 TO NN+1
2050  T=(I-1)*DT
2060  S=SIN(T)
2070  C=COS(T)
2080  CLS:PRINT:PRINT:PRINT"INTEGRATING ON STEP ";I
2090  PRINT " ANGLE ";T*57.296;" DEGREES"
2100  AA=A+S2102  HH=H^2/(12*R^2)
2104  CC=C/AA
2106  CA=2*NU*CC
2108  CB=CC^2
2110  CD=2*NU*S/AA
2112  AX=2*S*C/AA^2
```

```
2130   FOR K = 1 TO MM
2140    MP(K)=MP(K)-J(I)*P(I,1,K)*AA
2150    SQ=1/AA^2
2160    TR=2*NU/AA
2170    BX=H^3*(1+2*NU*C/AA+(C/AA)^2)/(12*R^2)
2180    FOR M = 1 TO MM
2190    MD(K,M)=MD(K,M)+J(I)*((1+HH)*(2*P(I,3,K)*P(I,3,M)
                          +CA*P(I,3,K)*P(I,2,M)
                          +CB*2*P(I,2,K)*P(I,2,M)))*AA
2200    ME(K,M)=ME(K,M)+J(I)*(2*P(I,1,M)*P(I,3,K)
                          +CA*(P(I,3,K)*P(I,1,M)*S
                          +P(I,1,M)*P(I,2,K)*C)
                          +AX*P(I,2,K)*P(I,1,M)
                          -HH*(2*P(I,3,M)*P(I,3,K)
                               +CA*(P(I,3,K)*P(I,2,M)
                               +P(I,3,M)*P(I,2,K))
                               +2*CB*P(I,2,K)*P(I,2,M)))*AA
2210    MF(K,M)=MF(K,M)+J(I)*(2*P(I,3,M)*P(I,1,K)
                          +2*NU*(P(I,1,K)*P(I,3,M)*S
                          +P(I,2,M)*P(I,1,K)*C)/AA
                          +AX*P(I,2,M)*P(I,1,K)
                          -HH*(2*P(I,3,M)*P(I,3,K)
                               +CA*(P(I,3,M)*P(I,2,K)
                               +P(I,3,K)*P(I,2,M))
                               +2*CB*P(I,2,M)*P(I,2,K)))*AA
2220    MG(K,M)=MG(K,M)+J(I)*(2*P(I,1,M)*P(I,1,K)
                          +2*CD*P(I,1,M)*P(I,1,K)
                          +2*P(I,1,M)*P(I,1,K)*S^2/AA^2
                          +HH*(2*P(I,3,M)*P(I,3,K)
                               +CA*(P(I,3,M)*P(I,2,K)
                               +P(I,3,K)*P(I,2,M))
                               +2*CB*P(I,2,M)*P(I,2,K)))*AA
2230     NEXT M
2240    NEXT K
2250 NEXT I
2260 '
2270 '
2280 FOR K = 1 TO MM
2290  MP(K)=PC*MP(K)
2300  FOR M = 1 TO MM
2310   MD(K,M)=IC*MD(K,M)
2320   ME(K,M)=IC*ME(K,M)
2330   MF(K,M)=IC*MF(K,M)
2340   MG(K,M)=IC*MG(K,M)
2350  NEXT M
2360 NEXT K


3000 'SOLVE THE MATRICES FOR wm AND um
3010 CLS:PRINT:PRINT:PRINT "SOLVING"
3020 '
3030 FOR K = 1 TO MM:FOR M = 1 TO MM:MX#(K,M)=MD(K,M):NEXT
M:NEXT K
3040 GOSUB 10000 '            ****** Invert MD  *****
3050 FOR K = 1 TO MM:FOR M = 1 TO MM
3060  MY#(K,M)=ME(K,M):MX#(K,M)=-MI#(K,M)
3070 NEXT M:NEXT K
3080 GOSUB 11000 '            *****  (-MD^-1)*ME *****
3090 FOR K = 1 TO MM:FOR M = 1 TO MM
3100  MY#(K,M)=MZ#(K,M):MX#(K,M)=MF(K,M)
3110  MS(K,M)=MZ#(K,M) '      *****  Saving this for Um
solution  *****
3120 NEXT M:NEXT K
```

```
3130 GOSUB 11000 '              *****  MF*(-MD^-1)*ME *****
3140 FOR K = 1 TO MM:FOR M = 1 TO MM
3150  MY#(K,M)=MG(K,M):MX#(K,M)=MZ#(K,M)
3160 NEXT M:NEXT K
3170 GOSUB 12000 '              *****  MG - MF*(-MD^-1)*ME  *****
3180 FOR K = 1 TO MM:FOR M = 1 TO MM:MX#(K,M)=MZ#(K,M)
        :NEXT M:NEXT K
3190 '
3200 'ROW REDUCTION
3210 'Input is MX#(K,M) and MP(K)
3220 'Output is the altered MX#(K,M) and MP(K)
3230 FOR S = 1 TO MM-1
3240  FOR K = S+1 TO MM
3250   Q=-MX#(K,S)/MX#(S,S)
3260   MP(K)=Q*MP(S)+MP(K)
3270   FOR M = 1 TO MM
3280    MX#(K,M)=Q*MX#(S,M)+MX#(K,M)
3290   NEXT M
3300  NEXT K
3310 NEXT S
3320 '

3330 'SOLVE FOR MW(M)
3340 'Continuing on using MX#(K,M) and MP(K) from before
3350 FOR S = MM TO 1 STEP -1
3360  MW(S)=MP(S)
3370  FOR M = S+1 TO MM
3380   MW(S)=MW(S)-MW(M)*MX#(S,M)
3390  NEXT M
3400  MW(S)=MW(S)/MX#(S,S)
3410 NEXT S
3420 '

3430 ' SOLVE FOR MU(K)
3440 FOR K = 1 TO MM
3450  FOR M = 1 TO MM
3460   MU(K)=MU(K)+MS(K,M)*MW(M)
3470  NEXT M
3480 NEXT K
3490 ' PRINT OUT RESULTS
3500 PRINT "#"," Wm "," Um "
3510 FOR M = 1 TO MM
3520  PRINT M,MW(M),MU(M):PRINT
3530 NEXT M
3540 OPEN "O",#1,A$    *This section puts the output onto
                         disk.
3550 PRINT #1,"NN = ";NN;"      MM = ";MM
3555 PRINT #1,"RADIUS =";A*R;"        radius =";R
3560 PRINT #1," "
3570 PRINT #1,"#"," Wm "," Um "
3580 FOR M = 1 TO MM
3590  PRINT #1, M,MW(M),MU(M):PRINT
3600 NEXT M
3610 CLOSE
3620 END
```

This section contains routines used in the main program.

```
10000 'INVERSION OF A MATRIX
10010 'Input is MX#(K,M)
10020 'Output is MI#(K,M)
10030 FOR K = 1 TO MM
10040  MI#(K,K)=1
10050 NEXT K
10060 FOR S = 1 TO MM-1
10070  FOR K = S+1 TO MM
10080   Q=-MX#(K,S)/MX#(S,S)
10090   FOR M = 1 TO MM
10100    MX#(K,M)=Q*MX#(S,M)+MX#(K,M)
10110    MI#(K,M)=Q*MI#(S,M)+MI#(K,M)
10120   NEXT M
10130  NEXT K
10140 NEXT S
10150 FOR S = MM TO 1 STEP -1
10160  FOR K = S-1 TO 1 STEP -1
10170   Q=-MX#(K,S)/MX#(S,S)
10180   FOR M=MM TO 1 STEP -1
10190    MX#(K,M)=Q*MX#(S,M)+MX#(K,M)
10200    MI#(K,M)=Q*MI#(S,M)+MI#(K,M)
10210   NEXT M
10220  NEXT K
10230 NEXT S
10240 FOR K = 1 TO MM
10250  FOR M = 1 TO MM
10260   MI#(K,M)=MI#(K,M)/MX#(K,K)
10270  NEXT M
10280 NEXT K
10290 RETURN

11000 'MULTIPLY TWO MATRICES
11010 'Multiply MX#(K,M) times MY#(K,M) in that order!
11020 'The output is MZ#(K,M)
11030 FOR K = 1 TO MM
11040  FOR M = 1 TO MM11050  MZ#(K,M)=0
11060   FOR S = 1 TO MM
11070    MZ#(K,M)=MZ#(K,M)+MX#(K,S)*MY#(S,M)
11075    IF ABS(MZ#(K,M))<.0000005 THEN MZ#(K,M)=0
11080   NEXT S
11090  NEXT M
11100 NEXT K
11110 RETURN

12000 'ADD TWO MATRICIES
12010 'Add MX#(K,M) to MY#(K,M)
12020 'Output is MZ#(K,M)
12030 FOR K = 1 TO MM
12040  FOR M = 1 TO MM
12050   MZ#(K,M)=MX#(K,M)+MY#(K,M)
12060  NEXT M
12070 NEXT K
12080 RETURN
```

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